Algebraic & Geometric Topology 6 (2006) 171–194

Sweepouts of amalgamated 3–manifolds

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We show that if two 3–manifolds with toroidal boundary are glued via a "sufficiently complicated" map then every Heegaard splitting of the resulting 3–manifold is weakly reducible. Additionally, suppose $X \cup_F Y$ is a manifold obtained by gluing X and Y, two connected small manifolds with incompressible boundary, along a closed surface F. Then the following inequality on genera is obtained:

$$g(X \cup_F Y) \geq \frac{1}{2}\left(g(X) + g(Y) - 2g(F)\right).$$

Both results follow from a new technique to simplify the intersection between an incompressible surface and a strongly irreducible Heegaard splitting.

57N10, 57M99; 57M27

1 Introduction

It is a consequence of the Haken Lemma [4] and the Uniqueness of Prime Decompositions, Kneser [8], that Heegaard genus is well behaved under connected sum. In particular, 3–manifold genus is additive:

$$g(X \# Y) = g(X) + g(Y)$$

Here we discuss the Heegaard splittings of a manifold obtained by gluing together manifolds along boundary components of higher genus.

To this end let X and Y be 3–manifolds with incompressible boundary homeomorphic to a connected surface F. It is not difficult to show that if H_X and H_Y are Heegaard surfaces in X and Y then we can amalgamate these splittings to obtain a Heegaard surface in $X \cup_F Y$ with genus equal to $g(H_X) + g(H_Y) - g(F)$ (see, for example, Schultens [14]). Letting $g(X)$, $g(Y)$, and $g(X \cup_F Y)$ denote the minimal genus among all Heegaard surfaces in the respective 3–manifolds, we find:

$$g(X \cup_F Y) \leq g(X) + g(Y) - g(F) \tag{1}$$

Published: 24 February 2006 DOI: 10.2140/agt.2006.6.171

Bounds in the other direction are harder to obtain. When $F \cong S^2$ it follows from the Haken Lemma [4] that the above inequality may be replaced by an equality. In Section 4 we examine the case where F is a torus. We assume here that the map which identifies ∂X to ∂Y is "sufficiently complicated," in a sense to be made precise in Section 4.

Theorem 4.1 *Suppose that X and Y are knot manifolds and $\varphi\colon \partial X \to \partial Y$ is a sufficiently complicated homeomorphism. Then the manifold $M(\varphi) = X \cup_\varphi Y$ has no strongly irreducible Heegaard splittings.*

In particular it follows from this result that every Heegaard splitting of $X \cup_F Y$ is an amalgamation of splittings of X and Y. In this situation Inequality (1) becomes an equality.

In the case where the genus of F is at least two there is the following result of Lackenby [9]:

Theorem *Let X and Y be simple 3–manifolds, and let $h\colon \partial X \to F$ and $h'\colon F \to \partial Y$ be homeomorphisms with some connected surface F of genus at least two. Let $\psi\colon F \to F$ be a psuedo-Anosov homeomorphism. Then, provided $|n|$ is sufficiently large,*

$$g(X \cup_{h'\psi^n h} Y) = g(X) + g(Y) - g(F).$$

Furthermore, any minimal genus Heegaard splitting for $X \cup_{h'\psi^n h} Y$ is obtained from splittings of X and Y by amalgamation, and hence is weakly reducible.

If ψ fails to be "sufficiently complicated" then there is no hope of an exact equality, as in the previous theorem. Previous known lower bounds were obtained by Johannson [7] when X and Y are simple

$$g(X \cup_F Y) \geq \frac{1}{5}(g(X) + g(Y) - 2g(F)).$$

Schultens has generalized this result to allow essential annuli [13].

By assuming the component manifolds X and Y are *small* we get a new bound. The following statement is one case of Theorem 5.1:

Theorem 5.1$'$ *Suppose X and Y are compact, orientable, connected, small 3–manifolds with incompressible boundary homeomorphic to a surface F. Then*

$$g(X \cup_F Y) \geq \frac{1}{2}(g(X) + g(Y) - 2g(F)).$$

Both of our results follow from showing that a strongly irreducible Heegaard surface H can be isotoped to meet the gluing surface F in a particularly nice fashion. Often in these types of arguments one simplifies the intersection by making every loop of $H \cap F$ essential in both surfaces. In this paper, rather than focusing on the intersection set $H \cap F$, we focus on the complimentary pieces $H \setminus N(F)$. Our result is that H and F may always be arranged so that *almost* every component H' of $H \setminus N(F)$ is *incompressible*. On such a component every loop which is essential in H' is essential in $M \setminus N(F)$. There is at most one component H'' which is compressible. In this case we find that H'' is *strongly irreducible*, in the sense that every essential loop which bounds a disk on one side meets every essential loop bounding a disk on the other. See Lemma 3.3.

2 Definitions

In this section we give some of the standard definitions that will be used throughout paper.

2.1 Essential loops, arcs, and surfaces

A loop γ embedded in the interior of a compact, orientable surface F is called *essential* if it does not bound a disk in F. If F is embedded in a 3–manifold, M, a *compressing disk* for F is a disk, $D \subset M$, such that $F \cap D = \partial D$, and such that ∂D is essential on F. If we identify a thickening of D in $M \setminus N(F)$ with $D \times I$ then to *compress F along D* is to remove $(\partial D) \times I$ from F and replace it with $D \times \partial I$.

A properly embedded arc α on F is *essential* if there is no subarc β of ∂F such that $\alpha \cup \beta$ is the boundary of a subdisk of F. If F is properly embedded in a 3–manifold, M, a *boundary-compressing disk* is a disk, D, such that $\partial D = \alpha \cup \beta$, where $F \cap D = \alpha$ is an essential arc on F and $D \cap \partial M = \beta$. If we identify a thickening of D in $M \setminus N(F)$ with $D \times I$ then to *boundary-compress F along D* is to remove $\alpha \times I$ from F and replace it with $D \times \partial I$.

A properly embedded surface is *incompressible* if there are no compressing disks for it. A properly embedded, separating surface is *strongly irreducible* if there are compressing disks for it on both sides, and each compressing disk on one side meets each compressing disk on the other side.

A compact, orientable 3–manifold is said to be *irreducible* if every embedded 2–sphere bounds a 3–ball. A 3–manifold is said to be *small* if it is irreducible and every incompressible surface is parallel to a boundary component.

2.2 Heegaard and generalized Heegaard Splittings.

A *compression body* is a 3–manifold C constructed in one of two different ways. The first way is to begin with a collection of zero–handles and attach one–handles to their boundaries, resulting in a manifold that may or may not be connected. In this case we say the *spine* of C is a 1–complex Σ in C such that C is homeomorphic to a thickening of Σ. We set $\partial_- C = \varnothing$ and $\partial_+ C = \partial C$.

The second way to construct a compression body is to begin with a closed (possibly disconnected) orientable surface F with no sphere components, and let C be the manifold obtained by attaching one–handles to the surface $F \times \{1\} \subset F \times I$. In this case we say $\partial_- C = F \times \{0\}$ and $\partial_+ C = \partial C \setminus \partial_- C$. The spine Σ is then the union of $\partial_- C$ and a collection of arcs which are properly embedded in C, such that C is a thickening of Σ.

