

A volume form on the $SU(2)$ –representation space of knot groups

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For a knot K in S^3 we construct according to Casson—or more precisely taking into account Lin’s [13] and Heusener’s [10] further works—a volume form on the $SU(2)$ –representation space of the group of K (see Section 3). We next prove that this volume form is a topological knot invariant (see Section 4) and explore some of its properties (see Section 5).

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Motivation and main ideas

In 1985, A. Casson constructed an integer valued invariant of integral homology 3–spheres. The original definition of Casson’s invariant is based on $SU(2)$ –representation spaces. Informally speaking, the Casson invariant of an homology 3–sphere M counts algebraically the number of conjugacy classes of irreducible $SU(2)$ –representations of $\pi_1(M)$ in the same sense that the Lefschetz number of a map counts the number of fixed points, see Akbulut–McCarthy [1] or Guillou–Marin [9]. In 1992, X-S Lin used an analogue of Casson’s original construction to define an integer valued knot invariant [13], where he indirectly proved that this invariant is equal to half the signature of the knot. At first sight, this equality between two apparently different quantities seems mysterious. In 2003, M Heusener explained Lin’s result using an orientation on the representation space of knot groups. More precisely, given a knot $K \subset S^3$ we let M_K denote its exterior and $G_K = \pi_1(M_K)$ its group. In general the $SU(2)$ –representation space of G_K has singularities; to avoid this difficulty, Heusener and Klassen introduced the notion of a regular representation in [11]. An irreducible representation ρ of G_K in $SU(2)$ is called regular if the real vector space $H_\rho^1(M_K)$ is 1–dimensional (over \mathbb{R}). Here $H_\rho^*(M_K)$ denotes the $(Ad \circ \rho)$ –twisted cohomology of M_K . We let $\mathcal{R}eg(K)$ denote the set of conjugacy classes of regular representations of G_K in $SU(2)$. Heusener proved that $\mathcal{R}eg(K)$ is a canonically oriented 1–dimensional manifold (see [10, Section 1]).

In this article, we investigate a volume form on $\mathcal{R}eg(K)$. We prove that it is an invariant of K and we explore some of its properties. The main result of this paper is the following theorem.

Main Theorem *The 1–dimensional manifold $\mathcal{R}eg(K)$ carries a well-defined “canonical” 1–volume form ω^K . This volume form is a knot invariant. It does not depend on the orientation of K ; and if K^* denotes the mirror-image of K , then as oriented manifolds equipped with the canonical volume form, we have*

$$(\mathcal{R}eg(K^*), \omega^{K^*}) = (-\mathcal{R}eg(K), -\omega^K).$$

The construction which enables us to define the “canonical” volume form ω^K on $\mathcal{R}eg(K)$ is motivated by the original construction of Casson’s invariant. In our construction the Heegaard splitting will be replaced by a plat presentation of K and we will not only compare orientations but “natural” volume forms on some appropriate representation subspaces of G_K (see [Section 3](#)).

It is possible to prove that the definition of ω^K can be reformulated as a “combinatorial invariant”, ie, using the CW–complex structure of the exterior of K . Informally speaking, this “combinatorial invariant” is a Reidemeister torsion form on $H_\rho^1(M_K) \cong T_{[\rho]} \widehat{R}(M_K)$. Its definition needs Turaev’s sign-determined Reidemeister torsion of CW–complexes (see for example Turaev’s monograph [\[17\]](#)) and certain distinguished bases for the twisted cohomology groups of M_K . This property of ω^K is discussed by the author in [\[8\]](#). This equality between two apparently different topological invariants—one by means of Reidemeister torsion and another using Casson’s original construction—can be considered as an analogue of a result of E Witten about the moduli space of a Riemannian surface. In [\[18\]](#), E Witten obtained a remarkable formula to compute the volume of the moduli space of a Riemannian surface in terms of a combinatorial invariant, namely a Reidemeister torsion form on the first twisted cohomology group of the Riemannian surface. Finally, note that this reformulation of the volume form ω^K using the Reidemeister torsion of M_K gives another proof of its invariance.

Organization

The paper is organized as follows. Sections [1–2](#) consist of a review of the notions of $SU(2)$ –representation spaces, volume forms and regularity for a representation of a knot group. In [Section 3](#) we describe in detail the construction of the “canonical” volume form on $\mathcal{R}eg(K)$. [Section 4](#) contains the proof of the first part of the Main Theorem ie, the invariance of the volume form (see [Theorem 4.1](#)). In [Section 5](#) we

finish the proof of the Main Theorem by proving some basic properties of the volume form, we also establish a connected sum formula. We end the paper by an explicit computation of the volume form associated to torus knots.

1 Preliminaries

In this section, we collect some well-known results about $SU(2)$ -representation spaces, volume forms and introduce the notation used throughout this paper.

1.1 Some notation

The fundamental group $\pi_1(W)$ of a connected CW-complex W is considered without specifying a base point since all the constructions we perform are invariant under conjugation (see Porti [16, page 9]).

The Lie group $SU(2)$ acts on its Lie algebra $\mathfrak{su}(2)$ via the adjoint representation $Ad_A: \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$ defined by $Ad_A(x) = Ax A^{-1}$, where $A \in SU(2)$. As a manifold $SU(2)$ is identified with the 3-sphere S^3 . Furthermore we identify the 2-sphere S^2 with the set of zero-trace matrices of $SU(2)$: $S^2 = \{A \in SU(2) \mid \text{Tr}(A) = 0\}$. Recall that for each $A \in SU(2)$ there are $\theta \in [0, \pi]$ and $P \in S^2$ such that $A = \cos(\theta) + \sin(\theta)P$. Moreover the pair (θ, P) is unique if and only if $A \neq \pm \mathbf{1}$. Note that Ad_A is the rotation of angle 2θ which fixes P . We always think of $SO(3)$ as the base space of the usual two-fold covering $SU(2) \rightarrow SO(3)$ given by $A \mapsto Ad_A$.

The Lie algebra $\mathfrak{su}(2)$ is equipped with the usual scalar product defined by $\langle x, y \rangle = -1/2 \cdot \text{Tr}(xy)$; and we identify $\mathfrak{su}(2)$ with the *pure quaternions*, ie, with the quaternions of the form $q = ai + bj + ck$.

1.2 Representation spaces

Given a finitely generated group G we let $R(G) = \text{Hom}(G; SU(2))$ denote the space of $SU(2)$ -representations of G . Observe that $R(G)$ is a topological space endowed with the compact-open topology. Here G is assumed to have the discrete topology and $SU(2)$ the usual one. A representation $\rho \in R(G)$ is called *abelian* (resp. *central*) if its image $\rho(G)$ is an abelian subgroup of $SU(2)$ (resp. is contained in the center $\{\pm \mathbf{1}\}$ of $SU(2)$). In the case of $SU(2)$, remark that a representation is abelian if and only if it is *reducible* in the usual sense: ρ is reducible if there exists a non-trivial proper subspace $U \subset \mathbb{C}^2$ such that $\rho(g)(U) \subset U$, for all $g \in G$. A representation is called *irreducible* if it is not abelian. We let $\widetilde{R}(G)$ (resp. $A(G)$, $C(G)$) denote the subspace

of irreducible (resp. abelian, central) representations. One can prove that $\widetilde{R}(G)$ is open in $R(G)$.

The compact Lie group $SU(2)$ acts on $R(G)$ by conjugation. We write $[\rho]$ for the conjugacy class of the representation $\rho \in R(G)$ and we let $SU(2)(\rho)$ denote its orbit. The action by conjugation factors through $SO(3) = SU(2)/\{\pm 1\}$ as a free action on the open subspace $\widetilde{R}(G)$ and we set $\widehat{R}(G) = \widetilde{R}(G)/SO(3)$. In this way, we can think of the map $\widetilde{R}(G) \rightarrow \widehat{R}(G)$ as a principal $SO(3)$ -bundle, see Guillou–Marin [9, Section 3.A].

Notation For a connected CW-complex W , we write $R(W) = R(\pi_1(W))$, $\widetilde{R}(W) = \widetilde{R}(\pi_1(W))$, $\widehat{R}(W) = \widehat{R}(\pi_1(W))$ etc.

1.3 Representation space of knot groups

For a knot $K \subset S^3$ let $M_K = S^3 \setminus N(K)$ denote its exterior and $G_K = \pi_1(M_K)$ its group. Here $N(K)$ is an open tubular neighbourhood of K . Recall that M_K is a compact 3-dimensional manifold whose boundary consists in a single 2-torus. The meridian m of K is only defined up to conjugation and if K is oriented then m is oriented by the convention $lk(K, m) = +1$, where lk denotes the linking number.

The abelianization $G_K/G'_K \cong H_1(M_K; \mathbb{Z})$ is generated by the meridian m of K . As a consequence, each abelian representation of G_K is conjugate to one and only one of the $\varphi_\theta: G_K \rightarrow SU(2)$ defined by $\varphi_\theta(m) = \cos(\theta) + \sin(\theta)\mathbf{i}$, with $0 \leq \theta \leq \pi$.

1.4 Volume forms

Here we collect some well-known facts about volume forms, see Milnor [14, Section 3] and [15, Section 1] for details and proofs.

1.4.1 Volume forms and compatibility Let E be a n -dimensional real vector space. A *volume form* v on E is a generator of the n^{th} exterior power $\bigwedge^n E^*$, where $E^* = \text{Hom}_{\mathbb{R}}(E, \mathbb{R})$ is the dual space of E . Let E', E'' be two real finite dimensional vector spaces and let v', v'' be volume forms on E', E'' respectively. The direct sum $E' \oplus E''$ inherits a canonical volume form denoted $v' \wedge v''$.

Consider now a short exact sequence $0 \rightarrow E' \xrightarrow{i} E \xrightarrow{j} E'' \rightarrow 0$ of real finite dimensional vector spaces. Let v', v and v'' be volume forms on E', E and E'' respectively. Let s denote a section of j so that $i \oplus s: E' \oplus E'' \rightarrow E$ is an isomorphism. We say that the previous three volume forms are *compatible* with each other if $v' \wedge v'' = (i \oplus s)^*(v)$. It is easy to verify that the notion of compatibility does not depend on the chosen section s . Finally, the following very useful lemma is quite clear.

Lemma 1.1 *Let $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ be an exact sequence of real finite dimensional vector spaces. If any two of the vector spaces E' , E and E'' are endowed with a volume form, then the third is endowed with a unique well-defined volume form which is compatible with the two others.*

In particular, if v, v'' are volume forms on E, E'' respectively, we will write $v' = v/v''$ the unique compatible volume form on E' to indicate its dependence on v and v'' . Compatibility will be used in order to build up “new” volume forms (see Section 3.4).

1.4.2 The “base \wedge fiber” condition A volume form v on a n -dimensional manifold is a nowhere vanishing differential n -form. In the sequel, we will make use of the “base \wedge fiber” condition, which is the following. Given two volume forms v and w on the manifolds M^m and N^n respectively, a submersion $f: M \rightarrow N$ and a point $y \in N$, then the subspace $f^{-1}(y) \subset M$ is a submanifold of dimension $m - n$, the tangent space $T_x f^{-1}(y)$ is the kernel of $D_x f$ and we have the short exact sequence

$$0 \rightarrow T_x f^{-1}(y) \xrightarrow{i} T_x M \xrightarrow{D_x f} T_y N \rightarrow 0.$$

The submanifold $f^{-1}(y)$ is endowed with the unique volume form ω such that, for each $x \in f^{-1}(y)$ one has $\omega_x = v_x/w_y$ ie, $\omega_x \wedge w_y = (i \oplus s)^*(v_x)$, s being a section of $D_x f$.

2 Notion of regularity

In this paper we will not consider the singular points of the semi-algebraic set $\widehat{R}(M_K)$ and only focus on the so-called regular representations. This section recalls the definition of regularity for $SU(2)$ -representations of knot groups, see Boyer–Zhang [3], Heusener–Klassen [11], Porti [16, Definition 3.21] and Heusener [10, Section 1] for more details on this notion.

Notation For a CW-complex W and a representation $\rho: \pi_1(W) \rightarrow SU(2)$, let $H_\rho^*(W) = H^*(W; \mathfrak{su}(2)_{Ad \circ \rho})$ denote the cohomology of W with coefficients in the adjoint representation $Ad \circ \rho$. This cohomology is called the $(Ad \circ \rho)$ -twisted cohomology of W . When $H_\rho^*(W) = 0$ we say that ρ is acyclic.

2.1 Non-acyclicity of representation of knot groups

The long exact sequence in $(Ad \circ \rho)$ -twisted cohomology corresponding to the pair $(M_K, \partial M_K)$ and Poincaré duality imply $\dim H_\rho^1(M_K) \geq 1$. So a $SU(2)$ -representation of a knot group is never acyclic.

