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HKR–type invariants of 4–thickenings of 2–dimensional CW complexes

Ivelina Bobtcheva
Maria Grazia Messia

Abstract The HKR (Hennings–Kauffman–Radford) framework is used to construct invariants of 4–thickenings of 2–dimensional CW complexes under 2–deformations (1– and 2– handle slides and creations and cancellations of 1–2 handle pairs). The input of the invariant is a finite dimensional unimodular ribbon Hopf algebra A and an element in a quotient of its center, which determines a trace function on A. We study the subset $\mathcal{T}^4$ of trace elements which define invariants of 4–thickenings under 2–deformations. In $\mathcal{T}^4$ two subsets are identified : $\mathcal{T}^3 \subset \mathcal{T}^4$, which produces invariants of 4–thickenings normalizable to invariants of the boundary, and $\mathcal{T}^2 \subset \mathcal{T}^4$, which produces invariants of 4–thickenings depending only on the 2–dimensional spine and the second Whitney number of the 4–thickening. The case of the quantum $sl(2)$ is studied in details. We conjecture that $sl(2)$ leads to four HKR–type invariants and describe the corresponding trace elements. Moreover, the fusion algebra of the semisimple quotient of the category of representations of the quantum $sl(2)$ is identified as a subalgebra of a quotient of its center.

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1 Introduction

1.1 The (generalized) Andrews–Curtis conjecture [1] asserts that any simple homotopy equivalence of 2–complexes can be obtained by deformation through 2–complexes (expansions and collapses of disks of dimension at most two and changing the attaching maps of the 2–cells by homotopy), to which we refer here as a 2–deformation. This conjecture is expected to be false and different proposals for counterexamples have been made, but there seem to be a lack of tools for actually detecting them as such. An extensive reference for all the problems connected with the Andrews–Curtis conjecture is [6].

To any 2–dimensional CW complex P, there corresponds a presentation of its fundamental group, which can be obtained by selecting a vertex as a base point b and a spanning tree T in the one-skeleton P_1 on the complex. Then any 1–cell x_i which is not in T, with a choice of orientation determines an element in $\pi_1(P_1, b)$ and the attaching map of any 2–cell defines a word R_j in the x_i's which represents a trivial element in $\pi_1(P, b)$. The presentation of $\pi_1(P, b)$ obtained in this way, $\hat{P} = \langle x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m \rangle$, depends on the choices made, but this dependence can be explicitly described. In [6] (theorem 2.4), it is shown that the correspondence $P \to \hat{P}$ induces a bijection between the 2–deformation types of connected 2–dimensional CW complexes and the equivalence classes of finite presentations under the following moves:

(i) The places of R_1 and R_s are interchanged;

(ii) R_1 is replaced with gR_1g^{-1}, where g is any element in the group, or the reverse of such a move;

(iii) R_1 is replaced with R_1^{-1};

(iv) R_1 is replaced with R_1R_2;

(v) Adding of an additional generator y and an additional relator yR, where R is any word in the x_i's, or the reverse of such a move;

We will refer to these six operations as AC–moves and, hopefully without causing confusion, changing a presentation with a sequence of AC–moves will be called again a 2–deformation of this presentation. The inverse $\hat{P} \to P$ of the bijection above is obtained by taking one-point union of n circles and attaching on them m 2–cells as described by the relations.

If two complexes X and Y are simple homotopy equivalent, then for some k there exists a 2–deformation from the one-point union of X with k copies of S^2 to the one-point union of Y with k copies of S^2. In particular, if an invariant of 2–complexes under a 2–deformation is multiplicative under one point union, in order to have some hope of detecting a counterexample of the AC–conjecture, its value on S^2 should not be a unit. Since, using the correspondence above, we will talk instead about invariants of presentations under the AC–moves, a multiplicative invariant would be considered potentially interesting for the AC–conjecture if its value for $\langle \emptyset \mid 1 \rangle$ is not a unit.

Such invariants were introduced by Quinn in [16] and studied in [2]. The input for their construction is a finite semisimple *symmetric* monoidal category, which is taken to be one of the Lie families described by Gelfand and Kazhdan in [4], obtained as subquotients of mod p representations of simple Lie algebras. Unfortunately, extensive numerical study of Quinn's invariants (described in

[24]) indicated that, in all numerically generated examples, the invariants come from a representation of the free group on the generators into a subgroup of $GL_N(Z/p)$ for some N, and in this representation every word has order p. Consequently, it was shown in [14] that any invariant possessing this property can't detect counterexamples to the AC–conjecture.

In the present work we use the framework of Hennings–Kauffman–Radford (HKR) [5, 10] to construct invariants of 4–dimensional thickenings of 2–complexes under 2–deformations, i.e. 1– and 2– handle slides and creations or cancellations of 1–2 handle pairs. The construction is based on a presentation of a 4–thickening by a framed link in S^3 (where the 1–handles are described by dotted components) and the input data is a finite dimensional unimodular ribbon Hopf algebra and an element in a quotient of its center which determines a trace function on the algebra.

As Hennings points out, any trace function on the algebra, and therefore any trace element, leads to an invariant of links, but very few trace elements lead to invariants of links which are also invariants under the band-connected sum of two link components (corresponding to 2–handle slides). Let $\mathcal{T}_s$ be the subset of these special trace elements. Then $\mathcal{T}_s$ contains always at least two elements which are 1 (one) and the algebra integral Λ. Moreover, when the Hopf algebra is the finite dimensional quantum enveloping algebra at root of unity of some simple Lie group, $\mathcal{T}_s$ contains at least one more element z_{RT} which corresponds to the Reshetikhin–Turaev invariant. This fact was first observed by Hennings, and then, for the quantum $sl(2)$, z_{RT} was made explicit by Kerler in [8] (for completeness, in the appendix we present the derivation of z_{RT}). In an analogous way (though it won't be done here), one can see that Quinn's invariant can be derived in the HKR–framework from a triangular Hopf algebra over Z/p and a central element $z_Q \neq 1$ in it. Moreover, the invariant corresponding to 1 is less interesting than Quinn's invariant. These facts imply that it is important not to restrict to the trace function corresponding to 1 (as it is done in [5, 10]), and rise the question what is the possible relationship between the different invariants derived from the same Hopf algebra. To answer this question, one needs to study the structure of $\mathcal{T}_s$ and we hope that the present work sets the framework for such study.

In particular, we determine a subset $\mathcal{T}$ of trace elements which lead to invariants of 4–thickenings under 2–handle slides, i.e. $\mathcal{T} \subset \mathcal{T}_s$. By adding the requirement for invariance under 1–2 handle cancellations, inside $\mathcal{T}$, it is described a subset $\mathcal{T}^4$ of trace elements which lead to invariants of 4–thickenings under 2–deformations. Then we study when the invariant of a 4–thickening reduces to un invariant of the boundary and when it reduces to an invariant of the spine. This leads to the description in $\mathcal{T}^4$ of two subsets:

- $\mathcal{T}^3 \subset \mathcal{T}^4$, whose elements lead to invariants which factor as a product of a 3-manifold invariant and a multiplicative invariant which depends on the signature and the Euler characteristic of the 4–thickening, and

- $\mathcal{T}^2 \subset \mathcal{T}^4$ whose elements lead to invariants which only depend on the 2–dimensional spine and the second Whitney number of the 4–thickening.

The definition of $\mathcal{T}$ allows to make some interesting conclusions about its structure. In particular, $\mathcal{T}$ carries two different monoidal structures and it is invariant under the action of the $\mathcal{S}$ operator defined in (2.55) of [8]. But for now we don't know a practical way of calculating the elements of $\mathcal{T}$ for a given algebra and this is quite unsatisfactory. The only partial remedy we can offer, is that by weakening "slightly" the defining conditions on $\mathcal{T}$, one can define a subset $\mathcal{T}_Z$, containing $\mathcal{T}$, such that its elements are relatively easy to determine since the calculations are entirely restricted to the center of the algebra. We make this calculation explicit for the case of the quantum $sl(2)$ and show that in this case $\mathcal{T}_Z$ consists of 4 elements, three of which are exactly 1, Λ and z_{RT}. This fact leads to the conjecture that $\mathcal{T}_Z = \mathcal{T}$ for the quantum $sl(2)$. Under this assumption we show that the invariant corresponding to the forth element in $\mathcal{T}_Z$ is the ratio of the Hennings and the Reshetikhin–Turaev invariants.

The paper is organized as follows. In section 2 we present the main definitions and results. Section 3 contains some notations and preliminaries on Hopf algebras. Section 4 is dedicated to the study of the structure of $\mathcal{T}$. Section 5 introduces the notion of K–links and K–tangles. Section 6 defines the invariant of 4–thickenings and shows that, when the trace element is in $\mathcal{T}^2$, the invariant depends only on the two dimensional spine of the 4–thickening and its second Whitney number. Section 7 studies the reducibility of the invariant to a 3–manifold invariant and section 8 illustrates the construction with two examples: the case of a group algebra and the case of the quantum $sl(2)$. At the end we list some open questions. In the appendix, always for the quantum $sl(2)$, we show that the Reshetikhin–Turaev invariant is a HKR–type invariant and calculate the corresponding trace element.

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2 Main Results

2.1 Let $(A, m, \Delta, S, \epsilon, e)$ be a finite dimensional unimodular ribbon Hopf algebra over a field k with an integral $\Lambda \in A$ and a right integral $\lambda \in A^*$ such that $\lambda(\Lambda) = 1$. We define a linear map $\star\colon A \otimes A \to A$ given by

$$a \star b = \sum_b \lambda(S(a)b_{(1)})b_{(2)}, \quad \text{where} \quad \Delta(b) = \sum_b b_{(1)} \otimes b_{(2)}.$$

Let $Z(A)$ be the center of A and let $K(A)$ be the null space of the pairing on $Z(A)$ induced by λ, i.e.

$$K(A) = \{a \in Z(A) \mid \text{for any} \ \ b \in Z(A), \ \lambda(ab) = 0\}.$$

Then $K(A)$ is an algebra ideal in $Z(A)$, and let $\hat{Z}(A) = Z(A)/K(A)$ be the quotient algebra. Given any $a \in Z(A)$, we will denote by $[a]$ its equivalence class in $\hat{Z}(A)$. Let also $\hat{Z}^S(A) = \{[a] \in \hat{Z}(A) \mid [S(a)] = [a]\}$ (this will be shown to be well defined in 4.4).

Lemma 2.2 *Let A be a finite-dimensional unimodular ribbon Hopf algebra over a field k as above. Then*

(a) $\star\colon\ Z(A) \otimes Z(A) \to Z(A)$ *defines an associative product on* $Z(A)$ *with an identity* Λ *and for any* $a, b \in Z(A)$, $S(a \star b) = S(b) \star S(a)$;

(b) $\star$ *defines an associative and commutative product on* $\hat{Z}(A)$.

2.3 Let $C^n \subseteq A^{\otimes n}$, $n > 1$, be the centralizer of the action of A on $A^{\otimes n}$ given by the comultiplication, i.e. $a \in C^n$ iff for any $b \in A$, $\Delta^{n-1}(b)a = a\Delta^{n-1}(b)$. Define also $C^1 = Z(A)$.

C^2 contains the commutative subalgebra C^2_Z generated by the elements of the form $(a \otimes b)\Delta c$, where $a, b, c \in Z(A)$. Let $\mu\colon C^2_Z \otimes C^2 \to k$, be given by

$$\mu(\sum_i a_i \otimes b_i, \sum_j c_j \otimes d_j) = \sum_{i,j} \lambda(a_i c_j)\lambda(b_i d_j),$$

and let $\bar{\mu}\colon C^2_Z \otimes C^2_Z \to k$ be the corresponding restriction of μ. Define

$$K^2_Z = \{x \in C^2_Z \mid \mu(x, y) = 0 \ \text{ for any } y \in C^2\} \ \text{ and}$$
$$\overline{K}^2_Z = \{x \in C^2_Z \mid \mu(x, y) = 0 \ \text{ for any } y \in C^2_Z\}.$$

Obviously K^2_Z and $\overline{K}^2_Z$ are ideals in C^2_Z and $K^2_Z \subset \overline{K}^2_Z$. This induces a surjective homomorphism

$$\pi_Z\colon\ C^2_Z/K^2_Z \to C^2_Z/\overline{K}^2_Z.$$

Define $\delta\colon Z(A)\otimes Z(A)\to C_Z^2$ as $\delta(w,z)=z\otimes w-(1\otimes w)\Delta(z)$.

Proposition 2.4 *δ factors through a well defined map $\hat{\delta}\colon \hat{Z}(A)\otimes\hat{Z}(A)\to C_Z^2/K_Z^2$.*

2.5 Let $\mathcal{T}\subset\mathcal{T}_Z\subset\hat{Z}^S(A)$ be $\mathcal{T}=\{[z]\in\hat{Z}^S(A)\mid\hat{\delta}([z],[z])=0\}$ and $\mathcal{T}_Z=\{[z]\in\hat{Z}^S(A)\mid\pi_Z\cdot\hat{\delta}([z],[z])=0\}$. Observe that $[z]\in\mathcal{T}_Z$ if and only if for any $a,b,c\in Z(A)$, $\lambda(zc(bz\star a))=\lambda(zc(b\star(za)))$. Hence

Proposition 2.6 *$[z]\in\mathcal{T}_Z$ if and only if for any $[a],[b]\in\hat{Z}(A)$, $[z(a\star zb)]=[z(az\star b)]$.*

2.7 Let $J\colon Z(A)\to Z(A)$, be defined as

$$J(z)=(\lambda\otimes 1)(z\otimes 1)R^{21}R=\sum_{i,j}\lambda(z\beta_i\alpha_j)\alpha_i\beta_j.$$

This operator is related to the image of one of the generators, $\mathcal{S}$, in the action of the torus group on $Z(A)$ (see [8], (2.55)) and it is essential in understanding when the invariant of the 4–thickening reduces to an invariant of the boundary. Let $Z_\star(A)$ denote the algebra which has $Z(A)$ as a vector space and the $\star$ product structure. Then

Proposition 2.8 (a) *$J\colon Z_\star(A)\to Z(A)$ is an algebra homomorphism, i.e. for any $a,b\in Z(A)$, $J(a\star b)=J(a)J(b)$.*

(b) *$J^2(a)=S(a)\star J(1)$;*

(c) *J factors through an algebra homomorphism map $\hat{J}\colon\hat{Z}_\star(A)\to\hat{Z}(A)$, and maps $\hat{Z}^S(A)$ into itself.*

Observe that, if $J(1)=\gamma\Lambda$, where $\gamma\in k$ is a unit, 2.8 (b) and the fact that on the center of a ribbon algebra S^2 acts as the identity, imply that J is bijective with an inverse $J^{-1}=\gamma^{-1}(S\circ J)$. Then from 2.8 (a) and 2.2 (a) one obtains

$$\begin{aligned}J(ab)&=J(J\circ J^{-1}(a)J\circ J^{-1}(b))=J^2(J^{-1}(b)\star J^{-1}(a))\\&=\gamma^{-1}S(S\circ J(b)\star S\circ J(a))=\gamma^{-1}J(a)\star J(b).\end{aligned}$$

Therefore we have proved the following:

Corollary 2.9 *If $J(1)=\gamma\Lambda$, where $\gamma\in k$ is a unit, then $\gamma^{-1}J\colon Z(A)\to Z_\star(A)$ is an algebra isomorphism. In particular, the algebra $Z_\star(A)$ is commutative.*

Definition 2.10 A quasitriangular unimodular ribbon Hopf algebra for which $J(1) = \gamma\Lambda$, where $\gamma \in k$ is a unit, will be called Λ*–factorizable.*[1]

Lemma 2.11 *$\mathcal{T}$ is a commutative monoid with respect to the usual and the $\star$–product on $\hat{Z}^S(A)$. Moreover $\hat{J}$ sends $\mathcal{T}$ into itself.*

We observe that proposition 2.8 implies that when the algebra is Λ–factorizable, $\hat{J}\colon\ \mathcal{T} \to \mathcal{T}$ is a bijection whose square is a multiple of the identity.

2.12 Let M be an orientable 4–dimensional manifold which possesses a decomposition as a handlebody with 0–, 1– and 2–handles. We remind that an n–handle is a product $D^n \times D^{4-n}$ and the choice of radial coordinates in D^{4-n} gives a description of the product as the mapping cylinder of a projection $D^n \times S^{3-n} \to D^n$. Then $D^n \times \{0\}$ is called the core, $S^{n-1} \times \{0\}$ is called the attaching sphere and $\{0\} \times S^{3-n}$ is called the belt sphere of the handle. When another handle is attached on top of this one the intersection of the attaching map with the handle lies in $D^n \times S^{3-n}$ and using the mapping cylinder coordinates the core of the upper handle can be extended in the lower handle. This extends the upper cores to a disk whose boundary lies on the lower cores. The union of these extended cores forms a 2–dimensional CW complex which will be called the *spine* of the handlebody. The mapping cylinder contractions also combine to give a standard deformation retraction of the handlebody to the spine.

A pair of $(n+1)$–handle and an n–handle is called a *cancelling pair* if the attaching sphere of the $(n+1)$–handle intersects the belt sphere of the n–handle in a single point.

Then a *4–thickening* M of a 2–dimensional CW complex P, denoted with (M, P), is an orientable 4–dimensional manifold together with a decomposition as a handlebody with 0–, 1– and 2–handles and an identification (as CW complexes) of the spine of the handlebody structure with P through an embedding $\iota_{M,P}\colon\ P \to M$. In particular, $\iota_{M,P}$ induces isomorphism on homology. We will restrict ourselves to 4–thickenings with a single 0–handle. A 2–deformation of such 4–thickenings is given by a sequence of the following handle moves:

(a) creation or cancellation of a cancelling 1–2 handle pair;

(b) changing the attaching maps of the 1– and 2– handles by isotopy.

[1] A quasitriangular Hopf algebra is called factorizable if $\bar{J}\colon\ A^* \to A$, given by $\bar{J}(f) = (f \otimes 1)(R^{21}R)$ is bijective.

Observe that these moves induce a 2–deformation on the spine.

