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The space of intervals in a Euclidean space

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Abstract For a path-connected space X, a well-known theorem of Segal, May and Milgram asserts that the configuration space of finite points in $\mathbb{R}^n$ with labels in X is weakly homotopy equivalent to $\Omega^n\Sigma^n X$. In this paper, we introduce a space $I_n(X)$ of intervals suitably topologized in $\mathbb{R}^n$ with labels in a space X and show that it is weakly homotopy equivalent to $\Omega^n\Sigma^n X$ without the assumption on path-connectivity.

AMS Classification 55P35; 55P40

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1 Introduction

G.Segal [8] introduced the configuration space $C(\mathbb{R}^n, X)$ of finite number of points in $\mathbb{R}^n$ with labels in a space X and showed that $C(\mathbb{R}^n, X)$ is weakly homotopy equivalent to $\Omega^n\Sigma^n X$ if X is path-connected. When X is not path-connected, it follows from Segal's result that $\Omega^n\Sigma^n X$ is a group-completion of $C(\mathbb{R}^n, X)$, i.e. that $H_*(C(\mathbb{R}^n, X); k)[\pi^{-1}]$ is isomorphic to $H_*(\Omega^n\Sigma^n X; k)$ for any field k, where $[\pi^{-1}]$ denotes the localization of the Pontrjagin ring $H_*(C(\mathbb{R}^n, X); k)$ with respect to a sub-monoid $\pi = \pi_0(C(\mathbb{R}^n, X))$. (This was also shown independently by F.Cohen [2].) On the other hand, in [6], D.McDuff considered the space $C^\pm(M)$ of positive and negative particles in a manifold M and showed that it is weakly equivalent to some space of vector fields on M. The topology of $C^\pm(M)$ is given so that two particles cannot collide if they have the same parity, but they can collide and annihilate if they are oppositely charged. When $M = \mathbb{R}^n$, we can think of $C^\pm(\mathbb{R}^n)$ as a H-space obtained by adjoining homotopy inverses to $C(\mathbb{R}^n, S^0)$. Since adjoining homotopy inverses to a H-space is a sort of group completion, one might hope that $C^\pm(\mathbb{R}^n)$ is weakly equivalent to $\Omega^n\Sigma^n S^0 = \Omega^n S^n$, but in fact, $C^\pm(\mathbb{R}^n) \simeq_w \Omega^n(S^n \times S^n/\Delta)$, where Δ is the diagonal subspace [6]. The aim of this paper is to construct a configuration space model which is a group-completion of $C(\mathbb{R}^n, X)$, thus is weakly homotopy equivalent to $\Omega^n\Sigma^n X$ for any X.

Caruso and Waner [1] constructed such a group-completion model based on the space of little cubes [5]. They constructed the space of "signed cubes merged along the first coordinate" and showed that it approximates $\Omega^n\Sigma^n X$ without the assumption on path-connectivity of X. In this paper, we introduce a space $I_n(X)$ of intervals suitably topologized in $\mathbb{R}^n$ and show that it gives another model for the group-completion. Our construction is, in some sense, a direct generalization of $C(\mathbb{R}^n, X)$ and simpler than the Caruso-Waner model. More precisely, $I_n(X)$ is the space of intervals ordered along parallel axes in $\mathbb{R}^n$ with labels in X. In this space the topology is such that "cutting and pasting" and "birth and death" of intervals are allowed; i.e. cutting and pasting means that an interval with an open end and another interval with a closed end can be attached at those ends to constitute one interval if they have the same label in X; birth and death means that any half-open interval can vanish when its length tends to zero.

Now we can state our main theorem as follows:

Theorem 1.1 *There is a weak homotopy equivalence* $I_n(X) \simeq_w \Omega^n\Sigma^n X$.

As contrasted with particles, intervals have two obvious features: firstly, they are stretched in a direction, thus have a length; secondly, any interval can be supposed to have a charge (p,q) where p (resp. q) is $+1$ or -1 depending on whether the interval contains or not contains the left (resp. right) endpoint. The former feature results in the gradual interaction of our objects. For example, a possible process of annihilation of a closed and an open interval is that: firstly, they are attached into one half-open interval and then its length decreases and finally it vanishes — i.e. they gradually annihilate. The latter feature of the interval plays an important role when we construct an analogue of the electric field map [8] from a"thickened version" of $I_n(X)$ into $\Omega C(\mathbb{R}^{n-1}, X)$. The main step of our proof of Theorem 1.1 is first deforming $I_n(X)$ into this thickened but equivalent version, then constructing the map in question and showing it is a weak equivalence using quasifibration techniques. We then conclude using Segal's classical result as applied to ΣX which is now path-connected.

This paper can be considered as a first step of a larger project proposed by K.Shimakawa. His idea is to use manifolds in G-vector spaces to approximate $\Omega^{V^\infty}\Sigma^{V^\infty}X$ equivariantly, where G is a compact Lie group and V^∞ is an orthogonal G-vector space which contains all the irreducible G-representations infinitely many times as direct summands.

In §2, we settle the notation for the configuration space with labels in a partial abelian monoid and observe some of its properties. The definition of $I_n(X)$ is given in §3. In the same section, we construct modifications of $I_n(X)$ which is needed to prove Theorem 1.1. In §4, we construct a map $\alpha\colon \widetilde{I}_n(X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X)$ and state Proposition 4.2, which is the key to prove Theorem 1.1. A proof of Theorem 1.1 is given in the same section. We give a proof of Proposition 4.2 in §5. Throughout the paper, X is assumed to be a space with non-degenerate base point $*$.

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2 Configuration space and partial abelian monoid

The notion of the configuration space with summable labels appear in several papers [7],[4],[9]. In this section, we introduce one form of such notion adapted to the purpose of this paper. Remark that our definition of partial abelian monoid is a special case of that given in [9], of which we shall call 'two-generated' partial abelian monoid.

Definition 2.1 A partial abelian monoid (PAM for short) is a space M equipped with a subspace $M_2 \subset M \times M$ and a map $\mu\colon M_2 \to M$ such that

(1) $M \vee M \subset M_2,\ \mu(a, *_M) = \mu(*_M, a) = a$,

(2) $(a, b) \in M_2 \Leftrightarrow (b, a) \in M_2, \mu(a, b) = \mu(b, a)$, and

(3) $(\mu(a, b), c) \in M_2 \Leftrightarrow (a, \mu(b, c)) \in M_2, \mu(\mu(a, b), c) = \mu(a, \mu(b, c))$.

We write $\mu(a, b) = a + b$. An element in M_2 is called a summable pair. Let M_k denote the subspace of M^k which consists of those k-tuples $(a_1, \ldots, a_k)$ such that $a_1 + \cdots + a_k$ is defined. A map between PAMs are called a PAM homomorphism if it sends summable pairs to summable pairs and preserves the sum.

Definition 2.2 Let $Z_n^{(k)}(M)$ denote the subspace of $(\mathbb{R}^n \times M)^k$ given by:

$$Z_n^{(k)}(M) = \left\{ ((v_1, a_1), \ldots, (v_k, a_k)) \;\middle|\; \begin{array}{r} \text{for any } i_1, \ldots, i_r \text{ such that} \\ v_{i_1} = \cdots = v_{i_r}, \\ (a_{i_1}, \ldots, a_{i_r}) \in M_r \end{array} \right\}$$

Then we define a space $C(\mathbb{R}^n, M)$ as $C(\mathbb{R}^n, M) = (\coprod_{k \geq 0} Z_n^{(k)}(M))/\sim$, where $\sim$ denotes the least equivalence relation which satisfies (R1)$\sim$(R3) below.