If $\rho: G_K \rightarrow \mathrm{SU}(2)$ is irreducible, then $H_\rho^0(M_K) = 0$ (because $H_\rho^0(M_K)$ is equal to the subgroup of $\mathfrak{su}(2)$ consisting of elements fixed by $\mathrm{Ad} \circ \rho(G_K)$). Moreover, we have $\dim H_\rho^2(M_K) = \dim H_\rho^1(M_K)$ because

$$\sum_i (-1)^i \dim H_\rho^i(M_K) = 3\chi(M_K) = 0.$$

2.2 Regular representations

An irreducible representation ρ of G_K is called *regular* if $\dim H_\rho^1(M_K) = 1$. One can easily prove that this notion is invariant by conjugation. In the sequel, we let $\mathcal{R}\mathrm{eg}(K)$ denote the set of conjugacy classes of regular representations of G_K in $\mathrm{SU}(2)$.

Example 1 If K denotes a torus knot or the figure eight knot, then any irreducible representation of G_K in $\mathrm{SU}(2)$ is regular (see [7, Example 1.43] or [8]).

M Heusener and E Klassen proved [11, Proposition 1] that $\mathcal{R}\mathrm{eg}(K)$ is a 1–dimensional manifold (which may be empty). If ρ is regular, then its conjugacy class $[\rho]$ is a smooth point of $\widehat{R}(M_K)$, $\dim \widehat{R}(M_K) = 1$ in a neighbourhood of $[\rho]$ and $T_{[\rho]} \widehat{R}(M_K)$ is isomorphic to $H_\rho^1(M_K)$ (see [8, Section 4]). In Section 3.3, we will see another formulation of the concept of regularity in terms of transversality of some appropriate representation subspaces of $\widehat{R}(M_K)$.

3 Construction of the volume form

In this section, we explain in details the Casson-type construction of the “canonical” volume form on $\mathcal{R}\mathrm{eg}(K)$, see [6] and [7, Chapter 3]. First, let us make a digression on the construction of Casson’s invariant.

Let M be an oriented integral homology 3–sphere. The original construction of Casson’s invariant is based on $\mathrm{SU}(2)$ –representation spaces. More precisely, consider a Heegaard splitting $M = H_1 \cup_F H_2$ of M , where H_i is a handlebody and $F = H_1 \cap H_2$ is a surface of genus g . The construction of Casson’s invariant is based on the fact that such a splitting of M gives rise to embeddings $\widehat{R}(M) \hookrightarrow \widehat{R}(H_i)$ and $\widehat{R}(H_i) \hookrightarrow \widehat{R}(F)$. Furthermore, $\widehat{R}(M)$ can be viewed as the intersection of the images of $\widehat{R}(H_1)$ and $\widehat{R}(H_2)$ inside $\widehat{R}(F)$. The crucial point is that the spaces $\widehat{R}(H_i)$ and $\widehat{R}(F)$ are canonically oriented manifolds. Informally speaking, the Casson invariant $\lambda(M)$ is the “algebraic intersection number” of $\widehat{R}(H_1)$ and $\widehat{R}(H_2)$ in $\widehat{R}(F)$. The technical difficulties of the construction are to make sense of the algebraic intersection number

of these proper open submanifolds and to show that it is independent of the Heegaard splitting (see [9] for details).

In our construction the Heegaard splitting will be replaced by a splitting of the knot exterior induced by a plat presentation of the knot and the orientations will be replaced by volume forms. Moreover, we will not only consider groups but marked groups. A *marked group* is a pair (G, \mathcal{G}) where G is a finitely generated group and \mathcal{G} is a fixed finite set of generators of G . We will associate to marked groups appropriate representation subspaces. All this material will be discuss in the following subsections.

Convention The 3-sphere S^3 is assumed to be oriented.

3.1 Plat presentation and splitting of knot exteriors

Each knot $K \subset S^3$ can be presented as a $2n$ -plat $\hat{\zeta}$, where $\hat{\zeta}$ is obtained from the $2n$ -braid $\zeta \in B_{2n}$ by closing it with $2n$ half circles as in Figure 1. Explicitly, assume that the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is oriented. We choose $\epsilon \in \{\pm 1\}$ such that the basis (e_1, e_2, e_3) represents the induced orientation of \mathbb{R}^3 , where $e_1 = \epsilon \mathbf{i} = (\epsilon, 0, 0)$, $e_2 = \mathbf{j} = (0, 1, 0)$, $e_3 = \mathbf{k} = (0, 0, 1)$. For $j = 1, \dots, 2n$, set

$$p_j = \begin{cases} (j, 0) \in \mathbb{R}^2 & \text{if } \epsilon = 1, \\ (2n + 1 - j, 0) \in \mathbb{R}^2 & \text{if } \epsilon = -1. \end{cases}$$

Let $J = [1, 2]$, $H_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z \leq 1\}$, $H_2 = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 2\}$ and let Q denote the cube $[0, 2n + 1] \times [-1, 1] \times J \subset \mathbb{R}^2 \times J$. We assume that ζ is contained in Q and in a small neighbourhood of the plane $\mathbb{R} \times \{0\} \times \mathbb{R}$. We also assume that $\zeta \cap (\mathbb{R}^2 \times \{i\}) = \mathbf{p} \times \{i\}$, $i = 1, 2$, where $\mathbf{p} = (p_1, \dots, p_{2n})$. The $2n$ -plat $\hat{\zeta}$ is obtained from ζ by closing it with two systems of n half-circles $C_i = \{c_k^{(i)}\}_{1 \leq k \leq n} \subset H_i \cap (\mathbb{R} \times \{0\} \times \mathbb{R})$, $i = 1, 2$. We assume that the endpoints of $c_k^{(i)}$ are exactly $p_{2k-1} \times \{i\}$, $p_{2k} \times \{i\}$ in ∂H_i , see Figure 1.

Such a presentation of K as a $2n$ -plat $\hat{\zeta}$ gives rise to a splitting of its exterior of the form $M_{\hat{\zeta}} = B_1 \cup_S B_2$, where B_1, B_2 are two handlebodies of genus n and $S = B_1 \cap B_2 = S^2 \setminus N(\hat{\zeta})$ is a $2n$ -punctured 2-sphere (cf [10, Section 3]). To be more precise, $S = S^2 \setminus N(\hat{\zeta}) = (\mathbb{R}^2 \setminus N(\mathbf{p}) \times \{1\}) \cup \{\infty\}$, $B_1 = (H_1 \setminus N(C_1)) \cup \{\infty\}$ and $B_2 = ((H_2 \cup \mathbb{R}^2 \times J) \setminus N(C_2 \cup \zeta)) \cup \{\infty\}$, see Figure 1.

This decomposition is similar to the Heegaard splitting used in the construction of Casson's invariant. It also gives rise to *special systems of generators* (depending on the orientation of S^3) for $\pi_1(B_i)$, $i = 1, 2$, and $\pi_1(S)$ respectively denoted

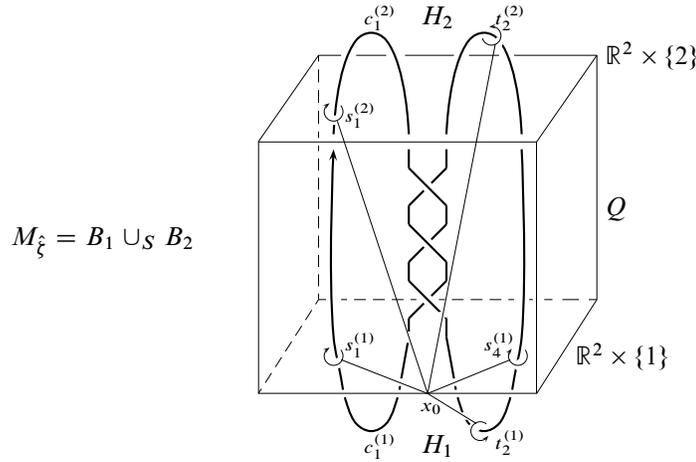


Figure 1: Special systems of generators with $\varepsilon = +1$.

$\mathcal{T}_i = \{t_j^{(i)}, 1 \leq j \leq n\}$ and $\mathcal{S} = \{s_j^{(1)}, 1 \leq j \leq 2n\}$, see Figure 1. We will precisely define these generators in Section 3.2.

With the notation above, the group $\pi_1(B_i)$ is the free group with basis \mathcal{T}_i and the group $\pi_1(\mathcal{S})$ admits the finitely presentation:

$$\pi_1(\mathcal{S}) = \langle s_1, \dots, s_{2n} \mid s_1 \cdots s_{2n} \rangle,$$

here $s_i = s_i^{(1)}$. Furthermore, each element of \mathcal{T}_i and \mathcal{S} is a meridian of K . In particular all these elements are conjugate in G_K . This obvious remark will be crucial in the sequel.

The inclusions $\mathcal{S} \hookrightarrow B_i$ and $B_i \hookrightarrow M_K, i = 1, 2$, give rise to the following commutative diagram:

$$(1) \quad \begin{array}{ccc} & \pi_1(B_1) & \\ \kappa_1 \nearrow & & \searrow p_1 \\ \pi_1(\mathcal{S}) & & \pi_1(M_K) = G_K \\ \kappa_2 \searrow & & \nearrow p_2 \\ & \pi_1(B_2) & \end{array}$$

Each homomorphism of diagram (1) is onto. The Seifert–Van Kampen Theorem and diagram (1) combine to yield the following presentation of G_K :

$$(2) \quad G_K = \langle t_i^{(1)}, t_i^{(2)}, 1 \leq i \leq n \mid p_1 \circ \kappa_1(s_j) = p_2 \circ \kappa_2(s_j), 1 \leq j \leq 2n - 1 \rangle,$$

Observe that presentation (2) is a particular Wirtinger presentation for G_K .

3.2 Choices of generators

Here we introduce the appropriate orientation conventions and we set up the special systems of generators corresponding to a plat presentation of the knot K .

Assume that the cube Q is endowed with the induced orientation of the one of \mathbb{R}^3 and choose $x_0 = (n, -1, 1) \in \partial Q$ as base point. We obtain the *special systems of generators* for the fundamental groups of B_1 , B_2 and S as follows.

The generator $s_j^{(i)}$ of $\pi_1((\mathbb{R}^2 \setminus N(\mathbf{p})) \times \{i\})$ is represented by a loop in $\mathbb{R}^2 \times \{i\}$ consisting of a small circle around $p_j \times \{i\}$ and the shortest arc in ∂Q connecting it to x_0 . The circle is oriented according to the rule: $\ell k(s_j^{(i)}, L_j) = 1$, where L_j denotes the oriented line $p_j \times \mathbb{R}$ (the orientation points in negative z -direction), see Figure 1. With these choices,

$$\pi_1 \left((\mathbb{R}^2 \setminus N(\mathbf{p}) \times \{i\}) \cup \{\infty\} \right) = \left\langle s_1^{(i)}, \dots, s_{2n}^{(i)} \mid s_1^{(i)} \cdots s_{2n}^{(i)} \right\rangle.$$

In order to define the other generators we choose an orientation for the plat $\hat{\zeta}$. We shall see later on that the construction does not depend on this choice (see Corollary 3.5 (3)). The generator $t_k^{(i)}$ of $\pi_1(H_i \setminus N(C_i))$ is represented by a loop consisting of a small circle around $c_k^{(i)}$ and the shortest arc in \mathbb{R}^3 connecting it to x_0 . The orientation of the circle is given by the rule: $\ell k(t_k^{(i)}, \hat{\zeta}) = 1$, see Figure 1.

Consider the homomorphism $\lambda_i: \pi_1((\mathbb{R}^2 \setminus N(\mathbf{p})) \times \{i\}) \rightarrow \pi_1(H_i \setminus N(C_i))$ induced by inclusion. We have

$$(3) \quad \lambda_i(s_{2k-1}^{(i)}) = (t_k^{(i)})^{\varepsilon_k^{(i)}}, \quad \lambda_i(s_{2k}^{(i)}) = (t_k^{(i)})^{-\varepsilon_k^{(i)}},$$

where $\varepsilon_k^{(i)} \in \{\pm 1\}$ depends on the orientation of $\hat{\zeta}$. Observe that the $\varepsilon_k^{(i)}$ change sign simultaneously if the orientation of $\hat{\zeta}$ is reversed.

The braid group B_{2n} can be viewed as a subgroup of $\text{Aut}(F_{2n})$. Here F_{2n} is identified to the fundamental group $\pi_1(Q \setminus N(\zeta))$. The braid ζ induces the automorphism $\phi_\zeta: \pi_1(Q \setminus N(\zeta)) \rightarrow \pi_1(Q \setminus N(\zeta))$ defined by $s_j^{(2)} \mapsto s_j^{(1)}$. Hence, $s_j^{(1)}$ can be viewed as a word in the generators $s_1^{(2)}, \dots, s_{2n}^{(2)}$. One can prove (cf diagram (1)) that

$$\kappa_1: s_j^{(1)} \mapsto \lambda_1(s_j^{(1)}) \text{ and } \kappa_2: s_j^{(1)} \mapsto \lambda_2 \circ \phi_\zeta(s_j^{(2)}).$$

3.3 Representation subspaces

Corresponding to a plat presentation of K —which gives rise to the splitting $M_K = B_1 \cup_S B_2$ and to special systems of generators for $\pi_1(B_i)$ and $\pi_1(S)$ —we introduce some *special representation subspaces* for $R(B_i)$, $i = 1, 2$, and $R(S)$, see [10, Section 3].