The word 4–thickening is supposed to stress not only the fact that a spine has been fixed, but also that we have weakened the equivalence relations on the objects with respect to 4–manifolds.[2]

2.13 The monoid $\mathcal{T}$ will be shown to correspond to invariants under 2–handle slides. An invariance under 2–deformations requires in addition invariance under 1–2 handle cancellations, and the center elements which lead to such invariants form the following subset of $\mathcal{T}$:

$$\mathcal{T}^4 = \{[z] \in \mathcal{T} \mid \text{there exits } [w] \in \hat{Z}^S(A) \text{ and } [zw] = [\Lambda]\}.$$

Let also $\mathcal{T}^3 = \{[z] \in \mathcal{T}^4 \mid [zJ(z)] = X_z[\Lambda] \text{ for some unit } X_z \in k\}$ and

$$\mathcal{T}^2 = \{[z] \in \mathcal{T}^4 \mid [z] = [z_1 J(z_2)] \text{ and } \hat{\delta}([z_1],[z_2]) = 0 \\ \text{for some } [z_1],[z_2] \in \hat{Z}^S(A)\}.$$

Theorem 2.14 *Given any* $[z] \in \mathcal{T}^4$ *and* $[w] \in \hat{Z}^S(A)$ *such that* $[zw] = [\Lambda]$*, there exists a HKR–type invariant of 4–thickenings under 2–deformations, denoted with* $\mathcal{Z}_{[z]}(M)$*, such that*

$$\mathcal{Z}_{[z]}(S^2 \times D^2) = \lambda(z) \text{ and } \mathcal{Z}_{[z]}(S^1 \times D^3) = \epsilon(w).$$

Obviously for any finite dimensional unimodular ribbon Hopf algebra A, the elements $[1],[\Lambda] \in \mathcal{T}^4$. The choice $[z] = [\Lambda]$ brings to the trivial invariant which is 1 for any M. On another hand $[z] = [1]$ gives the Hennings invariant (in the 3–manifold case):

Corollary 2.15 *Any finite-dimensional unimodular ribbon Hopf algebra* A *over a field* k*, determines an invariant* $\mathcal{Z}_A$ *of 4–thickenings under 2–deformations, such that*

$$\mathcal{Z}_A(S^2 \times D^2) = \lambda(1), \text{ and } \mathcal{Z}_A(S^1 \times D^3) = \epsilon(\Lambda),$$

In particular, $\mathcal{Z}_A(S^2 \times D^2) \neq 0$ *if and only if* A *is cosemisimple (*A^* *is semisimple), and* $\mathcal{Z}_A(S^1 \times D^3) \neq 0$ *if and only if* A *is semisimple.*

Given a 4–manifold M, let $w_2(M) \in H^2(M; Z/2)$ denote the second Whitney class of M.

[2]While changing the attaching map of a 2–handle by isotopy is equivalent to the creation and cancellation of cancelling 2–3 handle pairs, isotoping the attaching map of a 3–handle is not a 2–deformation.

Lemma 2.16 *Let P be a 2–dimensional CW complex and (M_1, P), (M_2, P) be two 4–thickenings of P such that $\iota^*_{M_1,P}(w_2(M_1)) = \iota^*_{M_2,P}(w_2(M_2))$. If $[z] \in \mathcal{T}^2$ then $\mathcal{Z}_{[z]}(M_1) = \mathcal{Z}_{[z]}(M_2)$.*

Corollary 2.17 *Let A be a triangular Hopf algebra and let $[z] \in \mathcal{T}^4$. If (M_1, P_1) and (M_2, P_2) are two 4–thickenings such that P_1 and P_2 are related by a 2–deformation, then $\mathcal{Z}_{[z]}(M_1) = \mathcal{Z}_{[z]}(M_2)$.*

Hence, if A is a triangular Hopf algebra any $[z] \in \mathcal{T}^4$ defines an invariant of 2–complexes under 2–deformations, and this invariant is denoted by $\mathcal{Z}^2_{[z]}(P)$. Then it is natural to expect that for triangular algebras $\mathcal{T}^4 = \mathcal{T}^2$. Actually, in this case for any $z \in Z(A)$, $J(z) = \lambda(z)1$. In particular,

$$\mathcal{T}^2 = \{[z] \in \mathcal{T}^4 \mid \text{ there exists } [w] \in \hat{Z}^S(A) \text{ with } \hat{\delta}([z],[w]) = 0 \text{ and } \lambda(w) \neq 0\}.$$

And since for any $z \in Z(A)$, $\delta(z, 1) = 0$, it follows that if A is triangular and cosemisimple (i.e. $\lambda(1) \neq 0$) then $\mathcal{T}^2 = \mathcal{T}^4$. We don't know if this is true for any triangular algebra.

2.18 Let M be a 4–thickening represented with a Kirby diagram L (see section 5) and let σ_+, σ_- and σ_0 be the numbers of positive, negative and zero eigenvalues of the linking matrix of L.

Corollary 2.19 *If $[z] \in \mathcal{T}^3$ then $C_+ = \mathcal{Z}_{[z]}(CP^2)$ and $C_- = \mathcal{Z}_{[z]}(\overline{CP^2})$ are units in k. Moreover, if M is a 4–thickening with n 1–handles, then*

$$C_+^{n-\sigma_+} C_-^{n-\sigma_-} \mathcal{Z}_{[z]}(M)$$

only depends on the boundary ∂M of M and is denoted by $\mathcal{Z}^\partial_{[z]}(\partial M)$.

3 Basic facts about Hopf algebras

Here, we introduce some notations assuming that the reader is familiar with the axioms of a Hopf algebra. A possible reference about Hopf algebras is [22]. Let $(A, m, \Delta, S, \epsilon, e)$ be a Hopf algebra over a field k, where:

$m\colon A \otimes A \to A$ multiplication map
$\Delta\colon A \to A \otimes A$ comultiplication map
$S\colon A \to A^{opp}$ antipode
$\epsilon\colon A \to k$ counit
$e\colon k \to A$ unit

Note also that there are natural isomorphisms; $k \otimes A \to A$ and $A \otimes k \to A$ which we will often omit, identifying $A \otimes k$ and $k \otimes A$ with A.

3.1 The maps above need to satisfy a list of compatibility conditions, out of which we only mention the following:

(a) $\Delta(\Delta \otimes 1) = \Delta(1 \otimes \Delta)\colon A \to A \otimes A \otimes A$ (coassociativity),

(b) $\Delta m = (m \otimes m)(1 \otimes T \otimes 1)(\Delta \otimes \Delta)\colon A \otimes A \to A \otimes A, \quad \Delta(1) = 1 \otimes 1,$

(c) $m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = e\epsilon\colon A \to A,$

where 1 denotes both the identity element $e(1_k)$ in A and the identity map $A \to A$, and $T\colon A \otimes A \to A \otimes A$ is the transposition map $a \otimes b \to b \otimes a$. An easy consequence of the definition of the antipode is that

(d) $T \circ (S \otimes S)\Delta(a) = \Delta(S(a))$.

Let $\Delta^n = (\Delta \otimes 1^{\otimes(n-1)})(\Delta \otimes 1^{\otimes(n-2)}) \ldots \Delta\colon A \to A^{\otimes(n+1)}$. We use Sweedler's notation $\Delta^{(n-1)}(a) = \sum_a a_{(1)} \otimes a_{(2)} \otimes \ldots a_{(n-1)} \otimes a_{(n)}$. Then (d) implies that

(e) $\Delta^{n-1}(S(a)) = \sum_a S(a_{(n)}) \otimes S(a_{(n-1)}) \ldots \otimes S(a_{(1)})$.

3.2 An element $\lambda_L \in A^*$ is called a *left integral for* A^* if

$$(f \otimes \lambda_L)\Delta(a) = \lambda_L(a)f(1), \text{ for any } a \in A \text{ and } f \in A^*.$$

An element $\lambda_R \in A^*$ is called a *right integral for* A^* if

$$(\lambda_R \otimes f)\Delta(a) = \lambda_R(a)f(1), \text{ for any } a \in A \text{ and } f \in A^*.$$

When A is finite-dimensional, the Hopf algebra isomorphism $A \simeq A^{**}$ implies that one can define a left (right) integral for A as an element $\Lambda \in A$, such that $a.\Lambda = \epsilon(a)\Lambda$ $(\Lambda a = \epsilon(a)\Lambda)$ for any $a \in A$.

3.3 The following results ([22, 19, 18]) concern the existence of integrals when A is a finite-dimensional Hopf algebra over a field k.

(a) The subspaces $\int_L^*, \int_R^* \subset A^*$ of left (right) integrals for A^* and the subspaces $\int_L, \int_R \subset A$ of left (right) integrals for A are one dimensional;

(b) The antipode map is bijective;

(c) For any nonzero $\lambda \in \int_R^*$ there exists $\Lambda \in \int_L$ such that

$$\lambda(\Lambda) = \lambda(S(\Lambda)) = 1;$$

(d) Given any nonzero $\lambda \in \int_R^*$ the map $\Phi\colon A \to A^*$ given by $\Phi(a)(b) = \lambda(ab)$ is a bijection;

3.4 Note that, if A is a finite-dimensional Hopf algebra and $\Lambda \in \int_R$, then $S(\Lambda)$, $S^{-1}(\Lambda) \in \int_L$. Moreover, if $\lambda \in \int_R^*$, then $\lambda \circ S, \lambda \circ S^{-1} \in \int_L^*$. A is called *unimodular* if $\int_R = \int_L$ and if A is unimodular then for any $\lambda \in \int_R^*$, $\lambda(ab) = \lambda(S^2(b)a)$.

3.5 A *quasitriangular* Hopf algebra is a Hopf algebra A endowed with invertible element $R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A$ such that

(a) $T \circ \Delta(a) = R\Delta(a)R^{-1}$ for any $a \in A$;

(b) $(\Delta \otimes 1)R = R^{13}R^{23}$;

(c) $(1 \otimes \Delta)R = R^{13}R^{12}$,

where as usual $R^{(kl)} \in A^{\otimes n}$ indicates the image of R under the injective homomorphism of the group of invertible elements in $A \otimes A$ into the group of invertible elements of $A^{\otimes n}$ where the first factor is mapped into k-th position and the second into l-th position.

If (A, R) is a quasitriangular Hopf algebra, the following relations hold:

(d) $R^{(12)}R^{(13)}R^{(23)} = R^{(23)}R^{(13)}R^{(12)}$;

(e) $(S \otimes 1)R = (1 \otimes S^{-1})R = R^{-1}$, and $(S \otimes S)R = R$;

(f) $(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1$;

(g) Let $u = \sum_i S(\beta_i)\alpha_i$, then u is invertible and $S^2(a) = uau^{-1}$, moreover,

$$\Delta(u) = (u \otimes u)(R^{(21)}R)^{-1}.$$

3.6 A quasitriangular Hopf algebra is called *triangular* if $R^{-1} = R^{(21)} = \sum_i \beta_i \otimes \alpha_i$. In this case u is a group-like element, i.e. $\Delta(u) = u \otimes u$, which, in the terminology below, implies that any triangular Hopf algebra is ribbon with ribbon element u.

A Hopf algebra A is called *cocommutative* if it possesses triangular structure with $R = 1 \otimes 1$, i.e. if $T \circ \Delta = \Delta$.

3.7 A quasitriangular Hopf algebra A is called *ribbon* if it is endowed with a grouplike element $g \in A$ such that $S^2(a) = gag^{-1}$, called the special grouplike element of A (grouplike means that g is invertible and $\Delta g = g \otimes g$). It can be shown (see for example [20, 10]) that if A is ribbon,

$$\theta = gu^{-1} = u^{-1}g = \sum_i \alpha_i g^{-1} \beta_i = \sum_i \beta_i g \alpha_i$$

is a central element in A such that

(a) $S(\theta) = \theta$;

(b) θ is invertible with inverse $\theta^{-1} = \sum_i \alpha_i S(\beta_i) g = \sum_i S(\beta_i)\alpha_i g^{-1}$;

(c) $\Delta(\theta) = (\theta \otimes \theta)(R^{(21)}R)^{-1}$.

θ is called the *ribbon element of* A.

A *trace function* on A is an element $f \in A^*$ such that, for any $a, b \in A$, $f(ab) = f(ba)$ and $f(a) = f(S(a))$. In a finite dimensional unimodular ribbon Hopf algebra there is a bijection between the set of S–invariant central elements in A and the space of trace functions on A given by $z \to \lambda_{zg}$, where $\lambda_{zg}(a) = \lambda(zga)$ ([5, 19]).

4 The center of a unimodular finite dimensional ribbon Hopf algebra

In the rest of the paper, unless specified otherwise, $(A, m, \Delta, S, \epsilon, e)$ will be a unimodular Hopf algebra over a field k with an integral $\Lambda \in A$, a right integral $\lambda \in A^*$ and a left integral $\lambda^S = \lambda \circ S$, such that $\lambda(\Lambda) = \lambda^S(\Lambda) = 1$. Moreover, we assume that A carries a ribbon structure given by an R–matrix $R = \sum_i \alpha_i \otimes \beta_i$ and a group like element g such that $gag^{-1} = S^2(a)$ for any $a \in A$. Many of the statements here can be easily illustrated using the diagrammatic language in the later chapters, but because of their purely algebraic significance we decided that it is better to prove them in a self-contained way.

4.1 Generating elements in C^n

(i) The first way to generate elements in C^n, is by "going up", i.e. by applying some of the following embeddings on C^{n-1}:

$$\eta_r^{(n-1)} \colon C^{n-1} \to C^n,\ a \to 1 \otimes a;$$
$$\eta_l^{(n-1)} \colon C^{n-1} \to C^n,\ a \to a \otimes 1;$$
$$1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(n-i-1)} \colon C^{n-1} \to C^n,\ i = 1, \dots, n-1.$$

The subalgebra of C^n generated inductively in this way, starting with $C^1 = Z(A)$, will be denoted with C^n_Z.

(ii) The second way to generate new elements in C^n is through the action of the braid group on C^n as follows. If B_n is the braid group on n strings and $q_n \colon B_n \to \mathbf{S_n}$ is its homomorphism onto the symmetric group $\mathbf{S_n}$, let $I_n = q_n^{-1}(id)$. The relation 3.5 (d) implies that one can define a representation of $\phi\colon B_n \to End(A^{\otimes n})$ by defining the image of the generator which interchanges the i-th and the $(i+1)$-st strings to be

$$\phi(\sigma_{i,i+1}) = 1^{\otimes(i-1)} \otimes (T \circ R) \otimes 1^{\otimes(n-i-1)},$$

where we first multiply the corresponding element in $A^{\otimes n}$ on the left with $1^{\otimes(i-1)} \otimes R \otimes 1^{\otimes(n-i-1)}$ and then apply the permutation. Suppose that $s, s' \in B_n$ are such that $q_n(s) = q_n(s')^{-1}$. Then the condition 3.5 (a) implies that given any $a \in C^n$, $\phi(s)\circ a \circ \phi(s')$ act on $A^{\otimes n}$ by multiplication with an element in C^n. We write this fact as $\phi(s) \circ C^n \circ \phi(s') \subset C^n$. For example, if $\sum_i c_i \otimes d_i \in C^2$ then $\sum_{i,k,j} \beta_k d_i \alpha_j \otimes \alpha_k c_i \beta_j \in C^2$. The statement implies in particular that $\phi(I_n) \subset C^n$. [3]

(iii) The third way to obtain elements in C^n is by "going down", i.e. by applying the integrals to the elements in C^{n+k}:

Proposition 4.2 *Let $\mathcal{L}_{n+1}\colon A^{\otimes(n+1)} \to A^{\otimes n}$ be the map which applies λ on the leftmost factor in $A^{\otimes(n+1)}$ and let $\mathcal{R}_{n+1}\colon A^{\otimes(n+1)} \to A^{\otimes n}$ be the map which applies λ^S on the rightmost factor in $A^{\otimes(n+1)}$. Then $\mathcal{L}_{n+1}$ and $\mathcal{R}_{n+1}$ map C^{n+1} into C^n.*

Proof The proof is standard, but for completeness we will show the first part of the statement and the second is analogous. Given any $\sum_i a_i \otimes b_i \in C^{n+1}$, where $a_i \in A,\ b_i \in A^{\otimes n}$ and any $c \in A$,

$$\begin{aligned}\textstyle\sum_i \lambda(a_i) b_i \Delta^{n-1}(c) &= \sum_{i,c} \lambda(a_i c_{(2)} S^{-1}(c_{(1)})) b_i \Delta^{n-1}(c_{(3)}) \\ &= \sum_{i,c} \lambda(c_{(2)} a_i S^{-1}(c_{(1)})) \Delta^{n-1}(c_{(3)}) b_i \\ &= \sum_{i,c} \lambda(S(c_{(1)}) c_{(2)} a_i) \Delta^{n-1}(c_{(3)}) b_i = \sum_i \lambda(a_i) \Delta^{n-1}(c) b_i,\end{aligned}$$

hence $\sum_i \lambda(a_i) b_i \in C^n$. □

By induction the last proposition implies that for any $0 \le k < l \le n$

$$\lambda^{\otimes k} \otimes 1^{\otimes(l-k)} \otimes (\lambda^S)^{\otimes(n-l)}\colon C^n \to C^{l-k}.$$

Proposition 4.3 *For any $a \in C^n$ and any partition $n' + n'' = n$, $\lambda^{\otimes n}(a) = (\lambda^{\otimes n'} \otimes (\lambda^S)^{\otimes n''})(a)$. In particular, $\lambda(a) = \lambda^S(a)$ for any $a \in Z(A)$.*

[3]Using 3.7 (c) one can show that actually $\phi(I_n) \subset C^n_Z$.

Proof First we will prove the statement for $n=1$. Suppose that $a \in Z(A)$. Then, using 3.7, it follows that

$$\begin{aligned} \lambda(a) &= \sum_{i,j} \lambda(a\beta_j\beta_i S^{-1}(\alpha_i)\alpha_j) = \sum_{i,j} \lambda(a\beta_i S^{-1}(\alpha_i)\alpha_j S^{-2}(\beta_j)) \\ &= \sum_{i,j} \lambda(gagg^{-1}\beta_i S^{-1}(\alpha_i)\alpha_j g^{-1}\beta_j) = \lambda(gagS^{-1}(\theta^{-1})\theta) \\ &= \lambda(gag) = \lambda(S(a)). \end{aligned}$$

Let now $a \in C^n$, $n > 1$. If $n'' = 0$, the statement is trivial. Suppose then that it is true for some $n'' \geq 0$. Then proposition 4.2 implies that $(\lambda^{\otimes(n'-1)} \otimes 1 \otimes (\lambda^S)^{\otimes n''})(a) \in Z(A)$ and hence the statement with $n''+1$ follows from the one for n'' and from the statement with $n=1$. □

This proposition implies that if $a \in K(A)$ then for any $b \in Z(A)$, $\lambda(bS(a)) = \lambda(S^2(a)S(b)) = \lambda(S(b)a) = 0$, i.e. $S(a) \in K(A)$. Hence

Corollary 4.4 *The algebra $\hat{Z}^S(A)$ in 2.1 is well defined.*

4.5 Proof of lemma 2.2 First observe that proposition 4.2 implies that, for any $a, b \in Z(A)$, $a \star b \in Z(A)$. To see the associativity of the product, let $a, b, c \in Z(A)$. Then

$$\begin{aligned} (a \star b) \star c &= \sum_{c,b} \lambda(S(a)b_{(1)})\lambda(S(b_{(2)})c_{(1)})c_{(2)} \\ &= \sum_{c,b} \lambda(S(a)b_{(1)}S(b_{(2)})c_{(2)})\lambda(S(b_{(3)})c_{(1)})c_{(3)} \\ &= \sum_{c} \lambda(S(a)c_{(2)})\lambda(S(b)c_{(1)})c_{(3)} = a \star (b \star c). \end{aligned}$$

To complete the proof of 2.2(a) we observe that for any $a, b \in Z(A)$,

$$\begin{aligned} S(a \star b) &= \sum_{b} \lambda(S(a)b_{(1)})S(b_{(2)}) = \sum_{S(a),b} \lambda(S(a)_{(1)}b_{(1)})S(a)_{(2)}b_{(2)}S(b_{(3)}) \\ &= \sum_{S(a)} \lambda(S(a)_{(1)}b)S(a)_{(2)} = S(b) \star S(a), \end{aligned}$$

which together with the definition of Λ implies that $a = \Lambda \star a = a \star \Lambda$. This completes the proof of proposition 2.2 (a). Now, for any $a, b, c \in Z(A)$, define

$$\sigma(a, b, c) = \lambda(S(a)(b \star c)).$$

Then 2.2 (b) follows from 4.3 and the following proposition.

