*ISSN 1472-2739 (on-line) 1472-2747 (printed)* **905**

*Algebraic & Geometric Topology*
Volume 3 (2003) 905–920
Published: 29 September 2003

# On the slice genus of links

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**Abstract** We define Casson-Gordon $\sigma$-invariants for links and give a lower bound of the slice genus of a link in terms of these invariants. We study as an example a family of two component links of genus $h$ and show that their slice genus is $h$, whereas the Murasugi-Tristram inequality does not obstruct this link from bounding an annulus in the 4-ball.

**AMS Classification** 57M25; 57M27

**Keywords** Casson-Gordon invariants, link signatures

## 1 Introduction

A knot in $S^3$ is slice if it bounds a smooth 2-disk in the 4-ball $B^4$. Levine showed [Le] that a slice knot is algebraically slice, i.e. any Seifert form of a slice knot is metabolic. In this case, the Tristram-Levine signatures at the prime power order roots of unity of a slice knot must be zero. Levine showed also that the converse holds in high odd dimensions, i.e. any algebraically slice knot is slice. This is false in dimension 3: Casson and Gordon [CG1, CG2, G] showed that certain two-bridge knots in $S^3$, which are algebraically slice, are not slice knots. For this purpose, they defined several knot and 3-manifold invariants, closely related to the Tristram-Levine signatures of associated links. Further methods to calculate these invariants were developed by Gilmer [Gi3, Gi4], Litherland [Li], Gilmer-Livingston [GL], and Naik [N]. Lines [L] also computed some of these invariants for some fibered knots, which are algebraically slice but not slice. The slice genus of a link is the minimal genus for a smooth oriented connected surface properly embedded in $B^4$ with boundary the given link.

The Murasugi-Tristram inequality (see Theorem 2.1 below) gives a lower bound on the slice genus of a link in terms of the link's Tristram-Levine signatures and related nullity invariants. The second author [Gi1] used Casson-Gordon invariants to give another lower bound on the slice genus of a knot. In particular he gave examples of algebraically slice knots whose slice genus is arbitrarily large. We apply these methods to restrict the slice genus of a link.

We study as an example a family of two component links, which have genus $h$ Seifert surfaces. Using Theorem 4.1, we show that these links cannot bound a smoothly embedded surface in $B^4$ with genus lower than $h$, while the Murasugi-Tristram inequality does not show this. In fact there are some links with the same Seifert form that bound annuli in $B^4$. We work in the smooth category.

The second author was partially supported by NSF-DMS-0203486.

## 2 Preliminaries

### 2.1 The Tristram-Levine signatures

Let $L$ be an oriented link in $S^3$, with $\mu$ components, and $\theta_S$ be the Seifert pairing corresponding to a connected Seifert surface $S$ of the link. For any complex number $\lambda$ with $|\lambda| = 1$, one considers the hermitian form $\theta_S^\lambda := (1 - \lambda)\theta_S + (1 - \overline{\lambda})(\theta_S)^T$. The Tristram signature $\sigma_L(\lambda)$ and nullity $n_L(\lambda)$ of $L$ are defined as the signature and nullity of $\theta_S^\lambda$. Levine defined these same signatures for knots [Le]. The Alexander polynomial of $L$ is $\Delta_L(t) := \mathrm{Det}(\theta_S - t(\theta_S)^T)$. As is well-known, $\sigma_L$ is a locally constant map on the complement in $S^1$ of the roots of $\Delta_L$ and $n_L$ is zero on this complement. If $\Delta_L = 0$, it is still true that the signature and nullity are locally constant functions on the complement of some finite collection of points.

The Murasugi-Tristram inequality allows one to estimate the slice genus of $L$, in terms of the values of $\sigma_L(\lambda)$ and $n_L(\lambda)$.

**Theorem 2.1** [M, T] *Suppose that $L$ is the boundary of a properly embedded connected oriented surface $F$ of genus $g$ in $B^4$. Then, if $\lambda$ is a prime power order root of unity, we have*

$$|\sigma_L(\lambda)| + n_L(\lambda) \leq 2g + \mu - 1.$$

### 2.2 The Casson-Gordon $\sigma$-invariant

In this section, for the reader convenience, we review the definition and some of the properties of the simplest kind of Casson-Gordon invariant. It is a reformulation of the Atiyah-Singer $\alpha$-invariant.

Let $M$ be an oriented compact three manifold and $\chi\colon H_1(M) \to \mathbb{C}^*$ be a character of finite order. For some $q \in \mathbb{N}^*$, the image of $\chi$ is contained a cyclic subgroup of order $q$ generated by $\alpha = e^{2i\pi/q}$. As $\mathrm{Hom}(H_1(M), C_q) = [M, B(C_q)]$, it follows that $\chi$ induces $q$-fold covering of $M$, denoted $\widetilde{M}$, with a canonical deck transformation. We will denote this transformation also by $\alpha$. If $\chi$ maps onto $C_q$, the canonical deck transformation sends $x$ to the other endpoint of the arc

that begins at $x$ and covers a loop representing an element of $(\chi)^{-1}(\alpha)$.

As the bordism group $\Omega_3(B(C_q)) = C_q$, we may conclude that $n$ disjoint copies of $M$ , for some integer $n$, bounds bound a compact 4-manifold $W$ over $B(C_q)$. Note $n$ can be taken to be $q$. Let $\widetilde{W}$ be the induced covering with the deck transformation, denoted also by $\alpha$, that restricts to $\alpha$ on the boundary. This induces a $\mathbb{Z}[C_q]$- module structure on $C_*(\widetilde{W})$, where the multiplication by $\alpha \in \mathbb{Z}[C_q]$ corresponds to the action of $\alpha$ on $\widetilde{W}$.