Consider a representation $\rho: G_K \rightarrow \mathrm{SU}(2)$ and look at its restrictions $\rho_i = \rho \circ p_i$ and $\rho_S = \rho \circ p_i \circ \kappa_i$. In this way we do not obtain all the representations of the groups $\pi_1(B_i)$ or $\pi_1(S)$ because all the generators of these groups are conjugate to the meridian of K . For this reason we introduce some appropriate representation subspaces associated to the marked groups $(\pi_1(B_i), \mathcal{T}_i)$ and $(\pi_1(S), \mathcal{S})$.

Corresponding to the marked group (G, \mathcal{G}) we define the subset $R^{\mathcal{G}}(G)$ of $R(G) \setminus C(G)$ by setting

$$(4) \quad R^{\mathcal{G}}(G) = \{\rho \in R(G) \setminus C(G) \mid \mathrm{Tr}(\rho(s)) = \mathrm{Tr}(\rho(t)) \forall s, t \in \mathcal{G}\}.$$

Write $\widetilde{R}^{\mathcal{G}}(G) = R^{\mathcal{G}}(G) \cap \widetilde{R}(G)$. The spaces $R^{\mathcal{G}}(G)$ and $\widetilde{R}^{\mathcal{G}}(G)$ explicitly depend on the choice of the system of generators \mathcal{G} . However the action by conjugation leaves $R^{\mathcal{G}}(G)$ and $\widetilde{R}^{\mathcal{G}}(G)$ invariant because the trace-function is invariant by conjugation. Thus, the quotient $\widehat{R}^{\mathcal{G}}(G) = \widetilde{R}^{\mathcal{G}}(G)/\mathrm{SO}(3)$ is well-defined. Observe that $\widetilde{R}^{\mathcal{G}}(G)$ can be identified with the total space of a principal $\mathrm{SO}(3)$ -bundle with $\widehat{R}^{\mathcal{G}}(G)$ as base space.

Here is a concrete example. Assume that G is the group G_K of the knot $K \subset S^3$. Assume that \mathcal{G} is a finite system of generators of G_K such that each element in \mathcal{G} is a meridian of K . Then $R^{\mathcal{G}}(G) = R(G) \setminus C(G)$.

Let (G, \mathcal{G}) and (G', \mathcal{G}') be two marked groups. A homomorphism $\phi: G \rightarrow G'$ is called *compatible* with \mathcal{G} and \mathcal{G}' if $\phi(g)$ is conjugate to an element of $\mathcal{G}' \cup (\mathcal{G}')^{-1}$ for all $g \in \mathcal{G}$. If $\phi: G \rightarrow G'$ is compatible with \mathcal{G} and \mathcal{G}' , then ϕ induces a transformation $\widehat{\phi}: \widehat{R}^{\mathcal{G}}(G) \rightarrow \widehat{R}^{\mathcal{G}'}(G')$.

Notation We write $\widehat{R}^{\mathcal{T}_i}(B_i) = \widehat{R}^{\mathcal{T}_i}(\pi_1(B_i))$, $i = 1, 2$, $\widehat{R}^{\mathcal{S}}(S) = \widehat{R}^{\mathcal{S}}(\pi_1(S))$ etc.

All epimorphisms in diagram (1) are compatible with the special systems of generators \mathcal{T}_i , $i = 1, 2$, and \mathcal{S} described above (because all the elements in \mathcal{T}_i , $i = 1, 2$, and in \mathcal{S} are conjugate to the meridian). Thus corresponding to diagram (1) we obtain the

commutative diagram:

$$(5) \quad \begin{array}{ccc} & \widehat{R}^{\mathcal{T}_1}(B_1) & \\ \widehat{\kappa}_1 \swarrow & & \searrow \widehat{p}_1 \\ \widehat{R}^S(S) & & \widehat{R}(M_K) \\ \widehat{\kappa}_2 \swarrow & & \searrow \widehat{p}_2 \\ & \widehat{R}^{\mathcal{T}_2}(B_2) & \end{array}$$

All arrows in diagram (5) are inclusions. Therefore, we can see $\widehat{R}(M_K)$ as the intersection of the images of $\widehat{R}^{\mathcal{T}_1}(B_1)$ and $\widehat{R}^{\mathcal{T}_2}(B_2)$ inside $\widehat{R}^S(S)$. The commutative diagram (5) is the main ingredient to define (generically) the volume form on the $SU(2)$ -representation space of G_K .

M Heusener [10] proved that $\widehat{R}^{\mathcal{T}_i}(B_i)$ is a $(2n-2)$ -dimensional manifold and $\widehat{R}^S(S)$ is a $(4n-5)$ -dimensional manifold. We furthermore prove in the following proposition that they also carry “natural” volume forms. Here, “natural” means that the volume forms are deduced from the usual ones on $SU(2)$ and on $(-2, 2)$.

Proposition 3.1 *The manifold $\widehat{R}^{\mathcal{T}_i}(B_i)$ (resp. $\widehat{R}^S(S)$) carries a “natural” $(2n-2)$ -volume form denoted $v_{\widehat{R}^{\mathcal{T}_i}(B_i)}$ (resp. a $(4n-5)$ -volume form denoted $v_{\widehat{R}^S(S)}$).*

Proof We explicitly describe the volume forms on the manifolds $\widehat{R}^{\mathcal{T}_i}(B_i)$ and $\widehat{R}^S(S)$. Their constructions are based on Lemma 1.1 and require the following steps.

- The Lie group $SU(2)$ is endowed with the 3-volume form η induced by the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Similarly we let η denote the 3-volume form on $SO(3) = SU(2)/\{\pm \mathbf{1}\}$ deduced from the one of $SU(2)$. Considering the fact that the trace-function $\text{Tr}: SU(2) \setminus \{\pm \mathbf{1}\} \rightarrow (-2, 2)$ is a submersion, we define a 2-volume form ν on the 2-sphere $S^2 = \{A \in SU(2) \mid \text{Tr}(A) = 0\}$ exploiting the “base \wedge fiber” condition. Explicitly, for each $A \in S^2$ we have the short exact sequence

$$0 \longrightarrow T_A S^2 \longrightarrow T_A SU(2) \longrightarrow T_0(-2, 2) \longrightarrow 0.$$

In this sequence, $SU(2)$ is endowed with the 3-volume form induced by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $(-2, 2)$ is endowed with the usual 1-volume form. The 2-volume form ν on S^2 is the unique compatible volume form with the two others.

- We define a natural $(2n+1)$ -volume form on $R^{\mathcal{T}_i}(B_i)$ as follows. Using the system of generators \mathcal{T}_i , the map $R(B_i) \rightarrow SU(2)^n$ defined by $\rho \mapsto$

$(\rho(t_1^{(i)}), \dots, \rho(t_n^{(i)}))$ is an isomorphism. This isomorphism allows us to identify $R(B_i)$ with $SU(2)^n$. Using the natural inclusion $\iota: (-2, 2) \times (S^2)^n \rightarrow SU(2)^n$ given by

$$\iota(2 \cos(\theta), P_1, \dots, P_n) = (\cos(\theta) + \sin(\theta) P_i)_{1 \leq i \leq n},$$

we identify $R^{T_i}(B_i)$ with the product $(-2, 2) \times (S^2)^n$. As a consequence, the $(2n + 1)$ -dimensional manifold $R^{T_i}(B_i)$ is endowed with a natural $(2n + 1)$ -volume form $v^{R^{T_i}(B_i)}$, namely the one induced by the product volume form on $(-2, 2) \times (S^2)^n$.

- We define a natural $(4n - 2)$ -volume form on $\tilde{R}^S(S)$ as follows. Let D^* be the $2n$ -punctured disk $S \setminus \{\infty\}$. The fundamental group of D^* is the free group of rank $2n$ with basis $\mathcal{S} = \{s_1, \dots, s_{2n}\}$ (see Figure 1). Let U be the subgroup normally generated by the product $s_1 \cdots s_{2n}$. We have $\pi_1(S) = \pi_1(D^*)/U$. Consider the map $\varphi: R^S(D^*) \rightarrow SU(2)$ defined by $\varphi(\rho) = \rho(s_1 \cdots s_{2n})$ and observe that $R^S(S) = \varphi^{-1}(\mathbf{1})$. In [10, Lemma 3.1], Heusener proved that φ is surjective and that the set of critical points of φ coincides exactly with the set of abelian $SU(2)$ -representations of $\pi_1(D^*)$. Thus, we have the short exact sequence

$$0 \rightarrow T_\rho \tilde{R}^S(S) \rightarrow T_\rho \tilde{R}^S(D^*) \rightarrow \mathfrak{su}(2) \rightarrow 0.$$

In this sequence, $\mathfrak{su}(2)$ is endowed with the 3-volume form induced by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $\tilde{R}^S(D^*)$ is endowed with a natural $(4n + 1)$ -volume form (observe that $\tilde{R}^S(D^*)$ is an open subset of $R^S(D^*) \cong (-2, 2) \times (S^2)^{2n}$ which is endowed with the product volume form). Then, $\tilde{R}^S(S)$ is endowed with the unique $(4n - 2)$ -volume form $v^{\tilde{R}^S(S)}$ which is compatible with the two others.

- Finally, if (G, \mathcal{G}) is one of the marked groups $(\pi_1(B_1), \mathcal{T}_1)$, $(\pi_1(B_2), \mathcal{T}_2)$ or $(\pi_1(S), \mathcal{S})$, then the map $\tilde{R}^{\mathcal{G}}(G) \rightarrow \hat{R}^{\mathcal{G}}(G)$ is a principal $SO(3)$ -bundle. As a consequence, we define a volume form on $\hat{R}^{\mathcal{G}}(G)$ exploiting the “base \wedge fiber” condition. To be more precise, we have the short exact sequence

$$0 \rightarrow T_\rho(SU(2)(\rho)) \rightarrow T_\rho \tilde{R}^{\mathcal{G}}(G) \rightarrow T_{[\rho]} \hat{R}^{\mathcal{G}}(G) \rightarrow 0,$$

where $SU(2)(\rho) = \{Ad_A \circ \rho \mid A \in SU(2)\}$. Observe that $SU(2)(\rho) \cong SU(2)$ is endowed with the 3-volume form η . So $\hat{R}^{\mathcal{G}}(G)$ is endowed with the unique volume form which is compatible with η and with the natural volume form we have just constructed on $\tilde{R}^{\mathcal{G}}(G)$. □

3.4 Construction of the volume form

In this subsection, we fix a $2n$ -plat presentation $\hat{\zeta}$ of the oriented knot K . Having in mind the result obtained in Proposition 3.1, we are ready to complete the construction of the volume form on $\mathcal{R}eg(\hat{\zeta})$.

3.4.1 Regularity and transversality Let $\rho: G_K \rightarrow SU(2)$ be a non-central representation and consider $\rho_i = \rho \circ p_i$ (resp. $\rho_S = \rho \circ p_i \circ \kappa_i$) its restriction to $\pi_1(B_i)$ (resp. $\pi_1(S)$). One can prove that ρ is regular (ie, $\dim H_\rho^1(M_K) = 1$) if and only if the images in $\hat{R}^S(S)$ of the manifolds $\hat{R}^{T_1}(B_1)$ and $\hat{R}^{T_2}(B_2)$ intersect transversally at $[\rho]$, see [10, Proposition 3.3]. Informally speaking, the proof of this fact is essentially based on a dimensional argument.

3.4.2 Definition The construction of the “natural” volume form on $\mathcal{R}eg(\hat{\zeta})$ combines the previous result and Proposition 3.1. It is based on the following fact: if ρ is a regular representation, then

$$(6) \quad 0 \longrightarrow T_{[\rho]} \hat{R}(M_K) \xrightarrow{D_{[\rho]} \hat{P}} T_{[\rho_1]} \hat{R}^{T_1}(B_1) \oplus T_{[\rho_2]} \hat{R}^{T_2}(B_2) \xrightarrow{D_{[\rho]} \hat{K}} T_{[\rho_S]} \hat{R}^S(S) \longrightarrow 0$$

is a short exact sequence (cf diagram (5) and the fact that $\hat{R}^{T_1}(B_1)$ and $\hat{R}^{T_2}(B_2)$ intersect transversally at $[\rho]$). Using the exactness of sequence (6), we define a 1-volume form $\omega_{[\rho]}^{\hat{\zeta}}$ on $T_{[\rho]} \hat{R}(M_K)$ by setting (see the notation of Section 1.4.2):

$$(7) \quad \omega_{[\rho]}^{\hat{\zeta}} = (-1)^n \left(v_{[\rho_1]} \hat{R}^{T_1}(B_1) \wedge v_{[\rho_2]} \hat{R}^{T_2}(B_2) \right) / v_{[\rho_S]} \hat{R}^S(S).$$

In this way we locally construct a 1-volume form $\omega^{\hat{\zeta}}: [\rho] \mapsto \omega_{[\rho]}^{\hat{\zeta}}$ on the 1-dimensional manifold $\mathcal{R}eg(\hat{\zeta})$.