(R1) If $a_i = *_M$ then

$$((v_1,a_1),\dots,(v_k,a_k)) \sim ((v_1,a_1),\dots,\widehat{(v_i,a_i)},\dots,(v_k,a_k)).$$

(R2) For any permutation $\sigma \in \Sigma_k$,

$$((v_1,a_1),\dots,(v_k,a_k)) \sim ((v_{\sigma^{-1}(1)},a_{\sigma^{-1}(1)}),\dots,(v_{\sigma^{-1}(k)},a_{\sigma^{-1}(k)})).$$

(R3) If $v_1 = v_2 = v$,

$$((v_1,a_1),\dots,(v_k,a_k)) \sim ((v,a_1+a_2),(v_3,a_3),\dots,(v_k,a_k)).$$

We regard $C(\mathbb{R}^n, M)$ as a PAM as follows. Let $C(\mathbb{R}^n, M)_2 \subset C(\mathbb{R}^n, M)^2$ denote the subspace which consists of pairs (ξ,η) which have representatives $((v_1,a_1),\dots,(v_k,a_k))$ and $((v_1,b_1),\dots,(v_k,b_k))$ such that v_i's are distinct and $(a_i,b_i) \in M_2$ for every i. (Note that some of a_i's or b_j's may be zero.) Then $\mu\colon C(\mathbb{R}^n, M)_2 \to C(\mathbb{R}^n, M)$ is defined by setting $\mu(\xi,\eta) = [(v_1,a_1+b_1),\dots,(v_k,a_k+b_k)]$. Note that $C(\mathbb{R}^n,-)$ is a self-functor on the category of partial abelian monoids.

Example 2.3 Any space X is regarded as a PAM by setting $X_2 = X \vee X$. Then $C(\mathbb{R}^n, X)$ is nothing but the configuration space of finite points in $\mathbb{R}^n$ labelled by X.

Example 2.4 Let $M = \{-1,0,1\}$ with $1+(-1)=0$ as the only non-trivial partial sum. Then $C(\mathbb{R}^n, M)$ is homeomorphic to $C^{\pm}(\mathbb{R}^n)$, the configuration space of positive and negative particles in $\mathbb{R}^n$, given in [6]. Furthermore, we give $X \wedge M$ a PAM structure by setting $(X\wedge M)_2 = \{(x,a;x,b) \mid (a,b) \in M_2\}$ and $(x,a)+(x,b) = (x,a+b)$. Then $C(\mathbb{R}^{n-1}, X\wedge M)$ is the labelled version of $C^{\pm}(\mathbb{R}^n)$ [1].

Lemma 2.5 below states that the functor $C(\mathbb{R}^n,-)$ preserves homotopies. Homotopy in the category of PAMs is defined as follows: Let M be a PAM. We regard $M \times I$ as a PAM by setting

$$(M\times I)_2 = \{(m,t;n,t) \mid (m,n) \in M_2\}$$

and
$$\mu_{M\times I}(m,t;n,t) = (\mu_M(m,n),t).$$

Homomorphisms $f,g\colon M \to N$ between PAMs are called homotopic via PAM homomorphisms if there exists a homomorphism $H\colon M \times I \to N$ such that $H_0 = f, H_1 = g$.

Lemma 2.5 *If $f,g\colon M \to N$ are homotopic via PAM homomorphisms then $C(\mathbb{R}^n, f)$ and $C(\mathbb{R}^n, g)$ are homotopic via PAM homomorphisms.*

Proof Let $H\colon M \times I \to N$ be a homotopy between f and g. Observe that we have a homomorphism $C(\mathbb{R}^n, M) \times I \to C(\mathbb{R}^n, M \times I)$ by setting

$$([(v_1, a_1), \dots, (v_k, a_k)], t) \mapsto [(v_1, (a_1, t)), \dots, (v_k, (a_k, t))].$$

Then this map followed by $C(\mathbb{R}^n, H)$ is a homotopy between $C(\mathbb{R}^n, f)$ and $C(\mathbb{R}^n, g)$ via PAM homomorphisms. □

Recall that the Moore loop space on a space X is defined by

$$\Omega(X) = \cup_{s \geq 0} \Omega_s(X) \times \{s\},$$

where $$\Omega_s(X) = \{l\colon [0, s] \to X \mid l(0) = l(s) = *\}$$

is the space of loops of length s. Recall also that $\Omega(X)$ is topologized as the subspace of $\mathrm{Map}([0, \infty), X) \times [0, \infty)$. Let M be a PAM. We give $\Omega_s(M)$ a PAM structure by setting

$$(\Omega_s(M))_2 = \{(l_1, l_2) \mid (l_1(t), l_2(t)) \in M_2 \text{ for all } t \in [0, s]\}$$

and $$(l_1 + l_2)(t) = l_1(t) + l_2(t) \in \Omega_s(M),\ t \in [0, s].$$

It is clear that Ω_s is a self-functor on the category of PAMs. We have a map $C(\mathbb{R}^n, \Omega_s(M)) \to \Omega_s C(\mathbb{R}^n, M)$ defined by

$$[(v_1, l_1), \dots, (v_k, l_k)] \mapsto (t \mapsto [(v_1, l_1(t)), \dots, (v_k, l_k(t))]),$$

which will be used in the construction of the map $\alpha\colon \widetilde{I}_n(X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X)$ in §4.

3 The space of intervals in $\mathbb{R}^n$

Let $\mathcal{I}$ denote the subspace of $\mathbb{R}^2 \times \{\pm 1\}^2$ consisting of all quadruples (u, v, p, q) such that $u < v$ if $p = q$ and $u \leq v$ if $p \neq q$. When $u < v$, $J = (u, v, p, q) \in \mathcal{I}$ can be identified with an interval in $\mathbb{R}$ whose endpoints are u and v. It contains (resp. not contains) the left endpoint if $p = 1$ (resp. $p = -1$). Similarly, it contains (resp. not contains) the right endpoint if $q = 1$ (resp. $q = -1$). Thus, for example, $J = (u, v, -1, 1)$ with $u < v$ is identified with the half-open interval $(u, v]$. For any $J = (u, v, p, q) \in \mathcal{I}$, we put $l(J) = u, r(J) = v, p_L(J) = p$ and $p_R(J) = q$. If $J, K \in \mathcal{I}$ and $r(J) \leq l(K)$ then we write $J \leq K$. If, moreover, $r(J) < l(K)$ then we write $J < K$.

Let U be a connected subset of $\mathbb{R}$. In applications, U is one of $(-s, s)$ or $(0, s)$ for $s > 0$ or $s = \infty$. Let $\mathcal{I}(U)$ denote the subspace of $\mathcal{I}$ consisting of those $J \in \mathcal{I}$ such that $l(J), r(J) \in U$. Remark that all the intervals in $\mathcal{I} = \mathcal{I}(\mathbb{R})$ are bounded.

Definition 3.1 We define $I_{(k)}(X)_U$ as the subspace of $(\mathcal{I}(U)\times X)^k$ consisting of those k-tuples $((J_1,x_1),\dots,(J_k,x_k))$ such that

(1) $J_1 \le J_2 \le \cdots \le J_k$,

(2) $x_{i-1} \ne x_i$ implies $J_{i-1} < J_i$, and

(3) $p_R(J_{i-1}) = p_L(J_i)$ implies $J_{i-1} < J_i$.

In other words, $J_1,\dots,J_k$ are disjoint ordered intervals of U with labels $x_1,\dots,x_k$ and J_{i-1} and J_i can have common endpoints only if the labels in X coincide and the given endpoints are of the opposite sign.