The cyclotomic field $\mathbb{Q}(C_q)$ is a natural $\mathbb{Z}[C_q]$-module and the twisted homology $H^t_*(W; \mathbb{Q}(C_q))$ is defined as the homology of

$$C_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Since $\mathbb{Q}(C_q)$ is flat over $\mathbb{Z}[C_q]$, we get an isomorphism

$$H^t_*(W; \mathbb{Q}(C_q)) \simeq H_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Similarly, the twisted homology $H^t_*(M; \mathbb{Q}(C_q))$ is defined as the homology of

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Let $\widetilde{\phi}$ be the intersection form on $H_2(\widetilde{W}; \mathbb{Q})$ and define

$$\phi_\chi(W)\colon\ H^t_2(W; \mathbb{Q}(C_q)) \times H^t_2(W; \mathbb{Q}(C_q)) \to \mathbb{Q}(C_q)$$

so that, for all $a, b$ in $\mathbb{Q}(C_q)$ and $x, y$ in $H_2(\widetilde{W})$,

$$\phi_\chi(W)(x \otimes a, y \otimes b) = \overline{a} b \sum_{i=1}^{q} \widetilde{\phi}(x, \alpha^i y) \overline{\alpha}^i,$$

where $a \to \bar{a}$ denotes the involution on $\mathbb{Q}(C_q)$ induced by complex conjugation.

**Definition 2.2** The Casson-Gordon $\sigma$-invariant of $(M, \chi)$ and the related nullity are

$$\sigma(M, \chi) := \frac{1}{n}\big(\mathrm{Sign}(\phi_\chi(W)) - \mathrm{Sign}(W)\big)$$
$$\eta(M, \chi) := \dim\ H^t_1(M; \mathbb{Q}(C_q)).$$

If $U$ is a closed 4-manifold and $\chi\colon H_1(U) \to C_q$ we may define $\phi_\chi(U)$ as above. One has that modulo torsion the bordism group $\Omega_4(B(C_q))$ is generated by the constant map from $CP(2)$ to $B(C_q)$. If $\chi$ is trivial, one has that $\mathrm{Sign}(\phi_\chi(U)) = \mathrm{Sign}(U)$. Since both signatures are invariant under cobordism, one has in general that $\mathrm{Sign}(\phi_\chi(U)) = \mathrm{Sign}(U)$. The independence of $\sigma(M,\chi)$ from the choice of $W$ and $n$ follows from this and Novikov additivity. One may see directly that these invariants do not depend on the choice of $q$. In this way Casson and Gordon argued that $\sigma(M,\chi)$ is an invariant. Alternatively one may use the Atiyah-Singer G-Signature theorem and Novikov additivity [AS].

We now describe a way to compute $\sigma(M,\chi)$ for a given surgery presentation of $(M,\chi)$.

**Definition 2.3** Let $K$ be an oriented knot in $S^3$. Let $A$ be an embedded annulus such that $\partial A = K \cup K'$ with $lk(K,K') = f$. A *$p$-cable on $K$ with twist $f$* is defined to be the union of oriented parallel copies of $K$ lying in $A$ such that the number of copies with the same orientation minus the number with opposite orientation is equal to $p$.

Let us suppose that $M$ is obtained by surgery on a framed link $L = L_1 \cup \cdots \cup L_\mu$ with framings $f_1, \ldots, f_\mu$. One shows that the linking matrix $\Lambda$ of $L$ with framings in the diagonal is a presentation matrix of $H_1(M)$ and a character on $H_1(M)$ is determined by $\alpha^{p_i} = \chi(m_{L_i}) \in C_q$ where $m_{L_i}$ denotes the class of the meridian of $L_i$. Let $\vec{p} = (p_1, \ldots, p_\mu)$. We use the following generalization of a formula in [CG2, Lemma (3.1)], where all $p_i$ are assumed to be 1, that is given in [Gi2, Theorem(3.6)].

**Proposition 2.4** *Suppose $\chi$ maps onto $C_q$. Let $L'$ with $\mu'$ components be the link obtained from $L$ by replacing each component by a non-empty algebraic $p_i$-cable with twist $f_i$ along this component. Then, if $\lambda = e^{2ir\pi/q}$, for $(r,q) = 1$, one has*

$$\sigma(M,\chi^r) = \sigma_{L'}(\lambda) - \mathrm{Sign}(\Lambda) + 2\frac{r(q-r)}{q^2}\vec{p}^{\top}\Lambda\vec{p},$$

$$\eta(M,\chi^r) = \eta_{L'}(\lambda) - \mu' + \mu.$$

The following proposition collects some easy additivity properties of the $\sigma$-invariant and the nullity under the connected sum.