As it stands $\omega^{\hat{\zeta}}$ is defined in terms of the plat decomposition of the knot and apparently depends on it. In fact, we will prove in Section 4 (see in particular Theorem 4.1) that it is not the case. Further observe that the normalisation given by the sign $(-1)^n$ in formula (7) is needed to ensure the invariance of $\omega^{\hat{\zeta}}$ (see in particular Lemma 4.3). We will also prove in Section 5 that $\omega^{\hat{\zeta}}$ does not depend on the orientation of $\hat{\zeta}$ (see Proposition 5.1). Finally remark that the order taken for the handlebodies B_1 and B_2 in formula (7) has no importance in the definition (because of the parity of the order of the $(2n - 2)$ -volume form $v \hat{R}^{T_i}(B_i)$).

3.4.3 An important remark We obtain an other splitting of the exterior of the plat $\widehat{\zeta}$ by choosing the punctured sphere $S' = (\mathbb{R}^2 \setminus N(\mathbf{p}) \times \{2\}) \cup \{\infty\}$ and the handlebodies $B'_1 = ((H_1 \cup \mathbb{R}^2 \times J) \setminus N(C_1 \cup \zeta)) \cup \{\infty\}$, $B'_2 = (H_2 \setminus N(C_2)) \cup \{\infty\}$, see Figure 1. Using the notation of Section 3.2, the epimorphisms corresponding to this splitting are given by:

$$\kappa'_1: s_j^{(1)} \mapsto \lambda_1 \circ \phi_\zeta^{-1}(s_j^{(1)}) \text{ and } \kappa'_2: s_j^{(2)} \mapsto \lambda_2(s_j^{(2)}).$$

Thus $\kappa_i = \kappa'_i \circ \phi_\zeta$, for $i = 1, 2$. Set $\widehat{Q}'_i = \widehat{\kappa}'_i(\widehat{R}^{\mathcal{T}_i}(B'_i))$, $i = 1, 2$, see diagram (5).

The volume form $\omega^{\widehat{\zeta}}$ on $\text{Reg}(\widehat{\zeta})$ can be defined using arbitrarily one of the two splittings $M_{\widehat{\zeta}} = B_1 \cup_S B_2$ or $M_{\widehat{\zeta}} = B'_1 \cup_{S'} B'_2$, because $\phi_\zeta: s_j^{(2)} \mapsto s_j^{(1)}$ induces a volume preserving diffeomorphism from the regular part of $\widehat{Q}'_1 \cap \widehat{Q}'_2$ to the one of $\widehat{Q}_1 \cap \widehat{Q}_2$. This result will be proved in the following subsection, see in particular Corollary 3.5.

3.5 Dependence of the volume forms on the generator systems

In this subsection, we analyse the dependence of the volume forms $v^{\widehat{R}^{\mathcal{T}_i}(B_i)}$ and $v^{\widehat{R}^{\mathcal{S}}(S)}$ in terms of the generator systems \mathcal{T}_i and \mathcal{S} respectively.

Let F_{2n} denote the free group of rank $2n$ with basis \mathcal{S} . A braid $\zeta \in B_{2n}$ induces an automorphism $\phi_\zeta: F_{2n} \rightarrow F_{2n}$, given by $\phi_\zeta: s_i \mapsto g_i s_{\pi(i)} g_i^{-1}$, where $g_i \in F_{2n}$ and $\pi \in \mathfrak{S}_{2n}$ is a permutation. Observe that $\prod_{i=1}^{2n} \phi_\zeta(s_i) = \prod_{i=1}^{2n} s_i$. As a consequence, ϕ_ζ is compatible with the system \mathcal{S} and thus induces a diffeomorphism $\widehat{\phi}_\zeta: \widehat{R}^{\mathcal{S}}(S) \rightarrow \widehat{R}^{\mathcal{S}}(S)$.

3.5.1 A “transfer lemma” We begin our study by establishing a useful technical lemma.

Let $F_m = \langle S_1, \dots, S_m \mid - \rangle$ be a free group. Each endomorphism $\phi: F_m \rightarrow F_m$ induces a transformation $\phi^\sharp: R(F_m) \rightarrow R(F_m)$ and a map $\phi^{\text{ab}}: H^1(F_m; \mathbb{Z}) \rightarrow H^1(F_m; \mathbb{Z})$, where $H^1(F_m; \mathbb{Z}) \cong \mathbb{Z}^m$. In fact, using the identification $R(F_m) \cong \text{SU}(2)^m$ induced by the presentation $\langle S_1, \dots, S_m \mid - \rangle$ of F_m , ϕ^\sharp is explicitly given by

$$(\rho(S_1), \dots, \rho(S_m)) \mapsto (\rho \circ \phi(S_1), \dots, \rho \circ \phi(S_m)).$$

For each $g \in F_m$ consider the evaluation map $\text{ev}_g: R(F_m) \rightarrow \text{SU}(2)$ given by $\text{ev}_g(\rho) = \rho(g)$. If $\phi: F_m \rightarrow F_m$ is an endomorphism, then $\text{ev}_g \circ \phi^\sharp = \text{ev}_{\phi(g)}$. With this notation, we have:

Lemma 3.2 *If $\phi: F_m \rightarrow F_m$ is an endomorphism, then*

$$\text{ev}_{\phi(S_1)}^*(\eta) \wedge \dots \wedge \text{ev}_{\phi(S_m)}^*(\eta) = \det(\phi^{\text{ab}}) \text{ev}_{S_1}^*(\eta) \wedge \dots \wedge \text{ev}_{S_m}^*(\eta).$$

Proof We just explain the main ideas which are similar to [1, Proposition 3.4].

The volume forms $\text{ev}_{\phi(S_1)}^*(\eta) \wedge \dots \wedge \text{ev}_{\phi(S_m)}^*(\eta)$ and $\text{ev}_{S_1}^*(\eta) \wedge \dots \wedge \text{ev}_{S_m}^*(\eta)$ are completely determined by their value at the trivial representation $\theta: F_m \rightarrow SU(2)$ defined by $\theta(g) = \mathbf{1}$, for all $g \in F_m$ (because they are right-invariant). At the representation θ , we have:

$$(\text{ev}_{\phi(S_1)}^*(\eta) \wedge \dots \wedge \text{ev}_{\phi(S_m)}^*(\eta))(\theta) = \det(D_\theta \phi^\#) (\text{ev}_{S_1}^*(\eta) \wedge \dots \wedge \text{ev}_{S_m}^*(\eta))(\theta).$$

Thus the proof reduces to the evaluation of $\det(D_\theta \phi^\#)$.

Let $\text{aug}: \mathbb{Z}[F_m] \rightarrow \mathbb{Z}$ be the augmentation map $\text{aug}(S_i) = 1$. Consider the Fox-matrix $A = (a_{ij})_{i,j}$ where $a_{ij} = \text{aug} \left(\frac{\partial \phi(S_i)}{\partial S_j} \right) \in \mathbb{Z}$. With this notation we have $\phi^{\text{ab}}(S_i) = \sum_{j=1}^m a_{ij} S_j$. If $\partial: \mathfrak{su}(2)^m \rightarrow \mathfrak{su}(2)^m$ denotes the map given by $\partial(x_1, \dots, x_m) = \left(\sum_{j=1}^m a_{ij} x_j \right)_{1 \leq i \leq m}$ then the diagram

$$\begin{array}{ccc} T_{\theta(S_1)}SU(2) \times \dots \times T_{\theta(S_m)}SU(2) & \xrightarrow{D_\theta \phi^\#} & T_{\theta \circ \phi(S_1)}SU(2) \times \dots \times T_{\theta \circ \phi(S_m)}SU(2) \\ \downarrow = & & \downarrow = \\ \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2) & \xrightarrow{\partial} & \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2) \end{array}$$

commutes. Hence $\det(D_\theta \phi^\#) = \det(\partial) = \det(\phi^{\text{ab}})$ as required. □

3.5.2 Dependence of the volume form $v_{\widehat{R}^{\mathcal{T}_i}(B_i)}$ Let (F_n, \mathcal{T}) be the marked group $(\pi_1(B_i), \mathcal{T}_i)$, $i = 1, 2$. Consider an automorphism $\phi: F_n \rightarrow F_n$ and assume that there is a permutation $\pi \in \mathfrak{S}_n$ such that

$$(8) \quad \phi(t_j) = g_j t_{\pi(j)}^{\varepsilon_j} g_j^{-1} \text{ where } g_j \in F_n \text{ and } \varepsilon_j \in \{\pm 1\}.$$

It follows that \mathcal{T} and $\mathcal{T}' = \phi(\mathcal{T})$ are compatible sets and that $R^{\mathcal{T}'}(F_n) = R^{\mathcal{T}}(F_n)$. As a consequence, ϕ induces a diffeomorphism $\widehat{\phi}: \widehat{R}^{\mathcal{T}}(F_n) \rightarrow \widehat{R}^{\mathcal{T}'}(F_n)$.

Lemma 3.3 *Let $N = \#\{\varepsilon_j, 1 \leq j \leq n \mid \varepsilon_j = -1\}$. The induced diffeomorphism $\widehat{\phi}: \widehat{R}^{\mathcal{T}}(F_n) \rightarrow \widehat{R}^{\mathcal{T}'}(F_n)$ satisfies*

$$\widehat{\phi}^*(v_{\widehat{R}^{\mathcal{T}'}(F_n)}) = (-1)^N v_{\widehat{R}^{\mathcal{T}}(F_n)}.$$

Here $v^{\widehat{R}^T(F_n)}$ denotes the natural volume form on $\widehat{R}^T(F_n)$ constructed in [Proposition 3.1](#).

Proof Assume that $\rho(t_j) = \cos(\theta) + \sin(\theta)P_j^\rho$, with $0 < \theta < \pi$ and $P_j^\rho \in S^2$, for each $\rho \in \widetilde{R}^T(F_n)$. Introduce the following maps:

$$\text{tr}: R^T(F_n) \rightarrow (-2, 2), \rho \mapsto 2 \cos(\theta) \text{ and } \text{ax}_{t_j}: R^T(F_n) \rightarrow S^2, \rho \mapsto P_j^\rho.$$

Observe that ax_{t_j} is the composition of ev_{t_j} with the canonical projection onto S^2 . The automorphism ϕ induces a diffeomorphism $\phi^\dagger: R^T(F_n) \rightarrow R^T(F_n)$ such that $\text{ax}_{t_j} \circ \phi^\dagger = \text{ax}_{\phi(t_j)}$.

Further consider the following claims:

- The map $(\text{tr}, \text{ax}_{t_1}, \dots, \text{ax}_{t_n}): R^T(F_n) \rightarrow (-2, 2) \times (S^2)^n$ is a diffeomorphism. The volume form $v^{R^T(F_n)}$ on $R^T(F_n)$ is the pull-back of the product volume form on $(-2, 2) \times (S^2)^n$ by this isomorphism.
- Consider $t_i: F_n \rightarrow F_n$, given by $t_i \mapsto t_i^{-1}$, $t_j \mapsto t_j$, if $j \neq i$. This transformation induces a diffeomorphism $t_i^\dagger: R^T(F_n) \rightarrow R^T(F_n)$ such that

$$(t_i^\dagger)^*(v^{R^T(F_n)}) = -v^{R^T(F_n)}, \text{ for all } i.$$

- Consider $t_\tau: F_n \rightarrow F_n$, given by $t_j \mapsto t_{\tau(j)}$, where $\tau \in \mathfrak{S}_n$ is a transposition. The transformation t_τ induces a volume preserving diffeomorphism $t_\tau^\dagger: R^T(F_n) \rightarrow R^T(F_n)$.