Proposition 4.6 *If (a',b',c') is any permutation of (a,b,c) or $(S(a),S(b),S(c))$, then*

$$\sigma(a',b',c')=\sigma(a,b,c).$$

Proof First we observe that 2.2 (a) and 4.3 imply that

$$\sigma(a,b,c)=\sigma(S(a),S(c),S(b)).$$

Hence, it is enough to show that $\sigma(a',b',c')=\sigma(a,b,c)$ where (a',b',c') is one of the two permutations (b,a,c) or (c,b,a). Now we claim that $\lambda(a(S(b)\star c))=\lambda(b(S(a)\star c))$ which would imply that $\sigma(a,b,c)=\sigma(b,a,c)$. To see this, let $\sum_i \gamma_i\otimes\delta_i=R^{-1}$. Then

$$\begin{aligned}\lambda(b(S(a)\star c)) &= \sum_c \lambda(ac_{(1)})\lambda(bc_{(2)})=\sum_{c,i,j}\lambda(a\gamma_i\alpha_j c_{(1)})\lambda(b\delta_i\beta_j c_{(2)})\\ &=\sum_{c,i,j}\lambda(a\gamma_i c_{(2)}\alpha_j)\lambda(b\delta_i c_{(1)}\beta_j)=\sum_{c,i,j}\lambda(ac_{(2)}\alpha_j S^{-2}(\gamma_i))\lambda(bc_{(1)}\beta_j S^{-2}(\delta_i))\\ &=\sum_c \lambda(bc_{(1)})\lambda(ac_{(2)})=\lambda(a(S(b)\star c)).\end{aligned}$$

We complete the proof of the proposition as follows:

$$\sigma(a,b,c)=\sigma(S(a),S(c),S(b))=\sigma(S(c),S(a),S(b))=\sigma(c,b,a).$$

□

4.7 Proof of proposition 2.4 It is enough to show that for any $z\in K(A)$ and any $\sum_i a_i\otimes b_i\in C^2$, the following three statements hold:

(a) $\sum_i \lambda(a_i)\lambda(zb_i)=0$,

(b) $\sum_{z,i}\lambda(z_{(1)}a_i)\lambda(z_{(2)}b_i)=0$,

(c) $\sum_i \lambda(za_i)\lambda(b_i)=0$.

(a) and (c) follow directly from 4.3 and 4.2. On another hand to show (b), using 4.3 and the fact that $z=z\star\Lambda$, we obtain

$$\begin{aligned}\textstyle\sum_{z,i}\lambda(z_{(1)}a_i)\lambda(z_{(2)}b_i) &= \sum_{\Lambda,i}\lambda(S(z)\Lambda_{(1)})\lambda(\Lambda_{(2)}a_i)\lambda(\Lambda_{(3)}b_i)\\ &=\sum_{\Lambda}\lambda(z(\sum_i \lambda^S(\Lambda_{(2)}a_i)\lambda^S(\Lambda_{(3)}b_i)S(\Lambda_{(1)})))=0.\end{aligned}$$

4.8 Proof of proposition 2.8 Observe that J actually maps the center into itself since from 3.7 it follows that

$$J(z) = \sum_{\theta} \lambda(z\theta\theta^{-1}_{(1)})\theta\theta^{-1}_{(2)} = \theta((S(z)\theta) \star (\theta^{-1})).$$

This expression also implies (together with 2.2 (b)) that J factors through a map $\hat{Z}(A) \to \hat{Z}(A)$. Now we can complete the proof of 2.8 (c). Let $[a] \in \hat{Z}^S(A)$. Then using the fact that $S(\theta) = \theta$ and 2.2 (a) and (b) we obtain that

$$[S(J(a)) - J(a)] = [\theta(((S(a) - a)\theta) \star (\theta^{-1}))] = 0.$$

Hence $[J(a)] \in \hat{Z}^S(A)$.

It is left to show 2.8 (a) and (b).

(a) Let $J' = J \circ S$. Then (a) is equivalent to show that $J'\colon Z_\star(A) \to Z(A)$ is an algebra isomorphism, i.e. for any $a, b \in Z(A)$, $J'(a \star b) = J'(a)J'(b)$. From 3.5 (b) and (c) it follows that

$$\begin{aligned} J'(a)J'(b) &= \sum_{i,j} \lambda(S(a)\beta_i\alpha_j)\alpha_i J'(b)\beta_j \\ &= \sum_{i,j,i',j'} \lambda(S(a)\beta_i\alpha_j)\lambda(S(b)\beta_{i'}\alpha_{j'})\alpha_i\alpha_{i'}\beta_{j'}\beta_j \\ &= \sum_{i,j,\alpha_j,\beta_i} \lambda(S(a)\beta_{i,(2)}\alpha_{j,(2)})\lambda(S(b)\beta_{i,(1)}\alpha_{j,(1)})\alpha_i\beta_j \\ &= \sum_{i,j,b} \lambda(S(a)b_{(1)})\lambda(S(b_{(2)})\beta_i\alpha_j)\alpha_i\beta_j = J'(a \star b). \end{aligned}$$

(b) From 3.5 (b) and (c) it follows that

$$\begin{aligned} S(a) \star J(1) &= \sum_{i,j,k,l} \lambda(\beta_i\beta_k\alpha_l\alpha_j)\lambda(a\alpha_i\beta_j)\alpha_k\beta_l \\ &= \lambda(\alpha_j\beta_i\beta_k\alpha_l)\lambda(a\beta_j\alpha_i)\alpha_k\beta_l = J^2(a). \end{aligned}$$

4.9 Proof of lemma 2.11 It is obvious that $\mathcal{T}$ is a monoid under the usual multiplication in $\hat{Z}^S(A)$.

First we will show that if $\hat{\delta}([z],[z]) = 0$ and $\hat{\delta}([w],[w]) = 0$, then $\hat{\delta}([z \star w],[z \star w]) = 0$. This is equivalent to say that for any $\sum_k a_k \otimes b_k \in C^2$,

$$\lambda(x(z \star w)) = \sum_k \lambda((z \star w)a_k)\lambda((z \star w)b_k),$$

where $x = \sum_{k,w} \lambda(S(z)w_{(1)})\lambda(w_{(2)}a_k)w_{(3)}b_k \in Z(A)$. From 4.6 it follows that the left hand side is actually equal to $\sigma(S(x), z, w) = \sigma(S(w), S(z), x)$. Hence

$$\begin{aligned} l.h.s &= \sum_{k,w} \lambda(S(z)w_{(1)})\lambda(w_{(2)}a_k)\lambda(zw_{(3)}b_{k,(1)})\lambda(ww_{(4)}b_{k,(2)}) \\ &= \sum_{k,w,i,j} \lambda(S(z)w_{(1)})\lambda(\alpha_i w_{(2)}a_k S(\alpha_j))\lambda(z\beta_i w_{(3)}b_{k,(1)}\beta_j)\lambda(ww_{(4)}b_{k,(2)}) \\ &= \sum_{k,w,i,j} \lambda(S(z)w_{(1)})\lambda(w_{(3)}\alpha_i a_k S(\alpha_j))\lambda(zw_{(2)}\beta_i b_{k,(1)}\beta_j)\lambda(ww_{(4)}b_{k,(2)}). \end{aligned}$$

The criteria established in 4.1 and 4.2 imply that

$$\sum_{k,w,i,j} \lambda(S(z)w_{(1)})\lambda(zw_{(2)}\beta_i b_{k,(1)}\beta_j)w_{(3)}\alpha_i a_k S(\alpha_j) \otimes ww_{(4)}b_{k,(2)} \in C^2.$$

Hence from proposition 4.3 it follows that

$$l.h.s = \sum_{k,w,i,j} \lambda(S(z)w_{(1)})\lambda(zw_{(2)}\beta_i b_{k,(1)}\beta_j)\lambda^S(w_{(3)}\alpha_i a_k S(\alpha_j))\lambda^S(ww_{(4)}b_{k,(2)}).$$

Now the S–invariance of $[z]$ together with the fact that $\hat{\delta}([z],[z]) = 0$ imply that

$$\begin{aligned} l.h.s &= \sum_{k,w,i,j,z} \lambda(z_{(1)}w_{(1)})\lambda(zz_{(2)}w_{(2)}\beta_i b_{k,(1)}\beta_j)\lambda^S(w_{(3)}\alpha_i a_k S(\alpha_j))\lambda^S(ww_{(4)}b_{k,(2)}) \\ &= \sum_{k,w,i,j,z} \lambda(zw_{(1)})\lambda(z\beta_i b_{k,(1)}\beta_j)\lambda^S(w_{(2)}\alpha_i a_k S(\alpha_j))\lambda^S(ww_{(3)}b_{k,(2)}) \\ &= \sum_{k,w,i,j} \lambda(z\beta_i b_{k,(1)}\beta_j)\lambda(zS((\alpha_i a_k S(\alpha_j))_{(1)}))\lambda^S(w_{(1)}(\alpha_i a_k S(\alpha_j))_{(2)})\lambda^S(ww_{(2)}b_{k,(2)}) \\ &= \sum_{k,w,i,j} \lambda(z\beta_i b_{k,(1)}\beta_j)\lambda^S(z(\alpha_i a_k S(\alpha_j))_{(1)})\lambda^S(w_{(1)}(\alpha_i a_k S(\alpha_j))_{(2)})\lambda^S(ww_{(2)}b_{k,(2)}) \\ &= \sum_{k,w,i,j} \lambda(z\beta_i b_{k,(1)}\beta_j)\lambda(z(\alpha_i a_k S(\alpha_j))_{(1)})\lambda(w_{(1)}(\alpha_i a_k S(\alpha_j))_{(2)})\lambda(ww_{(2)}b_{k,(2)}), \end{aligned}$$

where the last two equalities follow from 4.1, 4.2 and 4.3. At this point we use the fact that $[w] \in \mathcal{T}$ and obtain:

$$\begin{aligned}
l.h.s. &= \sum_{k,i,j} \lambda(z\beta_i b_{k,(1)}\beta_j)\lambda(z(\alpha_i a_k S(\alpha_j))_{(1)})\lambda(w(\alpha_i a_k S(\alpha_j))_{(2)})\lambda(wb_{k,(2)}) \\
&= \sum_k \lambda(za_{k,(1)})\lambda(wa_{k,(2)})\lambda(zb_{k,(1)})\lambda(wb_{k,(2)}) \\
&= \sum_k \lambda^S(za_{k,(1)})\lambda^S(wa_{k,(2)})\lambda^S(zb_{k,(1)})\lambda^S(wb_{k,(2)}) \\
&= \sum_k \lambda^S((z\star w)a_k)\lambda^S((z\star w)b_k) = \sum_k \lambda((z\star w)a_k)\lambda((z\star w)b_k).
\end{aligned}$$

Together with the fact that $\Lambda \in \mathcal{T}$, this implies that $\mathcal{T}$ is a monoid with respect to the $\star$–product structure as well.

It is left to show that $\mathcal{T}$ is invariant under the action of $\hat{J}$, i.e for any $[z] \in \mathcal{T}$, and $\sum_k a_k \otimes b_k \in C^2$,

$$\sum_k \lambda(J(z)a_k)\lambda(J(z)b_k) = \sum_{k,J(z)} \lambda(J(z)_{(1)}a_k)\lambda(J(z)J(z)_{(2)}b_k).$$

For the left hand side one has

$$\begin{aligned}
l.h.s. &= \sum_{i,j,n,m,k} \lambda(z\beta_j\alpha_i)\lambda(z\beta_m\alpha_n)\lambda(\alpha_j\beta_i a_k)\lambda(\alpha_m\beta_n b_k) \\
&= \sum_{i,j,n,m,k} \lambda(\beta_j z(\beta_m z\alpha_n)_{(2)}\alpha_i)\lambda((\beta_m z\alpha_n)_{(1)})\lambda(\alpha_j\beta_i a_k)\lambda(\alpha_m\beta_n b_k) \\
&= \sum_{i,j,n,m,k} \lambda(zz_{(2)}\alpha_{n,(2)}\alpha_i\beta_j\beta_{m,(2)})\lambda((z\alpha_n\beta_m)_{(1)})\lambda(\beta_i a_k\alpha_j)\lambda(\beta_n b_k\alpha_m).
\end{aligned}$$

Hence, from the fact that $[z] \in \mathcal{T}$ and 3.5 (b) and (c), it follows that

$$\begin{aligned}
l.h.s. &= \sum_{i,j,n,m,k} \lambda(z(\alpha_n\beta_m)_{(1)})\lambda(z\alpha_{n,(2)}\alpha_i\beta_j\beta_{m,(2)})\lambda(\beta_i a_k\alpha_j)\lambda(\beta_n b_k\alpha_m) \\
&= \sum_{i,j,n,m,k} \lambda(z(\beta_m\alpha_n)_{(1)})\lambda(z\beta_j\beta_{m,(2)}\alpha_{n,(2)}\alpha_i)\lambda(\alpha_j\beta_i a_k)\lambda(\alpha_m\beta_n b_k) \\
&= \sum_{i,j,n,m,k,n',m'} \lambda(z\beta_{m'}\alpha_n)\lambda(z\beta_j\beta_m\alpha_{n'}\alpha_i)\lambda(\alpha_j\beta_i a_k)\lambda(\alpha_m\alpha_{m'}\beta_n\beta_{n'}b_k) \\
&= \sum_{i,j,n',m,k} \lambda(z\beta_j\beta_m\alpha_{n'}\alpha_i)\lambda(\alpha_j\beta_i a_k)\lambda(\alpha_m J(z)\beta_{n'}b_k) \\
&= \sum_{k,J(z)} \lambda(J(z)_{(1)}a_k)\lambda(J(z)J(z)_{(2)}b_k).
\end{aligned}$$

5 K–links and K–tangles

Let M be an oriented 4–dimensional manifold together with a decomposition as a handlebody with a single 0–handle and a number of 1– and 2–handles. Then M can be represented by describing the attaching maps of the 1– and 2–handles in S^3 [12, 13]. The attaching map of a 1–handle is a pair of 3–balls in S^3 or equivalently it can be described as a unknot of framing 0 in S^3 (figure 1). In this last case the result of attaching the 1–handle is being thought as the manifold obtained by pushing into B^4 the disk bounded by the unknot and removing a neighborhood of it. We will use the second method putting a dot on the unknot to indicate that it describes a 1–handle. Then the attaching maps of the 2–handles are described by framed links in the 1–handlebody, where if a 2–handle goes over a 1–handle, the corresponding link component is drown to go through the dotted circle describing the 1–handle.

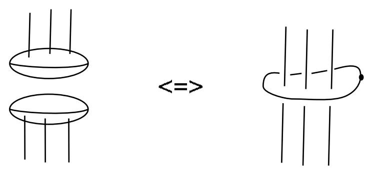

Figure 1: Representation of 1–handle with 2–handles which pass over it

5.1 Define a Kirby link (*K–link*) to be a framed link in S^3 where some of the unknotted components of framing 0, bounding disjoint Seifert surfaces, have been dotted. Then an oriented Kirby link (*OK–link*) is a K–link where an orientation of each link component has been fixed. A based oriented Kirby link (*BOK–link*) is an OK–link where one has fixed numbering and based points for the undotted components and a numbering and a set of disjoint Seifert surfaces for the dotted components.

Given a K–link (OK–link, BOK–link) L, we will denote with M_L the 4–dimensional handlebody described by L. If L is a BOK–link with n dotted and m undotted components, then it defines a unique presentation $\hat{P}_L = \langle x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m \rangle$ of $\pi_1(M_L)$, where $R_i = R_i(x_1, x_2, \ldots, x_n)$ is a (not freely reduced) word in the x_j's and shows in which order and with which sign the i-th undotted component intersects the Seifert surfaces of the dotted components starting from the base point. An example is shown in figure 2.

5.2 Two BOK–links are said to be 2–equivalent if and only if they can be deformed into each other through a sequence of the moves (a)–(f) below (cor-

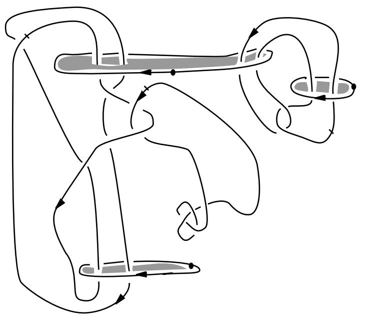

Figure 2: A BOK–link L with $P_L = \langle x, y, z \mid xy^{-1}xy, z^{-1}xx^{-1}z, 1\rangle$

responding to 1– and 2–handle moves of the underlying 4–manifold). Changing a BOK-link through such a sequence will be called a 2-deformation of this link:

(a) isotopy of framed links;

(b) any pair of one dotted component x and one undotted component y can be removed or added if the geometric intersection number of y and the Seifert surface S_x of x is ± 1, while S_x is disjoint from all other dotted and undotted components (1–2 handle cancellation or introduction);

(c) band-connected sum or difference of two undotted link components (sliding a 2–handle over another 2–handle);

(d) band-connected sum or difference of one undotted link component with one dotted link component ("sliding a 2–handle over 1–handle");

(e) band-connected sum or difference of two dotted link components (sliding an 1–handle over another 1–handle);

(f) change of numbering, base points, Seifert surfaces and orientation.

The moves are illustrated in figure 3.

Proposition 5.3 *If two BOK–links can be deformed into each other through the moves (a)–(f) above, then they can be deformed into each other via moves (a), (b), (c) and (f).*

The proof is sketched in figures 4 and 5.

Definition 5.4 Let L be a BOK–link and let $\sigma\colon \hat{P}_L \to \hat{P}'$ be a sequence of AC–moves. We say that σ can be lifted to L if there exists a 2-deformation $\tilde{\sigma}\colon L \to L'$ such that $\hat{P}_{L'} = \hat{P}'$.