**Proposition 2.5** *Suppose that $M_1$, $M_2$ are connected. Then, for all $\chi_i \in H^1(M_i; C_q)$, $i = 1, 2$, we have*

$$\sigma(M_1 \# M_2, \chi_1 \oplus \chi_2) = \sigma(M_1,\chi_1) + \sigma(M_2,\chi_2).$$

*If both $\chi_i$ are non-trivial, then*

$$\eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2) + 1.$$

*If one $\chi_i$ is trivial, then*

$$\eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2).$$

**Proposition 2.6** *For all $\chi \in H_1(S^1 \times S^2; C_q)$, we have*

$$\sigma(S^1 \times S^2, \chi) = 0$$

*If $\chi \neq 0$, then $\eta(S^1 \times S^2, \chi) = 0$. If $\chi = 0$, then $\eta(S^1 \times S^2, \chi) = 1$.*

Proposition 2.6 for non-trivial $\chi$ can be proved for example by the use of Proposition 2.4, since $S^1 \times S^2$ is obtained by surgery on the unknot framed $0$. However it is simplest to derive this result directly from the definitions.

## 2.3 The Casson-Gordon $\tau$-invariant

In this section, we recall the definition and some of the properties of the Casson-Gordon $\tau$-invariant. Let $C_\infty$ denote a multiplicative infinite cyclic group generated by $t$. For $\chi^+ \colon H_1(M) \to C_q \oplus C_\infty$, we denote $\bar{\chi} \colon H_1(M) \to C_q$ the character obtained by composing $\chi^+$ with projection on the first factor. The character $\chi^+$ induces a $C_q \times C_\infty$-covering $\widetilde{M}_\infty$ of $M$.

Since the bordism group $\Omega_3(B(C_q \times C_\infty)) = C_q$, bounds a compact 4-manifold $W$ over $B(C_q \times C_\infty)$ Again $n$ can be taken from to be $q$.

If we identify $\mathbb{Z}[C_q \times C_\infty]$ with the Laurent polynomial ring $\mathbb{Z}[C_q][t, t^{-1}]$, the field $\mathbb{Q}(C_q)(t)$ of rational functions over the cyclotomic field $\mathbb{Q}(C_q)$ is a flat $\mathbb{Z}[C_q \times C_\infty]$-module. We consider the chain complex $C_*(\widetilde{W}_\infty)$ as a $\mathbb{Z}[C_q \times C_\infty]$-module given by the deck transformation of the covering. Since $W$ is compact, the vector space $H_2^t(W; \mathbb{Q}(C_q)(t)) \simeq H_2(\widetilde{W}_\infty) \otimes_{\mathbb{Z}[C_q][t,t^{-1}]} \mathbb{Q}(C_q)(t)$ is finite dimensional.

We let $J$ denote the involution on $\mathbb{Q}(C_q)(t)$ that is linear over $\mathbb{Q}$ sends $t^i$ to $t^{-i}$ and $\alpha^i$ to $\alpha^{-i}$. As in [G], one defines a hermitian form, with respect to $J$,

$$\phi_{\chi^+} \colon H_2^t(W; \mathbb{Q}(C_q)(t)) \times H_2^t(W; \mathbb{Q}(C_q)(t)) \to \mathbb{Q}(C_q)(t),$$

such that

$$\phi_{\chi^+}(x \otimes a, y \otimes b) = J(a) \cdot b \cdot \sum_{i \in \mathbb{Z}} \sum_{j=1}^{q} \widetilde{\phi^+}(x, t^i \alpha^j y) \overline{\alpha}^j t^{-i}.$$

Here $\widetilde{\phi^+}$ denotes the ordinary intersection form on $\widetilde{W}_\infty$. Let $\mathcal{W}(\mathbb{Q}(C_q)(t))$ be the Witt group of non-singular hermitian forms on finite dimensional $\mathbb{Q}(C_q)(t)$ vector spaces. Let us consider $H_2^t(W;\mathbb{Q}(C_q)(t))/(\mathrm{Radical}(\phi_{\chi^+}))$. The induced form on it represents an element in $\mathcal{W}\ (\mathbb{Q}(C_q)(t))$, which we denote $w(W)$. Furthermore, the ordinary intersection form on $H_2(W;\mathbb{Q})$ represents an element of $\mathcal{W}(\mathbb{Q})$. Let $w_0(W)$ be the image of this element in $\mathcal{W}(\mathbb{Q}(C_q)(t))$.

**Definition 2.7** The Casson-Gordon $\tau$-invariant of $(M,\chi^+)$ is

$$\tau(M,\chi^+) := \frac{1}{n}\big(w(W) - w_0(W)\big) \in \mathcal{W}(\mathbb{Q}(C_q)(t)) \otimes \mathbb{Q}.$$

Suppose that $nM$ bounds another compact 4-manifold $W'$ over $B(C_q \times C_\infty)$. Form the closed compact manifold over $B(C_q \times C_\infty)$, $U := W \cup W'$ by gluing along the boundary. By Novikov additivity, we get $w(U) - w_0(U) = \big(w(W) - w_0(W)\big) - \big(w(W') - w_0(W')\big)$. Using [CF], the bordism group $\Omega_4(B(C_q \times C_\infty))$, modulo torsion, is generated by $CP(2)$, with the constant map to $B(C_q \times C_\infty)$. We have that $w(CP(2)) = w_0(CP(2))$. Since $w(U)$, and $w_0(U)$ only depend on the bordism class of $U$ over $B(C_q \times C_\infty)$, it follows that $w(U) = w_0(U)$ and $\tau(M,\chi^+)$ is independent of the choice of $W$. Using the above techniques, one may check $\tau(M,\chi^+)$ is independent of $n$.