From these observations, we conclude

$$(\phi^\dagger)^*(v^{R^T(F_n)}) = (-1)^N v^{R^T(F_n)}.$$

The diffeomorphism ϕ^\dagger induces a diffeomorphism $\widetilde{\phi}: \widetilde{R}^T(F_n) \rightarrow \widetilde{R}^T(F_n)$ which makes the following diagram commutative:

$$\begin{array}{ccccc} \text{SO}(3) & \longrightarrow & \widetilde{R}^T(F_n) & \longrightarrow & \widehat{R}^T(F_n) \\ \downarrow = & & \downarrow \widetilde{\phi} & & \downarrow \widehat{\phi} \\ \text{SO}(3) & \longrightarrow & \widetilde{R}^T(F_n) & \longrightarrow & \widehat{R}^T(F_n) \end{array}$$

This completes the proof. □

3.5.3 Dependence of the volume form $v_{\widehat{R}^S(S)}$ Consider the marked group (F_{2n}, S) where F_{2n} is the fundamental group of the $2n$ -punctured disk $S \setminus \{\infty\}$ and $S = \{s_1, \dots, s_{2n}\}$. Recall that $\pi_1(F) = F_{2n}/U$ where U is the subgroup normally generated by the product $s_1 \cdots s_{2n}$. We are interested in an automorphism ϕ of F_{2n} which satisfies:

- (1) there is a permutation $\pi \in \mathfrak{S}_{2n}$ such that $\phi(s_j) = g_j s_{\pi(j)}^{\varepsilon_j} g_j^{-1}$, where $g_j \in F_{2n}$ and $\varepsilon_j \in \{\pm 1\}$.
- (2) ϕ preserves the normal closure of $s_1 \cdots s_{2n} \in F_{2n}$, ie,

$$\phi(s_1 \cdots s_{2n}) = g(s_1 \cdots s_{2n})^\varepsilon g^{-1}, \text{ where } g \in F_{2n} \text{ and } \varepsilon \in \{\pm 1\}.$$

As before S and $S' = \phi(S)$ are compatible sets, we have $R^{S'}(S) = R^S(S)$ and ϕ induces a diffeomorphism $\widehat{\phi}: \widehat{R}^S(S) \rightarrow \widehat{R}^{S'}(S)$.

Lemma 3.4 Let $N = \#\{\varepsilon_j, 1 \leq j \leq 2n \mid \varepsilon_j = -1\}$. If $N_\phi = N + (\varepsilon - 1)/2$, then the induced diffeomorphism $\widehat{\phi}: \widehat{R}^S(S) \rightarrow \widehat{R}^{S'}(S)$ satisfies

$$\widehat{\phi}^*(v_{\widehat{R}^S(S)}) = (-1)^{N_\phi} v_{\widehat{R}^{S'}(S)}.$$

Here $v_{\widehat{R}^S(S)}$ denotes the natural volume form on $\widehat{R}^S(S)$ constructed in [Proposition 3.1](#).

Proof With the same notation as in the proof of [Lemma 3.3](#), we introduce

$$\text{tr}: R^S(F_{2n}) \rightarrow (-2, 2), \rho \mapsto 2 \cos(\theta) \text{ and } \text{ax}_{s_j}: R^S(F_{2n}) \rightarrow S^2, \rho \mapsto P_j^\rho.$$

An inner automorphism of F_{2n} induces the identity on $\widehat{R}^S(S)$. Therefore we may assume for simplicity that $\phi(s_1 \cdots s_{2n}) = (s_1 \cdots s_{2n})^\varepsilon$ where $\varepsilon \in \{\pm 1\}$.

The automorphism ϕ induces two diffeomorphisms $\phi^\ddagger: \widetilde{R}^S(F_{2n}) \rightarrow \widetilde{R}^{S'}(F_{2n})$ and $\widetilde{\phi}: \widetilde{R}^S(S) \rightarrow \widetilde{R}^{S'}(S)$. Moreover $\text{ax}_{s_j} \circ \phi^\ddagger = \text{ax}_{\phi(s_j)}$.

We first analyse the action of ϕ^\ddagger at the level of volume forms. As before $v_{R^S(F_{2n})}$ is the pull-back of the product volume form on $(-2, 2) \times (S^2)^{2n}$ by the diffeomorphism $(\text{tr}, \text{ax}_{s_1}, \dots, \text{ax}_{s_{2n}})$. Observe that by definition there is no central representation in $R^S(F_{2n})$. Let $i^S: R^S(F_{2n}) \rightarrow (SU(2) \setminus \{\pm 1\})^{2n}$ be the usual inclusion $i^S(\rho) = (\rho(s_1), \dots, \rho(s_{2n}))$ and let $T: (SU(2) \setminus \{\pm 1\})^{2n} \rightarrow (-2, 2)^{2n}$ be defined by

$$T(A_1, \dots, A_{2n}) = (\text{Tr}(A_1), \dots, \text{Tr}(A_{2n})).$$

Thus the diagram

$$\begin{array}{ccccc}
 R^S(F_{2n}) & \xrightarrow{i^S} & (\mathrm{SU}(2) \setminus \{\pm \mathbf{1}\})^{2n} & \xrightarrow{T} & (-2, 2)^{2n} \\
 \downarrow \phi^\ddagger & & \downarrow \phi^\# & & \downarrow \overline{\phi^{\mathrm{ab}}} \\
 R^S(F_{2n}) & \xrightarrow{i^S} & (\mathrm{SU}(2) \setminus \{\pm \mathbf{1}\})^{2n} & \xrightarrow{T} & (-2, 2)^{2n}
 \end{array}$$

commutes. If $\Delta = \{(x, \dots, x) \mid x \in (-2, 2)\}$, then $(T \circ i^S)^{-1}(\Delta) = R^S(F_{2n})$. Using Lemma 3.2, we conclude that

$$(\phi^\ddagger)^*(v^{R^S(F_{2n})}) = (-1)^N v^{R^S(F_{2n})}.$$

Next, consider the maps $\varphi: R^S(F_{2n}) \rightarrow \mathrm{SU}(2)$ defined by $\rho \mapsto \rho(s_1 \cdots s_{2n})$ and $\Phi: \mathrm{SU}(2) \rightarrow \mathrm{SU}(2)$ defined by $A \mapsto A^\varepsilon$. These maps make the diagram

$$\begin{array}{ccccc}
 \widetilde{R}^S(S) & \longrightarrow & \widetilde{R}^S(F_{2n}) & \xrightarrow{\varphi} & \mathrm{SU}(2) \\
 \downarrow \widetilde{\phi} & & \downarrow \phi^\ddagger & & \downarrow \Phi \\
 \widetilde{R}^S(S) & \longrightarrow & \widetilde{R}^S(F_{2n}) & \xrightarrow{\varphi} & \mathrm{SU}(2)
 \end{array}$$

commutative. From this observation, we deduce

$$\widetilde{\phi}^*(v^{\widetilde{R}^S(S)}) = (-1)^{N_{\phi}} v^{\widetilde{R}^S(S)}.$$

We finish the proof in the same way as in the one of Lemma 3.3. □

Some consequences of the preceding lemma are summarised in:

Corollary 3.5 *Let $\zeta \in B_{2n}$ be a braid such that $\widehat{\zeta}$ is a knot.*

- (1) *The diffeomorphism $\widehat{\phi}_\zeta: \widehat{R}^S(S) \rightarrow \widehat{R}^S(S)$ induced by the $2n$ -braid ζ is volume preserving.*
- (2) *The automorphism $\phi: F_{2n} \rightarrow F_{2n}$ defined by $\phi(s_j) = s_{2n-j+1}^{-1}$ (for all j) induces a diffeomorphism $\widehat{\phi}: \widehat{R}^S(S) \rightarrow \widehat{R}^S(S)$ which satisfies*

$$\widehat{\phi}^*(v^{\widehat{R}^S(S)}) = -v^{\widehat{R}^S(S)}.$$

- (3) *The volume form $\omega^{\widehat{\zeta}}$ does not depend on the orientation of $\widehat{\zeta}$.*
- (4) *If one changes the orientation of S^3 , then $\omega^{\widehat{\zeta}}$ changes into $-\omega^{\widehat{\zeta}}$.*

Proof

- (1) The automorphism ϕ_ζ induced by the braid ζ verifies $N_{\phi_\zeta} = 0$ and $\varepsilon = 1$. Thus, [Lemma 3.4](#) implies that $\widehat{\phi}_\zeta$ is volume preserving.
- (2) The automorphism ϕ defined by $\phi(s_j) = s_{2n-j+1}^{-1}$ verifies $N_\phi = 2n$ and $\varepsilon = -1$. Thus, [Lemma 3.4](#) implies that $\widehat{\phi}^*(v\widehat{R}^S(S)) = -v\widehat{R}^S(S)$ as required.
- (3) If we change the orientation of $\widehat{\zeta}$ then the $\varepsilon_k^{(i)} \in \{\pm 1\}$ are changing sign simultaneously. Hence the $(2n - 2)$ -volume form $v\widehat{R}^{\tau_1}(B_1)$ (resp. $v\widehat{R}^{\tau_2}(B_2)$) on $\widehat{R}^{\tau_1}(B_1)$ (resp. $\widehat{R}^{\tau_2}(B_2)$) is changed into $-v\widehat{R}^{\tau_1}(B_1)$ (resp. $-v\widehat{R}^{\tau_2}(B_2)$) and the $(4n - 5)$ -volume form $v\widehat{R}^S(S)$ is not affected. As a consequence, $\omega^{\widehat{\zeta}}$ does not change (see equation (7)).
- (4) If we change the orientation of S^3 , then as in (3) the volume form $v\widehat{R}^{\tau_1}(B_1)$ (resp. $v\widehat{R}^{\tau_2}(B_2)$) is changed into $-v\widehat{R}^{\tau_1}(B_1)$ (resp. $-v\widehat{R}^{\tau_2}(B_2)$). Moreover the generator $s_j^{(i)}$ in $\pi_1(S)$ is changed into $(s_{2n-j+1}^{(i)})^{-1}$, thus the $(4n - 5)$ -volume form $v\widehat{R}^S(S)$ on $\widehat{R}^S(S)$ is changed into $-v\widehat{R}^S(S)$. Hence $\omega^{\widehat{\zeta}}$ is changed into $-\omega^{\widehat{\zeta}}$. □

4 Invariance of the volume form

The aim of this section is to establish that the volume form $\omega^{\widehat{\zeta}}$ does not depend on the plat presentation used to define it and is a knot invariant.

Theorem 4.1 *Let $K \subset S^3$ be a knot and let $\zeta_i \in B_{2n_i}$ be given such that the plat $\widehat{\zeta}_i$ is isotopic to K for $i = 1, 2$. Consider the identification $\psi_i: \mathcal{R}eg(K) \rightarrow \mathcal{R}eg(\widehat{\zeta}_i)$ associated to the splitting of M_K induced by the plat presentation $\widehat{\zeta}_i$ of K for $i = 1, 2$. Then*

$$\psi_1^*(\omega^{\widehat{\zeta}_1}) = \psi_2^*(\omega^{\widehat{\zeta}_2}).$$

This result allows us to consider the 1-volume form on $\mathcal{R}eg(K)$ associated to the knot K . In the sequel, we let ω^K denote this volume form on $\mathcal{R}eg(K)$. The proof of [Theorem 4.1](#) is based on a theorem of Birman–Reidemeister. This theorem describes the relations between different braids which represent the same plat. It can be considered as an analogue for plats to Markov’s theorem for closed braids.

4.1 The Birman–Reidemeister Theorem

To state the Birman–Reidemeister theorem we need some more definition and notation.

The *subgroup of trivial half braid* is the subgroup H_{2n} of B_{2n} generated by the braids $\sigma_1, \sigma_2\sigma_1^2\sigma_2, \sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}, 1 \leq j \leq n-1$ (see Figure 2). It is clear that two braids of B_{2n} represent the same plat if they are in the same double coset of B_{2n} modulo the subgroup H_{2n} . Moreover, if $\zeta \in B_{2n}$ is such that $\widehat{\zeta}$ is a knot, then $\widehat{\zeta\sigma_{2n}}$ is a knot too; and it is evident that the plats $\widehat{\zeta}$ and $\widehat{\zeta\sigma_{2n}}$ are isotopic in S^3 (see Figure 3). The operation $\zeta \mapsto \zeta\sigma_{2n}$ described in Figure 3 is called an *elementary stabilisation*.

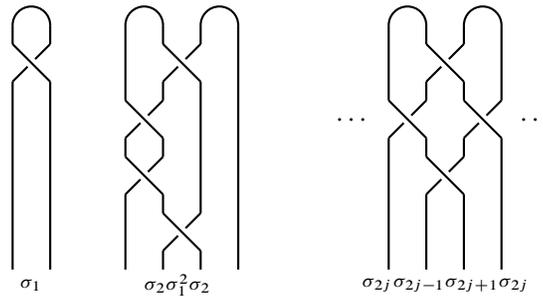


Figure 2: The braids $\sigma_1, \sigma_2\sigma_1^2\sigma_2$ and $\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$.