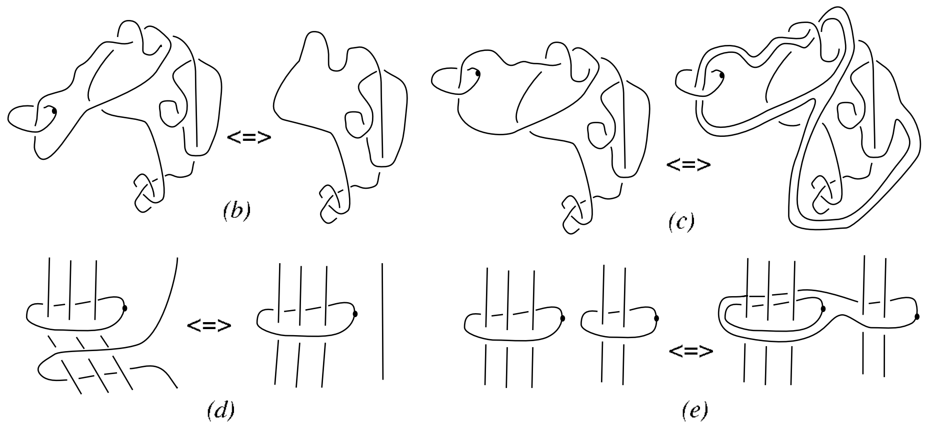

Figure 3: Illustration of the moves (b)–(e) of a 2–deformation of K–links

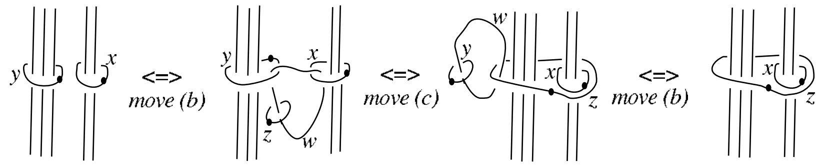

Figure 4: Move (e) is a consequence of moves (b) and (c).

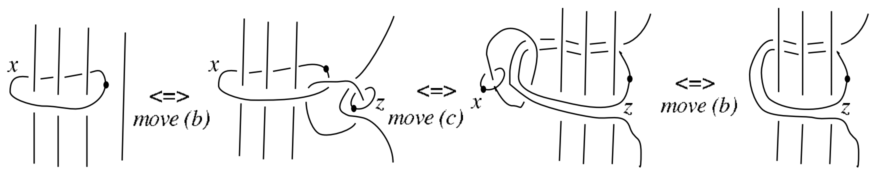

Figure 5: Move (d) is a consequence of (b) and (c).

Proposition 5.5 *Let L be a BOK–link. Then*

(a) *Any 2–deformation $L \to L'$ induces a 2–deformation (sequence of AC–moves) $\hat{P}_L \to \hat{P}_{L'}$;*

(b) *if $\sigma\colon \hat{P}_L \to \hat{P}'$ is a sequence of AC–moves then $\sigma = \xi \circ \sigma_0$, where σ_0 can be lifted to L and ξ is a sequence of cancellations of terms $x_i x_i^{-1}$ in the relations (considered as cyclic words in x_j's).*

Proof (a) is straightforward and for the case when the fundamental group of the 4–thickening is trivial, (b) is actually the statement of theorem 3.3 in [6]. In general one can prove (b) by induction on the length of the sequence of AC–moves σ. Suppose that σ consists of a single AC–move t. If t is not a cancellation of a term $x_i x_i^{-1}$ (i.e. the reverse direction (ii)$^{-1}$ in 1.1 (ii)), then it can be lifted to a single move $\tilde{t}\colon L \to L'$, of type (a)$\div$(f) in 5.2. Observe that this is not true if t is a cancellation of a term $x_i x_i^{-1}$ in a relation, since such term implies that the corresponding undotted component enters and then goes out of the i-th dotted component (without intersecting the Seifert surface of any other dotted component) but possibly linking with other undotted components or itself. Therefore, in general we can not pull it out of the i-th dotted component.

(b) will follow by induction, if we can show it for the case when $\sigma = t \circ w$, where w is a single AC–move of the type (ii)$^{-1}$ and t is any other single AC–move (since this would imply that the problematic moves can be shifted at the end of the sequence of AC–moves). Observe that if t is of the type (ii)$^{-1}$, the statement is trivial. If t is of the type 1.1 (i), (iii) or (v) or (v)$^{-1}$, it can be easily seen that σ_0 is a single AC–move of the the same type as t and hence we can define L' to be the BOK–link obtained by applying the move $\tilde{\sigma_0}$ on L.

Let t be of the type 1.1 (iv). Suppose that the first two relations of $\hat{P}_L$ are $R_1 = xR'_1x^{-1}$ and R_2 , where x, y, R'_1 are some words in the generators, and that w replaces R_1 with R'_1 and then t replaces R'_1 with R'_1R_2. Then define σ_0 to be the sequence of the following moves: conjugation of the second relation with x and then multiplication of the first relation with the second. These moves can be lifted to L and the resulting presentation has as first and second relations $xR'_1x^{-1}xR_2x^{-1}$ and xR_2x^{-1}. Obviously R'_1R_2, R_2 can be obtained from those by a sequence of moves of the type (ii)$^{-1}$.

If t is of the type 1.1 (ii), the only problem may arise if $R_1 = xR'_1x^{-1}$, w replaces R_1 in R'_1 and then t replaces R'_1 with yR'_1y^{-1}. Then define σ_0 to be the conjugation of R_1 with yx^{-1}. The statement follows. □

5.6 We will describe 4–thickenings via their BOK–links. In particular, there is a surjective map $\Psi\colon L \to (M_L, P_L)$ from the set of BOK–links onto the set of 4–thickenings, where $\hat{P}_L \to P_L$ is described in 1.1. Moreover changing L into L' by 2–deformation moves 5.2 (a)$\div$ (c) and (f) changes (M_L, P_L) into $(M_{L'}, P_{L'})$ by a 2–deformation and vice versa, i.e. Ψ induces a bijection between the 2-equivalence classes of BOK–links onto the 2-equivalence classes of 4-thickenings.

Given a presentation $\hat{P}$, with $[[\hat{P}]]$ we will denote the set of all BOK–links L such that $\hat{P}_L = \hat{P}$. Suppose now that P is a 2–complex realizing $\hat{P}$ under the bijection in 1.1 and fix an element $c \in H^2(P, Z/2)$. Then for any $L \in [[\hat{P}]]$, $P_L = P$, and there is an embedding $\iota_{M_L,P}\colon P \to M_L$. Denote with $[[\hat{P}, c]]$ the set of all BOK–links $L \in [[\hat{P}]]$ such that $\iota^*_{M_L,P}(w_2(M_L)) = c$. Observe that according to corollary 5.7.2 in [13], the second Whitney class $w_2(M) \in H^2(M; Z/2)$ of a 4–thickening M, represented by a K–link, is given by the cocycle in $H^2(M, M_1; Z/2)$ [4] whose value on each 2–handle is its framing coefficient modulo 2. Hence, if $\hat{P}$ has m relations and c is presented by a cocycle $\bar{c} \in H^2(P, P_1; Z/2) \simeq H^2(M, M_1; Z/2) \simeq (Z/2)^m$, $[[\hat{P}, c]]$ is the set of all BOK–links in $[[\hat{P}]]$ whose framing coefficient on the i-th undotted component is equal to $\bar{c}_i$ modulo 2.

5.7 We assume that the reader is familiar with the notion of a framed tangle, which intuitively is a slice of a framed link. A good reference is Shum [21], where it is called double tangle. Since all tangles with which we will work will be framed, in the future we will just call them tangles. A tangle with n incoming and m outgoing ends will be called an $n-m$ tangle.

A *K–tangle* will be a tangle in which some of the unknotted closed components of framing 0, bounding disjoint Seifert surfaces, have been dotted. An *OK–tangle* is a K–tangle in which an orientation of any dotted or undotted component has been fixed, and a *BOK–tangle* is an OK–tangle equipped with a choice of numbering of the closed dotted, of the closed undotted and of the open components, a choice of a set of disjoint Seifert surfaces for all dotted components, and a choice of a basepoint on each undotted component s, where if the component is open, the basepoint is the positively oriented point in ∂s.

A BOK–tangle is being described by a plane diagram which decomposes into a combination of the segments presented on figure 6 and the ones obtained from them by changing the orientation of some components. We make the convention that the incoming ends will be drawn on the top and the outgoing ends will be drawn on the bottom. The tangle plane diagrams used here come with a standard choice of Seifert surfaces which in the future won't be drawn, while the choice of base points on the closed undotted components needs to be indicated.

5.8 Two OK–tangles are equivalent if and only if their plane diagrams can be obtained from each other via the moves on figure 7 and 8 where any double line represents a number of parallel segments and the unoriented dotted and

[4] If M_k denotes the k-handlebody, then the boundary operator $H_k(M_k, M_{k-1}; Z) \to H_{k-1}(M_{k-1}, M_{k-2}; Z)$ is defined by the long exact sequence on the triple (M_k, M_{k-1}, M_{k-2}) and the cochain complex is obtained by dualizing the chain complex (see 4.2 in [13]).

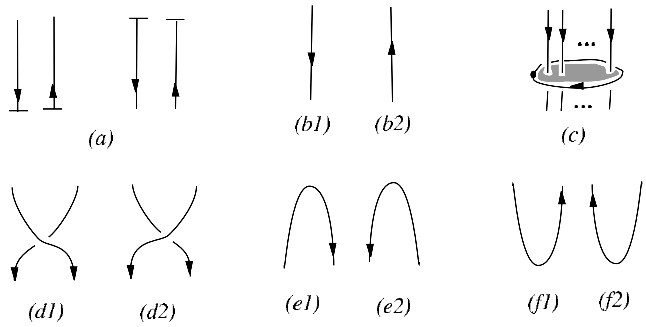

Figure 6: Elementary tangle plane diagrams

undotted components can be oriented in any way consistent on both sides of the identities. Two K–tangles are equivalent if and only if their plane diagrams can be obtained from each other via the moves on figure 7 and 8 where we have forgotten the information about orientation.

Observe that two K–links (i.e. 0-0 K–tangles) are equivalent if and only if the corresponding framed links are isotopic.

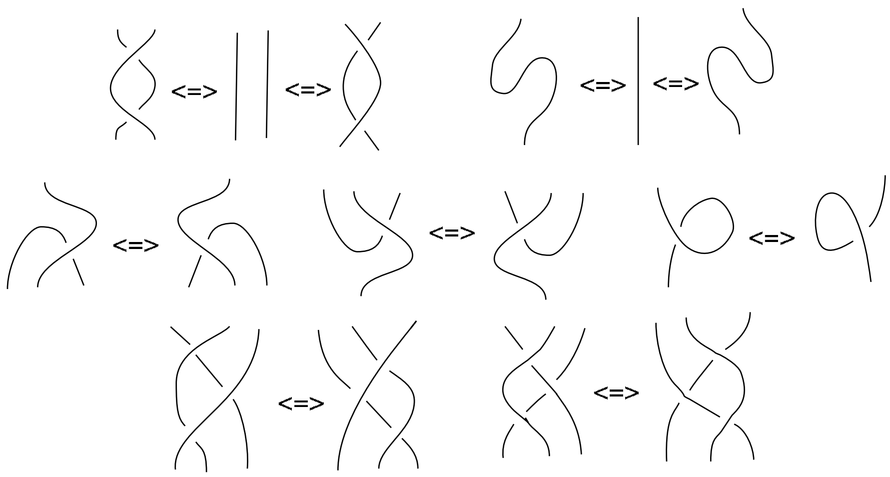

Figure 7: "Framed" Reidemeister moves

5.9 Let T be a $r-r$ K–tangle diagram with r open components $s_1, s_2, \ldots, s_r$ and let $A_1, A_2, \ldots, A_t$ be the incoming ends and $B_1, B_2, \ldots, B_t$ be the outgoing ends of T all numbered from left to right. Then T is called a *string tangle*

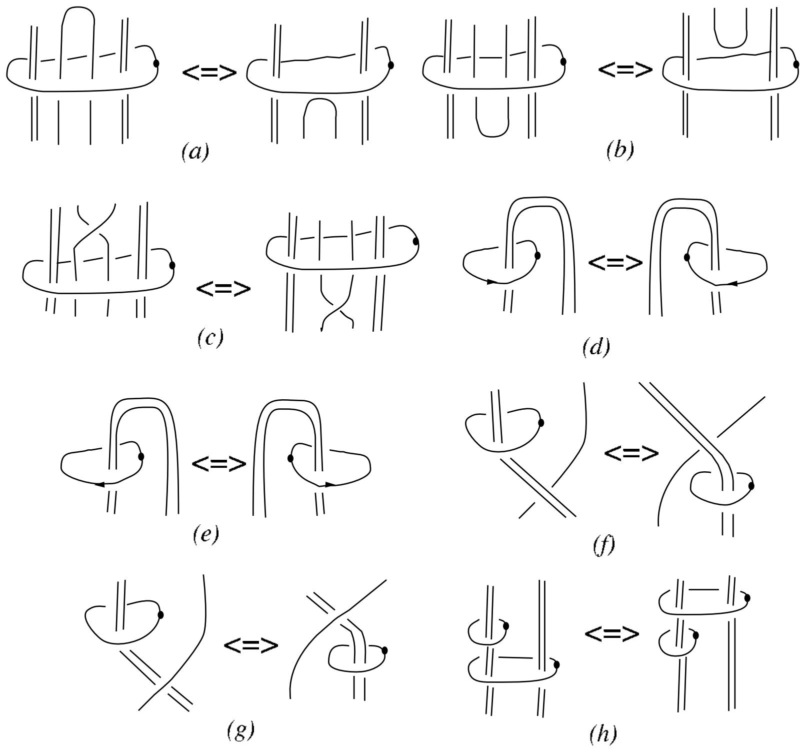

Figure 8: Additional isotopy moves

diagram if there exists an element σ in the symmetric group on r elements $\mathbf{S_r}$ such that $\partial s_i = A_i \cup B_{\sigma(i)}$. σ is called the underlying permutation of T. If T is an OK–tangle then we add the requirement that A_i is the positively oriented end of s_i, i.e. the strings "point down".

6 Definition of the invariant

6.1 Let T be a BOK–tangle with n dotted components, m closed undotted components and r open ones. Without loss of generality, we assume that if there are dotted components such that no undotted component intersects their Seifert surfaces, these are the first l components. By analogy with the definition of the Hennings invariant [5], extended to the presence of 1–handles (see for example in [7]), we define a map

$$\mathcal{Z}(T)\colon\ A^{\otimes(n+m)} \to A^{\otimes r}$$

as follows.

Let $z_j, w_i \in A,\ i = 1, \ldots, n,\ j = 1, \ldots, m$. We refer to z_j as the color of the j-th undotted component, and to w_i as the color of the i-th dotted component of T.

(a) Represent the BOK–tangle by plane diagram as above;

(b) Label the undotted components of each elementary plane diagram as follows:

- "cups" and "caps" as presented on figure 9;
- at each crossing of two undotted components pointing downwards, label the various segments of the plane diagram according to the Hennings rules presented in figure 9. Any other crossing is obtained from those presented in the figure by changing the orientation of some component y. Then the label of y changes by applying S^{-1};

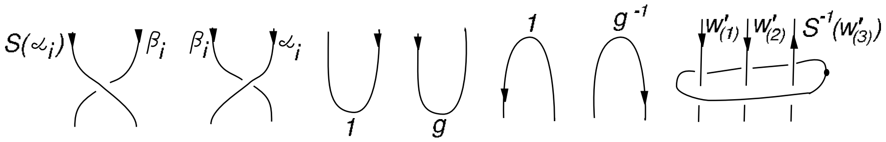

Figure 9: Hennings type rules for labeling extended plane diagrams

- Let x be a dotted component with color w and a Seifert surface S_x and let v_x be the normal vector of S_x. Let $w' = w$ if v_x points up, and $w' = S^{-1}(w)$ if v_x points down. Then, if $s_1, s_2, \ldots, s_t$ are the oriented segments intercepting S_x , and if $\Delta^{(t-1)}(w') = \sum_{w'} w'_{(1)} \otimes w'_{(2)} \otimes \ldots \otimes w'_{(t)}$, s_i gets labeled with $S^{-1}(w'_{(i)})$ if it points up, and with $w'_{(i)}$ otherwise as presented in figure 9.

(c) For each undotted component, starting from the base point, multiply on the right the various labeling elements, in the order they are found according to the orientation of the component. In this way, one obtains an element $\sum_i a_{1,i} \otimes a_{2,i} \otimes \ldots a_{m,i} \otimes b_{1,i} \otimes b_{2,i} \otimes \ldots b_{r,i} \in A^{\otimes(m+r)}$, where $a_{j,i}$ represents the product of the labelings of the j-th closed component and $b_{k,i}$ represents the product of the labelings of the k-th opened component. Then define

$$\begin{aligned} &\mathcal{Z}(T)(z_1, \ldots, z_m, w_1, \ldots, w_n) \\ &= \left(\prod_{j=1}^{l} \epsilon(w_j) \right) \sum_i \lambda(g z_1 a_{1,i}) \ldots \lambda(g z_m a_{m,i}) b_{1,i} \otimes \ldots \otimes b_{r,i} \in A^{\otimes r}. \end{aligned}$$

6.2 Remarks (a) The application of $\epsilon\colon A \to k$ to the label of the j-th open component gives exactly the invariant of the tangle T' obtained from T by removing the j-th open component.

(b) We have defined, somewhat arbitrary, the value of the invariant on a disjoint dotted component of color w to be $\epsilon(w)$. But as it will be shown in 6.9, this is the only choice consistent with the invariance under the cancellation of a dotted and undotted component (move 5.2 (b)).

6.3 We illustrate the definition with the example of an oriented extended tangle T presented in figure 10. If $w \in A$ is the color of the dotted component and $z \in A$ is the color of the undotted one then

$$\mathcal{Z}(T)(z,w) = \sum_{i,w} \lambda(gzg^{-1}w_{(2)}\alpha_i g^{-1}w_{(1)}\beta_i)S^{-1}(w_{(3)}) \in A.$$

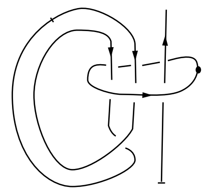

Figure 10: An example of a BOK–tangle

In the future, if we want to investigate the value of $\mathcal{Z}(T)$ for some particular color of dotted or undotted component, this color may be indicated on the plane diagram in a circle attached to the corresponding component as in figure 11 below.

6.4 Proof of theorem 2.14 The map defined so far obviously depends on the choices of numbering, base points and orientations. So we will start putting restrictions on the values of the colors in order to reduce this dependence and eventually obtain an invariant of BOK–links under the 2–deformation moves in 5.2.

The proof consists of showing the following statements:

(A) $\mathcal{Z}(T)\colon Z(A)^{\otimes(n+m)} \to A^{\otimes r}$ does not depend on the choice of base points and it is invariant under the moves of figures 7, 8;

Let now L be an BOK–link with n dotted and m undotted components. Then

(B) $\mathcal{Z}(L)\colon Z(A)^{\otimes(n+m)} \to k$ factors through a map $\hat{Z}(A)^{\otimes(n+m)} \to k$ which will be denoted in the same way;

(C) $\mathcal{Z}(L)\colon \hat{Z}^S(A)^{\otimes(n+m)} \to k$ doesn't depend on the choice of orientation of the components of the link;

(D) Let x be the first, and y be the second undotted component of L. Let also L' is being obtained from L by replacing y with a band connected sum of x and y. Then if $[z],[w] \in \hat{Z}^S(A)$ are such that $\hat{\delta}([w],[z]) = 0$, and $[c] \in \hat{Z}^S(A)^{\otimes(n+m-2)}$, we have

$$\mathcal{Z}(L)([z]\otimes[w]\otimes[c]) = \mathcal{Z}(L')([z]\otimes[w]\otimes[c]).$$

(E) For $[z],[w] \in \hat{Z}^S(A)$ let $\mathcal{Z}_{[z]}^{[w]}(L)$ denote the value of $\mathcal{Z}(L)$ where any undotted component is colored by $[z]$ and any dotted component is colored by $[w]$. Then if $[zw] = [\Lambda]$, $\mathcal{Z}_{[z]}^{[w]}(L)$ is invariant under move 5.2 (b). Moreover if $[z] \in \mathcal{T}^4$ and $[zw] = [zw'] = [\Lambda]$, then $\mathcal{Z}_{[z]}^{[w]}(L) = \mathcal{Z}_{[z]}^{[w']}(L)$. This common value will be denoted with $\mathcal{Z}_{[z]}(L)$.