A surface, H, in a 3–manifold, M, is a *Heegaard surface for M* if H separates M into two compression bodies, V and W, such that $H = \partial_+ V = \partial_+ W$.

A *generalized Heegaard splitting* of a 3–manifold M, Scharlemann–Thompson [12], is a sequence $\{H_i\}_{i=0}^{2n}$ of pairwise disjoint, closed surfaces in M such that

- $\partial M = H_0 \cup H_{2n}$ (if $\partial M = \varnothing$ then $H_0 = H_{2n} = \varnothing$) and
- for each odd i, the surface H_i is a Heegaard splitting of the submanifold cobounded by H_{i-1} and H_{i+1}.

We will call the set of surfaces with even index *thin levels* and the set with odd index *thick levels*.

Generalized Heegaard splittings are associated to handle structures in the following way. Given a generalized Heegaard splitting $\{H_i\}_{i=0}^{n}$ there is a sequence of submanifolds $\{M_i\}$ of M as follows:

- M_0 is a union of zero–handles and 1–handles.
- For odd i between 1 and n, M_i is obtained from M_{i-1} by attaching one–handles.
- For even i between 2 and $n-1$, M_i is obtained from M_{i-1} by attaching two–handles.
- $M_n = M$ is obtained from M_{n-1} by attaching two–handles and three–handles.

Conversely, given a handle structure for M there is an associated generalized Heegaard splitting as above.

Suppose H_X and H_Y are Heegaard surfaces in 3–manifolds X and Y. Suppose further that the boundaries of both X and Y are homeomorphic to a surface F. Then $\{\emptyset, H_X, F, H_Y, \emptyset\}$ is a generalized Heegaard splitting of $X \cup_F Y$. We may now choose a handle structure associated to this generalized Heegaard splitting, and re-arrange it so that handles are added in order of increasing index. The generalized Heegaard splitting associated to this new handle structure will be of the form $\{\emptyset, H, \emptyset\}$, where H is a Heegaard surface in $X \cup_F Y$. In this case the Heegaard surface H is the *amalgamation* of H_X and H_Y, as defined by Schultens [14].

2.3 Normal and almost normal surfaces.

A *normal disk* in a tetrahedron is a triangle or a quadrilateral, as in Figure 1. Let X be a 3–manifold equipped with a psuedo-triangulation. That is, X is expressed as a collection of tetrahedra, together with face pairings.

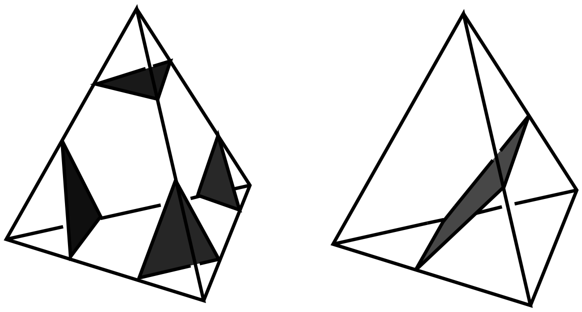

Figure 1: Normal disks

A properly embedded surface in X is *normal* if it intersects every tetrahedron in a collection of *triangles* and *quadrilaterals*. Normal surfaces were first introduced by Kneser [8], and later used to solve several important problems by Haken [3].

A properly embedded surface in X is *almost normal* if it is normal everywhere, with the exception of exactly one piece in one tetrahedron. The exceptional piece can either be an octagon, two normal disks connected by an unknotted tube, or two normal disks connected by a band along ∂X (see Figure 2). In the closed case, almost normal surfaces were introduced by Rubinstein [10]. They were later generalized to surfaces with non-empty boundary by the first author [1].

3 Labelling sweepouts

In this section we prove the technical lemmas on which Sections 4 and 5 rely.

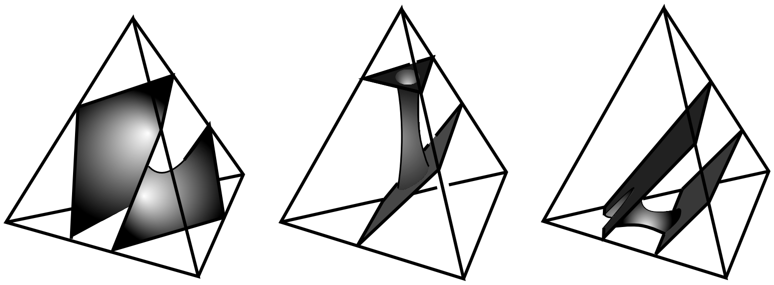

Figure 2: Exceptional disks in an almost normal surface

Lemma 3.1 (Scharlemann [11]) *Let H be a strongly irreducible Heegaard surface, and γ be an essential curve on H. Suppose γ bounds a disk $D \subset M$ such that D is transverse to H. Then γ bounds a compressing disk for H.*

Definition 3.2 Two surfaces H and F embedded in a 3–manifold are *almost transverse* if they have exactly one non-transverse intersection point, and it is a saddle point.

Lemma 3.3 *Let M be a compact, irreducible, orientable 3–manifold with ∂M incompressible, if non-empty. Suppose $M = V \cup_H W$, where H is a strongly irreducible Heegaard surface. Suppose further that M contains an incompressible, orientable, closed, non-boundary parallel surface F. Then either*

- *H may be isotoped to be transverse to F, with every component of $H \setminus N(F)$ incompressible in the respective submanifold of $M \setminus N(F)$,*
- *H may be isotoped to be transverse to F, with every component of $H \setminus N(F)$ incompressible in the respective submanifold of $M \setminus N(F)$ except for exactly one strongly irreducible component, or*
- *H may be isotoped to be almost transverse to F, with every component of $H \setminus N(F)$ incompressible in the respective submanifold of $M \setminus N(F)$.*

Remarks 3.4

(1) After applying the lemma every loop of $H \cap F$ must be essential on both surfaces. Otherwise there is such a loop that is inessential on F and essential on H. This loop, after a small isotopy, bounds a compressing disk D for a component H' of $H \setminus N(F)$. By the lemma, H' must then be strongly irreducible. But D is disjoint from every compressing disk for H' on the opposite side, a contradiction.

(2) In the case where $F \cong \mathbb{T}^2$ it will follow from the proof that H may actually be isotoped to be transverse to F. Here, only conclusions one or two of the lemma occur.

Proof of Lemma 3.3 Choose spines Σ_V of V and Σ_W of W.

Claim 3.5 The surface F meets both Σ_V and Σ_W.

Proof Suppose $F \cap \Sigma_V = \varnothing$. Then F lies in a compression body homeomorphic to W. As the only incompressible surfaces in W are components of $\partial_- W$, we conclude that F is boundary parallel in M. This violates the hypotheses of Lemma 3.3. □

Fix a *sweepout* of M: a continuous map $\Phi\colon H \times I \to M$ such that

- $H(0) = \Sigma_V$,
- $H(1) = \Sigma_W$, and
- the restriction of Φ to $H \times (0,1)$ is a smooth homeomorphism onto the complement of $\Sigma_V \cup \Sigma_W$.