If $A \in \mathcal{W}(\mathbb{Q}(C_q)(t))$, let $A(t)$ be a matrix representative for $A$. The entries of $A(t)$ are Laurent polynomials with coefficients in $\mathbb{Q}(C_q)$. If $\lambda$ is in $S^1 \subset \mathbb{C}$, then $A(\lambda)$ is hermitian and has a well defined signature $\sigma_\lambda(A)$. One can view $\sigma_\lambda(A)$ as a locally constant map on the complement of the set of the zeros of $\det A(\lambda)$. As in [CG1], we re-define $\sigma_\lambda(A)$ at each point of discontinuity as the average of the one-sided limits at the point.

We have the following estimate [Gi3, Equation (3.1)].

**Proposition 2.8** *Let $\chi^+ \colon H_1(M) \to C_q \oplus C_\infty$ and $\bar\chi \colon H_1(M) \to C_q$ be $\chi^+$ followed by the projection to $C_q$. We have*

$$|\sigma_1\big(\tau(M,\chi^+)\big) - \sigma(M,\bar\chi)| \leq \eta(M,\bar\chi).$$

## 2.4 Linking forms

Let $M$ be a rational homology 3-sphere with linking form

$$l \colon H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}.$$

We have that $l$ is non-singular, that is the adjoint of $l$ is an isomorphism $\iota \colon H_1(M) \to \mathrm{Hom}(H_1(M), \mathbb{Q}/\mathbb{Z})$. Let $H_1(M)^*$ denote $\mathrm{Hom}(H_1(M), \mathbb{C}^*)$. Let

$\nu$ denote the map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$ that sends $\frac{a}{b}$ to $e^{\frac{2\pi i a}{b}}$. So we have an isomorphism $\jmath\colon H_1(M) \to H_1(M)^*$ given by $x \mapsto \nu \circ \iota(x)$. Let $\beta\colon H_1(M)^* \times H_1(M)^* \to \mathbb{Q}/\mathbb{Z}$ be the dual form defined by $\beta(\jmath x, \jmath y) = -l(x, y)$.

**Definition 2.9** The form $\beta$ is metabolic with metabolizer $H$ if there exists a subgroup $H$ of $H_1(M)^*$ such that $H^\perp = H$.

**Lemma 2.10** [Gi1] *If $M$ bounds a spin 4-manifold $W$ then $\beta = \beta_1 \oplus \beta_2$ where $\beta_2$ is metabolic and $\beta_1$ has an even presentation with rank dim $H_2(W;\mathbb{Q})$ and signature Sign$(W)$. Moreover, the set of characters that extend to $H_1(W)$ forms a metabolizer for $\beta_2$.*

## 2.5 Link invariants

Let $L = L_1 \cup \cdots \cup L_\mu$ be an oriented link in $S^3$. Let $N_2$ be the two-fold covering of $S^3$ branched along $L$ and $\beta_L$ be the linking form on $H_1(N_2)^*$, see previous section.

We suppose that the Alexander polynomial of $L$ satisfies

$$\Delta_L(-1) \neq 0.$$

Hence, $N_2$ is a rational homology sphere. Note that if $\Delta_L(-1) \neq 1$, then $H_1(N_2;\mathbb{Z})$ is non-trivial.

**Definition 2.11** For all characters $\chi$ in $H_1(N_2)^*$, the Casson-Gordon $\sigma$-invariant of $L$ and the related nullity are (see Definition 2.2):

$$\sigma(L,\chi) := \sigma(N_2,\chi),$$

$$\eta(L,\chi) := \eta(N_2,\chi).$$

**Remark 2.12** If $L$ is a knot, then Definition 2.11 coincides with $\sigma(L,\chi)$ defined in [CG1, p.183].

# 3 Framed link descriptions

In this section, we study the Casson-Gordon $\tau$-invariants of the two-fold cover $M_2$ of the manifold $M_0$ described below.

Let $S^3 - T(L)$ be the complement in $S^3$ of an open tubular neighborhood of $L$ in $S^3$ and $P$ be a planar surface with $\mu$ boundary components.

Let $S$ be a Seifert surface for $L$ and $\gamma_i$ for $i = 1, \dots, \mu$ be the curves where $S$ intersects the boundary of $S^3 - T(L)$. We define $M_0$ as the result of gluing $P \times S^1$ to $S^3 - T(L)$, where $P \times 1$ is glued along the curves $\gamma_i$. Let $*$ be a point in the boundary of $P$.

A recipe for drawing a framed link description for $M_0$ is given in the proof of Proposition 3.1.

**Proposition 3.1**

$$H_1(M_0) \simeq \mathbb{Z} \oplus \mathbb{Z}^{\mu-1} \simeq \langle m \rangle \oplus \ \mathbb{Z}^{\mu-1},$$

*where $m$ denotes the class of $* \times S^1$ in $P \times S^1$.*

**Proof** Form a 4-manifold $X$ by gluing $P \times D^2$ to $D^4$ along $S^3$ in such a way that the total framing on $L$ agrees with the Seifert surface $S$. The boundary of this 4-manifold is $M_0$. We can get a surgery description of $M_0$ in the following way: pick $\mu - 1$ paths of $S$ joining up the components of $L$ in a chain. Deleting open neighborhoods of these paths in $S$ gives a Seifert surface for a knot $L'$ obtained by doing a fusion of $L$ along bands that are neighborhoods of the original paths. Put a circle with a dot around each of these bands (representing a 4-dimensional 1-handle in Kirby's [K] notation), and the framing zero on $L'$. This describes a handlebody decomposition of $X$.

