Categorical sequences

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We define and study the categorical sequence of a space, which is a new formalism that streamlines the computation of the Lusternik–Schnirelmann category of a space X by induction on its CW skeleta. The kth term in the categorical sequence of a CW complex X, $\sigma_X(k)$, is the least integer n for which $\text{cat}_X(X_n) \geq k$. We show that σ_X is a well-defined homotopy invariant of X. We prove that $\sigma_X(k+l) \geq \sigma_X(k)+\sigma_X(l)$, which is one of three keys to the power of categorical sequences. In addition to this formula, we provide formulas relating the categorical sequences of spaces and some of their algebraic invariants, including their cohomology algebras and their rational models; we also find relations between the categorical sequences of the spaces in a fibration sequence and give a preliminary result on the categorical sequence of a product of two spaces in the rational case. We completely characterize the sequences which can arise as categorical sequences of formal rational spaces. The most important of the many examples that we offer is a simple proof of a theorem of Ghienne: if X is a member of the Mislin genus of the Lie group $Sp(3)$, then $\text{cat}(X) = \text{cat}(Sp(3))$.

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Introduction

The *Lusternik–Schnirelmann category* of a topological space X is the least integer k for which X has an open cover $X = X_0 \cup X_1 \cup \cdots \cup X_k$ with the property that each inclusion map $X_j \hookrightarrow X$ is homotopic to a constant map; it is denoted $\text{cat}(X)$. This homotopy invariant of topological spaces was first introduced by Lusternik and Schnirelmann in 1934 as a tool to use in studying functions on (compact) manifolds: a smooth function $f \colon M \to \mathbb{R}$ must have at least $\text{cat}(M) + 1$ critical points.

If X is a CW complex, then $X_n = X_{n-1} \cup_\alpha (n-\text{cells})$, and therefore $\text{cat}(X_n) \leq \text{cat}(X_{n-1}) + 1$. Berstein and Hilton asked [3] what conditions must be placed on the attaching map α in order to guarantee that equality holds in this upper bound; the answer is that equality holds when a certain set of generalized Hopf invariants does

not contain the trivial map $*$. Thus it is possible, at least in principle, to compute the Lusternik–Schnirelmann category of a finite-dimensional CW complex inductively up its skeleta.

It was shown by the third author [34] that the Hopf sets for lower-dimensional skeleta partially determine the Hopf sets for high-dimensional skeleta. In actual computations, this makes it possible to 'bootstrap' up from relatively simple low-dimensional results to (apparently) difficult high-dimensional calculations. Our goal in this paper is to establish a convenient formalism for doing category calculations while making use of all low-dimensional information.

This is done via the *categorical sequence* of a space X, which is a function $\sigma_X\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by

$$\sigma_X(k) = \inf\{n \mid \mathrm{cat}_X(X_n) \geq k\},$$

where $\mathrm{cat}_X(X_n)$ is the category of X_n relative to X (see Definition 4)[1]. It is shown in Propositions 2.1 and 2.2 that σ_X is a well-defined homotopy invariant of X; ie, when n is larger than the connectivity of X, $\mathrm{cat}_X(X_n)$ depends only on n and the homotopy type of X, and not on any choices made in constructing a CW decomposition of X. If X is finite-dimensional, then σ_X determines $\mathrm{cat}(X)$; examples due to Roitberg [28] show that this is not true for infinite-dimensional spaces. In any case, the categorical sequence of X holds a wealth of useful information.

Though we are not directly concerned with the applications of Lusternik–Schnirelmann category to critical point theory in this paper, categorical sequences could play a useful role there. For example, in the study of the n–body problem, one is often interested in infinite-dimensional Sobolev spaces W; in order to apply the Lusternik–Schnirelmann method in this situation, it is necessary to find compact subsets $K \subseteq W$ such that the relative category $\mathrm{cat}_W(K)$ is large (see Ambrosetti and Zelati [1, Remarks 2.15 and 3.5], Fadell and Husseini [8, Theorem 4.6], or Rabinowitz [27], for example). The categorical sequence of W gives *lower* estimates on the dimension of such subsets. If $\sigma_W(k) = n$, then $\mathrm{cat}_W(W_{n-1}) < k$; if $\dim(K) < n$ then K can be deformed into W_{n-1}, and so $\mathrm{cat}_W(K) < k$.