Here $H(t) = \Phi(H \times t)$. The map Φ is a *sweepout* of M. (Note that this is a slightly different definition than the one introduced by Rubinstein). Let $V(t)$ and $W(t)$ denote the compression bodies bounded by $H(t)$ (where $\Sigma_V \subset V(t)$).

The sweepout Φ induces a height function $h\colon F \to I$ as follows. Define $h(x) = t$ if $x \in \Phi(H,t)$. Perturb F so that h is Morse on $F \smallsetminus (\Sigma_V \cup \Sigma_W)$. Let $\{t_i\}_{i=0}^n$ denote the set of critical values of h. It follows from Claim 3.5 that $t_0 = 0$ and $t_n = 1$. We now label each subinterval (t_i, t_{i+1}) with the letters $\mathbb{V}$ and/or $\mathbb{W}$ by the following scheme. If, for some $t \in (t_i, t_{i+1})$, there is a compressing disk for $H(t)$ in $V(t)$ with boundary disjoint from F then label this subinterval with the letter $\mathbb{V}$. See Figure 3. Similarly, if there is a compressing disk in $W(t)$ with boundary disjoint from F then label with the letter $\mathbb{W}$.

Claim 3.6 If the subinterval (t_i, t_{i+1}) is unlabelled then the first conclusion of Lemma 3.3 follows.

Proof Suppose $t \in (t_i, t_{i+1})$. First, we claim that all curves of $H(t) \cap F$ are essential on both or inessential on both. If not then, as F is incompressible, there is a loop $\delta \subset H(t) \cap F$ that is inessential on F but essential on $H(t)$. The loop δ bounds a disk $D \subset F$. Thus the hypotheses of Lemma 3.1 are satisfied. It follows that δ bounds a compressing disk in $V(t)$ or in $W(t)$. Finally, δ may be isotoped inside of $H(t)$ by a

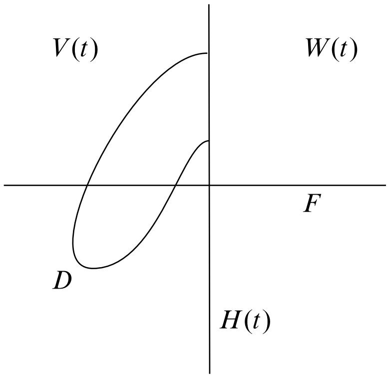

Figure 3: If D is a compressing disk for $H(t)$ in $V(t)$ with boundary disjoint from F then the interval containing t would get the label $\mathbb{V}$.

small pushout move to be disjoint from F. This violates the assumption that (t_i, t_{i+1}) is unlabelled. We deduce that all curves of $H(t) \cap F$ are essential or inessential on both.

As M is irreducible we may isotope $H(t)$ to remove those loops of $H(t) \cap F$ which are inessential on both surfaces, without affecting those loops of $H(t) \cap F$ which were essential on both. We now claim that after such an isotopy any essential loop of $H(t) \setminus N(F)$ is essential on $H(t)$. We prove the contrapositive: Suppose $E \subset H(t)$ is an embedded disk with $\partial E \cap F = \varnothing$. All curves of $E \cap F$ are inessential on both surfaces. Isotope E rel boundary to make $E \cap F = \varnothing$. We conclude $E \subset M \setminus N(F)$, and hence ∂E is inessential on $H(t) \setminus N(F)$.

Finally, we claim that the components of $H(t) \setminus N(F)$ are incompressible in the respective submanifolds of $M \setminus N(F)$. Suppose H' is a compressible component. Then there is an essential loop $\gamma \subset H'$ which bounds a compressing disk for H'. By the preceding remarks γ is essential on $H(t)$ as well. By Lemma 3.1 the loop γ bounds a compressing disk for $H(t)$, which must be in $V(t)$ or $W(t)$. This now contradicts the fact that (t_i, t_{i+1}) is unlabelled. □

Claim 3.7 If the subinterval (t_i, t_{i+1}) has both of the labels $\mathbb{V}$ and $\mathbb{W}$ then the second conclusion of Lemma 3.3 follows.

Proof Suppose $t \in (t_i, t_{i+1})$. We begin as in the proof of Claim 3.6 by asserting that all curves of $H(t) \cap F$ are either inessential or essential on both. If not, then as above there is a loop $\delta \subset H(t) \cap F$ which bounds a compressing disk for $H(t)$. Suppose δ bounds a compressing disk in $V(t)$. (The other case is similar.) Since (t_i, t_{i+1}) has the label $\mathbb{W}$ there is a loop γ on some component of $H(t)\smallsetminus N(F)$ which bounds a disk in $W(t)$. But then $\delta \cap \gamma = \varnothing$ contradicts the strong irreducibility of H.

As in the proof of Claim 3.6 it now follows that we may isotope $H(t)$, preserving the set of loops of $H(t) \cap F$ which are essential on both, so that any loop which is essential on $H(t)\smallsetminus N(F)$ is also essential on $H(t)$.

Let H' be a component of $H(t)\smallsetminus N(F)$ which contains a loop γ bounding a compressing disk for $H(t)$ in $W(t)$. By strong irreducibility of $H(t)$ any essential loop of $H(t)\smallsetminus N(F)$ which bounds a compressing disk in $V(t)$ must meet γ, and hence must also lie in H'. Furthermore, since the subinterval (t_i, t_{i+1}) has the label $\mathbb{V}$, there is at least one such loop ρ. By identical reasoning we conclude that any essential loop of $H(t)\smallsetminus N(F)$ which bounds a compressing disk in $W(t)$ must meet ρ, and hence must also be on H'. We conclude that there are no loops on any other component of $H(t)\smallsetminus N(F)$ which bound compressing disks. Hence all components of $(H(t)\smallsetminus N(F))\smallsetminus H'$ are incompressible in the respective submanifolds of $M \smallsetminus N(F)$. Furthermore, the strong irreducibility of H' follows from the existence of the $\mathbb{V}$ and $\mathbb{W}$ labels and strong irreducibility of $H(t)$. □

Claim 3.8 If the labelling of (t_{i-1}, t_i) is different from that of (t_i, t_{i+1}) then the critical value t_i corresponds to a saddle tangency between $H(t_i)$ and F. □

Claim 3.9 The subinterval $(0, t_1)$ is labelled $\mathbb{V}$ and the subinterval $(t_{n-1}, 1)$ is labelled $\mathbb{W}$.

Proof For sufficiently small ϵ the surface $H(\epsilon)$ looks like the frontier of a neighborhood of Σ_V. By Claim 3.5 the surface F meets Σ_V. Hence, F contains small compressions for $H(\epsilon)$ in $V(\epsilon)$. We can push these compressions off F, giving compressions with boundary on a component of $H(\epsilon)\smallsetminus N(F)$ in $V(\epsilon)$. Hence, the label of $(0, t_1)$ is $\mathbb{V}$. A symmetric argument completes the proof of the claim. □

Following Claims 3.6 and 3.7 we now assume that every subinterval has a label. Furthermore, we assume that every subinterval has exactly one label: either $\mathbb{V}$ or $\mathbb{W}$, but not both. It then follows from Claim 3.9 that there is some first critical value t_i where the labelling changes from $\mathbb{V}$ to $\mathbb{W}$. By Claim 3.8 this critical value must correspond to a saddle tangency.