One can then get a standard framed link description of $M_0$ by replacing the circle with dots with unknots $T_1, \dots, T_{\mu-1}$ framed zero. This changes the 4-manifold but not the boundary. Note also that $lk(T_i, T_j) = 0$ and $lk(T_i, L') = 0$ for all $i = 1, \dots, \mu - 1$. Hence $H_1(M_0) \simeq \mathbb{Z}^\mu$ and $m$ represents one of the generators. □

We now consider an infinite cyclic covering $M_\infty$ of $M_0$, defined by a character $H_1(M_0) \to C_\infty = \langle t \rangle$ that sends $m$ to $t$ and the other generators to zero. Let us denote by $M_2$ the intermediate two-fold covering obtained by composing this character with the quotient map $C_\infty \to C_2$ sending $t$ to $-1$. Let $m_2$ denote the loop in $M_2$ given by the inverse image of $m$. A recipe for drawing a framed link description for $M_2$ is given in the proof of Remark 3.3.

**Proposition 3.2** *There is an isomorphism between $H_1(N_2)$ and the torsion subgroup of $H_1(M_2)$, which only depends on $L$. Moreover*

$$H_1(M_2) \simeq H_1(N_2) \oplus \mathbb{Z}^{\mu} \simeq H_1(N_2) \oplus \langle m_2 \rangle \oplus \mathbb{Z}^{\mu-1}.$$

**Proof** Let $R$ be the result of gluing $P \times D^2$ to $S^3 \times I$ along $L \times 1 \subset S^3 \times 1$ using the framing given by the Seifert surface. Thus $R$ is the result of adding $\mu - 1$ 1-handles to $S^3 \times I$ and then one 2-handle along $L'$, as in the proof above. Then $X$ in the proof above can be obtained by gluing $D^4$ to $R$ along $S^3 \times 0$. Since $D^2$ is the double branched cover of itself along the origin, $P \times D^2$ is the double branched cover of itself along $P \times 0$. Let $R_2$ denote the double branched cover of $R$ that is obtained by gluing $P \times D^2$ to $N_2 \times I$ along a neighborhood of the lift of $L \times 1 \subset S^3 \times 1$. We have that $\partial R_2 = -N_2 \sqcup M_2$, where $R_2$ is the result of adding $\mu - 1$ 1-handles to $N_2 \times I$ and then one 2-handle along the lift $L'$. Moreover this lift of $L'$ is null-homologous in $N_2$. It follows that $H_1(R_2)$ is isomorphic to $H_1(N_2) \oplus \mathbb{Z}^{\mu-1}$, with the inclusion of $N_2$ into $R_2$ inducing an isomorphism $i_N$ of $H_1(N_2)$ to the torsion subgroup of $H_1(R_2)$. Turning this handle decomposition upside down we have that $R_2$ is the result of adding to $M_2 \times I$ one 2-handle along a neighborhood of $m_2$ and then $\mu - 1$ 3-handles. It follows that $H_1(R_2) \oplus \mathbb{Z} = H_1(R_2) \oplus \langle m_2 \rangle$ is isomorphic to $H_1(M_2)$ with the inclusion of $M_2$ in $R_2$ inducing an isomorphism $i_M$ of the torsion subgroup $H_1(M_2)$ to the torsion subgroup of $H_1(R_2)$. Thus $(i_M)^{-1} \circ i_N$ is an isomorphism from $H_1(N_2)$ to the torsion subgroup of $H_1(M_2)$ and this isomorphism is constructed without any arbitrary choices. □

**Remark 3.3** We could have proved Proposition 3.1 in a similar way to the proof of Proposition 3.2. We could have also proved Proposition 3.2 (except for the isomorphism only depending on $L$) in a similar way to the proof of Proposition 3.1 as follows. We can find a surgery description of $M_2$ from a surgery description of $N_2$. The procedure of how to visualize a lift of $L$ and the surface $S$ in $N_2$ is given in [AK]. One considers the lifts of the paths chosen in the proof of Proposition 3.1, on the lift of $S$. One then fuses the components of the lift of $L$ along these paths, obtaining a lift of $L'$. The surgery description of $M_2$ is obtained by adding to the surgery description of $N_2$ the lift of $L'$ with zero framing together with $\mu - 1$ more unknotted zero-framed components encircling each fusion. The linking matrix of this link is a direct sum of that of $N_2$ and a $\mu \times \mu$ zero matrix.

Let $i_T$ denote the inclusion of the torsion subgroup of $H_1(M_2)$ into $H_1(M_2)$, and let $\psi\colon\ H_1(N_2) \to H_1(M_2)$ denote the monomorphism given by $i_T \circ (i_M)^{-1} \circ i_N$.

**Theorem 3.4** *Let $\chi^+\colon\ H_1(M_2) \to C_q \oplus C_\infty$. Let $\chi\colon\ H_1(N_2) \to C_q$ be $\chi^+ \circ \psi$ composed with the projection to $C_q$. We have that:*

$$|\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| \leq \eta(L, \chi) + \mu.$$

**Remark 3.5** If $L$ is a knot, then $\tau(M_2, \chi^+)$ coincides with $\tau(L, \chi)$ defined in [CG1, p.189].

**Proof of Theorem 3.4** We use the surgery description of $M_2$ given in Remark 3.3. Let $P$ be given by the surgery description of $M_2$ but with the component corresponding to $L'$ deleted. Hence,

$$P = N_2 \sharp_{(\mu-1)} S^1 \times S^2.$$

$\chi+$ induces some character $\chi'$ on $H_1(P)$.