Definition 3.2 We define the space of intervals in U to be

$$I_1(X)_U = \coprod_{k\ge 0} I_{(k)}(X)_U/\sim,$$

where $\sim$ denotes the equivalence relation generated by the relation shown below. Suppose

$$\iota = ((J_1,x_1),\dots,(J_k,x_k)) \in I_{(k)}(X)_U$$

and
$$\iota' = ((K_1,y_1),\dots,(K_{k-1},y_{k-1})) \in I_{(k-1)}(X)_U.$$

Then $\iota' \sim \iota$ if one of the following holds:

(1) (cutting and pasting)

$$K_i = \begin{cases} J_i & \text{if } i<j \\ J_j \cup J_{j+1} & \text{if } i=j \\ J_{i+1} & \text{if } i>j \end{cases}, \quad y_i = \begin{cases} x_i & \text{if } i<j \\ x_j = x_{j+1} & \text{if } i=j \\ x_{i+1} & \text{if } i>j. \end{cases}$$

(2) (birth and death)

$$K_i = \begin{cases} J_i & \text{if } i<j \\ J_{i+1} & \text{if } i\ge j \end{cases}, \quad y_i = \begin{cases} x_i & \text{if } i<j \\ x_{i+1} & \text{if } i\ge j, \end{cases} \quad \text{and}$$

$x_j = *$ or $J_j = (u,u,p,-p)$ for some u and p.

For any $\iota \in I_1(X)_U$, we have a representative $((J_1,x_1),\dots,(J_k,x_k))$ such that $x_i \ne *$ for every i and $J_1 < J_2 < \cdots < J_k$, which is called the reduced representative.

We regard $I_1(X)_U$ as a partial monoid by considering a pair $(\xi,\eta) \in I_1(X)_U^2$ is summable if ξ and η have representatives such that the union of those satisfies the conditions (1)$\sim$(3) in Definition 3.1 after an appropriate change of the order of labelled intervals. The sum is given by the union of such representatives. It is clear that the only element in $I_{(0)}(X)_U$, denoted $\emptyset$, is the unit for the partial sum.

Remark Recall from Example 2.4 that when $M = \{-1, 0, 1\}$, $C(\mathbb{R}^n, X \wedge M)$ is the configuration space of positive and negative particles in $\mathbb{R}^n$ labelled by X. Remark that $I_1(X)$ can be embedded into $C(\mathbb{R}, X \wedge M)$ as a topological space in the following manner. Let $C'(\mathbb{R}, X \wedge M)$ denote the sub-PAM of $C(\mathbb{R}, X \wedge M)$ consisting of elements in $C(\mathbb{R}, X \wedge M)$ which have a representative

$$((v_1, x_1 \wedge a_1), \dots, (v_{2k}, x_{2k} \wedge a_{2k})) \in Z_1^{(2k)}(X \wedge M)$$

such that $v_1 \leq \cdots \leq v_{2k}$ and $x_{2i-1} = x_{2i}$ for all i. Then we have a homeomorphism $I_1(X) \to C'(\mathbb{R}, X \wedge M)$ defined by the correspondence

$$[(J_1, x_1), \dots, (J_k, x_k)] \mapsto [(u_1, p_1, x_1), (v_1, q_1, x_1), \dots, (u_k, p_k, x_k), (v_k, q_k, x_k)],$$

where $J_i = (u_i, v_i, p_i, q_i)$. We do not have such a relation between $I_n(X)$ and $C(\mathbb{R}^n, X \wedge M)$ for $n > 1$; the above homeomorphism does not help us about this since it is not a PAM homomorphism.

Definition 3.3 We define $I_n(X)_U = C(\mathbb{R}^{n-1}, I_1(X)_U)$. We denote $I_n(X) = I_n(X)_{\mathbb{R}}$ and $I_n(X)_s = I_n(X)_{(0,s)}$. Then $I_n(X)$ is homeomorphic to $I_n(X)_s$ for any $s > 0$. An element $[(v_1, \xi_1), \dots, (v_k, \xi_k)] \in I_n(X) = C(\mathbb{R}^{n-1}, I_1(X))$ can be thought of as intervals ordered along the lines parallel to the x_1-axis through v_i if we view $\mathbb{R}^{n-1}$ as the $(x_2, \dots, x_n)$ hyperplane in $\mathbb{R}^n$. Thus we call $I_n(X)$ the space of intervals in $\mathbb{R}^n$.

To relate $I_n(X)$ with $\Omega^n \Sigma^n X$, we construct an analogue of the electric field map in [8]. However, there is no direct analogue of the electric field map on $I_n(X)$ and we need a thickening of $I_n(X)$. We also need to modify $I_n(X)$ to get a space corresponding to the Moore loop space for the quasifibration argument given in Theorem 1.1 and the related lemmas.

Definition 3.4 We say that $\iota \in I_{(k)}(X)_U$ is ε-separated if

(1) $\iota \in I_{(k)}(X)_{Int(U - U^c_{\varepsilon/2})}$, where $U^c_{\varepsilon/2}$ denotes the $\varepsilon/2$-neighborhood of the complement of U,

(2) any two ends (of the same or distinct intervals) with the same parity are distant by at least ε, and

(3) any two intervals with the distinct labels in X are distant by at least ε.

We then say that $\xi \in I_1(X)_U$ is ε-separated if it is represented by some ε-separated $\iota \in I_{(k)}(X)_U$.

Let $I_1^\varepsilon(X)_U$ denote the subspace of $I_1(X)_U$ consisting of ε-separated elements. Then $I_1^\varepsilon(X)_U$ is given a PAM structure by regarding $(\xi,\eta)\in(I_1^\varepsilon(X)_U)^2$ as a summable pair if it is so as a pair of elements in $I_1(X)_U$ and the sum $\xi+\eta$ taken there is in $I_1^\varepsilon(X)_U$. We define

$$I_n^\varepsilon(X)_U = C(\mathbb{R}^{n-1}, I_1^\varepsilon(X)_U),$$

then define

$$\widetilde{I}_n(X) = \{(\xi,\varepsilon,s) \mid 0<\varepsilon\le 1,\ s\ge 0,\ \xi\in I_n^\varepsilon(X)_s\}$$

with the topology considered as the subspace of $I_n(X)_\infty \times (0,1]\times[0,\infty)$. A PAM structure on $\widetilde{I}_n(X)$ is defined so that (ξ,ε,s) and (η,τ,t) are summable if and only if $\varepsilon=\tau, s=t$ and (ξ,η) is a summable pair.

Lemma 3.5 *We have a weak homotopy equivalence* $\widetilde{I}_n(X)\simeq_w I_n(X)$.

Proof Since any homeomorphism $\mathbb{R}\to(0,\infty)$ induces a homeomorphism $I_n(X)\to I_n(X)_\infty$, it suffices to show that $\widetilde{I}_n(X)\simeq_w I_n(X)_\infty$. Note that we can embed $I_n^\varepsilon(X)_s$ into $I_n(X)_\infty$ using the inclusion $(0,s)\subset(0,\infty)$. Under this identification, let $p\colon \widetilde{I}_n(X)\to I_n(X)_\infty$ denote the map which assigns ξ to each (ξ,ε,s). Then $p_*\colon \pi_k(\widetilde{I}_n(X))\to\pi_k(I_n(X)_\infty)$ is an isomorphism for all $k\ge 0$. Indeed, for any map $f\colon S^k\to I_n(X)_\infty$, there exist ε and s such that $\mathrm{Im} f$ is contained in $I_n^\varepsilon(X)_s$ since S^k is compact. This proves that p_* is surjective. On the other hand, let $H\colon S^k\times I\to I_n(X)_\infty$ be a homotopy between $H_0=p\circ f$ and $H_1=p\circ g$. Since $S^k\times I$ is compact, we can restrict the codomain of H to $I_n^\varepsilon(X)_s$ for some ε and s. This proves that p_* is injective. □