Claim 3.10 There is a surface H_0, isotopic to $H(t_i)$, such that all components of $H_0\smallsetminus N(F)$ are incompressible.

Proof First, we claim that every component of $H(t_i) \cap F$ which is a loop is either essential or inessential on both surfaces. If not, then as in the proof of Claim 3.6 there is a loop component δ of $H(t_i) \cap F$ which bounds a compressing disk for $H(t_i)$. Assume that the compressing disk bounded by δ lies in $W(t_i)$, as the other case is similar. Pushing δ off of F along $H(t_i)$ then yields a loop on $H(t_i) \setminus N(F)$ bounding a compressing disk in $W(t_i)$. This implies that there is a loop on $H(t_i - \epsilon) \setminus N(F)$ that bounds a compressing disk for $H(t_i - \epsilon)$ in $W(t_i - \epsilon)$. This violates the fact that the subinterval (t_{i-1}, t_i) does not have the label $\mathbb{W}$.

Now let Γ_u denote the union of the inessential loops of $H(t_i) \cap F$ and Γ_e the union of the essential loops. The intersection set $H(t_i) \cap F$ thus consists of Γ_u, Γ_e, and a figure eight curve C. Let $N_H(C)$ denote a closed neighborhood of C on $H(t_i)$. If some component α of $\partial N_H(C)$ bounds a disk in $H(t_i)$ that contains C then we say C was *inessential*.

Let $\pi\colon H \times I \to H$ denote projection onto the first factor. Let $\pi_H = \pi \circ \Phi^{-1}$. Then, for each $t \in (0,1)$, the function $\pi_H | H(t)$ is a map from $H(t)$ to H.

The sets $\pi_H(\Gamma_u)$ and $\pi_H(\Gamma_e)$ are isotopic to subsets of $\pi_H(H(t_i - \epsilon) \cap F)$ and $\pi_H(H(t_i + \epsilon) \cap F)$, for sufficiently small ϵ. Such an isotopy induces an identification of Γ_u and Γ_e with subsets of $H(t_i - \epsilon) \cap F$ and $H(t_i + \epsilon) \cap F$. Furthermore the loop α (if it exists) can be identified with loops on $H(t_i - \epsilon)$ and $H(t_i + \epsilon)$ which are disjoint from F.

Let H_0, H_- and H_+ denote the surfaces obtained by isotoping $H(t_i)$, $H(t_i - \epsilon)$ and $H(t_i + \epsilon)$, preserving Γ_e, but removing Γ_u. In each case these isotopies can be achieved via a series of identical moves on innermost disks. Note that if the figure eight C is inessential and surrounded by some loop of Γ_u then it will disappear in the course of these isotopies.

Now suppose C was inessential but did not disappear (and is therefore not surrounded by some loop of Γ_u). By definition α bounds a disk D on H_0 (which can be identified with disks on H_- and H_+). As F is incompressible any intersection of D with F can be removed by a further isotopy of H_0, H_- and H_+. Henceforth, we will assume that if C is inessential then α bounds disks in H_0, H_- and H_+ which are disjoint from F.

Let V_0, W_0, V_-, W_-, V_+, and W_+ be the corresponding compression bodies bounded by H_0, H_-, and H_+. By assumption the interval (t_{i-1}, t_i) does not have the label $\mathbb{W}$. It thus follows that no essential loop of H_-, disjoint from F, bounds a compressing disk in W_-. This is because only inessential loops are effected in the passage from $H(t_i - \epsilon)$ to H_-. Similarly we may conclude that no essential loop of H_+, disjoint from F, bounds a compressing disk in V_+.

Assume, to obtain a contradiction, that E' is a compressing disk for a component H' of $H_0 \smallsetminus N(F)$. Since every loop of $H_0 \cap F$ is essential on H_0, and C was removed if it was inessential, it follows that $\partial E'$ is essential on H_0. Furthermore, as only the inessential intersection curves were effected in the passage from $H(t_i)$ to H_0 it follows that $\partial E'$ is an essential loop on $H(t_i)$, and is disjoint from F. It follows from Lemma 3.1 that there is a compressing disk E for $H(t_i)$ with $\partial E = \partial E'$. Hence ∂E is also disjoint from F.

The loop ∂E can be identified with essential loops of both $H(t_i - \epsilon) \smallsetminus N(F)$ and $H(t_i + \epsilon) \smallsetminus N(F)$ which bound similar compressing disks. We conclude the disk E may be identified with a compressing disk for both H_- and H_+ with boundary disjoint from F. If $E \subset W(t_i)$ then this violates the fact that there is no compressing disk for H_- in W_- with boundary disjoint from F. On the other hand, if $E \subset V(t_i)$, then we contradict the fact that there is no compressing disk for H_+ in V_+ with boundary disjoint from F.

We conclude that the components of $H_0 \smallsetminus N(F)$ are incompressible in the respective submanifolds of $M \smallsetminus N(F)$, as asserted by the third conclusion of the lemma. □

The third conclusion of Lemma 3.3 follows. This completes the proof of Lemma 3.3. □

We now use the above result to establish the following lemma.

Lemma 3.11 *Let M be a compact, irreducible, orientable 3–manifold with ∂M incompressible, if non-empty. Suppose $M = X \cup_F Y$, where F is essential, connected, and closed. Suppose $M = V \cup_H W$, where H is a Heegaard surface. Then either H is an amalgamation of splittings of X and Y or there are properly embedded surfaces $H_X \subset X$ and $H_Y \subset Y$ with boundaries on F such that at least one of the following holds:*

1. *The surfaces H_X and H_Y are incompressible, not boundary parallel, $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
2. *After possibly exchanging X and Y the surface H_X is incompressible, not boundary parallel, the surface H_Y is strongly irreducible, $\partial H_X = \partial H_Y$ and $\chi(H_X) + \chi(H_Y) \geq \chi(H)$.*
3. *The surfaces H_X and H_Y are incompressible, not boundary parallel, $\partial H_X \cap \partial H_Y = \varnothing$, and $\chi(H_X) + \chi(H_Y) - 1 \geq \chi(H)$.*

Remark 3.12 If H is assumed to be strongly irreducible then we will show that each of the above inequalities can be replaced by equalities.

Proof By Scharlemann–Thompson [12] we may *untelescope* the Heegaard splitting H. That is, there is a generalized Heegaard splitting $\{H_i\}_{i=0}^{2n}$ of M with thick and thin levels obtained from H by some number of compressions. Furthermore, we can find such a generalized Heegaard splitting such that each thick level H_i is strongly irreducible in the submanifold of M cobounded by H_{i-1} and H_{i+1}. It is shown in [12] that in such a generalized Heegaard splitting each thin level is incompressible in M.