Two braids are called *stably equivalent* if they represent, after a finite number of elementary stabilisations, the same double coset modulo the subgroup of trivial half braids. It is clear that two braids which are stably equivalent represent equivalent plats. The Birman–Reidemeister theorem asserts that the converse is also true:

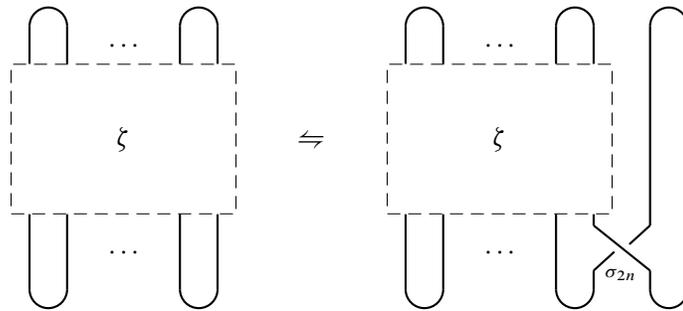


Figure 3: Elementary stabilisation

Theorem [2] Let $K_i \subset S^3$, $i = 1, 2$, be knots and let $\zeta_i \in B_{2n_i}$ be given such that the plat $\hat{\zeta}_i$ is isotopic to K_i . The knots K_1 and K_2 are isotopic if and only if there exists an integer $N \geq \max(n_1, n_2)$ such that for each $n \geq N$ the braids $\zeta'_i = \zeta_i \sigma_{2n_i} \sigma_{2n_i+2} \cdots \sigma_{2n} \in B_{2n+2}$, $i = 1, 2$, are in the same double coset of B_{2n+2} modulo the subgroup H_{2n+2} .

4.2 Proof of Theorem 4.1

Applying the Birman–Reidemeister Theorem, the proof of Theorem 4.1 splits into two steps: we first prove the invariance of $\omega_{\hat{\zeta}}$ under the change of the double coset representative (see Lemma 4.2) and next prove the invariance under elementary stabilisation (see Lemma 4.3).

4.2.1 Invariance under the change of the double coset representative

Write

$$D_i^* = (\mathbb{R}^2 \setminus N(\mathbf{p}) \times \{i\}) \text{ for } i = 1, 2 \text{ (see Figure 1).}$$

Keep the notation of Section 3.2. Let $F_{2n}^{(i)} = \pi_1(D_i^*) = \langle s_1^{(i)}, \dots, s_{2n}^{(i)} \mid - \rangle$ be the free group with basis $\mathcal{S}_i = \{s_1^{(i)}, \dots, s_{2n}^{(i)}\}$ and let $F_n^{(i)} = \pi_1(B_i) = \langle t_1^{(i)}, \dots, t_n^{(i)} \mid - \rangle$ be the one with basis \mathcal{T}_i , $i = 1, 2$, see Figure 1.

In [2], J. Birman proved that $\xi \in B_{2n}$ is in the subgroup H_{2n} if and only if ξ leaves the normal closure of $\{s_1^{(i)} s_2^{(i)}, \dots, s_{2n-1}^{(i)} s_{2n}^{(i)}\}$ invariant in $F_{2n}^{(i)}$. Therefore we have an automorphism $\phi_{\xi}^{(i)}: F_n^{(i)} \rightarrow F_n^{(i)}$ such that the following diagram commutes (cf equation (3)):

$$\begin{array}{ccc} F_{2n}^{(i)} & \xrightarrow{\lambda_i} & F_n^{(i)} \\ \downarrow \phi_{\xi} & & \downarrow \phi_{\xi}^{(i)} \\ F_{2n}^{(i)} & \xrightarrow{\lambda_i} & F_n^{(i)} \end{array}$$

For the generators of H_{2n} an elementary computation gives:

$$(9) \quad \phi_{\sigma_1}^{(i)}(t_1^{(i)}) = (t_1^{(i)})^{-1}, \quad \phi_{\sigma_1}^{(i)}(t_j^{(i)}) = t_j^{(i)} \text{ for } j \geq 2;$$

$$(10) \quad \phi_{\sigma_2 \sigma_1^2 \sigma_2}^{(i)}(t_1^{(i)}) = (t_2^{(i)})^{\varepsilon_2^{(i)}} t_1^{(i)} (t_2^{(i)})^{-\varepsilon_2^{(i)}}, \quad \phi_{\sigma_2 \sigma_1^2 \sigma_2}^{(i)}(t_j^{(i)}) = t_j^{(i)} \text{ for } j \geq 2;$$

$$(11) \quad \phi_{\sigma_{2k} \sigma_{2k-1} \sigma_{2k+1} \sigma_{2k}}^{(i)}(t_j^{(i)}) = t_{\tau_k(j)}^{(i)} \text{ for } 2 \leq j \leq n.$$

Here $\tau_k \in \mathfrak{S}_n$ denotes the transposition which permutes k and $k + 1$.

Let ζ be a $2n$ -braid such that $\widehat{\zeta}$ is a knot. If $\xi_1, \xi_2 \in H_{2n}$ then $\widehat{\xi_1 \zeta \xi_2}$ is a knot too. Let $\kappa_i: \pi_1(S) \rightarrow \pi_1(B_i)$, $\lambda_i: F_{2n}^{(i)} \rightarrow F_n^{(i)}$ ($i = 1, 2$) denote the epimorphisms induced by $\widehat{\zeta}$ and let $\kappa'_i: \pi_1(S) \rightarrow \pi_1(B_i)$, $\lambda'_i: F_{2n}^{(i)} \rightarrow F_n^{(i)}$ ($i = 1, 2$) denote the ones induced by $\widehat{\xi_1 \zeta \xi_2}$.

Lemma 4.2 *For all $\xi_1, \xi_2 \in H_{2n}$, the identification $\Gamma: \mathcal{R}eg(\widehat{\zeta}) \rightarrow \mathcal{R}eg(\widehat{\xi_1 \zeta \xi_2})$ associated to the plat presentation $\widehat{\xi_1 \zeta \xi_2}$ of the knot $\widehat{\zeta}$ is a volume preserving diffeomorphism.*

Proof It is sufficient to prove the lemma in the case where ξ_i is one of the generators of H_{2n} , namely: $\xi_i = \sigma_1, \xi_i = \sigma_2 \sigma_1^2 \sigma_2$ or $\xi_i = \sigma_{2k} \sigma_{2k-1} \sigma_{2k+1} \sigma_{2k}$ $1 \leq k \leq n - 1$. First observe that λ'_i differs from λ_i only if $\xi_i = \sigma_1$. Next we examine the different cases.

- If $\xi_i = \sigma_1$, then $\lambda'_i(s_1^{(i)}) = \lambda_i(s_1^{(i)})^{-1}$ and equation (9) implies $\kappa'_i = \kappa_i$.
- If $\xi_i = \sigma_2 \sigma_1^2 \sigma_2$, then equation (10) implies that $\phi_{\xi_i}^{(i)}: F_n^{(i)} \rightarrow F_n^{(i)}$ induces a volume preserving diffeomorphism $\widehat{\phi}_{\xi_i}^{(i)}: \widehat{R}^{\mathcal{T}_i}(F_n^{(i)}) \rightarrow \widehat{R}^{\mathcal{T}_i}(F_n^{(i)})$. Moreover, $\widehat{\kappa}'_i = \widehat{\kappa}_i \circ \widehat{\phi}_{\xi_i}^{(i)}$ (cf diagram (5)).
- If $\xi_i = \sigma_{2k} \sigma_{2k-1} \sigma_{2k+1} \sigma_{2k}$, then equation (11) implies that the automorphism $\phi_{\xi_i}^{(i)}: F_n^{(i)} \rightarrow F_n^{(i)}$ induces a volume preserving diffeomorphism

$$\widehat{\phi}_{\xi_i}^{(i)}: \widehat{R}^{\mathcal{T}_i}(F_n^{(i)}) \rightarrow \widehat{R}^{\mathcal{T}_i}(F_n^{(i)})$$

(because ν is a 2-volume form). We conclude as before.

We deduce from these facts that there is a volume preserving diffeomorphism $\psi_{\xi_i}: \widehat{R}^{\mathcal{T}_i}(B_i) \rightarrow \widehat{R}^{\mathcal{T}_i}(B_i)$ such that $\widehat{\kappa}'_i = \widehat{\kappa}_i \circ \psi_{\xi_i}$, for $i = 1, 2$. These observations complete the proof because the volume form on $\widehat{R}^S(S)$ is not affected. \square

4.2.2 Invariance under stabilisation Let ζ be a $2n$ -braid such that $\widehat{\zeta}$ is a knot. We are interested in the new braid $\zeta' = \zeta \sigma_{2n} \in B_{2n+2}$. The plat $\widehat{\zeta}'$ is a knot too, and this knot is isotopic to $\widehat{\zeta}$ (see Figure 3). The aim of this subsection is to establish the following lemma.

Lemma 4.3 *The identification $\Psi: \mathcal{R}eg(\widehat{\zeta}) \rightarrow \mathcal{R}eg(\widehat{\zeta \sigma_{2n}})$ associated to the plat presentation $\widehat{\zeta \sigma_{2n}}$ of the knot $\widehat{\zeta}$ is a volume preserving diffeomorphism.*

Proof One needs some more notation. Set $S' = S \cup \{s_{2n+1}, s_{2n+2}\}$. Consider the epimorphism $\kappa'_i: \pi_1(S') \rightarrow \pi_1(B'_i)$ induced by $\hat{\zeta}'$ for $i = 1, 2$ and write $Q_i = \kappa_i(R^{\mathcal{T}_i}(B_i))$, $\widehat{Q}_i = \widehat{\kappa}_i(\widehat{R}^{\mathcal{T}_i}(B_i))$ and $Q'_i = \kappa'_i(R^{\mathcal{T}'_i}(B'_i))$, $\widehat{Q}'_i = \widehat{\kappa}'_i(\widehat{R}^{\mathcal{T}'_i}(B'_i))$, see Figure 3.

We have $\kappa_1 = \lambda_1$, $\kappa_2 = \lambda_2 \circ \zeta$ and $\kappa'_1 = \lambda'_1 \circ \sigma_{2n}^{-1}$, $\kappa'_2 = \lambda'_2 \circ \zeta$, where $\lambda'_i: \pi_1(S') \rightarrow \pi_1(B_i)$ is given by:

$$\begin{aligned} \lambda'_1(s_j^{(1)}) &= \lambda_1(s_j^{(1)}) \text{ if } 1 \leq j \leq 2n, & \lambda'_2(s_j^{(2)}) &= \lambda_2(s_j^{(2)}) \text{ if } 1 \leq j \leq 2n, \\ \lambda'_1(s_{2n+1}^{(1)}) &= (t_{n+1}^{(1)})^{-\varepsilon_n^{(1)}}, & \lambda'_2(s_{2n+1}^{(2)}) &= (t_{n+1}^{(2)})^{-\varepsilon_n^{(2)}}, \\ \lambda'_1(s_{2n+2}^{(1)}) &= (t_{n+1}^{(1)})^{\varepsilon_n^{(1)}}, & \lambda'_2(s_{2n+2}^{(2)}) &= (t_{n+1}^{(2)})^{\varepsilon_n^{(2)}}. \end{aligned}$$

The map λ_i induces $\overline{\lambda}_i: (-2, 2) \times (S^2)^n \rightarrow R^S(S)$ given by

$$\overline{\lambda}_i(2 \cos(\theta), P_1, \dots, P_n) = \left(2 \cos(\theta), \varepsilon_1^{(i)} P_1, -\varepsilon_1^{(i)} P_1, \dots, \varepsilon_n^{(i)} P_n, -\varepsilon_n^{(i)} P_n \right).$$

Moreover $\overline{\kappa}_1 = \overline{\lambda}_1$, $\overline{\kappa}_2 = \overline{\phi}_\zeta \circ \overline{\lambda}_2$.

We use the following notation

$$\begin{aligned} (t, P_1^\zeta, \dots, P_{2n}^\zeta) &:= \overline{\kappa}_2(t, P_1, \dots, P_n) \\ &= \overline{\phi}_\zeta(t, \varepsilon_1^{(2)} P_1, -\varepsilon_1^{(2)} P_1, \dots, \varepsilon_n^{(2)} P_n, -\varepsilon_n^{(2)} P_n). \end{aligned}$$

We have

$$\begin{aligned} \overline{\kappa}_1(2 \cos(\theta), P_1, \dots, P_{n+1}) &= \left(2 \cos(\theta), \varepsilon_1^{(1)} P_1, -\varepsilon_1^{(1)} P_1, \dots, \varepsilon_n^{(1)} P_n, -\varepsilon_n^{(1)} P_{n+1}, \right. \\ &\quad \left. Ad_{\cos(\theta) + \varepsilon_n^{(1)} \sin(\theta) P_{n+1}}(-\varepsilon_n^{(1)} P_n), \varepsilon_n^{(1)} P_{n+1} \right) \end{aligned}$$

and

$$\overline{\kappa}'_2(2 \cos(\theta), P_1, \dots, P_{n+1}) = \left(2 \cos(\theta), P_1^\zeta, \dots, P_{2n}^\zeta, -\varepsilon_n^{(1)} P_{n+1}, \varepsilon_n^{(1)} P_{n+1} \right).$$

Consider the map $f: (-2, 2) \times (S^2)^{2n} \rightarrow (-2, 2) \times (S^2)^{2n+2}$ given by

$$(12) \quad f((2 \cos(\theta), P_1, \dots, P_{2n})) = (2 \cos(\theta), P_1, \dots, P_{2n}, P_{2n}, -P_{2n}).$$

We have $f(Q_1 \cap Q_2) = Q'_1 \cap Q'_2$ and f induces an immersion $\tilde{f}: \widetilde{R}^S(S) \rightarrow \widetilde{R}^{S'}(S')$, which itself induces an immersion $\widehat{f}: \widehat{R}^S(S) \rightarrow \widehat{R}^{S'}(S')$.