Now we proceed to construct another modification of $I_n(X)$, which models, as we shall see later, the Moore path space $PC(\mathbb{R}^{n-1},\Sigma X)$. Let s be a positive number, or $s=\infty$. For any element $J=(u,v,p,q)\in\mathcal{I}$, we put $-J=(-v,-u,-q,-p)\in\mathcal{I}$. Then we have an involution on $I_{(k)}(X)_{(-s,s)}$ by setting

$$(-1)\cdot((J_1,x_1),\ldots,(J_k,x_k)) = ((-J_k,x_k),\ldots,(-J_1,x_1)),$$

which induces an involution on $I_1(X)_{(-s,s)}$. We denote by $E_1(X)_s$, the subspace of $I_1(X)_{(-s,s)}$ invariant under the involution. Note that $E_1(X)_s$ has a PAM structure induced by that of $I_1(X)_{(-s,s)}$; we define $E_n(X)_s = C(\mathbb{R}^{n-1},E_1(X)_s)$. Since the involution on $I_1(X)_{(-s,s)}$ restricts to an involution on $I_n^\varepsilon(X)_{(-s,s)}$, we can define $E_1^\varepsilon(X)_s$ and $E_n^\varepsilon(X)_s = C(\mathbb{R}^{n-1},E_1^\varepsilon(X)_s)$ similarly. Now we define,

$$\widetilde{E}_n(X) = \{(\xi,\varepsilon,s) \mid 0<\varepsilon\le 1, s\ge 0, \xi\in E_n^\varepsilon(X)_s\},$$

with the topology considered as a subset of $E_n(X) \times (0,1] \times [0,\infty)$. A PAM structure on $\widetilde{E}_n(X)$ is defined so that (ξ,ε,s) and (η,τ,t) are summable if and only if $\varepsilon=\tau, s=t$ and (ξ,η) is a summable pair.

Any element of $E_1(X)$ have a representative of the form

$$((-J_k,x_k),\dots,(-J_1,x_1),(J_1,x_1),\dots,(J_k,x_k)) \in I_{(2k)}(X)_{(-s,s)}$$

for some $s>0$.

It is useful to denote this representative by $m((J_1,x_1),\dots,(J_k,x_k))$.

Lemma 3.6 *We have a weak homotopy equivalence* $\widetilde{E}_n(X) \simeq_w *$.

Proof We can prove that $\widetilde{E}_n(X) \simeq_w E_n(X)$ in a similar way to the proof of Lemma 3.5. So, by Lemma 2.5, it suffices to show that $E_1(X)$ is homotopy equivalent to $\{0\}$ via PAM homomorphisms. Since $E_1(X)$ is homeomorphic to $E_1(X)_s$ for any $s>0$, we prove that $E_1(X)_1 \simeq \{0\}$ via PAM homomorphisms. Let $h_t\colon (-1,1)\to(-1,1)$ $(0\le t\le 1)$ denote the homotopy defined by:

$$h_t(u) = \begin{cases} u-t & \text{if } u \ge t \\ 0 & \text{if } |u| < t \\ u+t & \text{if } u \le -t \end{cases}$$

Then the contracting homotopy $H_t\colon E_1(X)_1 \to E_1(X)_1$ is defined as follows. For $\xi \in E_1(X)$, we take a representative

$$m((J_1,x_1),\dots,(J_k,x_k)) \in I_{(2k)}(X)_{(-1,1)}.$$

Then $H_t\colon E_1(X)_1 \to E_1(X)_1$ is defined by setting $H_t(\xi)$ as the class represented by

$$m((h_t(J_r),x_r),\dots,(h_t(J_k),x_k)) \in I_{(2(k-r+1))}(X)_{(-1,1)},$$

where r is the least integer among $i>0$ such that $r(J_i)>t$ and $h_t(J_i)$ denotes the element $(h_t(l(J_i)),h_t(r(J_i)),p_L(J_i),p_R(J_i)) \in \mathcal{I}$. It is straightforward to show that H_t is the desired contracting homotopy. □

Consider the map $I_{(k)}(X)_s \to I_{(k)}(X)_{(-s,s)}$ given by

$$((J_1,x_1),\dots,(J_k,x_k)) \mapsto ((-J_k,x_k),\dots,(-J_1,x_1),(J_1,x_1),\dots,(J_k,x_k)).$$

These maps for all k induce a map $I_1(X)_s \to E_1(X)_s$, which restricts to a map $I_1^\varepsilon(X)_s \to E_1^\varepsilon(X)_s$. Thus we have an embedding $i\colon \widetilde{I}_n(X) \to \widetilde{E}_n(X)$. We regard $\widetilde{I}_n(X)$ as the subspace of $\widetilde{E}_n(X)$ via this embedding.

4 The map $\alpha\colon \widetilde{I}_n(X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X)$ and the proof of Theorem 1.1

We first construct a map $\alpha\colon \widetilde{I}_n(X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X)$. Let ξ be an element of $I_1^\varepsilon(X)_s$ and $((J_1, x_1), \dots, (J_k, x_k)) \in I_{(k)}(X)_s$ denote the reduced representative of ξ. Let $u_{2i-1} = l(J_i), u_{2i} = r(J_i), p_{2i-1} = p_L(J_i)$ and $p_{2i} = p_R(J_i)$. Then we define $N_i \subset [0, s]$ $(i = 1, \dots, 2k)$ as

$$N_1 = [u_1 - \varepsilon/2, \mathrm{Min}\,(u_1 + \varepsilon/2, u_2 - \varepsilon/2)]\,,$$

$$N_i = [\mathrm{Max}\,(u_i - \varepsilon/2, u_{i-1} + \varepsilon/2)\,, \mathrm{Min}\,(u_i + \varepsilon/2, u_{i+1} - \varepsilon/2)]\,,\ \text{for } 1 < i < 2k,$$

and
$$N_{2k} = [\mathrm{Max}\,(u_{2k} - \varepsilon/2, u_{2k-1} + \varepsilon/2)\,, u_{2k} + \varepsilon/2]\,.$$

Lemma 4.1 *There exists a map $f\colon [0, s] \to S^1 \wedge X$ such that*

(1) $f(t) = [p_i(\frac{t-u_i}{\varepsilon} + \frac{(-1)^i}{2})] \wedge x_{[\frac{i+1}{2}]}$, *if $t \in N_i$, where $[\frac{i+1}{2}]$ denotes the largest integer which does not exceed $\frac{i+1}{2}$,*

(2) *f is piecewise constant outside $\bigcup_{i=1}^{2k} N_i$, and*

(3) *$f(0) = f(s) = *$, the base point of $S^1 \wedge X$.*

Proof First of all, N_i's are non-empty. The only problem is to show that $u_{i-1} + \varepsilon/2 < u_{i+1} - \varepsilon/2$. However, at least two of three points located at u_{i-1}, u_i and u_{i+1} should have the same parity. Hence we have $u_{i+1} - u_{i-1} \geq \varepsilon$ by (2) of Definition 3.4. Since we took the reduced representative of ξ, it also follows that N_i's are intervals such that $N_1 \leq \cdots \leq N_{2k}$.

By the definition of N_i's, $N_i \cap N_{i+1} \neq \emptyset$ only if $r(N_i) = l(N_{i+1})$ i.e. $u_{i+1} - u_i = \varepsilon$. So suppose $u_{i+1} - u_i = \varepsilon$. If i is even, then we have $f(r(N_i)) = f(l(N_{i+1})) = * \in S^1 \wedge X$. On the other hand, if i is odd, then we have $f(r(N_i)) = f(l(N_{i+1})) = 0 \wedge x_{\frac{i+1}{2}}$. Thus f is well-defined on $\bigcup_{k=1}^{2k} N_i$.