Isotope F to meet the set of thin levels of $\{H_i\}$ in a minimal number of curves. Suppose first that for some i, the surface F is parallel to a component of the thin level H_{2i}. Then the components of $\{H_i\}$ which meet X form an untelescoped Heegaard splitting of X, and the components which meet Y form an untelescoped Heegaard splitting of Y. Telescoping (the operation which is the inverse of untelescoping) now produces Heegaard splittings of X and Y with amalgamation H. Hence, the first conclusion of Lemma 3.11 follows.

Now suppose F intersects the thin level H_{2i}. Then F divides H_{2i} into subsurfaces $H_X \subset X$ and $H_Y \subset Y$. We claim that H_X is incompressible in X and H_Y is incompressible in Y. If not, then there is some compressing disk D for H_X (say) in X. As H_{2i} is incompressible in M, ∂D bounds a disk E in H_{2i}. Since ∂D is essential in H_X but inessential in H_{2i} the surface F must intersect the disk $E \subset H_{2i}$. As M is irreducible we can now do a sequence of isotopies to remove all curves of $E \cap F$, reducing the number of times F meets the set of thin levels.

Since F meets all thin levels minimally it also follows that neither H_X nor H_Y are boundary parallel. Finally, since $H_{2i} = H_X \cup H_Y$, and H_{2i} is obtained from H be some number of compressions, we have $\chi(H_X) + \chi(H_Y) \geq \chi(H)$. Hence, Case (1) of the conclusion of Lemma 3.11 follows.

We are now reduced to the case where F misses all thin levels, and is parallel to none. Hence, F is completely contained in a submanifold with incompressible boundary which has a strongly irreducible Heegaard splitting, obtained from H by some number of compressions. It suffices, then, to prove Lemma 3.11 in the case where H is strongly irreducible.

Use Lemma 3.3 to isotope H so that it is transverse or almost transverse to F, and so that the conclusion of Lemma 3.3 follows. If H is transverse to F then let $H_X = H \cap X$ and $H_Y = H \cap Y$, and Case (1) or (2) of the lemma at hand follows.

The remaining case is when H meets F almost transversally. Let p denote the saddle point of $H \cap F$. Isotope H by pushing the point p slightly into Y, to obtain the surface H'. Hence, H' is transverse to F. Furthermore, any compressing disk for

$H_X = H' \cap X$ is a compressing disk for $H \cap X$, so there must be none by Lemma 3.3. We conclude H_X is a properly embedded, incompressible surface in X. Similarly, by pushing p slightly into X we may obtain from H a properly embedded, incompressible surface $H_Y \subset Y$.

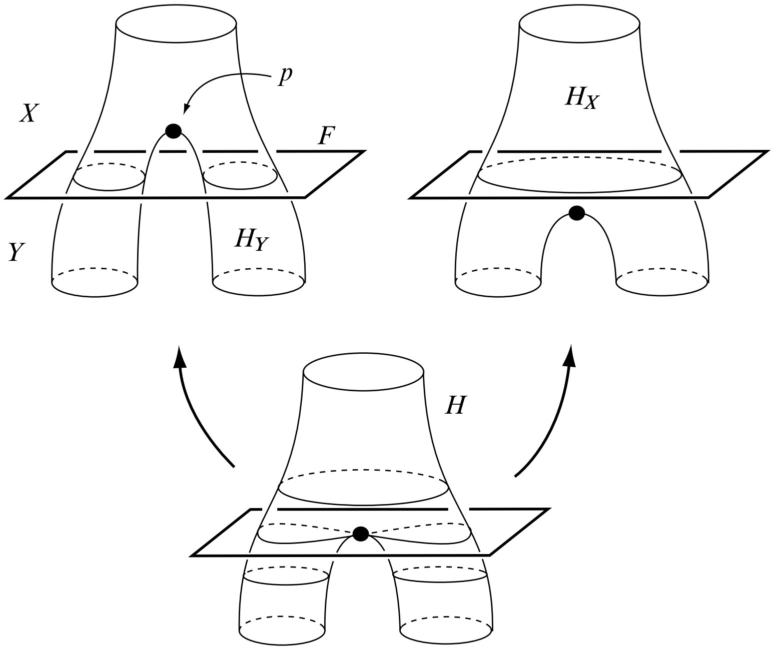

Figure 4: H differs from $H_X \cup H_Y$ by a pair of pants.

As H and F are orientable, it follows that $H_X \cap F$ may be made disjoint from $H_Y \cap F$. Furthermore, the only essential difference between $H_X \cup H_Y$ and H is a pair of pants, having Euler characteristic negative one (see Figure 4). Hence, Case (3) of the conclusion of Lemma 3.11 now follows. □

4 Manifolds with no strongly irreducible Heegaard splittings

A *knot manifold* is a compact, orientable, irreducible three–manifold with a single boundary component, which is incompressible and homeomorphic to a torus. The goal of this section is to prove the following theorem:

Theorem 4.1 *Suppose that X and Y are* knot manifolds *and $\varphi\colon \partial X \to \partial Y$ is a* sufficiently complicated *homeomorphism. Then the manifold $M(\varphi) = X \cup_\varphi Y$ has no strongly irreducible Heegaard splittings.*

Note the similarity of Theorem 4.1 to Cooper and Scharlemann's result [2]. That paper proves that if a 3–manifold is constructed by identifying the boundary components of $T^2 \times I$ via a "sufficiently complicated" map then there are no strongly irreducible Heegaard splitting of the resulting 3–manifold.

To make the statement of Theorem 4.1 precise we must give a reasonable definition of the term *sufficiently complicated.* To this end fix, once and for all, psuedo-triangulations of X and Y with one vertex. (A *psuedo-triangulation* is a decomposition into simplices where any two such simplices intersect in a collection of lower dimensional simplices.) Let $\Delta(X)$ be the set of slopes in ∂X which are the boundary of some normal or almost normal surface in X. Note that $\Delta(X)$ is finite, by a result of Jaco and Sedgwick [6] (see also Theorem 9.7 of Bachman [1] for a discussion of the almost normal case). Define $\Delta(Y)$ similarly.

Recall now the definition of the *Farey graph*, $\mathcal{F}(X)$. The vertices of $\mathcal{F}(X)$ are all slopes in ∂X. Two slopes are connected by an edge if they intersect once. The *distance* between two slopes is then defined to be the minimal number of edges required in a path connecting them. The *distance* between two sets of slopes is the minimal distance between their elements.

Definition 4.2 A map $\varphi\colon \partial X \to \partial Y$ is *sufficiently complicated* if the distance between $\Delta(X)$ to $\varphi^{-1}(\Delta(Y))$ inside of $\mathcal{F}(X)$ is at least two.

Remark 4.3 Note that, as $\Delta(X)$ and $\Delta(Y)$ are finite, "most" elements of $\mathcal{MCG}(\mathbb{T}^2) \cong SL(2,\mathbb{Z})$ are sufficiently complicated, in the above sense. In particular any sufficiently large power of an Anosov map is sufficiently complicated. The same holds for all but a finite number of Dehn twists.

Before giving the proof of Theorem 4.1 we must discuss boundary compressions. Suppose $G \subset N$ is a properly embedded, two–sided surface in a compact, orientable, irreducible three–manifold N. We suppose further that ∂N is incompressible in N. Suppose $D \subset N$ is a boundary compression for G.