According to Section 2.3, we let $\overline{\chi} \in H^1(M_2; C_q)$ and $\overline{\chi}' \in H^1(P; C_q)$ denote the characters $\chi^+$ and $\chi'$ followed by the projection $C_q \oplus C_\infty \to C_q$. Using Propositions 2.5 and 2.6, one has that

$$\sigma(P, \overline{\chi}') = \sigma(L, \chi) \text{ and } \eta(P, \overline{\chi}') = \eta(L, \chi) + \mu - 1.$$

Moreover, since $M_2$ is obtained by surgery on $L'$ in $P$, it follows from [Gi3, Proposition (3.3)] that

$$|\sigma(P, \overline{\chi}') - \sigma(M_2, \overline{\chi})| + |\eta(M_2, \overline{\chi}) - \eta(P, \overline{\chi}')| \leq 1 \text{ or}$$

$$|\sigma(L, \chi) - \sigma(M_2, \overline{\chi})| + |\eta(M_2, \overline{\chi}) - \eta(L, \chi) - \mu + 1| \leq 1.$$

Thus

$$|\sigma(L, \chi) - \sigma(M_2, \overline{\chi})| \leq \eta(L, \chi) + \mu - \eta(M_2, \overline{\chi}).$$

Finally, one gets, by Theorem 2.8,

$$\begin{aligned} |\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| &\leq |\sigma_1(\tau(M_2, \chi^+)) - \sigma(M_2, \overline{\chi})| + |\sigma(M_2, \overline{\chi}) - \sigma(L, \chi)| \\ &\leq \eta(M_2, \overline{\chi}) + \eta(L, \chi) + \mu - \eta(M_2, \overline{\chi}) = \eta(L, \chi) + \mu. \end{aligned}$$

□

# 4 The slice genus of links

See Section 2.5 for notations.

**Theorem 4.1** *Suppose $L$ is the boundary of a connected oriented properly embedded surface $F$ of genus $g$ in $B^4$, and that $\Delta_L(-1) \neq 0$. Then, $\beta_L$ can be written as a direct sum $\beta_1 \oplus \beta_2$ such that the following two conditions hold:*

1) *$\beta_1$ has an even presentation of rank $2g+\mu-1$ and signature $\sigma_L(-1)$, and $\beta_2$ is metabolic.*

2) *There is a metabolizer for $\beta_2$ such that for all characters $\chi$ of prime power order in this metabolizer,*

$$|\sigma(L,\chi)+\sigma_L(-1)| \leq \eta(L,\chi)+4g+3\mu-2.$$

**Proof** We let $b_i(X)$ denote the ith Betti number of a space $X$. We have $b_1(F)=2g+\mu-1$.

Let $W_0'$, with boundary $M_0'$, be the complement of an open tubular neighborhood of $F$ in $B^4$. By the Thom isomorphism, excision, and the long exact sequence of the pair $(B^4, W_0')$, $W_0'$ has the homology of $S^1$ wedge $b_1(F)$ 2-spheres. Let $W_2'$ with boundary $M_2'$ be the two-fold covering of $W_0'$. Note that if $F$ is planar, $M_0'=M_0$, and $M_2'=M_2$ (see Section 3).

Let $V_2$ be the two-fold covering of $B^4$ with branched set $F$. Note that $V_2$ is spin as $w_2(V_2)$ is the pull-up of a class in $H^2(B^4,\mathbb{Z}_2)$, by [Gi5, Theorem 7], for instance. The boundary of $V_2$ is $N_2$. As in [Gi1], one calculates that $b_2(V_2)=2g+\mu-1$. One has $\mathrm{Sign}(V_2)=\sigma_L(-1)$ by [V].

By Lemma 2.10, $\beta_L$ can be written as a direct sum $\beta_1\oplus\beta_2$ as in condition 1) above, such that the characters on $H_1(N_2)$ that extend to $H_1(V_2)$ form a metabolizer $H$ for $\beta_2$. We now suppose $\chi\in H$ and show that Condition 2) holds for $\chi$.

We also let $\chi$ denote an extension of $\chi$ to $H_1(V_2)$ with image some cyclic group $C_q$ where $q$ is a power of a prime integer (possibly larger than those corresponding to the character on $H_1(N_2)$). Of course $\chi\in H^1(V_2,C_q)$ restricted to $W_2'$ extends $\chi$ restricted to $M_2'$. We simply denote all these restrictions by $\chi$.

Let $W_\infty'$ denote the infinite cyclic cover of $W_0'$. Note that $W_2'$ is a quotient of this covering space. $\chi$ induces a $C_q$-covering of $V_2$ and thus of $W_2'$. If we pull the $C_q$-covering of $W_2'$ up to $W_\infty'$, we obtain $\widetilde{W_\infty'}$, a $C_q\times C_\infty$-covering of $W_2'$. If we identify properly $F\times S^1$ in $M_2'$, this covering restricted to $F\times S^1$ is given by a character $H_1(F\times S^1)\simeq H_1(F)\oplus H_1(S^1)\to C_q\times C_\infty$ that maps $H_1(F)$ to zero in $C_\infty$, $H_1(S^1)$ to zero in $C_q$ and isomorphically onto $C_\infty$. For this note: since $\mathrm{Hom}(H_1(F),\mathbb{Z})=H^1(F)=[F,S^1]$, we may define diffeomorphisms of $F\times S^1$ that induce the identity on the second factor of $H_1(F\times S^1)\approx H_1(F)\oplus\mathbb{Z}$, and send $(x,0)\in H_1(F)\oplus\mathbb{Z}$, to $(x,f(x))\in H_1(F)\oplus\mathbb{Z}$, for any $f\in\mathrm{Hom}(H_1(F),\mathbb{Z})$.