6.5 Proof of (A) First we remind Hennings' result ([5]) that if the colors of the undotted components are in the center of the algebra, $\mathcal{Z}(T)\colon Z(A)^{\otimes m} \otimes A^{\otimes n} \to A^{\otimes r}$ is independent of the choice of base points on the closed undotted components, and it is an invariant under the moves presented in figure 7. Moreover, from the defining identity 3.5 (a) for the R–matrix and the defining property of g, it is easy to see that it is also an invariant under the moves (a)$\div$(c) on figure 8.

Suppose now that the colors of the dotted components are in the center of the algebra as well. Then the identities (f), (g) and (h) are automatically satisfied. So, it is left to show that in this case (d) and (e) are satisfied as well. Let x be the dotted component which we want to slide over the cup, and let $w \in Z(A)$ be its color. Since w is in the center of a ribbon algebra, $S^2(w) = w$ and (d) and (e) become equivalent. So it is enough to show (e). Let $w' = S^{-1}(w)$. Suppose that n undotted segments pass through x. Then, depending on its orientation, under the move (e) the label of the i-th segment changes as $g^{-1}w_{(i)} \to S^{-1}(w'_{(n-i)})g^{-1}$ or $S^{-1}(w_{(i)}) \to w'_{(n-i)}$. But from 3.1 (d) it follows that $S^{-1}(w'_{(n-i)})g^{-1} = g^{-1}S(w'_{(n-i)}) = g^{-1}w_{(i)}$ and $w'_{(n-i)} = S^{-1}(w_{(i)})$.

6.6 Proof of (B) The proof is based on the following observation which is a version of the centrality result of the HKR–invariant in [11].

Let T be a $k-l$ BOK–tangle with $n+m$ closed and r open components. Let also T' be the BOK–tangle obtained from T by embracing all incoming ends (figure 11 (a)) with a dotted component x', and let T'' be the BOK–tangle obtained from T by embracing all outgoing ends with a dotted component x'' (figure 11 (b)). Fix the colors of x' and x'' to be the same element $a \in A$ and let $c \in Z(A)^{\otimes(n+m)}$ describe the coloring of the closed components of T. Then

$$\mathcal{Z}(T')(a \otimes c) = \mathcal{Z}(T'')(a \otimes c).$$

(a) (b)

Figure 11: Centrality of the invariant

This can be seen by decomposing the plane diagram of T into slices such that each slice contains only one subdiagram of the type crossing, cup, cap or dotted component. Then, since all colors of the components of T are in $Z(A)$, one can use moves (a), (b), (c) and (h) to slide the dotted component colored by a through.

The statement above implies that if T is an $r-r$ string tangle then $\mathcal{Z}(T)$ sends $Z(A)^{\otimes(n+m)}$ into C^r. In particular, if T is a 1-1 BOK–tangle with $(n+m)$ closed components, $\mathcal{Z}(T)$ sends $Z(A)^{\otimes(n+m)}$ into $Z(A)$.

Now we can show (B). Let $K(A) \subset Z(A)$ be the null space of the pairing on $Z(A)$ induced by λ as in 2.1. Suppose that an undotted component y of L has a color $z \in K(A)$. Then we can use isotopy moves to present L as a closure of a 1-1 string tangle T on y and $\mathcal{Z}(T)$ sends $Z(A)^{\otimes(n+m-1)}$ into $Z(A)$. Hence for any $a \in Z(A)^{\otimes(n+m-1)}$, $\mathcal{Z}(L)(z \otimes a) = \lambda(z\, \mathcal{Z}(T)(a)) = 0$ by the definition of $K(A)$.

Now suppose that a dotted component x of L has a color $w \in K(A)$. Since $w = w \star \Lambda$ without changing the value of the invariant we can introduce an undotted unknotted component y of color $S(w)$ which passes once through x and in the same time change the color of x to Λ as shown in figure 12. But since the new tangle has an undotted component of color $S(w) \in K(A)$ its invariant is 0 as shown previously.

6.7 Proof of (C) Observe that changing the orientation of a dotted component x with color $[w] \in \hat{Z}(A)$ has the same effect as leaving its orientation the

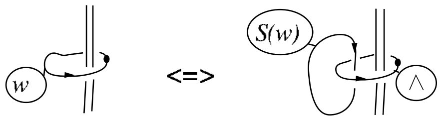

Figure 12: Replacing a dotted component of color w with a pair of dotted component of color Λ and undotted component of color $S(w)$

same but changing its color to $[S(w)]$ or $[S^{-1}(w)]$. Hence if $[w] \in \hat{Z}^S(A)$, the value of $\mathcal{Z}(L)$ remines unchanged.

The fact that changing the orientation of an undotted component doesn't change the invariant is a modification of Hennings' argument when there is no dotted components. The link plane diagram can be deformed via the regular isotopy moves of figures 7,8 and if necessary changing orientation of dotted components into one which is composed totally of segments of the types presented on figure 13. We do this by first pulling all dotted components on the

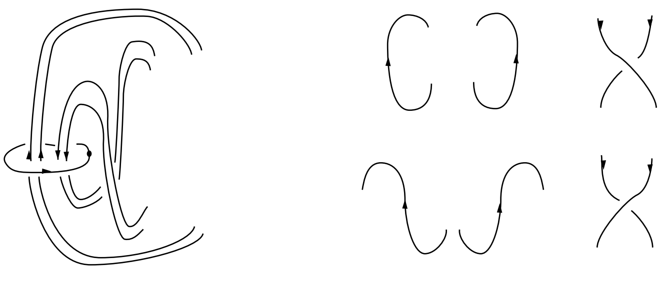

Figure 13: Elementary plane diagrams

left of the plane diagram using the moves (f) and (g) of figure 8. In this way, on the right there is left a tangle T which gets closed through the dotted components as shown in figure 14 (a). Then, using move (c) of figure 8 we pull all undotted segments, which pass through a dotted component and point down, to the right and absorb the resulting crossings into T obtaining another tangle T' as shown in figure 14 (b). Then we pull down the upper ends and pull up the lower ends of these undotted segments which point down as they pass through a dotted component. In this way the plane diagram is presented as the closure (through the dotted components) of a string tangle T'' with positively oriented ends as shown in figure 14 (c). At the end, by local deformations as the one on figure 14 (d) we obtain a plane diagram in which all crossings have the two segments pointing down. After doing some moves of the type of the second one in figure 7, we can assume that the segments of the undotted components in T'' between crossings and end points are of the type presented in figure 13 (b). Now we want to show that, under a change of orientation, the

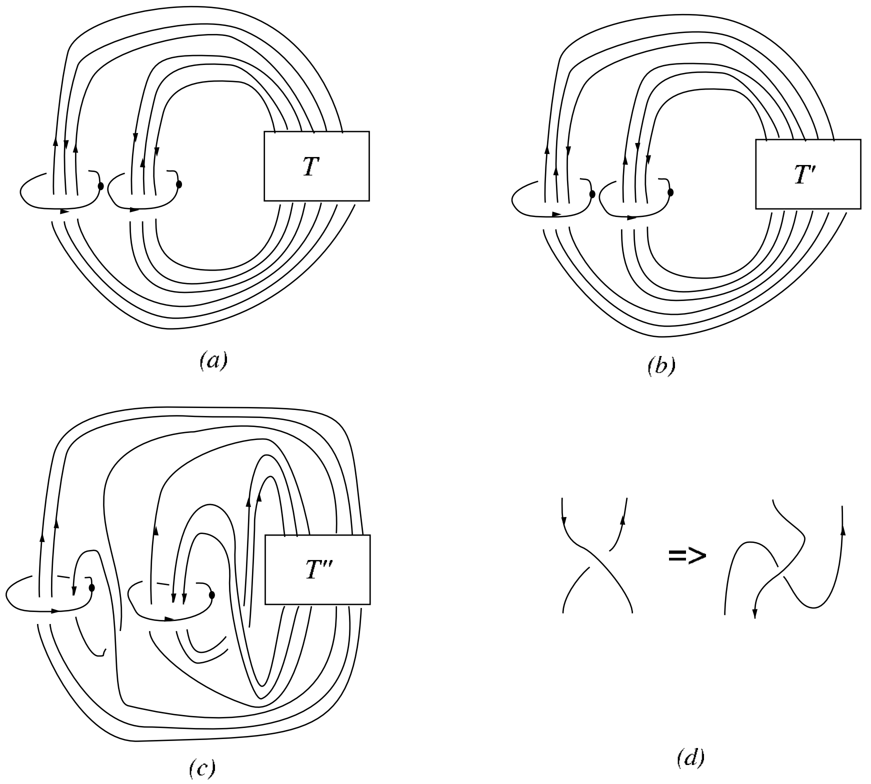

Figure 14: Deformation of a link plane diagram

label of an undotted component changes by application of S^{-1}. By definition, this is the case if we change the orientation of an undotted component in one of the segments presented on figures 13 (b). Then it is enough to show the same statement for the undotted components in figures 13 (a). The labeling of an undotted component which points up as it passes through a dotted circle of color w is of the type $a = S^{-1}(w_{(i)})g^{-1}$ and, after its orientation has been changed, becomes $w_{(i)}g = gS^{-2}(w_{(i)}) = S^{-1}(a)$. The label of an undotted component which points down as it passes through a dotted circle of color w is of the type $b = \alpha_{j,(k)}w_{(i)}S(\beta_{j,(k)})$, and after a change of the orientation, it becomes $\beta_{j,(k)}S^{-1}(w_{(i)})g^{-1}S(\alpha_{j,(k)})g = S^{-1}(b)$. Since $\lambda_{gz} \circ S = \lambda_{gz}$, the statement follows.

6.8 Proof of (D) First, using isotopy moves, deform the link plane diagram as the closure of a tangle T on y and x, where x is oriented downwards and y is oriented upwards as shown in figure 15 (a). Without loss of generality, we may assume that the band connected sum is like the one presented on 15 (b). Let $\mathcal{Z}(T)([c]) = \sum_i a_i \otimes b_i \in A \otimes A$. Then,

$$\mathcal{Z}(L)([z] \otimes [w] \otimes [c]) = \sum_i \lambda(za_i)\lambda(wb_i).$$

On another hand, $\mathcal{Z}(L')([z] \otimes [w] \otimes [c]) = \sum_i \lambda(za_{i,(1)})\lambda(wb_i a_{i,(2)})$. Moreover, $a_{i,(1)} \otimes b_i a_{i,(2)} \in C^2$, since it represents the invariant of a 2-2 string tangle. Hence,

$$\sum_{i,a_i} \lambda(za_{i,(1)})\lambda(wb_i a_{i,(2)}) = \sum_{i,a_i,z} \lambda(z_{(1)}a_{i,(1)})\lambda(wz_{(2)}b_i a_{i,(2)})$$
$$= \sum_{i,a_i,z} \lambda(a_{i,(1)}z_{(1)})\lambda(wb_i a_{i,(2)}z_{(2)}) = \sum_i \lambda(za_i)\lambda(wb_i).$$

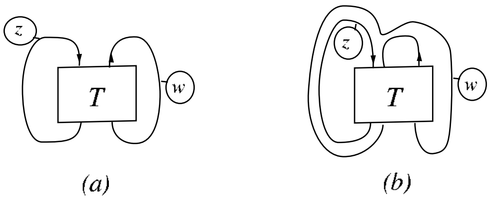

Figure 15: On the proof of 6.4 (D)

6.9 Proof of (E) The invariance under the cancellation of a pair of dotted and undotted component (move 5.2 (b)) is a straightforward consequence of the definition of Λ and the fact that $\lambda(\Lambda) = 1$ with the exception of the case when $L = L' \sqcup K$, where K is a dotted component whose Seifert surface is disjoint from the rest of the link, and we have added a cancelling pair of dotted and undotted components such that the new undotted component passes through K, obtaining in this way a new BOK–link L''. Then by definition $\mathcal{Z}^{[w]}_{[z]}(L) = \epsilon(w)\mathcal{Z}^{[w]}_{[z]}(L')$. On another hand, since $[zww] = \epsilon(w)[zw]$, $\mathcal{Z}^{[w]}_{[z]}(L'') = \epsilon(w)\mathcal{Z}^{[w]}_{[z]}(L')$. Hence $\mathcal{Z}^{[w]}_{[z]}(L'') = \mathcal{Z}^{[w]}_{[z]}(L)$ as requested.

Assume now that $[z] \in \mathcal{T}^4$ and $[w], [w'] \in \hat{Z}^S(A)$ are such that $[zw] = [zw'] = [\Lambda]$. Starting with $\mathcal{Z}^{[w]}_{[z]}(L)$ we will show that one can change the color of all

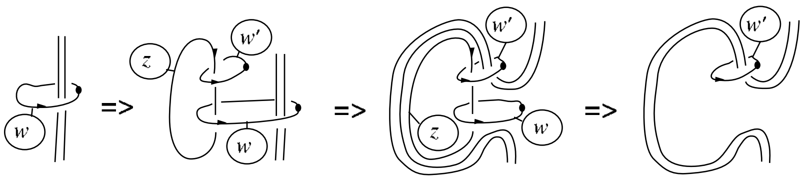

Figure 16: On the proof of 6.4 (E)

dotted components from $[w]$ to $[w']$ without changing the value of the invariant. Suppose that x_1 is a dotted component of color $[w]$. Since $[w] = [\Lambda \star w] = [(zw')\star w]$, we can add a canceling pair of dotted component x_2 of color $[w']$ and an undotted component y of color $[z]$ which passes once through x as shown in figure 16. Then, using 6.4 (D), slide the components which pass through x_1 over y and since $[(zw)\star w'] = [w']$, cancel the pair x_1, y. Now (E) follows from the fact that $[S(w')] = [w']$.

We have shown that $\mathcal{Z}_{[z]}(M_L) = \mathcal{Z}_{[z]}(L)$ defines an invariant of 4–thickenings. To complete the proof of theorem 2.14 it is left to observe that $S^2 \times D^2$ is represented by an undotted unknot of framing 0 and hence $\mathcal{Z}_{[z]}(S^2 \times D^2) = \lambda(z)$, while $S^1 \times D^3$ is represented by one dotted component and hence $\mathcal{Z}_{[z]}(S^1 \times D^3) = \epsilon(w)$.

Observe that if $[z] \in \mathcal{T}^4$, and $[zw] = [\Lambda]$, then for any unit $\gamma \in k$, $[z'] = [\gamma z] \in \mathcal{T}^4$ and $[z'w'] = [\Lambda]$ where $[w'] = \frac{1}{\gamma}[w]$. Hence

Corollary 6.10 *For any unit* $\gamma \in k$, $\mathcal{Z}_{[\gamma z]}(M) = \gamma^{\chi(M)-1}\mathcal{Z}_{[z]}(M)$, *where* $\chi(M)$ *is the Euler characteristic of* M.

6.11 Factorization properties of the link invariant Suppose that $L = L' \sqcup L''$ is a link (without dotted components), and L' and L'' are sublinks of L which don't have common components. Then let $\mathcal{Z}_{[z],[w]}(L' \sqcup L'') \in k$ denote the value of $\mathcal{Z}(L)$ where all components of L' have been labeled with $[z]$ and all components of L'' have been labeled with $[w]$.

Corollary 6.12 (a) *If* $[z], [w] \in \hat{Z}^S(A)$ *are such that* $\hat{\delta}([w],[z]) = 0$, *then* $\mathcal{Z}_{[w],[J(z)]}(L' \sqcup L'') = \mathcal{Z}_{[w]}(L')\mathcal{Z}_{[J(z)]}(L'')$;

(b) *If* $[z] \in \mathcal{T}$ *then* $\mathcal{Z}_{[z\star J(z)]}(L) = \mathcal{Z}_{[z]}(L)\mathcal{Z}_{[J(z)]}(L)$.

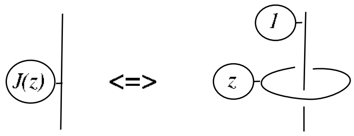

Figure 17: Replacing an undotted component of color $J(z)$ with an undotted component of color 1 embraced by an undotted component of color z

Proof The definition of J in 2.7 implies that coloring a component $x \in L''$ with $[J(z)]$ is equivalent to coloring x with 1 and embracing it with a small undotted unknot x' of color $[z]$ as showed in figure 17. But since any component $y \in L'$ has color $[w]$, according to 6.4 (D), y can be slided over x' and it is a basic fact from the Kirby calculus, that in this way y can be unlinked from x. Hence we can unlink any component of L' from any component of L'' and move them apart. This shows (a).

Let now $L^{\#} = L \sqcup L'$ be the double of L, i.e. L' is a copy of L, and $L^{\#}$ is obtained from L by adding a parallel to each component of L, using the framing. Then (b) would follow from (a) if we could show that for any $[z], [w] \in \hat{Z}^S(A)$,

$$\mathcal{Z}_{[z\star w]}(L) = \mathcal{Z}_{[z],[w]}(L' \sqcup L).$$

Let x be a component of L colored by $[z \star w]$. L can be presented as a closure of a 1-1 string tangle T on x with $\mathcal{Z}_{[z\star w]}(T) = c \in Z(A)$. Then

$$\mathcal{Z}_{[z\star w]}(L) = \lambda((z \star w)c) = \lambda(w(S(z) \star c)) = \lambda(zc_{(1)})\lambda(wc_{(2)}),$$

where in the last two equalities we have used 4.6 and 4.3. But the last expression is exactly the invariant of a link obtained from L by adding a parallel component x' of x and coloring x by $[w]$ and x' by $[z]$. □

6.13 Proof of lemma 2.16 Let P be a 2–dimensional CW complex, $c \in H^2(P, Z/2)$, and let

$$\hat{P} = \langle x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m \rangle.$$

From 5.6 it follows that in order to prove lemma 2.16 it is enough to show that, if L_0 is a standard representative in $[[\hat{P}, c]]$, then for any other $L \in [[\hat{P}, c]]$ and any $[z] \in \mathcal{T}^2$, $\mathcal{Z}_{[z]}(L) = \mathcal{Z}_{[z]}(L_0)$. So, we proceed with the description of L_0.