Definition 4.4 The boundary compression D is *honest* if $D \cap \partial N$ is essential as a properly embedded arc in $\partial N \setminus \partial G$. If D is not honest it is *dishonest.*

Definition 4.5 Let N be a knot manifold. We now define the *banding*, $\widehat{D}$, of a boundary compression D for G. First assume D is honest. Then $D \cap \partial N$ meets distinct boundary components of ∂G, as G is orientable. These components of ∂G cobound an annulus $A \subset \partial N$ such that $D \cap \partial N \subset A$. Let D' denote the disk obtained from A by removing a neighborhood of $D \cap \partial N$ and attaching two parallel copies of D. Isotope D' to be disjoint from ∂N while maintaining $\partial D' \subset G$. The resulting disk is the desired banding $\widehat{D}$ of D.

Now suppose D is dishonest. Then the arc $D \cap \partial N$ cobounds, with a subarc of ∂G, a subdisk D' of ∂N. The disk $\widehat{D}$ is obtained by pushing $D'' = D \cup D'$ into the interior of N, while maintaining $\partial D'' \subset G$.

Note that when C is a compressing disk and D is a boundary-compressing disk (honest or dishonest) if $C \cap D = \varnothing$ then $C \cap \widehat{D} = \varnothing$.

Lemma 4.6 *If D is a boundary compression for G and $\partial N = \mathbb{T}^2$ then G is either compressible or the component of G meeting D is a boundary parallel annulus.* □

Recall that by a *strongly irreducible* surface we mean a properly embedded, two–sided surface which compresses on both sides and all pairs of compressing disks on opposite sides must meet. We now strengthen this definition to account for boundary compressions, as in Bachman [1].

Definition 4.7 A properly embedded, separating surface is *∂–strongly irreducible* if

(1) every compressing and boundary-compressing disk on one side meets every compressing and boundary-compressing disk on the other side, and
(2) there is at least one compressing or boundary-compressing disk on each side.

Lemma 4.8 *Let N be a knot manifold. Let G be a separating, properly embedded, connected surface in N which is strongly irreducible, has non-empty boundary, and is not peripheral. Then either G is ∂–strongly irreducible or ∂G is at most distance one from the boundary of some properly embedded surface which is both incompressible and boundary-incompressible.*

Proof Suppose G divides N into V and W. If G is not ∂–strongly irreducible then there are disjoint disks $D \subset V$ and $E \subset W$ such that at least one, say D, is a boundary-compressing disk. The disk E is either a compression or a boundary compression.

Since G is not a boundary parallel annulus we know by Lemma 4.6 that the banding disk $\widehat{D}$ is a compressing disk for G. If E is a compressing disk then $E \cap D = \varnothing$ implies that $E \cap \widehat{D} = \varnothing$, contradicting strong irreducibility. We conclude E is a boundary compression.

Let G' denote the result of boundary-compressing G along both D and E. Let V' and W' denote the sides of G' which correspond to V and W. We now claim that G' is incompressible. Suppose D' is a compressing disk for G' in V'. Then D' must have been a compressing disk for G in V which was disjoint from E, and hence disjoint from $\widehat{E}$. This contradicts the strong irreducibility of G. By symmetry we conclude G' is incompressible.

We now claim G' is boundary incompressible as well. Suppose C is a boundary-compressing disk for G'. Since G' is incompressible we know $\widehat{C}$ is not a compressing disk, so it follows from Lemma 4.6 that G' must be a boundary parallel annulus. It follows that all of G was isotopic into a neighborhood of ∂N, contradicting our hypotheses.

It remains only to show that ∂G is at a distance of at most one from $\partial G'$. In order for the slope of $\partial G'$ to be different from the slope of ∂G all of the loops of ∂G must meet either D or E. This immediately implies $|\partial G| \leq 4$. The possibility that $|\partial G|$ is one or three is ruled out by the fact that G is separating. The fact that D and E are on opposite sides of G rules out $|\partial G| = 4$, since we are assuming that every component of ∂G meets either D or E.

If $|\partial G| = 2$, both D and E are dishonest, and each meets different components of ∂G then $\widehat{D} \cap \widehat{E} = \varnothing$. This violates the strong irreducibility of G.

There are three remaining cases. In each of these cases $|\partial G| = 2$ and both boundary loops are affected by the transition to G'. See Figure 5. In the top picture both D and E are honest. The two loops of ∂G are transformed into two loops, both distance one from the original. In the middle picture exactly one of the disks D or E is dishonest, and the boundary slope remains unchanged. The configuration depicted at the bottom of Figure 5 cannot happen, since it represents a situation in which $\widehat{D}$ is disjoint from $\widehat{E}$, contradicting the strong irreducibility of G. □

We conclude with:

Proof of Theorem 4.1 Suppose that X and Y are triangulated knot manifolds, as above. Fix a gluing $\varphi\colon \partial X \to \partial Y$. Suppose that $H \subset M(\varphi) = X \cup_\varphi Y$ is a strongly irreducible Heegaard splitting surface. Let $F \cong \mathbb{T}^2$ be the image of ∂X inside of $M(\varphi)$.

Now apply Lemma 3.3 and Remark 3.4 to the pair H and F in $M(\varphi)$. Let H_X be a component of $H \cap X$ which is incompressible and not a boundary parallel annulus, if such exists. If no such component exists take H_X to be the non-boundary parallel component of $H \cap X$. In this case H_X is strongly irreducible. (At least one component

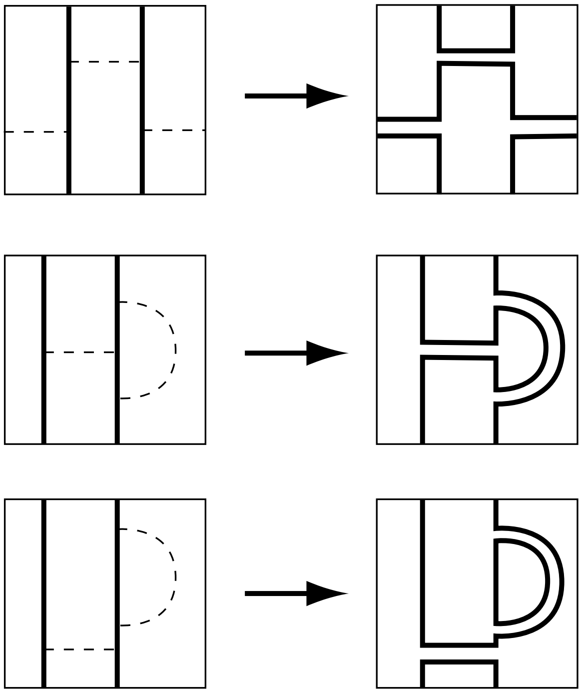

Figure 5: Possible effects of boundary-compression on ∂G

of $H \cap X$ is not boundary parallel. Otherwise H is isotopic into Y, a contradiction.) Choose H_Y similarly and note that, by Lemma 3.3, not both of H_X and H_Y are strongly irreducible. Note that ∂H_X and $\varphi^{-1}(\partial H_Y)$ have the same slope.