Next, we show that $f : \bigcup_{k=1}^{2k} N_i \to S^1 \wedge X$ can be extended to $[0, s]$ so that it is piecewise constant outside $\bigcup_{i=1}^{2k} N_i$. To show this, it suffices to show that $f(r(N_i)) = f(l(N_{i+1}))$ even for $u_{i+1} - u_i \neq \varepsilon$. If $u_{i+1} - u_i > \varepsilon$, $f(r(N_i)) = f(l(N_{i+1}))$ follows by the same argument as above. Suppose $u_{i+1} - u_i < \varepsilon$. By (2) of Definition 3.4, we have $p_i = -p_{i+1}$. We also have $x_{[\frac{i+1}{2}]} = x_{[\frac{i+2}{2}]}$, by (3) of Definition 3.4 if i is even. Now a direct calculation shows that $f(r(N_i)) = f(l(N_{i+1}))$.

Finally $f(0) = f(s) = *$ since, by (1) of Definition 3.4, we have $l(N_1) \geq 0$ and $r(N_{2k}) \leq s$. □

By setting $\alpha_s^\varepsilon(\xi) = f$, we obtain a PAM homomorphism

$$\alpha_s^\varepsilon\colon\ I_1^\varepsilon(X)_s \to \Omega_s(\Sigma X),$$

where Ω_s is the "loops of length s" functor defined in §2. Then we define a map $\alpha\colon\ \widetilde{I}_1(X) \to \Omega\Sigma X$ by $(\xi, \varepsilon, s) \mapsto \alpha_s^\varepsilon(\xi)$, which is also a PAM homomorphism. Now we define a map $\alpha\colon\ \widetilde{I}_n(X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X)$ by the composite

$$\widetilde{I}_n(X) \to C(\mathbb{R}^{n-1}, \widetilde{I}_1(X)) \stackrel{C(\mathbb{R}^{n-1},\alpha)}{\longrightarrow} C(\mathbb{R}^{n-1}, \Omega\Sigma X) \to \Omega C(\mathbb{R}^{n-1}, \Sigma X).$$

where the first map is defined similarly to the map given in the proof of Lemma 2.5 and the last map is the one given in the end of §2.

Next, we construct a map $p\colon\ \widetilde{E}_n(X) \to C(\mathbb{R}^{n-1}, \Sigma X)$. We define

$$\alpha_{(-s,s)}^\varepsilon\colon\ I_1^\varepsilon(X)_{(-s,s)} \to \mathrm{Map}((-s,s); \Sigma X)$$

similarly to the definition of α_s^ε. Let p_s^ε denote the composite

$$E_1^\varepsilon(X)_s \to I_1^\varepsilon(X)_{(-s,s)} \stackrel{\alpha_{(-s,s)}^\varepsilon}{\longrightarrow} \mathrm{Map}((-s,s); \Sigma X) \stackrel{e_0}{\longrightarrow} \Sigma X,$$

where e_0 denotes the evaluation at $0 \in (-s, s)$. The map p_s^ε induces a map

$$p\colon\ \widetilde{E}_n(X) \to C(\mathbb{R}^{n-1}, \Sigma X),$$

which is surjective.

Remark By definition, a configuration $p(\xi, \varepsilon, s) \in C(\mathbb{R}^{n-1}, \Sigma X)$ has a particle located at $v \in \mathbb{R}^{n-1}$ with non-trivial label in X if and only if the configuration $\xi \in E_n^\varepsilon(X)_s = C(\mathbb{R}^{n-1}, E_1^\varepsilon(X)_s)$ has a particle located at v labelled by some $\iota = [m((J_1, x_1), \ldots, (J_k, x_k))] \in E_1^\varepsilon(X)_s$ such that $l(J_1) < \varepsilon$. Thus the fiber of p at $\emptyset \in C(\mathbb{R}^{n-1}, \Sigma X)$ is $\widetilde{I}_n(X)$.

Proposition 4.2 *The sequence* $\widetilde{I}_n(X) \stackrel{i}{\to} \widetilde{E}_n(X) \stackrel{p}{\to} C(\mathbb{R}^{n-1}, \Sigma X)$ *is a quasi-fibration.*

Proof of Proposition 4.2 is given in the next section.

Proof of Theorem 1 We define $\beta\colon\ \widetilde{E}_n(X) \to PC(\mathbb{R}^{n-1}, \Sigma X)$ by

$$\beta(\xi, \varepsilon, s) = \alpha_{(-s,s)}^\varepsilon(\xi)|_{[0,s]}.$$

By the definition of α and β the following diagram is commutative.

$$\begin{array}{ccccc} \widetilde{I}_n(X) & \stackrel{i}{\longrightarrow} & \widetilde{E}_n(X) & \stackrel{p}{\longrightarrow} & C(\mathbb{R}^{n-1}, \Sigma X) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| \\ \Omega C(\mathbb{R}^{n-1}, \Sigma X) & \longrightarrow & PC(\mathbb{R}^{n-1}, \Sigma X) & \longrightarrow & C(\mathbb{R}^{n-1}, \Sigma X), \end{array}$$

where the lower horizontal row is the path-loop fibration on $C(\mathbb{R}^{n-1}, \Sigma X)$. Since β is a weak homotopy equivalence by Lemma 3.6, so is α. Now the theorem follows from the Segal's theorem [8] since ΣX is path-connected. □

5 Proof of Proposition 4.2

Before stating the proof of Proposition 4.2, we need some observations on the filtration.

Definition 5.1 Let M be a PAM and A a closed sub-PAM of M. The filtration on $C(\mathbb{R}^n, M)$ associated to A is

$$F_j^A C(\mathbb{R}^n, M) = \{[(v_1, a_1), \dots, (v_k, a_k)] \mid \#\{i | a_i \notin A\} \leq j\} \ (j \geq 0).$$

Example 5.2 We have "the number of points" filtration on $C(\mathbb{R}^{n-1}, \Sigma X)$ defined by

$$F_j C(\mathbb{R}^{n-1}, \Sigma X) = \coprod_{k \leq j} Z_{n-1}^{(k)}(\Sigma X)/\sim .$$

This coincides with $F_j^A C(\mathbb{R}^{n-1}, \Sigma X)$ if we put $A = *$.

Example 5.3 Let A_s^ε denote the closed sub-PAM of $E_1^\varepsilon(X)_s$ given by

$$A_s^\varepsilon = \{[m((J_1, x_1), \dots, (J_k, x_k))] \in E_1^\varepsilon(X)_s \mid l(J_1) \geq \varepsilon/2\}.$$

Then we have a filtration $F_j^{A_s^\varepsilon} C(\mathbb{R}^{n-1}, E_1^\varepsilon(X)_s)$ on $E_n^\varepsilon(X)_s = C(\mathbb{R}^{n-1}, E_1^\varepsilon(X)_s)$. Using this filtration, we get a filtration on $\widetilde{E}_n(X)$ by setting

$$F_j \widetilde{E}_n(X) = \left\{(\xi, \varepsilon, s) \mid \xi \in F_j^{A_s^\varepsilon} E_n^\varepsilon(X)_s\right\}.$$

Example 5.4 Let A_s^ε be as the above example. Then a closed sub-PAM $\widetilde{A}$ of $\widetilde{E}_1(X)$ can be given by

$$\widetilde{A} = \{(\xi, \varepsilon, s) \in \widetilde{E}_1(X) \mid \xi \in A_s^\varepsilon\}.$$

Then we have a filtration $F_j^{\widetilde{A}} C(\mathbb{R}^{n-1}, \widetilde{E}_1(X))$.

Remark By the reason explained in the Remark in §4, we see that the projection map $p\colon \widetilde{E}_n(X) \to C(\mathbb{R}^{n-1}, \Sigma X)$ defined in §4 preserves the filtrations given in Example 5.2 and Example 5.3 in the sense that $F_j \widetilde{E}_n(X) = p^{-1} F_j C(\mathbb{R}^{n-1}, \Sigma X)$.