As in [Gi1], choose inductively a collection of $g$ disjoint curves in the kernel of $\chi$ that form a metabolizer for the intersection form on $H_1(F)/H_1(\partial F)$. By taking a tubular neighborhood of these curves in $F$, we obtain a collection of $S^1\times I$ embedded in $F$. Using these embeddings we can attach round 2-handles $(B^2\times I)\times S^1$ along $(S^1\times I)\times S^1$ to the trivial cobordism $M_2'\times I$ and obtain a cobordism $\Omega$ between $M_2$ and $M_2'$.

Let $U=W_2'\cup_{M_2'}\Omega$ with boundary $M_2$. The $C_q\times C_\infty$-covering of $W_2'$ extends uniquely to $U$. Note that $\Omega$ may also be viewed as the result of attaching round 1-handles to $M_2\times I$.

As in [Gi1], $\mathrm{Sign}(W_2')=\mathrm{Sign}(V_2)$. Since the intersection form on $\Omega$ is zero, we get $\mathrm{Sign}(U)=\mathrm{Sign}(W_2')=\mathrm{Sign}(V_2)=\sigma_L(-1)$. The $C_q\times C_\infty$-covering of $\Omega$, restricted to each round 2-handle is $q$ copies of $B^2\times I\times\mathbb{R}$ attached to the trivial cobordism $\widetilde{M'_\infty}\times I$ along $q$ copies of $S^1\times I\times\mathbb{R}$. Using a Mayer-Vietoris sequence, one sees that the inclusion induces an isomorphism (which preserves the Hermitian form)

$$H_2^t(U;\mathbb{Q}(C_q)(t))\simeq H_2^t(W_2';\mathbb{Q}(C_q)(t)).$$

Thus, if $w(W_2')$ denotes the image of the intersection form on $H_2^t(W_2';\mathbb{Q}(C_q)(t))$ in $\mathcal{W}\ (\mathbb{Q}(C_q)(t))$, we get $\sigma_1(\tau(M_2,\chi^+))=\sigma_1(w(W_2'))-\sigma_L(-1)$.

If $q$ is a prime power, we may apply Lemma 2 of [Gi1] and conclude that $H_i(\widetilde{W'_\infty};\mathbb{Q})$ is finite dimensional for all $i\neq 2$. Thus, $H_i^t(W_2';\mathbb{Q}(C_q)(t))$ is zero for all $i\neq 2$. Since the Euler characteristic of $W_2'$ with coefficients in $\mathbb{Q}(C_q)(t)$ coincides with those with coefficients in $\mathbb{Q}$, we get $\dim\, H_2^t(W_2';\mathbb{Q}(C_q)(t))=\chi(W_2')=2\chi(W_0')=2(1-\chi(F))=2b_1(F)$. Thus $|\sigma_1(\tau(M_2,\chi^+)+\sigma_L(-1)|\le 2b_1(F)$. Hence,

$$\begin{aligned}&|\sigma(L,\chi)+\sigma_L(-1)|\le|\sigma(L,\chi)-\sigma_1(\tau(M_2,\chi)^+)|+|\sigma_1(\tau(M_2,\chi)^+)+\sigma_L(-1)|\\&\le\eta(L,\chi)+\mu+2(2g+\mu-1)=\eta(L,\chi)+4g+3\mu-2 \text{ by Theorem 3.4.}\end{aligned}$$ □

# 5 Examples

Let $L=L_1\cup L_2$ be the link with two components of Figure 1 and $S$ be the Seifert surface of $L$ given by the picture. The squares with $K$ denote two parallel copies with linking number 0 of an arc tied in the knot $K$. Note that $L$ is actually a family of examples. Specific links are determined by the choice of two parameters: a knot $K$ and a positive integer $h$. Since $S$ has genus $h$, the slice genus of $L$ is at most $h$.

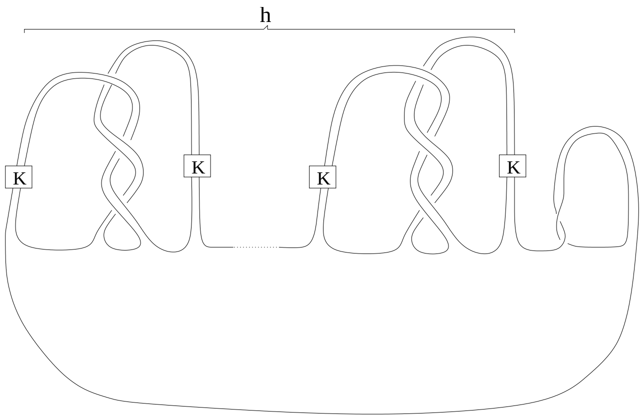


Figure 1: The link $L$

One calculates that $\sigma_L(\lambda) = 1$, and $n_L(\lambda) = 0$ for all $\lambda$. Thus, the Murasugi-Tristram inequality says nothing about the slice genus of $L$. In fact, if $K$ is a slice knot, then one can surger this surface to obtain a smooth cylinder in the 4-ball with boundary $L$. Thus there can be no arguments based solely on a Seifert pairing for $L$ that would imply that the slice genus is non-zero.