Suppose that H_X and H_Y are both incompressible. As $\partial X \cong \partial Y \cong \mathbb{T}^2$ it follows from Lemma 4.6 that H_X and H_Y are also boundary incompressible. So H_X and H_Y may be normalized with respect to the given triangulations Haken [3]. It follows that the sets $\Delta(X)$ and $\varphi^{-1}(\Delta(Y))$ intersect and thus φ is not sufficiently complicated.

Suppose now that H_X is incompressible and thus boundary incompressible. Suppose that H_Y is a strongly irreducible surface. Then, by Lemma 4.8, either H_Y is ∂–strongly irreducible or ∂H_Y intersects the boundary of some incompressible, boundary incompressible surface H'_Y at most once. In the latter case H'_Y may be normalized, and hence $\partial H'_Y \in \Delta(Y)$. In the former case it follows from work of the first author (Corollary 8.9 of [1]) that the surface H_Y is properly isotopic to an almost normal surface, and so $\partial H_Y \in \Delta(Y)$. In either case we see ∂H_X (an element of $\Delta(X)$) is within distance one from some element of $\varphi^{-1}(\Delta(Y))$ and hence φ is not sufficiently complicated. □

5 Amalgamating small manifolds

Let X be a manifold with boundary. The *tunnel number* of X, $t(X)$ is the minimal number of properly embedded arcs that need to be drilled out of X to obtain a handlebody; i.e. so that $X \smallsetminus N(\text{arcs})$ is a handlebody. The *handle number* of X is the minimal number of properly embedded arcs that need to be drilled out of X to obtain a compression body; i.e. so that $X \smallsetminus N(\text{arcs})$ is a compression body. If $|\partial X| = 1$ then $t(X) = h(X)$.

Let $M = X \cup_F Y$ be a manifold obtained by gluing X and Y, two connected small manifolds with incompressible boundary, along a collection of boundary components homeomorphic to a surface F. The goal of this section is to show that the Heegaard genera of X and Y are bounded in terms of the Heegaard genus of $M = X \cup_F Y$. More specifically, we establish:

Theorem 5.1 *Let M be a compact, orientable 3–manifold with incompressible boundary. Suppose M is obtained by gluing two connected, small manifolds along a union of incompressible boundary components, $M = X \cup_F Y$. Then the following statements hold:*

(1) $g(M) \geq \frac{1}{2}(h(X) + h(Y))$

(2) *if M is closed and F is connected,* $g(M) \geq \frac{1}{2}(t(X) + t(Y))$

(3) $g(M) \geq \frac{1}{2}(g(X) + g(Y) - 2g(F))$.

The theorem is motivated by the fact that a properly embedded, incompressible surface cuts a small manifold into one or two compression bodies.

We begin with the following definitions. Let F be an orientable surface, possibly with boundary components, and possibly disconnected. Let C be the manifold obtained by forming $F \times I$ and attaching one handles to the surface $F \times \{1\}$. Then C is a *relative compression body*. We label the boundary as follows: the *negative boundary* is $\partial_- C = F \times \{0\}$, the *vertical boundary* is $\partial_V C = \partial F \times I$, and the *positive boundary* is $\partial_+ C = \overline{\partial C \smallsetminus (\partial_- C \cup \partial_V C)}$. The vertical boundary is a collection of annuli. It is important to note that a given manifold may admit many relative compression body structures. For example, if F is a surface with boundary and $C = F \times I$, then C can be thought of as a relative compression body with $\partial_- C = F \times \{0\}$, or C can be thought of as a handlebody with $\partial_- C = \varnothing$. In fact, given a relative compression body C, it is always possible to think of C as a (non-relative) compression body by *promoting* all non-closed components of $\partial_- C$ and all components of $\partial_V C$ to the positive boundary.

A *relative Heegaard splitting* is the union of two relative compression bodies, identified along their positive boundaries. The splitting will be considered *non-trivial* if neither relative compression body is a product; i.e. both compression bodies have 1–handles.

Lemma 5.2 *Let X be a manifold that admits a non-trivial, strongly irreducible and relative Heegaard splitting $X = C_1 \cup C_2$. Then $\partial_- C_1$ and $\partial_- C_2$ are incompressible in X.*

Proof An examination of the proof of the Haken Lemma [4] (see also Jaco [5]) will reveal that it applies directly to the case of relative Heegaard splittings. In particular, if either $\partial_- C_1$ or $\partial_- C_2$ has compressible boundary, then there is a compressing disk D for the boundary component that meets the splitting surface in a single closed loop. The loop decomposes the compressing disk into a vertical annulus in one compression body, say C_1, and a disk $D_2 \subset C_2$. Since C_1 is not a product we can find a compressing disk D_1 for $\partial_+ C_1$, disjoint from the annulus, and hence disjoint from D_2. The pair (D_1, D_2) contradicts strong irreducibility of the relative Heegaard splitting. □

Lemma 5.3 *An irreducible connected small manifold with compressible boundary is a compression body.*

Proof Let X be a connected small manifold with compressible boundary. In an optimistic fashion, denote a compressible boundary component by $\partial_+ X$ and all other components by $\partial_- X$. Since $\partial_+ X$ is compressible it bounds a (not properly embedded) submanifold C of X which is a compression body, so that $\partial_+ C = \partial_+ X$. Choose C to be maximal in this regard. Precisely, choose C so that $\partial_- C$ contains no 2–spheres (X is irreducible) and so that $\sum(1 - \chi(S_i))$ is minimal, where $\{S_i\}$ are the components of $\partial_- C$.

If S is a component of $\partial_- C$ then S is incompressible in C. Suppose D is a compressing disk for S in $X \setminus C$. Then D is the core of a 2–handle that we can attach to C to obtain a new compression body with negative boundary "smaller" than that of C. This contradicts our minimality assumption. We conclude S is incompressible in $X \setminus C$. As X is small, S must be peripheral, and since C is not a product, it is parallel in $X \setminus C$ to a component of $\partial_- X$. The (possibly disconnected) surface $\partial_- C$ separates the components of $\partial_- X$ from $\partial_+ X$, so each component of $\partial_- X$ is in fact parallel to a component of $\partial_- C$. The parallelism yields an isotopy between X and C. X is therefore a compression body. Note that only one boundary component, $\partial_+ X$, is compressible. □

Theorem 5.4 *Let* H_X *be a non-peripheral, connected, incompressible surface that is properly embedded in a connected, small manifold* X*. Then* $h(X) \le 1 - \chi(H_X)$*. If* X *has a single boundary component or* H_X *meets every boundary component of* X*, then this applies to the tunnel number:* $t(X) \le 1 - \chi(H_X)$*.*

Proof Let $\partial_1 X$ denote those boundary components of X that meet H_X and $\partial_2 X$ denote those boundary components which do not meet H_X.