Lemma 5.5 *Let A be a closed sub-PAM of M. Suppose that there exists a map $u\colon A \to [0,1]$ such that $u^{-1}(0) = A$. Suppose also that $(M-A)\times(M-A)\cap M_2 = \emptyset$ and $f\colon M \to N$ is a function which preserves partial sums, continuous on A and $M-A$. Then the induced function $C(id,f)\colon C(\mathbb{R}^n,M) \to C(\mathbb{R}^n,N)$ is continuous on $F_j^A C(\mathbb{R}^n,M) - F_{j-1}^A C(\mathbb{R}^n,M)$ for any j.*

Proof Let $\pi\colon \coprod_{k\geq 0} Z_n^{(k)}(M) \to C(\mathbb{R}^n,M)$ denote the projection. By the definition of quotient topology, it suffices to show that $(id\times f)^k$ is continuous on $\pi^{-1}(F_j^A C(\mathbb{R}^n,M) - F_{j-1}^A C(\mathbb{R}^n,M)) \subset \coprod_{k\geq 0} Z_n^{(k)}(M)$. To do this, we express $\pi^{-1}(F_j^A C(\mathbb{R}^n,M) - F_{j-1}^A C(\mathbb{R}^n,M))$ as a disjoint union

$$\coprod_{k\geq j}\left(\bigcup_{[\sigma]} Z_n^{(k)}(M)\cap\sigma((M-A)^j\times A^{k-j})\right)$$

where $[\sigma]$ runs over the congruence class in $\Sigma_k/\Sigma_j\times\Sigma_{k-j}$ and $\sigma\in\Sigma_k$ acts on M^k by permutation. Then we see that it suffices to show that $f^k\colon M^k\to N^k$ is continuous on $\cup_{[\sigma]}\sigma((M-A)^j\times A^{k-j})$. By the hypothesis, f^k is continuous on each $\sigma((M-A)^j\times A^{k-j})$. As we shall show below, each $\sigma((M-A)^j\times A^{k-j})$ is open in $\cup_{[\sigma]}\sigma((M-A)^j\times A^{k-j})$ and the lemma follows.

To show that $\sigma((M-A)^j\times A^{k-j})$ is open, we set

$$Z = \{(a_1,\ldots,a_k) \mid \mathrm{Max}\{u(a_i)\mid i>j\} < \mathrm{Min}\{u(a_i)\mid i\leq j\}\}\subset A^k.$$

Then Z is an open neighborhood of $(M-A)^j\times A^{k-j}$ such that $\sigma Z\cap\sigma' Z=\emptyset$ if $[\sigma]\neq[\sigma']$ in $\Sigma_k/\Sigma_j\times\Sigma_{k-j}$. □

Now we prove Proposition 4.2. The proof reduces to two lemmas below.(We use May's form of the Dold-Thom criterion for a quasifibration [3],[5].) Our proof is similar to the argument given in §4 of [1], but we present the proof here since the construction of maps and homotopies are special to our setting and not obvious. Recall that a subset V of $C(\mathbb{R}^{n-1},\Sigma X)$ is said to be distinguished if $p\colon p^{-1}V\to V$ is a quasifibration.

Lemma 5.6 *Any open set $V\subset F_jC(\mathbb{R}^{n-1},\Sigma X) - F_{j-1}C(\mathbb{R}^{n-1},\Sigma X)$ is distinguished.*

Proof We show that $p^{-1}(V)\simeq V\times\widetilde{I}_n(X)$ for any open set $V\subset F_jC(\mathbb{R}^{n-1},\Sigma X) - F_{j-1}C(\mathbb{R}^{n-1},\Sigma X)$. Firstly, we construct a map $\varphi\colon p^{-1}V\to\widetilde{I}_n(X)$. Suppose ξ is an element of $E_1^\varepsilon(X)_s$ represented by $m((J_1,x_1),\ldots,(J_k,x_k))\in$

$I^{\varepsilon}_{(2k)}(X)_{(-s,s)}$. Let T_t denote the translation of intervals by t. Then we define a function $\varphi^{\varepsilon}_s\colon\ E^{\varepsilon}_1(X)_s \to I^{\varepsilon}_1(X)_{s+2\varepsilon}$ by

$$\varphi^{\varepsilon}_s(\xi) = \begin{cases} [T_{2\varepsilon}((J_1,x_1),\ldots,(J_k,x_k))] & \text{if } l(J_1) \geq \varepsilon/2 \\ [(K,x_1),T_{2\varepsilon}((J_1,x_1),\ldots,(J_k,x_k)] & \text{if } l(J_1) < \varepsilon/2 \end{cases}$$

where $K = (\varepsilon - l(J_1), 2\varepsilon - l(J_1), -1, -p_L(J_1))$. Note that the definition of φ^{ε}_s does not depend on the choice of the representative of ξ. Since φ^{ε}_s is continuous on A^{ε}_s and $E^{\varepsilon}_1(X)_s - A^{\varepsilon}_s$, the induced function $\varphi\colon\ E^{\varepsilon}_n(X)_s \to I^{\varepsilon}_n(X)_{s+2\varepsilon}$ is continuous on $F^{A^{\varepsilon}_s}_j E^{\varepsilon}_n(X)_s - F^{A^{\varepsilon}_s}_{j-1} E^{\varepsilon}_n(X)_s$ by Lemma 5.5. Moreover φ induces a function $\widetilde{\varphi}\colon\ \widetilde{E}_1(X) \to \widetilde{I}_1(X)$, which is continuous on $\widetilde{A}$ and $\widetilde{E}_1(X) - \widetilde{A}$. By Lemma 5.5, a function

$$C(id_{\mathbb{R}^{n-1}}, \widetilde{\varphi})\colon\ C(\mathbb{R}^{n-1}, \widetilde{E}_1(X)) \to C(\mathbb{R}^{n-1}, \widetilde{I}_1(X))$$

is continuous on $F^{\widetilde{A}}_j C(\mathbb{R}^{n-1}, \widetilde{E}_1(X)) - F^{\widetilde{A}}_{j-1} C(\mathbb{R}^{n-1}, \widetilde{E}_1(X))$. Note that $\widetilde{E}_n(X)$ can be embedded into $C(\mathbb{R}^{n-1}, \widetilde{E}_1(X))$ by the correspondence

$$(\{(v_i,\xi_i)\}_i, \varepsilon, s) \mapsto [\{(v_i,(\xi_i,\varepsilon,s))\}_i],$$

where $v_i \in \mathbb{R}^{n-1}$ and $\xi_i \in E^{\varepsilon}_1(X)_s$. Similarly, we can embed $\widetilde{I}_n(X)$ into $C(\mathbb{R}^{n-1}, \widetilde{I}_1(X))$. Then we can restrict $C(id_{\mathbb{R}^{n-1}}, \widetilde{\varphi})$ to a function $\varphi\colon\ \widetilde{E}_n(X) \to \widetilde{I}_n(X)$, which is continuous on $F_j\widetilde{E}_n(X) - F_{j-1}\widetilde{E}_n(X)$.