Without loss of generality we assume that, if $\hat{P}$ contains trivial relations, these are the last k relations. Then let

$$Q = R_1 R_2 \ldots R_{m-k} = x_{i_1}^{e_1} x_{i_2}^{e_2} \ldots x_{i_t}^{e_t}, \text{ where } e_i = \pm 1,$$

be the unreduced word obtained by putting together all nontrivial relations in $\hat{P}$. Let also t_i^+ (t_i^-) denote the absolute value of the sum of the positive (negative) exponents of x_i in Q and l_i denote the length of the relation R_i (the sum of the absolute values of the exponents of x_j's in R_i). Define σ_Q to be the permutation element in the symmetric group $\mathbf{S_t}$, such that $\sigma_Q(k) < \sigma_Q(l)$ if ($i_k < i_l$) or ($i_k = i_l$ and $e_k < e_l$) or ($i_k = i_l$, $e_k = e_l$ and $k < l$). Observe that applying the permutation σ_Q on the letters of Q gives the word $x_1^{-t_1^-} x_1^{+t_1^+} x_2^{-t_2^-} x_2^{t_2^+} \dots x_n^{-t_n^-} x_n^{t_n^+}$. Let also τ_Q be the following element in $\mathbf{S_t}$ presented as product of cycles:

$$\begin{aligned}\tau_Q = \quad & (\sigma_Q(1), \sigma_Q(2), \dots, \sigma_Q(l_1))(\sigma_Q(l_1+1), \dots, \sigma_Q(l_1+l_2)) \dots \\ & \dots (\sigma_Q(t-l_m+1), \dots, \sigma_Q(t)).\end{aligned}$$

Fix a braid B_Q on t strings oriented downwards, which has τ_Q as underlying permutation. Then the standard representative L_0 is defined to be the BOK–link in $[[P,c]]$ of the type presented in figure 14 (c), where the dotted components are ordered in increasing order from the right to the left and where $T'' = T_0$ is a string tangle which is obtained by putting next to B_Q k undotted unknots. The framing coefficients of all undotted components are chosen to be 0 or 1 depending on the corresponding value of the cocycle $\bar{c} \in H^2(P, P_1; Z/2)$.

Let L be another BOK–link in $[[\hat{P}, c]]$ and let $[z] \in \mathcal{T}^2$, i.e. there exist $[z_1], [z_2] \in \hat{Z}^S(A)$ such that $[z] = [z_1 J(z_2)]$ and $\hat{\delta}([z_1],[z_2]) = 0$. By the definition of $\hat{P}_L$, each letter $x_{i_j}^{e_j}$ in Q corresponds to an intersection point A_j in L of an undotted component with the Seifert surface of the i_j-th dotted component and e_j is the sign of this intersection. Use σ_Q to define an order of the set of points A_j, in particular $A_j \prec A_l$ if $\sigma_Q(k) < \sigma_Q(l)$. By isotopy moves as in 6.7, we deform L into a link L'' from the type presented in figure 14 (c) so that the points A_j are ordered in increasing order from the right to the left. Then $\mathcal{Z}_{[z]}(L) = \mathcal{Z}_{[z]}(L'')$. Of course, T'' in general will be different from T_0, but it is a string tangle and since $\hat{P}_L = \hat{P}_{L_0}$, T'' has the same underlying permutation τ_Q. Now, as shown in figure 17, labeling an undotted component $y \in L''$ with $[z_1 J(z_2)]$ is the same as labeling y with $[z_1]$ and embracing it with undotted component y' labeled by $[z_2]$. But since $\hat{\delta}([z_1],[z_2]) = 0$, any component labeled by $[z_1]$ can be slided over any component labeled by $[z_2]$. Therefore if x is any other undotted component in L'', we can use sliding of x over y' to change the sign of any crossing of y with x and by sliding y over y' we can add two positive or two negative twists on y, i.e. change the framing coefficient of y with ± 2. Since T'' and T_0 have the same underlying permutation, by applying a sequence of such operations T'' can be transformed into T_0. Hence $\mathcal{Z}_{[z]}(L_0) = \mathcal{Z}_{[z]}(L'') = \mathcal{Z}_{[z]}(L)$.

6.14 Proof of corollary 2.17 If A is a finite-dimensional unimodular triangular Hopf algebra then the positive and negative crossings of two undotted components have the same labeling. Therefore for any $[z] \in \mathcal{T}^4$ we can repeat the argument above and show that if $L_1, L_2 \in [[\hat{P}, c]]$ then $\mathcal{Z}_{[z]}(L_1) = \mathcal{Z}_{[z]}(L_2)$. Moreover, the ribbon element in a triangular algebra is $\theta = 1$. Hence the invariant in lemma 2.16 won't depend any more on the framings of the undotted components, in particular for any $L_1, L_2 \in [[\hat{P}]]$ and any $[z] \in \mathcal{T}^4$,

$$\mathcal{Z}_{[z]}(L_1) = \mathcal{Z}_{[z]}(L_2).$$

Now, let $\hat{P} \to \hat{P}'$ be an AC–move and $L \in [[\hat{P}]]$. By 5.5 (b) there exists a BOK–link L', 2–equivalent to L such that $\hat{P}'$ can be obtained from $\hat{P}_{L'}$ by cancellations of terms of the type $x_i x_i^{-1}$. But such term in L' corresponds to an undotted segment which enters into the i-th dotted component x_i, possibly links with other undotted components or itself (but doesn't pass through other dotted ones) and then goes out of x_i. Now by cross changes we can unlink any such undotted component and then by isotopy moves, pull it out of x_i without changing the value of the invariant. The result is an BOK–link $L'' \in [[\hat{P}']]$ and we have $\mathcal{Z}_{[z]}(L'') = \mathcal{Z}_{[z]}(L') = \mathcal{Z}_{[z]}(L)$.

7 Relation with the 3–manifold invariants

7.1 Suppose that we want an invariant of a 4–thickening to depend only on its boundary. This would imply (see [12]) invariance under two additional moves:

(i) Removing or adding a dot on an 0–framed unknot. This corresponds to replacing a one handle with its canceling 2–handle and vice versa;

(ii) Deleting or adding an unknot $U^{\pm 1}$ of framing ± 1, contained in a neighborhood disjoint from the rest of the link, which corresponds to taking a connected union with CP^2 or $\overline{CP^2}$.

In general, $\mathcal{Z}_{[z]}$ won't be invariant under these additional moves, but in many examples (including all the ones coming from the quantum $sl(2)$) $\mathcal{Z}_{[z]}$ can be normalized to depend only on the boundary. We will use the statement below only for $[z] \in \mathcal{T}$, but observe that it is true in the following weaker form:

Proposition 7.2 *Suppose that $[z] \in \mathcal{T}_Z$ and that $[zJ(z)] = X[\Lambda]$ for some unit $X \in k$. Then $X = \lambda(z\theta^{-1})\lambda(z\theta)$.*

Proof Since $\epsilon(\theta) = \sum_i \epsilon(\beta_i g\alpha_i) = 1$, $[\theta^{-1}zJ(z)] = X[\Lambda]$ and therefore $X = \lambda(\theta^{-1}zJ(z))$. Substituting here the expression for $J(z)$ from 4.8 we obtain that

$$X = \lambda(\theta^{-1}zJ(z)) = \lambda(z((S(z)\theta) \star \theta^{-1})).$$

Now since $[S(z)] = [z]$, applying 4.6 it follows that

$$X = \lambda((z\theta)(z \star \theta^{-1})) = \lambda((z\theta)(1 \star z\theta^{-1})) = \lambda(z\theta^{-1})\lambda(z\theta),$$

where in the second equality we have used the fact that $[z] \in \mathcal{T}_Z$. □

7.3 Proof of corollary 2.19 Since $C_\pm = \lambda(z\theta^{\pm 1})$, the first assertion follows from the proposition above. The rest follows from the observation that the ordered pair $(\sigma_+ - n, \sigma_- - n)$ is an invariant under 2–deformations of M since a 2–handle slide 5.2 (c) doesn't change the number of dotted components and the values of σ_+ and σ_-, while move 5.2 (b) reduces by one the number of dotted components, and in the same time reduces by one the values of σ_+ and σ_-. Moreover, the proposition 7.2 implies that under the moves 7.1(i) and (ii), $\mathcal{Z}_{[z]}(M)$ changes exactly as $C_+^{\sigma_+ - n} C_-^{\sigma_- - n}$ and therefore their quotient $\mathcal{Z}^\partial_{[z]}(\partial M)$ depends only on the boundary.

Proposition 7.4 *Let $[z] \in \mathcal{T}^3$. Then*

(a) *for any unit $\gamma \in k$, $[\gamma z] \in \mathcal{T}^3$ and $\mathcal{Z}^\partial_{[\gamma z]}(\partial M) = \gamma^{\sigma_0} \mathcal{Z}^\partial_{[z]}(\partial M)$;*

(b) *if $[J(z)], [z \star J(z)] \in \mathcal{T}^3$ then $\mathcal{Z}^\partial_{[z \star J(z)]}(\partial M) = \mathcal{Z}^\partial_{[z]}(\partial M) \mathcal{Z}^\partial_{[J(z)]}(\partial M)$.*

The proposition is a direct consequence of the corollaries 6.10 and 6.12.

Corollary 7.5 *If A is Λ–factorizable then for any $[z] \in \mathcal{T}^3$,*

$$\mathcal{Z}^\partial_{[z]}(\partial M) \mathcal{Z}^\partial_{[J(z)]}(\partial M) = X_z^{\sigma_0} \mathcal{Z}^\partial_{[1]}(\partial M).$$

Proof Since the algebra is Λ–factorizable, $J(1) = \gamma\Lambda$. Then $[z] \in \mathcal{T}^3$ implies that $[zJ(z)] = X_z[\Lambda]$. Applying $\frac{1}{\gamma}J$ on both sides of the equality and using 2.9, we obtain that $[J(z) \star J^2(z)] = \gamma X_z\,[1]$. But 2.8 (b) implies that $[J^2(z)] = \gamma[S(z)] = \gamma[z]$. Hence $[J(z) \star z] = \gamma X_z\,[1]$. Since in this case J is a bijection, we can reverse the argument and therefore obtain that, if the algebra is Λ–factorizable,

$$\mathcal{T}^3 = \{[z] \in \mathcal{T} \mid [z \star J(z)] = X_z[1] \text{ for some unit } X_z \in k\}.$$

In particular, if $[z] \in \mathcal{T}^3$ then $[J(z)], [z \star J(z)] = X_z[1] \in \mathcal{T}^3$. Now the statement follows from proposition 7.4. □

8 Examples

To illustrate the generality of the present framework we describe two examples. The first one is useful to get familiar with the framework, and the second one is the quantum $sl(2)$ case, which shows quite rich algebraic structure, but it is not interesting for the AC–conjecture. Indeed all $sl(2)$ theories are actually 3–dimensional.

8.1 The cocommutative case: $R = 1 \otimes 1$

Since this is a particular case of a triangular structure on A, we are talking about invariants of 2–complexes. First, observe that in this case $g = 1$ and $S^2 = 1$. As a consequence, the invariant has very simple definition, which is worth writing down. Let $[z] \in \mathcal{T}^4$ and choose $[w] \in \hat{Z}^S(A)$ such that $[zw] = [\Lambda]$. Let $\hat{P} = \langle x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m \rangle$ be a presentation, where $R_i = R_i(x_1, x_2, \ldots, x_n)$. Let also Q, σ_Q, $t_i^{\pm}$, l_j and t be as in 6.13 and $t_i = t_i^+ + t_i^-$ be the total exponent of x_i. Associated to Q, define a bijective map $S_Q \colon A^{\otimes t} \to A^{\otimes t}$ such that

$$S_Q(\sum_i a_{1,i} \otimes a_{2,i} \otimes \ldots \otimes a_{t,i}) = \sum_i S^{\epsilon_1}(a_{1,i}) \otimes S^{\epsilon_1}(a_{2,i}) \otimes \ldots \otimes S^{\epsilon_1}(a_{t,i}),$$

where $\epsilon_j = (1 - e_j)/2$ and $S^0 = id_A$, i.e. in case that the j-th exponent in Q is negative S_Q applies the antipode on the j-th factor in $A^{\otimes t}$.

Let $\bar{\sigma}_Q \colon A^{\otimes t} \to A^{\otimes t}$ be the permutation of factors induced by σ_Q and let

$$\sum_i a_{1,i} \otimes a_{2,i} \otimes \ldots \otimes a_{t,i} = S_Q \circ \bar{\sigma}_Q^{-1}(\Delta^{t_1 - 1} w \otimes \Delta^{t_2 - 1} w \otimes \ldots \otimes \Delta^{t_n - 1} w) \in A^{\otimes t}.$$

Then from the definition of $\mathcal{Z}^2_{[z]}$ in section 7 and the fact that we are in the case when $R = 1 \otimes 1$, it follows that

$$\begin{aligned} \mathcal{Z}^2_{[z]}(P) = \; & \sum_i \lambda(z a_{1,i} a_{2,i} \ldots a_{l_1,i}) \lambda(z a_{l_1+1,i} a_{l_1+2,i} \ldots a_{l_1+l_2,i}) \ldots \\ & \ldots \lambda(z a_{t-l_m+1,i} a_{t-l_m+2,i} \ldots a_{t,i}). \end{aligned}$$

We illustrate the technique with the case of a group algebra and $[z] = 1$. The result is a well known invariant which depends on the fundamental group of P. Let $A = k[G]$, where G is a finite group. Then the product on A is induced from the one in G, and for any $a \in G$, $\Delta(a) = a \otimes a$ and $S(a) = a^{-1}$. A is a unimodular algebra with $\Lambda = \sum_{a \in G} a$, and $\lambda \in A^*$ defined as $\lambda(1) = 1$, and $\lambda(a) = 0$ if $a \neq 1$. Hence the algebra is cosemisimple, and it is semisimple if and only if the characteristic of k doesn't divide the order of G. For $z = 1$ and $w = \Lambda$, the value of the invariant is:

$$\mathcal{Z}^2_{[1]}(P) = \sum_{\{a_j\}_{j=1}^n} \lambda(R_1(a_1,\dots,a_n))\lambda(R_2(a_1,\dots,a_n))\dots\lambda(R_m(a_1,\dots,a_n)),$$

where the sum is over all possible sequences $\{a_j\}_{j=1}^n$ of elements in G and $R_i(a_1,a_2,\dots,a_n)$ denotes the image of the word R_i under the group homomorphism of the free group on the generators $x_1, x_2,\dots,x_n$ into G given by $x_j \to a_j$. Hence $\mathcal{Z}^2_{[1]}(P)$ is equal to the number of all possible group homomorphisms $G \to \pi_1(P)$.

8.2 The quantum enveloping algebra of $sl(2)$

We use here the definition of the finite-dimensional quantum enveloping algebra of $sl(2)$ "at root of unity" as given in chapter 36 of the book of G. Lusztig [15], and we refer the reader to [15], chapters 23, 31, 32, 34 and 36, for the proof that the definition is consistent with the Hopf algebra axioms and that the category of representations of the algebra is the same as the one of the finite-dimensional quantum $sl(2)$, defined in a more familiar ways. For the $sl(2)$ case, many statements can actually be easily verified by direct computation as well.

8.3 Let $p > 3$ be a prime number and let $k' = Z[v]/\langle 1 + v + \dots + v^{p-1}\rangle$ and $k = Q[v]/\langle 1 + v + \dots + v^{p-1}\rangle$. For any $n, m \in Z$ such that $m \geq 0$ we will use the following common notations:

$$[n] = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{\prod_{s=0}^{m-1}(v^{n-s} - v^{-n+s})}{\prod_{s=1}^{m}(v^s - v^{-s})},$$
$$\{m\} = \prod_{i=1}^{m}(v^i - v^{-i}), \quad \{0\} = 1,$$

hoping that the double use of square bracket to denote equivalence classes in $\hat{Z}(A)$ and quantum integers will not bring to a confusion. Note that $\{p-1\} = p$. Define A to be the k algebra generated by the elements $1_cE^{(n)}$, $1_cF^{(n)}$ such that $c \in Z/p$ and $0 \le n \le p-1$ and relations:

$$1_c E^{(n)} 1_s E^{(m)} = \delta_{c,s+2n} \begin{bmatrix} n+m \\ n \end{bmatrix} 1_c E^{(n+m)};$$
$$1_c F^{(n)} 1_s F^{(m)} = \delta_{c,s-2n} \begin{bmatrix} n+m \\ n \end{bmatrix} 1_c F^{(n+m)};$$
$$1_c F^{(n)} 1_s E^{(m)} = \delta_{c,s-2n} \sum_{t=0}^{min(m,n)} \begin{bmatrix} m+n-s \\ t \end{bmatrix} 1_c E^{(m-t)} 1_{c-2(m-t)} F^{(n-t)};$$
$$1_c E^{(n)} 1_s F^{(m)} = \delta_{c,s+2n} \sum_{t=0}^{min(m,n)} \begin{bmatrix} m+n+s \\ t \end{bmatrix} 1_c F^{(m-t)} 1_{c+2(m-t)} E^{(n-t)}.$$

We introduce the notation $1_c E^{(n)} F^{(m)} = 1_c E^{(n)} 1_{c-2n} F^{(m)}$. Then A is a finite-dimensional algebra with identity $\mathbf{1} = \sum_{c \in Z/p} 1_c$ and basis $\{1_c E^{(n)} F^{(m)}\}$, where $c \in Z/p$, $0 \le n, m \le p-1$. A has a Hopf algebra structure with the following structure maps:

$$\epsilon(1_c E^{(n)}) = \epsilon(1_c F^{(n)}) = \delta_{c,0}\delta_{n,0};$$
$$\Delta(1_c E^{(n)}) = \sum_{a=0}^{n} \sum_{r \in Z/p} v^{a(a-n)+r(n-a)} 1_r E^{(a)} \otimes 1_{c-r} E^{(n-a)};$$
$$\Delta(1_c F^{(n)}) = \sum_{a=0}^{n} \sum_{r \in Z/p} v^{a(a-n)-(c-r)a} 1_r F^{(a)} \otimes 1_{c-r} F^{(n-a)};$$
$$S(1_c E^{(n)}) = (-1)^n v^{n(c-1-n)} 1_{-c+2n} E^{(n)};$$
$$S(1_c F^{(n)}) = (-1)^n v^{-n(c-1+n)} 1_{-c-2n} F^{(n)};$$

It is easy to check that A is a unimodular Hopf algebra with an integral $\Lambda = 1_0 E^{(p-1)} F^{(p-1)}$ and that A^* has as a right integral λ defined as

$$\lambda(1_c E^{(n)} F^{(m)}) = v^c \delta_{n,p-1} \delta_{m,p-1}.$$

Obviously, $\lambda(\Lambda) = 1$. A is a quasitriangular ribbon algebra with

$$R = \sum_{n=0}^{p-1} \sum_{r,s \in Z/p} v^{\frac{n(n-1)}{2} + \frac{rs}{2}} \{n\} 1_r F^{(n)} \otimes 1_s E^{(n)} \quad \text{and} \quad g = \sum_{c \in Z/p} v^{-c} 1_c.$$

8.4 The center of A is described in [8], where the following notations are used: $K = \sum_{s \in Z/p} v^s 1_s$, $\pi_s(K) = 1_{-2s}$, $E = (v - v^{-1}) \sum_{c \in Z/p} 1_c E^{(1)}$ and

$F = \sum_{c \in Z/p} 1_c F^{(1)}$. Following [8] we define

$$X = (v - v^{-1}) \sum_{s=0}^{p-1} 1_s E^{(1)} F^{(1)} + \sum_{k=1}^{p-1} b(k-1) 1_{2k} \in Z(A) \quad \text{and}$$
$$\phi_j(x) = \prod_{0 \le s \le p-1 : b(s) \ne b(j)} (x - b(s)) \in k[x],\ j = 0, \dots, q$$

where $b(s) = b(p-1-s) = \frac{v^{2s+1} + v^{-2s-1}}{v - v^{-1}}$. Let $q = \frac{p-1}{2}$ and let

$$P_j = \frac{1}{\phi_j(b(j))} \phi_j(X) - \frac{\phi_j'(b(j))}{\phi_j(b(j))^2} \phi_j(X)(X - b(j)),\ j = 0, \dots, q,$$
$$N_j = \frac{1}{\phi_j(b(j))} \phi_j(X)(X - b(j)),\ j = 0, \dots, q-1,$$
$$N_j^+ = T_j N_j,\ N_j^- = (1 - T_j) N_j,\ \text{where}\ T_j = \sum_{s=j+1}^{p-1-j} 1_{-2s}.$$

Lemma 18 in [8] allows to express the elements above in terms of the algebra basic elements $1_s E^{(i)} F^{(j)}$ as follows:

$$1_{-2s} \phi_k(X)(X - b(k)) = \sum_{j=0}^{p-1} \prod_{i=j+1}^{p-1} (b(k) - b(i+s))([j]!)^2 (v - v^{-1})^j 1_{-2s} E^{(j)} F^{(j)},$$

$$1_{-2s} \phi_k(X) = \sum_{j=0}^{p-2} \sum_{t=j+1}^{p-1} \prod_{i=j+1, i \ne t}^{p-1} (b(k) - b(i+s))([j]!)^2 (v - v^{-1})^j 1_{-2s} E^{(j)} F^{(j)},$$

$$\Phi_k(b(k)) = ([p-1]!)^2 \frac{(v - v^{-1})^{p-2}}{[2k+1]^2},$$
$$\Phi_k'(b(k)) = ([p-1]!)^2 \frac{(v - v^{-1})^{p-3} [2(2k+1)]}{[2k+1]^5},$$

for any $k = 0, \dots, q-1$, and

$$1_{-2s} \phi_q(X) = \sum_{j=0}^{p-1} \prod_{i=j+1}^{p-1} (b(q) - b(i+s))([j]!)^2 (v - v^{-1})^j 1_{-2s} E^{(j)} F^{(j)},$$
$$\Phi_q(b(q)) = ([p-1]!)^2 (v - v^{-1})^{p-1}.$$

From here one can see that $N_0^- = (v - v^{-1}) \Lambda$ and $\lambda(N_i^-) = (v - v^{-1})[2i+1]^3$. In particular $\lambda(N_i^-) \ne 0$ for any $i = 0, \dots, q-1$.