Our theoretical results establish formulas for calculation with categorical sequences. Some of the statements make use of another sequence, the *product length sequence* of a non-negatively graded commutative algebra $\mathcal{A}$, defined by setting $\sigma_{\mathcal{A}}(k)$ to be

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[1]In his encyclopedic paper [15], Fox defined a *categorical sequence* to be a certain kind of filtration of a space; this idea was used in his proof of the product inequality $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$. Our use of the term 'categorical sequence' for a completely different idea should carry no risk of confusion, since the earlier notion is no longer used, at least in the homotopy theory of Lusternik–Schnirelmann category (but see Cicortaş [5; 6] for an equivariant version).

the least dimension n for which the nth grading $\mathcal{A}^n$ of $\mathcal{A}$ contains a nontrivial k–fold product.

Proposition 3.2 *For any space X and any ring R, $\sigma_X \leq \sigma_{H^*(X;R)}$.*

We also estimate the categorical sequence of a rational space in terms of any of its models.

Proposition 3.7 *If X is a simply-connected rational space and $\mathcal{A}$ is any model for X, then $\sigma_X \geq \sigma_{\mathcal{A}}$.*

Recall that a simply-connected rational space X is *formal* if its cohomology algebra $H^*(X)$, with trivial differential, is a model for X. Thus we have the following computation for formal spaces.

Proposition 5.1 *If X is a simply-connected formal rational space, then $\sigma_X = \sigma_{H^*(X)}$.*

More generally, we completely determine the sequences σ which can arise as the categorical sequences of simply-connected formal rational spaces.

Theorem 5.2 *The following conditions on a sequence σ with $\sigma(1) > 1$ are equivalent:*

(a) $\sigma = \sigma_{\mathcal{A}}$ *for some commutative graded algebra $\mathcal{A}$,*

(b) $\sigma(k + 1) \geq \frac{k+1}{k} \sigma(k)$ *for each k,*

(c) $\sigma = \sigma_W$ *where $W = \bigvee P_i$ and $P_i = \prod S^{n_j}$ is a product of spheres, and*

(d) $\sigma = \sigma_X$ *for some formal rational space X.*

The keys to the computational power of categorical sequences, though, are the three properties listed in the following theorem. In order to prove parts (b) and (c) for *all* spaces (and not just spaces of finite type, say), we have to use a set-theoretical framework in which Whitehead's Problem (which asks: does $\text{Ext}(A, \mathbb{Z}) = 0$ imply A is free?) has a positive solution. See Section 1.1 and Lemma 1.3 for details.

Theorem 3.4 *For any space X,*

(a) $\sigma_X(k + l) \geq \sigma_X(k) + \sigma_X(l)$,

(b) *if X is simply-connected and $\sigma_X(k) = n$, then $H^n(X; A) \neq 0$ for some coefficient group A, and*

(c) *if equality occurs in (a) and X is simply-connected, then the cup product*

$$H^k(X; A) \otimes H^l(X; B) \to H^{k+l}(X; A \otimes B)$$

is nontrivial for some choice of coefficients.

The point we hope to make in this paper is that calculation with sequences is no harder than calculation of category; indeed, the extra information contained in the sequence, together with Theorem 3.4, can greatly facilitate computations. To illustrate this point, let X be any simply-connected space such that $H^*(X; \mathbb{Z}) \cong H^*(Sp(3); \mathbb{Z}) = \Lambda(x_3, x_7, x_{11})$. The categorical sequence σ_X clearly has $\sigma_X(1) = 3$ and $\sigma_X(2) \geq 7$ by Theorem 3.4(b). By Theorem 3.4(a), $\sigma_X(4) \geq \sigma_X(2) + \sigma_X(2) \geq 14$. Furthermore, $\sigma_X(4) > 14$ by Theorem 3.4(c), because the cup product $H^7(X) \otimes H^7(X) \to H^{14}(X)$ is trivial. Now we have $\sigma_X(4) \geq 18$ by Theorem 3.4(b), and hence $\sigma_X(5) \geq \sigma_X(4) + \sigma_X(1) = 21$ by Theorem 3.4(a). Since X is 21–dimensional (up to homotopy), this proves that $\mathrm{cat}(X) \leq 5$ by Theorem 3.4(b). We will see in Theorem 7.1 below that this calculation constitutes a simple proof of a result of Ghienne [18] about the Mislin genus of $Sp(3)$.