Let $\pi_m: (-2, 2) \times (S^2)^m \rightarrow \text{SU}(2)$ be the map given by

$$\pi_m(2 \cos(\theta), P_1, \dots, P_m) = \prod_{j=1}^m (\cos(\theta) + \sin(\theta) P_j).$$

Observe that $\pi_{2n+2} \circ f = \pi_{2n}$. As a consequence

$$D_{(t, \mathbf{P})} f|_{\pi_{2n}^*(\mathfrak{su}(2))}: \pi_{2n}^*(\mathfrak{su}(2)) \rightarrow \pi_{2n+2}^*(\mathfrak{su}(2))$$

is a volume preserving isomorphism. Here $\pi_m^*(\mathfrak{su}(2))$ denotes the pull-back.

If $(t, \mathbf{P}) = (2 \cos(\theta), P_1, \dots, P_{2n})$ corresponds to a representation in $\widetilde{R}^S(S)$, then

$$0 \longrightarrow T_{(t, \mathbf{P})} \widetilde{R}^S(S) \xrightarrow{D_{(t, \mathbf{P})} \widetilde{f}} T_{\widetilde{f}(t, \mathbf{P})} \widetilde{R}^{S'}(S') \xrightarrow{\text{pr}} T_{P_{2n}} S^2 \oplus T_{-P_{2n}} S^2 \longrightarrow 0$$

is a short exact sequence. Here $\text{pr} = (\text{pr}_{2n+1}, \text{pr}_{2n+2})$. Hence

$$(13) \quad T_{\widetilde{f}(t, \mathbf{P})} \widetilde{R}^{S'}(S') \cong D_{(t, \mathbf{P})} \widetilde{f}(T_{(t, \mathbf{P})} \widetilde{R}^S(S)) \oplus T_{P_{2n}} S^2 \oplus T_{-P_{2n}} S^2,$$

and if s denotes a section of pr we have, at the level of volume forms,

$$v_{(t, \mathbf{P})} \widetilde{R}^S(S) \wedge v_{2n+1} \wedge v_{2n+2} = (D_{(t, \mathbf{P})} \widetilde{f} \oplus s)^*(v_{\widetilde{f}(t, \mathbf{P})} \widetilde{R}^{S'}(S')).$$

If $(t, \mathbf{P}) \in Q_1 \cap Q_2$ then there exist $(t, \mathbf{P}^{(i)}) = (2 \cos(\theta), P_1^{(i)}, \dots, P_n^{(i)}) \in (-2, 2) \times (S^2)^n$ such that $\kappa_i(t, \mathbf{P}^{(i)}) = (t, \mathbf{P})$, for $i = 1, 2$. With these preliminaries in mind, we turn to the main ingredient of the proof of [Lemma 4.3](#).

Claim 4.4 *Let ζ be a $2n$ -braid such that $\widehat{\zeta}$ is a knot. If $\zeta' = \zeta \sigma_{2n} \in B_{2n+2}$, then $\widehat{f}: \widehat{R}^S(S) \rightarrow \widehat{R}^{S'}(S')$ restricts to a volume preserving diffeomorphism*

$$\widehat{f}: (-1)^n \widehat{Q}_1 \cap \widehat{Q}_2 \rightarrow (-1)^{n+1} \widehat{Q}'_1 \cap \widehat{Q}'_2$$

in a neighbourhood of each regular point.

Proof Assume that $(t, \mathbf{P}) \in Q_1 \cap Q_2$ is regular. Then $f((t, \mathbf{P}))$ is also regular in $Q'_1 \cap Q'_2$ (see [Section 3.4](#)). For $i = 1, 2$,

$$0 \longrightarrow T_{(t, \mathbf{P})} Q_i \xrightarrow{D_{(t, \mathbf{P})} \widetilde{f}} T_{\widetilde{f}(t, \mathbf{P})} Q'_i \xrightarrow{\text{pr}_i} \mathcal{V}_i \longrightarrow 0$$

is a short exact sequence where $\mathcal{V}_i \cong T_{P_n^{(i)}} S^2$. Thus, we have

$$T_{\widetilde{f}(t, \mathbf{P})} Q'_i \cong D_{(t, \mathbf{P})} \widetilde{f}(T_{(t, \mathbf{P})} Q_i) \oplus \mathcal{V}_i.$$

Moreover, if s_i denotes a section of pr_i , then at the level of volume forms we have

$$v_{(t,\mathbf{P})}^{Q_i} \wedge v_i = (D_{(t,\mathbf{P})} \tilde{f} \oplus s_i)^* (v_{\tilde{f}(t,\mathbf{P})}^{Q'_i}),$$

where v_i denotes the natural 2-volume form on \mathcal{V}_i . Equation (13) implies

$$T_{\tilde{f}(t,\mathbf{P})} \tilde{R}^{S'} \cong D_{(t,\mathbf{P})} \tilde{f} (T_{(t,\mathbf{P})} \tilde{R}^S(S)) \oplus \mathcal{W}$$

where $\mathcal{W} \cong T_{-\varepsilon P_n^{(1)}} S^2 \oplus T_{\varepsilon P_n^{(1)}} S^2$. As oriented vector spaces we have $-\mathcal{W} \cong \mathcal{V}_1 \oplus \mathcal{V}_2$ and \mathcal{W} is endowed with the 4-form induced by $-v_1 \wedge v_2$. Thus, the transformation $\tilde{f}: \tilde{R}^S(S) \rightarrow \tilde{R}^{S'}(S')$ induces a volume preserving diffeomorphism

$$\hat{f}: (-1)^n \hat{Q}_1 \cap \hat{Q}_2 \rightarrow (-1)^{n+1} \hat{Q}'_1 \cap \hat{Q}'_2$$

in a neighbourhood of each regular point. □

This last claim achieves the proof of the invariance of $\omega^{\hat{\zeta}}$ under elementary stabilisation. □

5 Some properties and applications

5.1 Basic properties

The next proposition collects some basic properties of the volume form ω^K .

Proposition 5.1 *Let K be an knot in S^3 . The volume form ω^K satisfies the following assertions:*

- (1) *the orientation on $\text{Reg}(K)$ induced by the 1-volume form ω^K is the orientation defined in [10],*
- (2) *the volume form ω^K does not depend on the orientation of K .*
- (3) *if K^* denotes the mirror image of K , then the canonical isomorphism $\Lambda: \text{Reg}(K) \rightarrow \text{Reg}(K^*)$ satisfies $\Lambda^*(\omega^{K^*}) = -\omega^K$.*

Proof Items (2) and (3) are immediate consequence of Corollary 3.5 (3) and (4) respectively.

Let us prove the first one. Take a $2n$ -plat presentation $\hat{\zeta}$ of the knot K and let $\mathfrak{D}^{\hat{\zeta}}$ denote the orientation on $\text{Reg}(\hat{\zeta})$ defined in [10]. Precisely the orientation $\mathfrak{D}^{\hat{\zeta}}$ is defined [10, Definition 3.5] by the rule $\text{Reg}(\hat{\zeta}) = (-1)^n (\hat{Q}_1 \cap \hat{Q}_2)$. We easily

observe that the orientation induced by $v\widehat{R}^{\tau_i}(B_i)$ (resp. $v\widehat{R}^s(S)$) on $\widehat{R}^{\tau_i}(B_i)$ (resp. $\widehat{R}^s(S)$) corresponds to the one defined in [10, Sections 3 & 4.1] (resp. in [10, Section 4.1]). Formula (7) implies that $\mathfrak{D}^{\hat{\zeta}}$ is precisely the orientation induced by $\omega^{\hat{\zeta}}$. \square

5.2 A connected sum formula

The aim of this subsection is to prove a connected sum formula to compute the volume form $\omega^{K_1 \# K_2}$ associated to a composite knot $K_1 \# K_2$ in terms of the volume forms ω^{K_1} and ω^{K_2} . The beginning of our study consists in a technical lemma in which we describe the regular part of the representation space of the group of $K_1 \# K_2$ in terms of the regular parts of the groups of K_1 and K_2 .

Recall that an abelian representation of G_K is conjugate to one and only one of the $\varphi_\theta: G_K \rightarrow \text{SU}(2)$ given by $\varphi_\theta(m) = \cos(\theta) + \sin(\theta)\mathbf{i}$ with $0 \leq \theta \leq \pi$. The abelian representation φ_θ is called *regular* if $e^{2\theta i}$ is not a zero of the Alexander polynomial of K . For such representation, E. Klassen proved in [12, Theorem 19] that $H_{\varphi_\theta}^0(M_K) \cong H^0(M_K; \mathbb{R}) \cong \mathbb{R}$ and $H_{\varphi_\theta}^1(M_K) \cong H^1(M_K; \mathbb{R}) \cong \mathbb{R}$.

Let $K = K_1 \# K_2$ be a composite knot. Let m_i (resp. G_i) denote the meridian (resp. the group) of K_i , for $i = 1, 2$. Let m denote the meridian of K . The group G_K of K is the amalgamated product $G_1 *_U G_2$. Here the subgroup U of amalgamation is the normal closure of $m_1 m_2^{-1}$ in the free product $G_1 * G_2$ (see [4, Proposition 7.10]). Observe that in G_K we have $m = m_1 = m_2$.

To each representation $\rho \in R(G_K)$ corresponds the restrictions $\rho_i = \rho|_{G_i} \in R(G_i)$ for $i = 1, 2$. We write $\rho = \rho_1 * \rho_2$. If $\rho_i \in R(G_i)$ then we can form the ‘‘composite’’ representation $\rho = \rho_1 * \rho_2$ of G_K if and only if $\rho_1(m_1) = \rho_2(m_2)$. In particular, to each representation $\rho_1 \in R(G_1)$ (resp. $\rho_2 \in R(G_2)$) corresponds, up to conjugation, a unique abelian representation $\alpha_2: G_2 \rightarrow \text{SU}(2)$ (resp. $\alpha_1: G_1 \rightarrow \text{SU}(2)$) such that $\rho_1(m_1) = \alpha_2(m_2)$ (resp. $\alpha_1(m_1) = \rho_2(m_2)$). Hence, we can consider the following two maps:

$$\begin{aligned} \iota_1: \widehat{R}(M_{K_1}) &\rightarrow \widehat{R}(M_K), [\rho_1] \mapsto [\rho_1 * \alpha_2], \\ \iota_2: \widehat{R}(M_{K_2}) &\rightarrow \widehat{R}(M_K), [\rho_2] \mapsto [\alpha_1 * \rho_2]. \end{aligned}$$

Using this notation, we prove:

Lemma 5.2 *Let $K = K_1 \# K_2$ be the connected sum of the knots K_1 and K_2 . If R_1 (resp. R_2) denotes the space of conjugacy classes of regular representations $\rho_1: G_1 \rightarrow \text{SU}(2)$ (resp. $\rho_2: G_2 \rightarrow \text{SU}(2)$) such that the associated abelian representation $\alpha_2: G_2 \rightarrow \text{SU}(2)$ (resp. $\alpha_1: G_1 \rightarrow \text{SU}(2)$) is also regular, then:*

- (1) R_1 (resp. R_2) is an open submanifold of $\mathcal{R}eg(K_1)$ (resp. $\mathcal{R}eg(K_2)$),
- (2) $\mathcal{R}eg(K) = \iota_1(R_1) \cup \iota_2(R_2)$,
- (3) $\iota_1: R_1 \rightarrow \mathcal{R}eg(K)$ (resp. $\iota_2: R_2 \rightarrow \mathcal{R}eg(K)$) is an immersion.

Proof First observe that R_i is obtained from $\mathcal{R}eg(K_i)$ by removing a finite number of points. Precisely the removed points coincide with the regular representations whose associated abelian representation corresponds to a zero of the Alexander polynomial of K_{3-i} for $i = 1, 2$. This is sufficient to prove the first assertion.

We next prove the second item. In [12, Proposition 12], E. Klassen gives the structure of the irreducible representation space of G_K . Any irreducible representation $\rho \in \widetilde{R}(G_K)$ is conjugate to one of the following “composite” representations: $\rho_1 * \alpha_2$, $\alpha_1 * \rho_2$ or $\rho_1 * \rho_2$, where $\rho_i \in \widetilde{R}(G_i)$ and $\alpha_i \in A(G_i)$, $i = 1, 2$. In Figure 4 we illustrate our observations and we give a picture of the representation space of the group of $K = K_1 \sharp K_2$, where K_1 is the torus knot of type $(2, 5)$ and K_2 is the trefoil knot (ie, the torus knot of type $(2, 3)$). The proof of the second assertion is based on the following two claims.