8.5 (**Kerler [8]**) $Z(A)$ is a $3q+1$ dimensional algebra with basis $\{P_i, N_j^{\pm}, i = 0, \dots, q,\ j = 0, \dots, q-1\}$ and products:

$$\begin{aligned} &P_i P_j = \delta_{i,j} P_j \\ &P_i N_j^{\pm} = \delta_{i,j} N_j^{\pm} \\ &N_l^{\pm} N_j^{\pm} = N_l^{\mp} N_j^{\pm} = 0. \end{aligned}$$

Moreover, the ribbon element in this basis is given by

$$\theta = v^q P_q + \sum_{j=0}^{q-1} v^{2j(j+1)} (P_j + \frac{2j+1}{[2j+1]} N_j - \frac{p}{[2j+1]} N_j^-).$$

Observe that since X and T_j are S–invariant, any element in $Z(A)$ is S–invariant and

$$K(A) = span\{P_q, N_j,\ j = 0, \dots, q-1\}.$$

Hence $\hat{Z}(A) = \hat{Z}^S(A)$ is generated by $[P_i], [N_j^-]$, $i, j = 0, \dots, q-1$ and the following relations:

(a) $[P_i][P_j] = \delta_{i,j}[P_j]$,

(b) $[P_i][N_j^-] = \delta_{i,j}[N_j^-]$,

(c) $[N_l^-][N_j^-] = 0$.

To be able to continue we need to understand also the $\star$ product structure of the algebra. An easy calculation shows that

(d) $\hat{J}([1]) = \gamma_p[\Lambda]$ and $\hat{J}([\Lambda]) = [1] = \sum_{i=1}^{q-1}[P_i]$,

where $\gamma_p = p^3$, i.e. the algebra is Λ–factorizable. Then according to corollary 2.9, $J^2 = \gamma_p \mathbf{1}$, $\gamma_p^{-1} J \colon Z(A) \to Z_\star(A)$ is an algebra isomorphism and therefore the $\star$ algebra structure can be derived from the knowledge of J.

Lemma 8.6 $\hat{J}([N_i^-]) = (v - v^{-1})[2i+1]^2 \sum_{k=0}^{q-1} \frac{[(2i+1)(2k+1)]}{[2k+1]}[P_k]$.

We will need the following proposition:

Proposition 8.7 *For any b such that $0 \le b \le p-2$, let $\Omega_b = Z(A) \cap span\{1_s E^{(a)} F^{(a)},\ s \in Z/p,\ 0 \le a \le b\}$. Then $\Omega_b \subset span\{P_i,\ N_j,\ 0 \le i \le q,\ 0 \le j \le q-1\}$.*

Proof We will show that $\Omega_b = span\{X^a,\ 0 \le a \le b\}$. Then the statement will follow from the observation in [8] that any polynomial in X is contained in the span of P_i, $i = 1, \dots, q$ and N_j, $j = 1, \dots, q-1$.

Let $Y = \sum_{s\in Z/p}\sum_{a=0}^{p-1}\tau^Y_{a,s}1_sE^{(a)}F^{(a)}$ be in $Z(A)$. Then for any $s\in Z/p$,

$$1_sE^{(1)}Y = Y1_sE^{(1)}$$

From here by direct computation one can see that for any $0\le a\le p-2$,

$$[a-s]\tau^Y_{a+1,s+2} = [a+1](\tau^Y_{a,s}-\tau^Y_{a,s+2}).$$

This implies that if $Y\in\Omega_b$ then $\tau^Y_{b,s}$ doesn't depend on s and we denote it with τ^Y_b. In particular, X^b is of this type, moreover $\tau^X_b\neq 0$ and therefore, there exists $r\in k$ such that if $b>0$ then $Y-rX^{b-1}\in\Omega_{b-1}$ and if $b=1$ then $Y=rX^0=r\mathbf{1}$. The proposition follows by induction. □

8.8 Proof of lemma 8.6 Now we continue with the proof of the lemma 8.6. Observe that since $[N_i^-+N_i^+]=0$, $\hat J([N_i^-])=-\hat J([N_i^+])$, so we will compute $\hat J([N_i^+])$. From the expressions in 8.4 one obtains:

$$P_j = 1_{-2j}+1_{2j+2}+\sum_{s\in Z/p}\sum_{a=1}^{p-1}\tau^j_{s,a}1_sE^{(a)}F^{(a)},\ 0\le j\le q-1$$

$$P_q = 1_1+\sum_{s\in Z/p}\sum_{a=1}^{p-1}\tau^q_{s,a}1_sE^{(a)}F^{(a)},$$

$$N_j=\sum_{a=0}^{p-1}\nu^j_{-2s,a}1_{-2s}E^{(a)}F^{(a)},\ N_j^+=\sum_{s=j+1}^{p-1-j}\sum_{a=0}^{p-1}\nu^j_{-2s,a}1_{-2s}E^{(a)}F^{(a)},$$

where $\nu^j_{-2s,0}=0$ and $\nu^j_{-2s,p-1}=(v-v^{-1})[2j+1]^2$. Given i,a such that $0\le i\le q-1$, $0\le a\le p-1$ and given $s\in Z/p$, let $\bar\nu^i_{-2s,a}\in k$ are the coefficients of the expansion of $J(N_i^+)$ in terms of the basis $1_{-2s}E^{(a)}F^{(a)}$, i.e.

$$J(N_i^+)=S\circ J(N_i^+)=\sum_{n,m}\lambda(\beta_nN_i^+\alpha_m)S(\alpha_n\beta_m)=\sum_{s\in Z/p}\sum_{a=0}^{p-1}\bar\nu^i_{-2s,a}1_{-2s}E^{(a)}F^{(a)}.$$

Substituting here the expression for the R–matrix and for N_i^+ we obtain

$$\bar\nu^i_{-2s,a}=v^{a(a+1)+2as}\{a\}^2\begin{bmatrix}p-1\\a\end{bmatrix}^2\sum_{l=i+1}^{p-1-i}v^{2l(a-2s-1)}\nu^i_{-2l,p-1-a}.$$

In particular $\bar\nu^i_{-2s,p-1}=0$ and

$$\bar\nu^i_{-2s,0}=-(v-v^{-1})[2i+1]^2\frac{[(2s+1)(2i+1)]}{[2s+1]}.$$

Then the lemma follows from proposition 8.7 and the expression for P_s.

8.9 For any $0 \le i,j \le q-1$, let $\omega_{i,j} = \frac{[(2j+1)(2i+1)]}{[2j+1]}$. Let also

$$\dot{N}_i = \frac{N_i}{(v-v^{-1})[2i+1]^2} \quad \text{and} \quad \dot{N}_i^{\pm} = \frac{N_i^{\pm}}{(v-v^{-1})[2i+1]^2}.$$

Observe that $[\dot{N}_0^-] = [\Lambda]$. Since $\hat{J}$ is injective, the $(q-1)\times(q-1)$ matrix ω is nondegenerate and the proposition above implies that

$$\hat{J}([\dot{N}_j^-]) = \sum_{i=1}^{q-1} \omega_{ji}[P_i] \quad \text{and} \quad \hat{J}([P_j]) = \gamma_p \sum_{i=1}^{q-1} (\omega^{-1})_{ji}[\dot{N}_i^-].$$

Proposition 8.10 (a) $\sigma_{ij}^k = \sigma([\dot{N}_i^-],[\dot{N}_j^-],[P_k]) = \lambda(\dot{N}_k^-)\sum_{s=0}^{q-1}\omega_{is}\omega_{js}\omega_{sk}^{-1}$, *and* $\sigma(a,b,c)=0$ *for any other triple of generators* a, b, c*;*

(b) $\sigma_{ij}^k/\lambda(\dot{N}_k^-) = 1$ *if all of the following four conditions are satisfied:*

$$i+j+k \le p-2, \quad i+j-k \ge 0, \quad k+i-j \ge 0, \quad k+j-i \ge 0.$$

Otherwise $\sigma_{ij}^k = 0$.

Proof of (a) Lemma 2.8 allows as to express the $\star$ product in the following way:

$$[\dot{N}_i^-] \star [\dot{N}_j^-] = \gamma_p^{-1}\hat{J}(\hat{J}([\dot{N}_i^-])\hat{J}([\dot{N}_j^-])) = \sum_{k,s=0}^{q-1} \omega_{is}\omega_{js}(\omega^{-1})_{sk}[\dot{N}_k^-];$$

$$[P_i] \star [P_j] = \gamma_p^{-1}\hat{J}(\hat{J}([P_i])\hat{J}([P_j])) = 0;$$

This implies that $\sigma(a,b,c)=0$ if all three elements are of the type $\dot{N}_i$, or if only one of them is such. For the only nonzero case we obtain

$$\sigma([P_k],[\dot{N}_i^-],[\dot{N}_j^-]) = \lambda(P_k,(\dot{N}_i^- \star \dot{N}_j^-)) = \lambda(\dot{N}_k^-)\sum_{s=0}^{q-1}\omega_{is}\omega_{js}(\omega^{-1})_{sk}. \qquad \square$$

Proof of (b) Using that for any primitive p-th root of unity v and any $a \in Z/p$,

$$\sum_{s=0}^{q-1} v^{a(2s+1)} = \frac{p\,\delta_{a,0} - v^{-a}}{1+v^{-a}},$$

one obtains that

$$\sum_{i=0}^{q-1}[(2j+1)(2i+1)][(2i+1)(2k+1)] = -\frac{p}{(v-v^{-1})^2}\delta_{j,k}.$$

Hence

$$(\omega^{-1})_{i,j} = -\frac{(v-v^{-1})^2}{p}[2i+1][(2i+1)(2j+1)],$$

and

$$\frac{\sigma_{ij}^k}{\lambda(\dot{N}_k^-)} = -\frac{(v-v^{-1})^2}{p}\sum_{s=0}^{q-1}\frac{[(2i+1)(2s+1)][(2j+1)(2s+1)][(2k+1)(2s+1)]}{[2s+1]}.$$

Substituting above the expression

$$\frac{[(2i+1)(2s+1)]}{[2s+1]} = \sum_{l=0}^{2i} v^{2(i-l)(2s+1)},$$

and expanding we obtain that

$$\begin{aligned} &p\frac{\sigma_{ij}^k}{\lambda(\dot{N}_k^-)} \\ &= \sum_{l=k-i-j}^{k+i-j}\sum_{s=0}^{q-1}(v^{2l(2s+1)}+v^{-2l(2s+1)}) - \sum_{l=j+k-i+1}^{j+k+i+1}\sum_{s=0}^{q-1}(v^{2l(2s+1)}+v^{-2l(2s+1)}) \\ &= p\left(\sum_{l=k-i-j}^{k+i-j}\delta_{\bar{l},0} - \sum_{l=j+k-i+1}^{j+k+i+1}\delta_{\bar{l},0}\right), \end{aligned}$$

where $\bar{l} = \mathrm{Mod}(l,p)$. This completes the proof of the proposition. □

Observe that the proof of proposition 8.10 above imply:

Corollary 8.11 *The subalgebra of $\hat{Z}_\star(A)$ spanned by $[\dot{N}_j^-]$, $0 \le j \le (q-1)$ is isomorphic to the fusion algebra $\mathcal{F}_p$ of the semisimple quotient of the representation category of A defined in 10.3.*

Finally we can describe all elements in $\mathcal{T}_Z$.

Theorem 8.12 *$\mathcal{T}_Z$ consists of the multiples of $[1]$, $[\Lambda]$, $\sum_{j=0}^{q-1}[2j+1][\dot{N}_j^-]$ and $[P_0]$. Moreover, $\hat{J}$ sends bijectively $\mathcal{T}_Z$ into itself.*

Proof Suppose that $[z] = \sum_{i=0}^{q-1} x_i[P_i] + \sum_{i=0}^{q-1} y_i[\dot{N}_i^-]$. According to 2.6 $[z] \in \mathcal{T}_Z$ if and only if for any $[a],[b],[c] \in \hat{Z}(A)$, $\sigma(zc,za,b) = \sigma(zc,a,zb)$.

Replacing here all possible choices of a, b, c we obtain that this condition is equivalent to the following system of equations for the coefficients x_i, y_i:

$$\begin{aligned}
&\text{(i)} \quad y_i y_k \sigma^j_{ik} = y_j y_k \sigma^i_{jk};\\
&\text{(ii)} \quad y_k(x_i - x_j)\sigma^i_{jk} = y_i x_k \sigma^k_{ji};\\
&\text{(iii)} \quad x_k(x_i - x_j)\sigma^k_{ji} = 0;\\
&\text{(iv)} \quad y_i x_k \sigma^j_{ik} = y_j x_k \sigma^i_{jk};\\
&\text{(v)} \quad x_k(x_i - x_j)\sigma^i_{jk} = 0,
\end{aligned}$$

for any $0 \le i, j, k \le q-1$. Now we want to show that $[z]$ is contained either in the span of the $[P_i]$'s or in the span of the $[\dot{N}^-_i]$'s. Observe that $\sigma^k_{0,j} = \delta_{kj}\lambda(\dot{N}^-_j)$. Hence equations (v) and (ii) with $j = 0$ become

$$x_i(x_i - x_0) = 0 \qquad y_i x_0 = 0.$$

Therefore either $x_i = 0$ for any i or $y_i = 0$ for any i. Suppose now that we are in the case when $y_i = 0$ for any i and let $\mathcal{I} \neq \emptyset$ be the subset of indices such that $x_i \neq 0$. Then, condition (v) implies that

(a) $0 \in \mathcal{I}$;

(b) for any other $i \in \mathcal{I}$, $x_0 = x_i$;

(c) if $\sigma^i_{jk} \neq 0$ and two of the indices i, j, k are in $\mathcal{I}$, then the third one must be in $\mathcal{I}$ as well.

Moreover, any subset $\mathcal{I}$ which satisfies these conditions corresponds to a solution of the form $[z_\mathcal{I}] = \sum_{i\in\mathcal{I}}[P_i]$. In particular, since $\sigma^k_{0,0} = \sigma^0_{0,k} = \delta_{k,0}$, $\mathcal{I} = \{0\}$ ($[z_\mathcal{I}] = [P_0]$) gives a solution of the problem.

Suppose now that $i \in \mathcal{I}$ and $i \neq 0$. Since $\sigma^1_{ii} \neq 0$ (8.10 (b)) it follows that 1 should be in $\mathcal{I}$ as well. But if $1, j \in \mathcal{I}$ where $j \le q-2$, then $j+1 \in \mathcal{I}$ (since $\sigma^{j+1}_{j,1} \neq 0$). Hence, if $\mathcal{I}$ contains one nonzero index, it must contain all indices, i.e. $\mathcal{I} = \{0, 1, \ldots, q-1\}$ and $[z_\mathcal{I}] = [\mathbf{1}]$.