Secondly we construct a map $\psi\colon\ V \times \widetilde{I}_n(X) \to p^{-1}V$. Suppose $y = [t] \wedge x \in S^1 \wedge X$ where $x \in X$, $t \in [-1,1]$ and $[t] \in S^1 = [-1,1]/\pm 1$. We define a map $s^{\varepsilon}\colon\ \Sigma X \to I^{\varepsilon}_1(X)_{(-2\varepsilon,2\varepsilon)}$ by

$$s^{\varepsilon}(y) = [m(L,x)] = [(-L,x),(L,x)],$$

where $L = (|t|\varepsilon/2, (|t|/2+1)\,\varepsilon, p, 1)$. Here, if $t \neq 0$, we put $p = -t/|t|$ while if $t = 0$ we may put either $p = +1$ or -1 since $s(y)$ can be represented by one labelled interval lying over the origin $0 \in \mathbb{R}$. Now suppose $v = [(c_1,y_1),\ldots,(c_j,y_j)] \in V$ and $y_i \in \Sigma X$. We define a map $\sigma^{\varepsilon}\colon\ V \to p^{-1}V$ by

$$\sigma^{\varepsilon}(v) = [(c_1, s^{\varepsilon}(y_1)),\ldots,(c_j, s^{\varepsilon}(y_j))].$$

Then $\psi\colon\ V \times \widetilde{I}_n(X) \to p^{-1}V$ is defined by

$$\psi(v,(\xi,\varepsilon,s)) = (\sigma^{\varepsilon}(v) + m(T_{2\varepsilon}(\xi)), \varepsilon, s+2\varepsilon).$$

Next we show that $\psi \circ (p \times \varphi)\colon\ p^{-1}V \to p^{-1}V$ is homotopic to $id_{p^{-1}V}$. Observe that $\psi \circ (p \times \varphi)$ is induced by the function $\Phi^{\varepsilon}_s\colon\ E^{\varepsilon}_1(X)_s \to E^{\varepsilon}_1(X)_{s+4\varepsilon}$ defined by

$$\Phi^{\varepsilon}_s(\xi) = \begin{cases} [\ m(T_{4\varepsilon}((J_1,x_1),\ldots,(J_k,x_k)))\ ] & \text{if } l(J_1) \geq \varepsilon/2 \\ [\ m((L,x_1), T_{2\varepsilon}(K,x_1), T_{4\varepsilon}((J_1,x_1),\ldots,(J_k,x_k)))\ ] & \text{if } l(J_1) < \varepsilon/2 \end{cases}$$

where $K = (\varepsilon - l(J_1), 2\varepsilon - l(J_1), -1, -p_L(J_1))$, $L = (l(J_1), l(J_1)+\varepsilon, p_L(J_1), 1)$. A homotopy $\psi\circ(p\times\varphi) \simeq id_{p^{-1}V}$ is induced by a deformation of Ψ_s^ε to $id_{E_1^\varepsilon(X)_s}$ in $\widetilde{E}_1(X)$, where $E_1^\varepsilon(X)_s$ is regarded as a subspace of $\widetilde{E}_1(X)$ by the correspondence $\xi \mapsto (\xi, \varepsilon, s)$. This deformation is given by a function

$$H\colon\ E_1^\varepsilon(X)_s \times I \to \widetilde{E}_1(X)$$

which coincides with Ψ_s^ε on $E_1^\varepsilon(X)_s \times 0$ and is the identity on $E_1^\varepsilon(X)_s \times 1$. Intuitively, the essential task of this homotopy (on the right hand side of the origin) is that

(1) Push $T_{2\varepsilon}(K)$ and $T_{4\varepsilon}((J_1),\dots,(J_k))$ to the left until $T_*(K)$ meets L. Then $T_*(K,x_1)$ and (L,x_1) are merged.

(2) Push $T_*(J_1,\dots,J_k)$ to the left until $T_{*'}(J_1)$ meets $L\cup T_*(K)$. Then $T_{*'}(J_1,x_1)$ and $(L\cup T_*(K),x_1)$ are merged.

(3) Push the right end of $L\cup T_*(K)\cup T_{*'}(J_1)$ and $T_{*'}((J_2,x_2),\dots,(J_k,x_k))$ to the left until the length of $L\cup T_*(K)\cup T_{*'}(J_1)$ coincides with that of original J_1.

More precisely, we consider a homotopy $h_t^\varepsilon\colon\ [0,\infty)\to[0,\infty)$ $(0\le t\le 1)$ given by the following formulae.

If $0 \le t \le \frac{1}{4}$:

$$h_t^\varepsilon(u) = \begin{cases} u & u \le (2-4t)\varepsilon \\ (2-4t)\varepsilon & (2-4t)\varepsilon < u \le (2+4t)\varepsilon \\ u - 8t\varepsilon & u > (2+4t)\varepsilon \end{cases}$$

If $\frac{1}{4} < t \le \frac{1}{2}$:

$$h_t^\varepsilon(u) = \begin{cases} u & u \le \varepsilon \\ \varepsilon & \varepsilon < u \le 3\varepsilon \\ u - 2\varepsilon & 3\varepsilon < u \le \left(\frac{9}{2}-2t\right)\varepsilon \\ \left(\frac{5}{2}-2t\right)\varepsilon & \left(\frac{9}{2}-2t\right)\varepsilon < u \le \left(\frac{7}{2}+2t\right)\varepsilon \\ u-(4t+1)\varepsilon & \left(\frac{7}{2}+2t\right)\varepsilon < u \end{cases}$$

If $\frac{1}{2} < t \le \frac{3}{4}$:

$$h_t^\varepsilon(u) = \begin{cases} u & u \le \varepsilon \\ \varepsilon & \varepsilon < u \le 3\varepsilon \\ u - 2\varepsilon & 3\varepsilon < u \le \left(\frac{9}{2}-2t\right)\varepsilon \\ \left(\frac{5}{2}-2t\right)\varepsilon & \left(\frac{9}{2}-2t\right)\varepsilon < u \le \left(\frac{11}{2}-2t\right)\varepsilon \\ u-3\varepsilon & u > \left(\frac{11}{2}-2t\right)\varepsilon \end{cases}$$

If $\frac{3}{4} < t \leq 1$:

$$h_t^\varepsilon(u) = \begin{cases} u & u \leq \left(\frac{5}{2} - 2t\right)\varepsilon \\ \left(\frac{5}{2} - 2t\right)\varepsilon & \left(\frac{5}{2} - 2t\right)\varepsilon < u \leq \left(\frac{5}{2} + 2t\right)\varepsilon \\ u - 4t\varepsilon & u > \left(\frac{5}{2} + 2t\right)\varepsilon \end{cases}$$

Then we define $H\colon \widetilde{E}_1(X) \times I \to \widetilde{E}_1(X)$ by $H(\xi, \varepsilon, s; t) = (h_{t*} \circ \Phi_s^\varepsilon(\xi), \varepsilon, s)$. Since H is continuous on $\widetilde{A} \times I$ and $\widetilde{E}_1(X) \times I - \widetilde{A} \times I$, we can apply Lemma 5.5 and get a map

$$F_j^{\widetilde{A}\times I} C(\mathbb{R}^{n-1}, \widetilde{E}_1(X) \times I) - F_{j-1}^{\widetilde{A}\times I} C(\mathbb{R}^{n-1}, \widetilde{E}_1(X) \times I) \to C(\mathbb{R}^{n-1}, \widetilde{E}_1(X)).$$

Consider the following sequence of embeddings

$$\widetilde{E}_n(X) \times I \hookrightarrow C(\mathbb{R}^{n-1}, \widetilde{E}_1(X)) \times I \hookrightarrow C(\mathbb{R}^{n-1}, \widetilde{E}_1(X) \times I).$$

Since these embeddings are compatible with the filtrations $F_j, F_j^{\widetilde{A}}$, and $F_j^{\widetilde{A}\times I}$, we can restrict H to $(F_j\widetilde{E}_n(X) - F_{j-1}\widetilde{E}_n(X)) \times I$. Since this restriction map has its image in $\widetilde{E}_n(X) \subset C(\mathbb{R}^{n-1}, \widetilde{E}_1(X) \times I)$, we obtain a map $H\colon (F_j\widetilde{E}_n(X) - F_{j-1}\widetilde{E}_n(X)) \times I \to \widetilde{E}_n(X)$. By the definition of h_t^ε, H is a fibre-preserving map with respect to $p\colon \widetilde{E}_n(X) \to C(\mathbb{R}^{n-1}, \Sigma X)$. Thus we obtain a homotopy $H\colon p^{-1}V \times I \to p^{-1}V$ between $\psi \circ (p \times \varphi)$ and $id_{p^{-1}V}$.