**Theorem 5.1** *If $\sigma_K(e^{2i\pi/3}) \geq 2h$ or $\sigma_K(e^{2i\pi/3}) \leq -2h-2$, then $L$ has slice genus $h$.*

**Proof** Using [AK], a surgery presentation of $N_2$ as surgery on a framed link of $2h+1$ components can be obtained from the surface $S$ (see Figure 2).

Let $Q$ be the 3-manifold obtained from the link pictured in Figure 2. Here $K'$ denotes $K$ with the string orientation reversed. Since $RP(3)$ is obtained by surgery on the unknot framed 2, we get:

$$N_2 = RP(3)\#_h Q.$$

The linking matrix of the framed link of the surgery presentation of $N_2$ is $\Lambda = [2] \bigoplus \oplus^h \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$. $\Lambda$ is a presentation matrix of $(H_1(N_2)^*, \beta_L)$; we obtain

$$H_1(N_2)^* \simeq \mathbb{Z}_2 \bigoplus \oplus^{2h} \mathbb{Z}_3$$

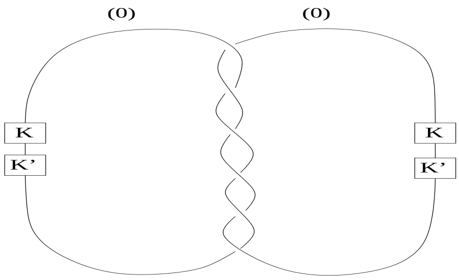


Figure 2: Surgery presentation of $Q$

and $\beta_L$ is given by the following matrix, with entries in $\mathbb{Q}/\mathbb{Z}$:

$$[1/2] \bigoplus \oplus^h \begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix}.$$

By Theorem 4.1, if $L$ bounds a surface of genus $h-1$ in $B^4$, then $\beta_L$ must be decomposed as $\beta_1 \oplus \beta_2$ where:

1) $\beta_1$ has an even presentation matrix of rank $2h-1$, and signature 1 (all we really need here is that it has a rank $2h-1$ presentation.)

2) $\beta_2$ is metabolic and for all characters $\chi$ of prime power order in some metabolizer of $\beta_2$, the following inequality holds:

$$(*) \qquad |\sigma(L,\chi)+1| - \eta(L,\chi) \leq 4h.$$

As $\mathbb{Z}_2 \bigoplus \oplus^{2h}\mathbb{Z}_3$ does not have a rank $2h-1$ presentation, $\beta_2$ is non-trivial. As metabolic forms are defined on groups whose cardinality is a square, $\beta_2$ is defined on a group with no 2-torsion. Thus the metabolizer contains a non-trivial character of order three satisfying $\beta_L(\chi,\chi) = 0$.

The first homology of $Q$ is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by, say, $m_1$ and $m_2$, positive meridians of these components. Each of these components is oriented counter-clockwise. We first work out $\sigma(Q,\chi)$ and $\eta(Q,\chi)$ for characters of order three. Let $\chi_{(a_1,a_2)}$ denote the character on $H_1(Q)$ sending $m_j$ to $e^{\frac{2i\pi a_j}{3}}$, where the $a_j$ take the values zero and $\pm 1$.

We use Proposition 2.4 to compute $\sigma(Q,\chi_{(1,0)})$ and $\eta(Q,\chi_{(1,0)})$ assuming that $K$ is trivial. For this, one may adapt the trick illustrated on a link with 2 twists between the components [Gi2, Fig (3.3), Remark (3.65b)]. In the case $K$ is the unknot, we obtain

$$\sigma(Q,\chi_{(1,0)}) = 1 \quad \text{and} \quad \eta(Q,\chi_{(1,0)}) = 0.$$

It is not difficult to see that inserting the knots of the type $K$ changes the result as follows (note that $K$ and $K'$ have the same Tristram-Levine signatures):

$$\sigma(Q,\chi_{(1,0)}) = 1 + 2\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q,\chi_{(1,0)}) = 0.$$

These same values hold for the characters $\chi_{(-1,0)}$ and $\chi_{(0,\pm 1)}$ by symmetry.

Using Proposition 2.4

$$\sigma(Q,\pm\chi_{(1,1)}) = -1 - 24/9 + 4\sigma_K(e^{2\pi i/3}), \qquad \eta(Q,\pm\chi_{(1,1)}) = 0$$

$$\sigma(Q,\pm\chi_{(1,-1)}) = 4 + 24/9 + 4\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q,\pm\chi_{(1,-1)}) = 1.$$

One also has

$$\sigma(Q,\chi_{(0,0)}) = 0 \quad \text{and} \quad \eta(Q,\chi_{(0,0)}) = 0.$$

Any order three character on $N_2$ that is self annihilating under the linking form is given as the sum of the trivial character on $RP(3)$ and characters of type $\chi_{(0,0)}$, $\chi_{(\pm 1,0)}$ and $\chi_{(0,\pm 1)}$ on $Q$ and characters of type $\pm\chi_{(1,1)} + \pm\chi_{(1,-1)}$ on $Q\#Q$. Using Proposition 2.5, one can calculate $\sigma(L,\chi)$ and $\eta(L,\chi)$ for all these characters $\chi$. It is now a trivial matter to check that for every non-trivial character with $\beta(\chi,\chi) = 0$, the inequality (*) is not satisfied. □

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