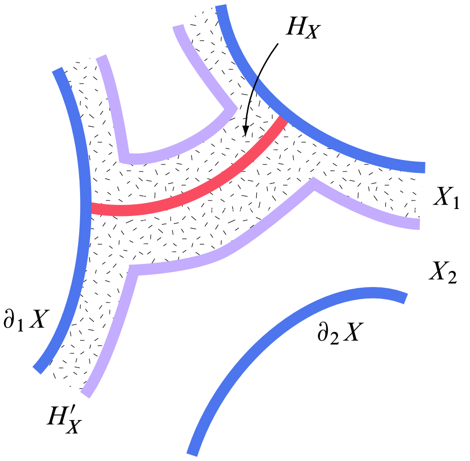

Figure 6: Labelling the boundary components of X

Let $X_1 = \overline{N(H_X \cup \partial_1 X)}$ and $X_2 = \overline{X \smallsetminus X_1}$. This decomposes X into $X = X_1 \cup_{H'_X} X_2$, where H'_X is the common boundary of X_1 and X_2. See the schematic in Figure 6. Note that $\partial_1 X$ and H_X are contained in X_1 and $\partial_2 X$ is contained in X_2. Since H_X is connected it follows that X_1 is connected. If H_X separates X then X_2 will have two components.

The surface H'_X will have two components if H_X separates and one component otherwise. Since X is a small manifold, each component of H'_X is either compressible in X or peripheral to a boundary component of X.

Claim *If a component of* H'_X *is compressible, it is compressible into* X_2*.*

Proof If there is a compressing disk for the compressible component of H'_X then there is one that is disjoint from H_X. This is because any intersection could be removed by surgery. If H'_X has two components then H_X separates them. Hence our chosen compressing disk does not meet the other component of H'_X. Therefore, our compressing disk is properly embedded in either $X_1 \smallsetminus N(H_X)$ or X_2. But, $X_1 \smallsetminus N(H_X)$

is a product and has incompressible boundary. It follows that a compressible component of H'_X is compressible into X_2. □

Claim *No component of H'_X is peripheral into $\partial_1 X$.*

Proof If this occurred, X_1 would be contained in a product neighborhood of a boundary component. This in turn implies that H_X was peripheral. □

Claim *Each component of X_2 is a compression body.*

Proof Suppose that a component X' of X_2 contains a closed non-peripheral essential surface G. Since X is small, G is either compressible in X or parallel to a component G' of $\partial X \setminus \partial X'$. In the latter case $G' \subset \partial_1 X$ or $G' \subset \partial_2 X \setminus \partial X'$. If $G' \subset \partial_2 X \setminus \partial X'$ then H_X separates G from G'.

Since H_X is incompressible, any compressing disk $D \subset X$ for G can be isotoped so that it does not intersect H_X, and so can be isotoped to miss X_1. Therefore G is compressible in X_2, contradicting the essentiality of G. If $G' \subset \partial_1 X$ or $G' \subset \partial_2 X \setminus \partial X'$ then there is a product containing H_X. In particular, this implies that H_X is contained in a product neighborhood of ∂X, contradicting the fact that H_X is not peripheral. Thus, X_2 is small.

Each component of H'_X is therefore compressible into X_2 or parallel to a component of $\partial_2 X$. In either case, by Lemma 5.3 or by parallelism, $H'_X = \partial_+ X_2$, where X_2 is either one or two compression bodies. □

It is now straightforward to build a handle system for X (tunnel system in the case that $\partial_1 X = \partial X$). Choose τ, a minimal collection of arcs that are properly embedded in H_X and that cut H_X into a single disk D. The collection τ contains $1 - \chi(H_X)$ arcs. Moreover, τ is a handle system that induces a Heegaard splitting, $X = C_1 \cup C_2$, where $C_1 = \overline{N(\partial_1 X \cup \tau)}$ and $C_2 = \overline{X \setminus C_1}$. Clearly C_1 is a compression body. C_2 is a compression body because it is formed by attaching a 1–handle (a neighborhood of the cocore of D) to the positive boundary of the compression body/bodies X_2. This completes the proof of Theorem 5.4. □

Theorem 5.5 *Let H_X be a non-peripheral, bi-compressible, connected, strongly irreducible surface properly embedded in a connected, small manifold X. Then $h(X) \leq 1 - \chi(H_X)$. If X has a single boundary component, then this applies to the tunnel number: $t(X) \leq 1 - \chi(H_X)$.*

Proof We may apply the previous theorem if X also contains a non-peripheral incompressible surface with boundary whose negative Euler characteristic is less than that of H_X. We may therefore assume that H_X is a separating surface; if not we may compress H_X to obtain such an incompressible surface. As before we will let $\partial_1 X$ denote those boundary components of X that meet H_X and $\partial_2 X$ denote those boundary components which do not meet H_X.

By compressing H_X maximally to both sides, we define a relative Heegaard splitting of a submanifold $X' = C_1 \cup_{H_X} C_2 \subset X$. Since we have compressed maximally, the negative boundary components of C_1 and C_2 are incompressible outside X'. They are incompressible inside X' by Lemma 5.2. If any component is non-peripheral, we have our conclusion via Theorem 5.4. Each component of $\partial_- C_i, i = 1, 2$ is therefore peripheral. It now follows from the fact that H_X is non-peripheral that X' is isotopic to X.

As in the earlier theorem, this structure defines a handle system for X. Choose τ, a minimal collection of arcs that are properly embedded in H_X and that cut H_X into a single disk D. Now, τ is a handle system for X that induces the Heegaard splitting, $X = C_1' \cup C_2'$, where $C_1' = \overline{N(\partial_1 X \cup \tau)}$ and $C_2' = \overline{X \smallsetminus C_1'}$. Clearly C_1' is a compression body. C_2' is a compression body because it can be obtained by first promoting the vertical and non-closed negative boundary components of C_1 and C_2 and then joining the positive boundary of these (non-relative) compression bodies with a 1–handle (a neighborhood of the cocore of D).

The handle number of X is thus bounded by $1 - \chi(H_X)$. □

Proof of Theorem 5.1 Let H be a minimal genus splitting of M. If H is an amalgamation of splittings of X and Y, then the result holds trivially. Otherwise, by Lemma 3.11 we can construct properly embedded non-boundary parallel surfaces $H_X' \subset X$ and $H_Y' \subset Y$ so that each is either incompressible or strongly irreducible. As neither surface is boundary-parallel they contain components $H_X \subset H_X'$ and $H_Y \subset H_Y'$ which are non-boundary parallel and either incompressible or strongly irreducible. Furthermore, $\chi(H_X) + \chi(H_Y) \geq \chi(H) = 2 - 2g(M)$, or equivalently, $g(M) \geq \frac{1}{2}(2 - \chi(H_X) - \chi(H_Y))$.

By either Theorem 5.4 or Theorem 5.5, X and Y admit handle systems that are attached to components of F and so that the number of handles is at most $1 - \chi(H_X)$ and $1 - \chi(H_Y)$, respectively. The first two assertions of Theorem 5.1 follow.

Our induced splitting of X is obtained by attaching $1 - \chi(H_X)$ handles to F. The genus of X is therefore bounded by

$$g(X) \leq g(F) + 1 - \chi(H_X).$$

Since a symmetric bound holds for $g(Y)$ we obtained the third conclusion of Theorem 5.1. $\square$

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Received: 26 July 2005 Revised: 18 January 2006