Claim 5.3 *If $\rho_i \in \widetilde{R}(G_i)$, then $\rho = \rho_1 * \rho_2$ is not regular.*

Proof of the claim Corresponding to the representation $\rho = \rho_1 * \rho_2$ and to the amalgamated group $G_K = G_1 *_U G_2$ is the Mayer–Vietoris sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta} H^1_\rho(G_K) \xrightarrow{\kappa^*} H^1_{\rho_1}(G_1) \oplus H^1_{\rho_2}(G_2) \xrightarrow{i^*} \mathbb{R} \longrightarrow H^2_\rho(G_K) \longrightarrow H^2_{\rho_1}(G_1) \oplus H^2_{\rho_2}(G_2) \longrightarrow 0.$$

For $i = 1, 2$, we have $\dim H^1_{\rho_i}(G_i) \geq 1$ (see Section 2.1). Hence $\text{rk } \kappa^* = \dim \ker i^* \geq 1$. As a consequence, $2 \leq 1 + \text{rk } \kappa^* = \dim H^1_\rho(G_K)$. Thus ρ is not regular. \square

Claim 5.4 *The representation $\rho_1 * \alpha_2$ (resp. $\alpha_1 * \rho_2$) is regular if and only if ρ_1 and α_2 (resp. α_1 and ρ_2) are regular.*

Proof of the claim Corresponding to the irreducible representation $\rho = \rho_1 * \alpha_2$ and to the amalgamated group $G_K = G_1 *_U G_2$, the Mayer–Vietoris sequence in twisted cohomology reduces to

$$0 \longrightarrow \mathbb{R} \xrightarrow{\cong} \mathbb{R} \xrightarrow{0} H^1_\rho(G_K) \xrightarrow{\kappa^{(1)*}} H^1_{\rho_1}(G_1) \oplus H^1_{\alpha_2}(G_2) \xrightarrow{i^*} \mathbb{R} \longrightarrow H^2_\rho(G_K) \xrightarrow{\kappa^{(2)*}} H^2_{\rho_1}(G_1) \oplus H^2_{\alpha_2}(G_2) \longrightarrow 0.$$

Here $\kappa_{(2)}^*$ is onto. Hence $\dim H_\rho^2(G_K) \geq \dim H_{\rho_1}^2(G_1) + \dim H_{\alpha_2}^2(G_2)$.

Assume that ρ_1 is not regular or that α_2 is not abelian regular. Thus we have $\dim H_{\rho_1}^2(G_1) \geq 2$ or $\dim H_{\alpha_2}^2(G_2) \geq 1$. Hence, in each case, $\dim H_\rho^2(G_K) \geq 2$, which proves that $\rho_1 * \alpha_2$ is not regular.

Now, if ρ_1 is assumed to be regular and if α_2 is assumed to be abelian regular, then $\dim H_{\rho_1}^1(G_1) = \dim H_{\alpha_2}^1(G_2) = 1$ and $H_{\alpha_2}^2(G_2) = 0$. Thus, $1 = \text{rk } i^* = 2 - \text{rk } \kappa_{(1)}^*$. Hence $\dim H_\rho^1(G_K) = \text{rk } \kappa_{(1)}^* = 1$. As a result, $\rho_1 * \alpha_2$ is regular.

The same arguments prove that $\alpha_1 * \rho_2$ is regular if and only if ρ_2 is regular and α_1 is abelian regular. \square

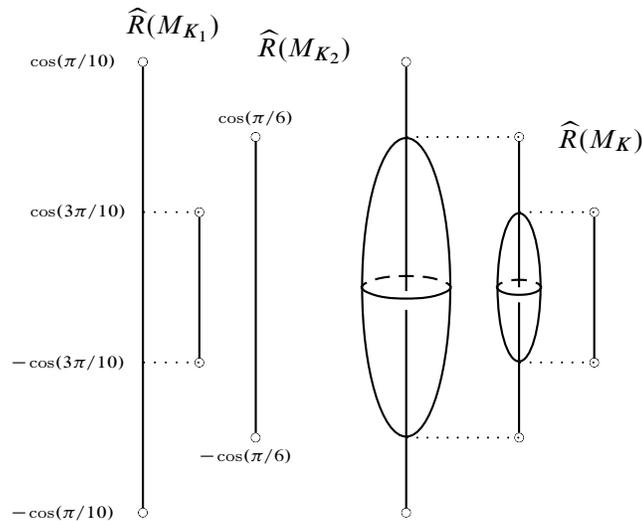


Figure 4: Irreducible representations of the group of $K = K_1 \# K_2$.

Let us establish the last point. The epimorphism $G_1 *_{U} G_2 \twoheadrightarrow G_1 *_{U} U \cong G_1$ induces the injection $H_{\rho_1}^1(G_1) \hookrightarrow H_{\rho}^1(G_K)$. Hence, $\iota_1: R_1 \rightarrow \mathcal{R}\text{eg}(K)$ is an immersion. The same argument proves that $\iota_2: R_2 \rightarrow \mathcal{R}\text{eg}(K)$ is also an immersion. \square

Proposition 5.5 *Let $K = K_1 \# K_2$ be a composite knot. With the same notation as in Lemma 5.2, we have*

$$\iota_1^*(\omega^K) = \omega^{K_1} \text{ and } \iota_2^*(\omega^K) = \omega^{K_2}.$$

Proof Assume that $K_1 = \widehat{\zeta}_1$ is presented as a $2n$ -plat and $K_2 = \widehat{\zeta}_2$ as a $2m$ -plat. If δ^{n-1} denotes the $(n-1)$ -shift operator, then $\widehat{\zeta_1 \delta^{n-1} \zeta_2}$ is a $2(n+m-1)$ -plat presentation of the connected sum $K_1 \sharp K_2$ (see Figure 5). Lemma 5.2 gives the form of the regular representations of G_K in terms of the regular ones of G_1 and G_2 . To prove $\iota_1^*(\omega^K) = \omega^{K_1}$, we use the same method as in Lemma 4.3. Consider the punctured 2-sphere $S' = S \cup \{s_{2n+1}, \dots, s_{2n+2(m-1)}\}$ as in Figure 5. This sphere allows us to split the exterior of the knot K . Use Section 3.4.3 and let

$$f: (-2, 2) \times (S^2)^{2n} \rightarrow (-2, 2) \times (S^2)^{2n+2(m-1)}$$

be the map defined by

$$f((2 \cos(\theta), P_1, \dots, P_{2n})) = (2 \cos(\theta), P_1, \dots, P_{2n}, \varepsilon_1 P_{2n}, -\varepsilon_1 P_{2n}, \dots, \varepsilon_{m-1} P_{2n}, -\varepsilon_{m-1} P_{2n}).$$

Here $\varepsilon_i \in \{\pm 1\}$ depends on the braid ζ_2 . As in the proof of Lemma 4.3, f induces a transformation $\widetilde{f}: \widetilde{R}^S(S) \rightarrow \widetilde{R}^{S'}(S')$.

The same ideas as in the proof of Lemma 4.3 imply that \widetilde{f} induces a volume preserving diffeomorphism $(-1)^n \widehat{Q}_1 \cap \widehat{Q}_2 \rightarrow (-1)^{n+m-1} \widehat{Q}'_1 \cap \widehat{Q}'_2$ in a neighbourhood of each regular representation of G_1 . We prove equality $\iota_2^*(\omega^K) = \omega^{K_2}$ by the same arguments. \square

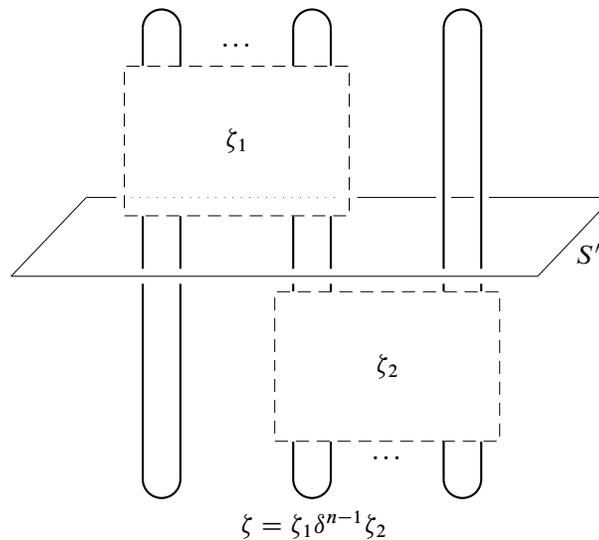


Figure 5: Connected sum of plats.

5.3 An explicit computation

In this subsection, we give without a proof an explicit formula for the volume form $\omega^{K_{2,q}}$ associated to the torus knot $K_{2,q}$ of type $(2, q)$. Here $q \geq 3$ denotes an odd integer. Other explicit computations of the volume form ω^K are given in [5] for fibered knots—and in particular for all torus knots—using the non abelian Reidemeister torsion. Recall that the group of the torus knot $K_{2,q}$ admits the well-known presentation $G_{K_{2,q}} = \langle x, y \mid x^2 = y^q \rangle$.

Proposition 5.6 *Let $K_{2,q}$ be the (left-handed) torus knot of type $(2, q)$. Each irreducible representation of $G_{K_{2,q}}$ into $SU(2)$ is conjugate to one and only one of the $\rho_{\ell,t}: G_{K_{2,q}} \rightarrow SU(2)$, where*

$$\begin{aligned} \rho_{\ell,t}(x) &= \mathbf{i} \\ \rho_{\ell,t}(y) &= \cos((2\ell - 1)\pi/q) + \sin((2\ell - 1)\pi/q) (\cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j}), \end{aligned}$$

for $\ell \in \{1, \dots, (q - 1)/2\}$ and $0 < t < 1$.

One has

$$(14) \quad \mathcal{R}eg(K_{2,q}) = \bigcup_{\ell=1}^{(q-1)/2} \{[\rho_{\ell,t}]; 0 < t < 1\}$$

and

$$(15) \quad \omega_{[\rho_{\ell,t}]}^{K_{2,q}} \left(\left. \frac{d\rho_{\ell,s}}{ds} \right|_{s=t} \right) = \frac{8}{q} \sin^2 \left(\frac{(2\ell - 1)\pi}{q} \right) \left. \frac{d\theta_m^{\rho_{\ell,t}}}{ds} \right|_{s=t}$$

where $\theta_m^{\rho_{\ell,t}} = \arccos \left((-1)^{\ell-1} \cos((2\ell - 1)\pi/2q) \cos(\pi t) \right)$.

Equation (14) is due to E Klassen [12]. One can find a proof of Proposition 5.6 in [8].

5.4 Remarks on the volume of the manifold $\mathcal{R}eg(K)$

In the introduction, we mentioned a result of Witten about the symplectic volume of the moduli space of a Riemannian surface. To close this paper we collect some remarks about the volume of $\mathcal{R}eg(K)$ (with respect to the canonical volume form ω^K).

Because of the non-compactness of $\mathcal{R}eg(K)$, it is unclear if $\int_{\mathcal{R}eg(K)} \omega^K$ always exists in general. In the case of the torus knot of type $(2, q)$, the integral exists and we have, using Proposition 5.6,

$$\int_{\mathcal{R}eg(K_{2,q})} \omega^{K_{2,q}} = \sum_{\ell=1}^{\frac{q-1}{2}} (-1)^{\ell-1} \frac{8\pi(q - 2\ell + 1)}{q^2} \sin^2 \left(\frac{(2\ell - 1)\pi}{q} \right).$$

This formula is obtained by an elementary computation based on the knowledge of explicit descriptions of the representation space of the group of the torus knot $K_{2,q}$ and of the volume form $\omega^{K_{2,q}}$.

Suppose that $\int_{\mathcal{R}\text{eg}(K)} \omega^K$ exists, then $\int_{\mathcal{R}\text{eg}(K^*)} \omega^{K^*}$ also exists and using [Proposition 5.1 \(3\)](#), we have $\int_{\mathcal{R}\text{eg}(K)} \omega^K = \int_{\mathcal{R}\text{eg}(K^*)} \omega^{K^*}$.

Suppose that $\int_{\mathcal{R}\text{eg}(K_i)} \omega^{K_i}$ exists for $i = 1, 2$. [Proposition 5.5](#) implies that

$$\int_{\mathcal{R}\text{eg}(K_1 \# K_2)} \omega^{K_1 \# K_2}$$

also exists and that

$$\int_{\mathcal{R}\text{eg}(K_1 \# K_2)} \omega^{K_1 \# K_2} = \int_{\mathcal{R}\text{eg}(K_1)} \omega^{K_1} + \int_{\mathcal{R}\text{eg}(K_2)} \omega^{K_2}.$$

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