Finally we show that $(p \times \varphi) \circ \psi \simeq id_{V \times \widetilde{I}_n(X)}$. It suffices to show that $\pi_i \circ (p \times \varphi) \circ \psi \simeq \pi_i$ $(i = 1, 2)$, where π_1 and π_2 are the projections onto V and $\widetilde{I}_n(X)$ respectively. For any $y = [t] \wedge x \in \Sigma X, t \in [-1, 1], x \in X$, we put $t^\varepsilon(y) = [(K, x), (L, x)] \in I_1^\varepsilon(X)_{4\varepsilon}$, where

$$K = ((1 - |t|/2)\,\varepsilon, (2 - |t|/2)\,\varepsilon, -1, t/|t|)\,,$$

and
$$L = ((2 + |t|/2)\,\varepsilon, (3 + |t|/2)\,\varepsilon, -t/|t|, 1)\,.$$

Suppose $v = [(c_1, y_1), \ldots, (c_j, y_j)] \in V, y_i = [t_i] \wedge x_i \in \Sigma X, t_i \in [-1, 1]$, and $x_i \in X$. We put $\tau^\varepsilon(v) = [(c_1, t^\varepsilon(y_1)), \ldots, (c_j, t^\varepsilon(y_j))] \in I_n^\varepsilon(X)_{4\varepsilon}$. Then we have $((p \times \varphi) \circ \psi)(v, (\xi, \varepsilon, s)) = (v, (\tau^\varepsilon(v) + T_{4\varepsilon}(\xi), \varepsilon, s + 4\varepsilon))$. From this formula, it follows that $\pi_1 \circ (p \times \varphi) \circ \psi$ coincides with π_1. We prove $\pi_2 \circ (p \times \varphi) \circ \psi \simeq \pi_2$ by the "push to the left argument." Consider a homotopy $k_t\colon (0, \infty] \to (0, \infty]$ defined by

$$k_t^\varepsilon(u) = \begin{cases} u & 0 \leq u \leq 2\varepsilon(1 - t) \\ 2\varepsilon(1 - t) & 2\varepsilon(1 - t) < u \leq 2\varepsilon(1 + t) \\ u - 4\varepsilon t & 2\varepsilon(1 + t) \leq u. \end{cases}$$

Then we define $K\colon V \times \widetilde{I}_n(X) \times I \to \widetilde{I}_n(X)$ by $K(v, \xi, \varepsilon, s; t) = (k_{t*}^\varepsilon(\xi), \varepsilon, s)$. Thus we constructed a homotopy $K\colon V \times \widetilde{I}_n(X) \times I \to \widetilde{I}_n(X)$ between $\pi_2 \circ (p \times \varphi) \circ \psi$ to π_2. This completes the proof of the lemma. □

Lemma 5.7 *There exist an open neighborhood U of $F_{j-1}C(\mathbb{R}^{n-1}, \Sigma X)$ in $F_jC(\mathbb{R}^{n-1}, \Sigma X)$ and homotopies $h_t \colon U \to U$ and $H_t \colon p^{-1}U \to p^{-1}U$ such that*

(1) $h_0 = id_U$ *and* $h_1(U) \subset F_{j-1}C(\mathbb{R}^{n-1}, \Sigma X)$,

(2) $H_0 = id_{p^{-1}U}$ *and* $pH_t = h_t p$ *for all* t, *and*

(3) $H_1 \colon p^{-1}z \to p^{-1}h_1(z)$ *is a homotopy equivalence for all* $z \in U$.

Proof Let $u \colon X \to I$ and a homotopy $k_t \colon X \to X$ $(0 \le t \le 1)$ represent $(X, *)$ as a NDR-pair. Thus $u^{-1}(0) = *$, $k|_{X \times 0} = id_X$, $k(*, t) = *$ and $k(W, 1) = *$, where $W = u^{-1}[0, 1)$. We take $U \subset F_jC(\mathbb{R}^{n-1}, \Sigma X)$ to be a neighborhood of $F_{j-1}C(\mathbb{R}^{n-1}, \Sigma X)$ which consists of elements represented by $((c_1, y_1), \ldots, (c_j, y_j))$ such that there exist one or more i with $|t_i| > \frac{1}{2}$ or $x_i \in W$, where $y_i = t_i \wedge x_i$. We define a homotopy $h'_t \colon [-1, 1] \to [-1, 1]$, $(0 \le t \le 1)$ by

$$h'_t(u) = \begin{cases} -1 & -1 \le u \le \frac{t}{2} - 1 \\ \frac{2u}{2-t} & \frac{t}{2} - 1 \le u \le 1 - \frac{t}{2} \\ 1 & 1 - \frac{t}{2} \le u \le 1. \end{cases}$$

Then we can define a homotopy $h_t \colon U \to U$ by

$$[(c_1, t_1 \wedge x_1), \ldots, (c_j, t_j \wedge x_j)] \mapsto [(c_1, h_t(t_1) \wedge k_t(x_1)), \ldots, (c_j, h_t(t_j) \wedge k_t(x_j))].$$

Note that $h_0 = id_U$ and $h_1(U) \subset F_{j-1}C(\mathbb{R}^{n-1}, \Sigma X)$.

Next we construct a homotopy $H_t \colon p^{-1}U \to p^{-1}U$ which covers h_t. Let $H'_t \colon (0, 1] \to (0, 1]$ denote the homotopy defined by $H'_t(u) = \left(1 - \frac{t}{2}\right) u$ and $K_t = E_n(k_t) \colon E_n(X) \to E_n(X)$. Then we can define a homotopy $H_t \colon p^{-1}U \to p^{-1}U$ by $(\xi, \varepsilon, s) \mapsto (K_t(\xi), H'_t(\varepsilon), s)$. It is straightforward to check that $H_0 = id_{p^{-1}U}$ and $pH_t = h_t p$ for all t.

To show that $H_1 \colon p^{-1}z \to p^{-1}h_1(z)$ is a homotopy equivalence for all $z \in U$, a homotopy inverse map $G \colon p^{-1}h_1(z) \to p^{-1}z$ is defined as follows: Suppose $z = [(c_1, y_1), \ldots, (c_j, y_j)] \in U$ and $y_i = t_i \wedge x_i \in \Sigma X$. We put $g^\varepsilon(y) = [m(K, x)] \in E_1^\varepsilon(X)_{2\varepsilon}$, where $K = (|t|\varepsilon/2, (2 - |t|)\varepsilon, -t/|t|, t/|t|)$, then we put $\gamma^\varepsilon(z) = [(c_1, g^\varepsilon(y_1)), \ldots, (c_k, g^\varepsilon(y_k))] \in E_n(X)$. Now $G \colon p^{-1}h_1(z) \to p^{-1}z$ is defined by $G([\xi, \varepsilon, s]) = (\gamma^\varepsilon(z) + T_{2\varepsilon}(\xi), \varepsilon, s + 2\varepsilon)$, where $T_{2\varepsilon}(\xi)$ is understood to be an element of $E_n(X)$ given by translation 2ε on the right and -2ε on the left; this construction is ambiguous and not continuous as it is, but G is well-defined and continuous by virtue of the insertion of $\gamma^\varepsilon(\xi)$. A proof that G is the homotopy inverse of $H_1 \colon p^{-1}z \to p^{-1}h_1(z)$ is again by the "push to the left" argument. □

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