Suppose now that $x_0 = 0$ and $\mathcal{I} \neq \emptyset$ is the subset of indices such that $y_i \neq 0$ i.e. $[z] = \sum_{i\in\mathcal{I}} y_i[\dot{N}^-_i]$. From 8.10 (b) it follows that $\sigma^i_{jk} = \lambda(\dot{N}_i)\epsilon_{ijk}$ where ϵ_{ijk} is symmetric with respect to the three indices. Then equation (i) becomes:

$$y_k(\lambda(\dot{N}_j)y_i - \lambda(\dot{N}_i)y_j)\epsilon_{ijk} = 0.$$

In particular for $i = 0$ and $j = k$ we have $y_k(\lambda(\dot{N}_k)y_0 - y_k) = 0$. Hence $\mathcal{I}$ satisfies the conditions (a)–(c) above and therefore either $\mathcal{I} = \{0\}$ or $\mathcal{I} =$

$\{0, 1, \ldots, q-1\}$ and $y_k = \lambda(\dot{N}_k)y_0$ for any $k \neq 0$. The corresponding solutions for $[z]$ are $[z] = y_0[N_0^-] = y_0\gamma_p[J(\mathbf{1})]$ and

$$[z] = y_0 \sum_{j=0}^{q-1} [2j+1][\dot{N}_j^-] = -\frac{y_0 \hat{J}([P_0])}{(v-v^{-1})^2 p^2}.$$

This completes the proof of the theorem. □

8.13 The $sl(2)$ HKR–type invariants

We remind that $\mathcal{T}_s$ denotes the subset of elements in $\hat{Z}^S(A)$ which define invariants of links under the band-connected sum of two distinct components. Then $\mathcal{T}^3 \subset \mathcal{T} \subset \mathcal{T}_s$. We can not offer a way to calculate the elements in $\mathcal{T}$ and even less a way to study its maximality, i.e. if it coincides with $\mathcal{T}_s$. But since $\mathcal{T}_Z \supset \mathcal{T}$, a hypothetical search for the elements in $\mathcal{T}$ could start by calculating the elements in $\mathcal{T}_Z$ as it has been done above for the $sl(2)$ case. The surprise is that $\mathcal{T}_Z$ is already very restrictive: up to multiplication by an element in k, it consists of four elements and, using proposition 10.6 in the appendix, we see that three of them are in $\mathcal{T}_s$:

$[z_H] = [\mathbf{1}]$ gives the Hennings invariant;

$[z_{RT}^*] = [P_0]$

$[z_H^*] = [\Lambda]$ gives the trivial invariant (equal to 1 for any manifold);

$[z_{RT}] = -\dfrac{[J(P_0)]}{(v-v^{-1})^2p^2} = \sum_{j=0}^{q-1}[2j+1][\dot{N}_j^-]$ gives the RT–invariant;

So, it seems reasonable to make the following conjecture:

Conjecture 8.14 *If A is a finite dimensional, unimodular, ribbon, Λ–factorizable algebra, then $\mathcal{T}_Z = \mathcal{T}$.*

If the conjecture holds then $sl(2)$ produces exactly four HKR–type invariants, all normalizable to 3–manifold invariants. Moreover, since $[P_0 z_{RT}] = -p^2[\Lambda]$ and $[P_0 \star z_{RT}] = [1]$, proposition 7.5 implies:

Corollary 8.15 $\mathcal{Z}^{\partial}_{[z_H]}(\partial M) = \mathcal{Z}^{\partial}_{[z_{RT}]}(\partial M)\mathcal{Z}^{\partial}_{[z_{RT}^*]}(\partial M)$.

To support the conjecture, we show that the statement of corollary 8.15 holds for the values of the three invariants for $S^2 \times S^1$ and the Lens spaces. Directly from the definition for $S^2 \times S^1$ we have:

$$\mathcal{Z}^{\partial}_{[z_H]}(S^2 \times S^1) = \lambda(1) = 0;$$
$$\mathcal{Z}^{\partial}_{[z^*_{RT}]}(S^2 \times S^1) = \lambda(P_0) = 0;$$
$$\mathcal{Z}^{\partial}_{[z_{RT}]}(S^2 \times S^1) = \lambda(z_{RT}) = \sum_{j=1}^{q-1} = [2j+1]^2.$$

Observe that $\mathcal{Z}^{\partial}_{[z]}(L(1,n)) = \lambda(z\theta^n)/\lambda(z\theta)$. Then from 8.5 one obtains that $[\theta]^n = \sum_{j=0}^{q-1} v^{2nj(j+1)}([P_j] - n\,p(v - v^{-1})[2j+1][\dot{N}_j^-])$. Hence,

$$\lambda(z^*_{RT}\theta^n) = -pn(v - v^{-1});$$
$$\lambda(z_{RT}\theta^n) = \sum_{j=0}^{q-1} v^{2nj(j+1)}[2j+1]^2;$$
$$\lambda(\theta^n) = -pn(v - v^{-1}) \sum_{j=0}^{q-1} v^{2nj(j+1)}[2j+1]^2.$$

Therefore the statement of corollary 8.15 holds for the values of the three invariants for the Lens spaces as well.

9 Questions

9.1 If the conjecture 8.14 is false, this would imply that the condition $[z] \in \mathcal{T}$ in theorem 2.14 is too strong and needs to be weakened. Then one may ask if it can be replaced with $[z] \in \mathcal{T}_Z$.

9.2 In the case of the quantum $sl(2)$ we saw that the fusion algebra of the semisimple quotient of the representation category is a subalgebra of $\hat{Z}^S_\star(A)$ generated by nilpotent elements. What is in general the relationship between $\hat{Z}^S_\star(A)$ and the representation theory of A?

9.3 Observe that if the Hopf algebra is triangular, then $\mathcal{T}^3 = \{X[\Lambda] \mid X \in k\}$, i.e. such algebra doesn't produce nontrivial 3–manifold invariants. On another hand if $\mathcal{T}^3 = \mathcal{T}^4$ (i.e. any 4–invariant is normalizable to a 3–manifold invariant) then $\mathcal{T}^2 = \{X[\Lambda] \mid X \in k\}$, i.e. such algebra doesn't produce nontrivial invariants of 2–complexes. This seems to be the example of the quantum $sl(2)$. It would be interesting to know if there exists a Hopf algebra for which $\mathcal{T}^4$

doesn't reduce to $\mathcal{T}^2$ or to $\mathcal{T}^3$ and if a similar algebra exists, one may ask if as $\star$–monoid $\mathcal{T}$ is generated by $\mathcal{T}^2$ and $\mathcal{T}^3$. This is related to the following purely topological question:

9.4 Let (M,P) and (M',P') be two 4–thickenings such that $\text{index}(M) = \text{index}(M')$, P is 2–equivalent to P' and ∂M is diffeomorphic to $\partial M'$. Then is it true that M is diffeomorphic to M'? Is M 2–equivalent to M'? The results in [17] seem to support the affirmative answer.

10 Appendix: The Reshetikhin–Turaev $sl(2)$–invariant as HKR–type invariant

Before starting working on this project, the first author asked T.Kerler why the Reshetikhin–Turaev $sl(2)$–invariant is a HKR–type invariant. For completeness we give here Kerler's explanation and the evaluation of the corresponding trace element $[z_{RT}] \in \hat{Z}(A)$. In somewhat different form this evaluation has been done in [7].

We use the definition of the Reshetikhin–Turaev invariant as given in [20]. But since the definition of the quantum $sl(2)$ here is slightly different from the one in [20], the reader is referred to the work of Gelfand and Kazhdan [4] for the proof that, the full linear category generated from the "small" representations used below, satisfies the requirements in paragraph 3.1 of [20].

Let A, k and g be as in 8.3. For any finite dimension left A–module V define the dual representation V^* of V to be representation with linear space $Hom(V,k)$ and action of $a \in A$ given by $S(a)^*$. Define also the quantum trace $\text{tr}_V\colon A \to k$ of V to be

$$\text{tr}_V(a) = \sum_{i=1}^{dim(V)} e_i^*(gae_i) \text{ for any } a \in A,$$

where $\{e_i\}_{i=1}^{dim(V)}$ is a basis for V and $\{e_i^*\}_{i=1}^{dim(V)}$ its the dual basis for V^*.

Proposition 10.1 *For any finite dimensional left A–module V there exists $z_V \in Z(A)$ such that for any $a \in A$, $tr_V(a) = \lambda(g^2 z_V a)$.*

Proof First observe that for any $a, b \in A$,

$$\text{tr}_V(ab) = \sum_{i=1}^{dim(V)} e_i^*(gabe_i) = \sum_{i=1}^{dim(V)} e_i^*(bgae_i) = \text{tr}_V(bS^2(a)).$$

Now, since g is invertible, from 3.3 (d) it follows that there exists an element $z_V \in A$ such that $\mathrm{tr}_V(a) = \lambda(g^2 z_V a)$. Moreover, 3.4 and 3.7 imply that for any $a, b \in A$,

$$\begin{aligned}\lambda(g^2(z_V a - a z_V)b) &= \lambda(g^2 z_V ab - S^4(a) g^2 z_V b) \\ &= \lambda(g^2 z_V ab - g^2 z_V b S^2(a)) = \mathrm{tr}_V(ab - bS^2(a)) = 0.\end{aligned}$$

Then the statement follows from 3.3 (d). □

10.2 Let $\Sigma = \{0, 1, \ldots, q-1\}$ and let V_n, $n \in \Sigma$ be the simple left A-module with highest weight $2n$. Then V_n has a basis $\{e_i^n\}_{i=-n}^n$ and the action of the algebra generators is as follows:

$$1_{2i-2} F^{(1)} e_i^n = \begin{cases} 0 & \text{if } i = -n \\ e_{(i-1)}^n & \text{otherwise} \end{cases}$$

$$1_{2i+2} E^{(1)} e_i^n = \begin{cases} 0 & \text{if } i = n \\ [n+i+1] e_{(i+1)}^n & \text{otherwise} \end{cases}$$

$$1_c e_i^n = \begin{cases} e_i^n & \text{if } c = 2i \ (mod\ p) \\ 0 & \text{otherwise} \end{cases}$$

When it is clear which one is the representation, we will use e_i instead of e_i^n. Moreover, $d_n = 2n+1$ will denote the dimension of V_n and $z_n = z_{V_n}$.

Given a sequence $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ of elements in Σ, define

$$V(\mathbf{i}) = V_{i_1} \otimes V_{i_2} \otimes \ldots \otimes V_{i_k} \quad \text{and} \quad r(\mathbf{i}) = tr_{V(\mathbf{i})}(id).$$

Observe that $r(n) = [2n+1]$ and since g is a group-like element, $r(\mathbf{i}) = \prod_{s=1}^k r(i_s)$.

10.3 As it is shown in [4], the full linear category $\mathcal{C}_p$ generated by V_n, $n \in \Sigma$, is equivalent to the semisimple quotient of the category of integral representation of A, and this equivalence induces a braided monoidal structure on $\mathcal{C}_p$. In particular there is a product structure on $\mathcal{C}_p$ given by

$$V_i \diamond V_j = \oplus_{s \in \Sigma} k^{\epsilon_{ij}^s} \otimes V_s.$$

The essence of this product structure is encoded in the fusion algebra $\mathcal{F}_p$ which is defined as the vector space $Z[x_0, x_1, \ldots, x_{q-1}]$ and product structure given by

$$x_i \diamond x_j = \sum_{s \in \Sigma} \epsilon_{ij}^s \, x_s,$$

for any $i, j \in \Sigma$. The (non negative) integers ϵ_{ij}^s are called the fusion coefficients of $\mathcal{C}_p$. The fusion coefficients for the quantum $sl(2)$ have been calculated in

[20, 4] and are the following: $\epsilon^s_{ij} = 1$ if all of the following four conditions are satisfied

$$i + j + s \leq p - 2, \quad i + j - s \geq 0, \quad s + i - j \geq 0, \quad s + j - i \geq 0,$$

and $\epsilon^s_{ij} = 0$ otherwise.

10.4 Given an oriented $k - l$ tangle T, represented with a tangle diagram, one associates to the incoming and the outgoing ends of T the sequences $\underline{\epsilon} = \{\epsilon_1, \dots, \epsilon_k\}$ and $\overline{\epsilon} = \{\epsilon^1, \dots, \epsilon^l\}$ where $\epsilon_i = 1$ ($\epsilon^i = 1$) if in a neighborhood of the point the tangle component points down and $\epsilon_i = -1$ ($\epsilon^i = -1$) otherwise.

A *coloring* $\mathbf{n} = (n_1, n_2, \dots, n_m) \in \Sigma^{\times m}$ of an oriented $k - l$ tangle T with m components, is a map which associates to the i-th connected component of T an element $n_i \in \Sigma$. A coloring of the tangle induces colorings $\underline{i}(\mathbf{n}) = \{i_1, i_2, \dots, i_k\}$ and $\overline{i}(\mathbf{n}) = \{i^1, i^2, \dots, i^l\}$ of the incoming and the outgoing ends of the tangle.

The colored tangles form a category $\mathcal{H}$ with objects the set $\mathcal{S}$ of sequences $\{(\epsilon_s, i_s)\}_{s=1}^k$, where $\epsilon_s = \pm 1$ and $i_s \in \Sigma$. If $\eta, \eta' \in \mathcal{S}$ then a morphism $\eta \to \eta'$ is a colored tangle considered up to isotopy such that the sequence of signs and colors of the outgoing ends is equal to η and the one of the incoming ends is equal to η' (This is not a mistake. While in the HKR framework we were multiplying the algebra elements on the right, in the Reshetikhin–Turaev framework one considers the left action of the algebra on a representation and this leads to the necessity of reversing the idea of incoming and outgoing). The composition of two tangles $T' \circ T$ is obtained by placing T' on the top of T and gluing the ends. The category can also be provided with tensor product by defining $T' \otimes T$ to be the tangle obtained by placing T' to the left of T.

10.5 Theorem 2.5 in [20] states that there exists a unique covariant functor $F\colon \mathcal{H} \to \mathit{Rep}\, A$ such that for any object η in $\mathcal{H}$, $F(\eta) = V_{i_1}^{\epsilon_1} \otimes V_{i_2}^{\epsilon_2} \otimes \dots V_{i_k}^{\epsilon_k}$, where $V_n^1 = V_n$ and $V_n^{-1} = V_n^*$. Moreover, F preserves the tensor product and if $F(T; \mathbf{n})$ denotes the value of F on an oriented tangle T with coloring $\mathbf{n}$, on the elementary colored tangles presented in figure 6 this value is as follows:

$$F(b1;i) = id_{V_i}, \quad F(b2;i) = id_{V_i^*}$$
$$F(d1;i,j)\colon\ x\otimes y \to \sum_n \beta_n.y\otimes\alpha_n.x\colon\ V_i\otimes V_j \to V_j\otimes V_i;$$
$$F(d2;i,j)\colon\ x\otimes y \to \sum_n S(\alpha_n).y\otimes\beta_n.x\colon\ V_i\otimes V_j \to V_j\otimes V_i,$$
$$F(e1;i)\colon\ x\otimes y \to x(y)\colon\ V_i^*\otimes V_i \to k;$$
$$F(e2;i)\colon\ y\otimes x \to x(g.y)\colon\ V_i\otimes V_i^* \to k;$$
$$F(f1;i)\colon\ 1 \to \sum_{k=1}^{d_i} e_k\otimes e_k^*\colon\ k \to V_i\otimes V_i^*;$$
$$F(f2;i)\colon\ 1 \to \sum_{k=1}^{d_i} e_k^*\otimes g^{-1}.e_k\colon\ k \to V_i^*\otimes V_i,$$

where with "." denotes the left action of A on the corresponding left A–module.

Let L be a link with m components. Fix an orientation of L and define $\{L\} = \sum_{\mathbf{n}} r(\mathbf{n})F(L;\mathbf{n})$, where the sum is over all possible colorings $\mathbf{n} = \{n_1,\ldots,n_m\}$ of L. Then theorem 3.3.2 in [20] states that $\{L\}$ doesn't depend on the orientation of the components of L. Moreover, $\{L\}$ is an invariant of the link under isotopy and under taking the band connected sum of two different components.

Proposition 10.6 *$\{L\} = \mathcal{Z}_{[z_{RT}]}(L)$, where $z_{RT} = \sum_{n=0}^{q-1} r(n)z_n$. In particular, $[z_{RT}] \in \mathcal{T}_s$.*

Proof We can represent L as the closure of a braid B on k strings oriented downwards as in the example in figure 18. Let σ be the underlying permutation

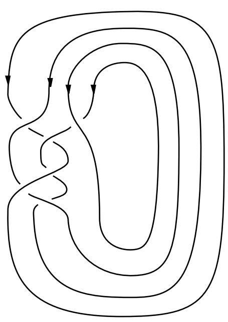

Figure 18: Presenting a link as the closure of a braid

of B, i.e. the boundary of the i-th component of B consists of the i-th incoming and the $\sigma(i)$-th outgoing ends (counted from the left to the right). Then σ is the product of m cycles:

$$\sigma = (j^1_1, j^1_2, \ldots j^1_{s_1}) \ldots (j^m_1, j^m_2, \ldots j^m_{s_m}).$$

For the example of figure 18, $\sigma = (1)(2,3,4)$. Let $\mathcal{Z}(B) = \sum_i a_{1,i} \otimes a_{2,i} \ldots \otimes a_{k,i}$ be the element in $A^{\otimes k}$ as defined in 6.1. Let also

$$\sum_j c_{1,j} \otimes c_{2,j} \ldots \otimes c_{m,j} = \sum_i (a_{j^1_1,i} g a_{j^1_2,i} \ldots g a_{j^1_{s_1},i}) \otimes \ldots \otimes (a_{j^m_1,i} g a_{j^m_2,i} \ldots g a_{j^m_{s_m},i}).$$

Then from 10.5 it follows that for any coloring $\mathbf{n} = \{n_1, n_2, \ldots, n_m\}$ of L,

$$F(L;\mathbf{n}) = \sum_j \mathrm{tr}_{V_{n_1}}(c_{1,j}) \ldots \mathrm{tr}_{V_{n_m}}(c_{m,j}) = \sum_j \lambda(g z_{n_1} c_{1,j} g) \ldots \lambda(g z_{n_m} c_{m,j} g).$$

Here we have used the fact that for any $a, b \in A$ and $-n \leq s, l \leq n$,

$$\sum_{i=-n}^{n} e^*_l(a.e_i) e^*_i(b.e_s) = e^*_l(ab.e_s).$$

Making the confrontation with the expression for $\mathcal{Z}$ in 6.1, we see that $F(L;\mathbf{n}) = \mathcal{Z}(L)(z_{n_1}, \ldots, z_{n_m})$. The statement of the proposition follows by linearity. □

Proposition 10.7 *For any* $0 \leq n \leq q-1$, $[z_n] = [\dot{N}^-_n]$.

Proof From 8.5 it follows that $z_n = \sum_{i=0}^{q-1} (x_i P_i + y_i \dot{N}_i^{\,-} + w_i \dot{N}_i) + x_q P_q$. Let $0 \leq j \leq q$ and $a_j = 1_{-2j} E^{(p-1)} F^{(p-1)}$. Then 10.2 implies that $\mathrm{tr}_{V_n}(a_j) = 0$ for any j. On another hand, from the expressions for P_j and N_j in 8.8 it follows that

$$\lambda(g z_n a_j g) = v^{2j} x_j.$$

Hence $x_j = 0$ for any $0 \leq j \leq q$. On another hand, for every $0 \leq j \leq n$, $\mathrm{tr}_{V_n}(1_{-2j}) = v^{2j}$ and from 8.8 it follows that

$$\lambda(g z_n 1_{-2j} g) = v^{4j} \sum_{i=0}^{q-1} (y_i \lambda(\dot{N}_i^{\,-} 1_{-2j}) + w_i \lambda(\dot{N}_i 1_{-2j})).$$

Hence we obtain the following system of $q+1$ equations for the coefficients y_i, w_i:

$$\sum_{i=j}^{q-1} y_i + \sum_{s=0}^{q-1} w_s = 1, \ 0 \leq j \leq n;$$

$$\sum_{i=j}^{q-1} y_i + \sum_{s=0}^{q-1} w_s = 0, \ n+1 \leq j \leq q;$$

The solution is $y_i = \delta_{i,n}$ and $\sum_{s=0}^{q-1} w_s = 0$. Hence $z_n = \dot{N}_n^- + \sum_{s=0}^{q-1} w_s \dot{N}_s$ and $[z_n] = [\dot{N}_n^-]$. □

As a consequence of the last two propositions it follows that

$$[z_{RT}] = \sum_{n=0}^{q-1} [2n+1][\dot{N_i}^-].$$

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Dipartimento di Scienze Matematiche, Università di Ancona
Via Brece Bianche 1, 60131, Ancona, Italy

Email: `bobtchev@dipmat.unian.it`