Theorem 3.4 can also be used to prove a generalization of a somewhat obscure result of Ganea [17].

Corollary 3.6 *Let X be simply-connected and of finite type with $\sigma_X(k) = n$. If there are integers $0 < a_1 < a_2 < \cdots < a_l$ such that*

$$\{n \mid \widetilde{H}^n(X; G) \neq 0 \text{ for some } G\} \subseteq I_1 \cup I_2 \cup \cdots \cup I_l$$

where $I_j = [a_j, a_j + (n-1)]$ (brackets denote closed intervals in $\mathbb{R}$), then $\mathrm{cat}(X) < k(l+1)$.

The importance of Corollary 3.6 is not the result as such. Rather, it is the fact that, since it simply encodes an elementary computation with sequences, the result can be safely disregarded without losing computational power. Our proof is *completely different* from the one given in [17]. Ganea's proof makes use of the Blakers–Massey theorem: certain cofiber sequences are treated as fibration sequences. Our argument uses Theorem 3.4, which in turn rests on a much more elementary fact: the factorization $\Delta_{k+l} = (\Delta_k \times \Delta_l) \circ \Delta_2$ of diagonal maps. However, Ganea's theorem also applies to the strong category of X, while ours only works for ordinary category.

One of our most pleasing general results gives formulas relating the categorical sequences of the spaces in a fibration sequence.

Theorem 4.2 *Let* $F \xrightarrow{q} E \xrightarrow{p} B$ *be a fibration sequence and write* $a = \mathrm{cat}(q) \leq \mathrm{cat}(F)$ *and* $b = \mathrm{cat}(p) \leq \mathrm{cat}(B)$. *Then*

(a) $\sigma_E(k(a+1)) \geq \sigma_B(k)$, *and*

(b) $\sigma_E(k(b+1)) \geq \sigma_F(k)$.

As a corollary to Theorem 4.2 we obtain the following elaboration of the celebrated Mapping Theorem from the rational theory of Lusternik–Schnirelmann category.

Proposition 4.7 *Let* $f : X \to Y$ *be a map between simply-connected rational spaces which induces an injective map* $f_* : \pi_*(X) \to \pi_*(Y)$. *Then* $\sigma_X \geq \sigma_Y$.

Finally, we address products. To state our result (and our conjectures), we construct, for sequences σ and τ, a 'product sequence' $\sigma * \tau$ defined by $\sigma * \tau(k) = \min\{\sigma(i) + \tau(j) \mid i + j = k\}$. It is not hard to see, using Proposition 3.7, that if X and Y are simply-connected formal rational spaces, then $\sigma_{X \times Y} = \sigma_X * \sigma_Y$. We conjecture that this equation holds in general for simply-connected rational spaces. So far however, the best we have been able to do is an inequality.

Theorem 6.2 *For simply-connected rational X and Y, $\sigma_{X \times Y} \leq \sigma_X * \sigma_Y$.*

This inequality is certainly not true in general for non-rational spaces, as the examples of Iwase [22] show. However, we conjecture that the *reverse* inequality $\sigma_X * \sigma_Y \leq \sigma_{X \times Y}$ is valid, not only for rational spaces, but for *all* spaces.

1 Preliminaries

In this section we establish the basic notation and concepts that will be used in the body of the paper.

1.1 Basics

We work with pointed spaces and maps; we use $*$ to denote the one point space or any trivial map. We use $\mathrm{id}_X : X \to X$ to denote the identity map and $\Delta_k : X \to X^k$ to denote the diagonal map $\Delta_k(x) = (x, x, \ldots, x)$. The symbol $\simeq$ denotes homotopy equivalence of spaces or homotopy of maps. All solid arrow diagrams in this paper are (homotopy) commutative.

If S is a set of real numbers, then $\inf(S)$ is the infimum of S. We adopt the usual convention that $\inf(\emptyset) = \infty$.

Set theory Whitehead's Problem asks: if A is an abelian group such that $\text{Ext}(A, \mathbb{Z}) = 0$, does it follow that A is free? The answer is 'yes' if A is finitely generated. Shelah has shown that the general problem is undecidable in ordinary ZFC set theory, but the answer is 'yes' if Gödel's Axiom of Constructibility is assumed (see Shelah [32]). In order to avoid 'unnecessary' hypotheses in Lemma 1.3 and Theorem 3.4 below,

    ★   we work in a set theory where $\text{Ext}(A, \mathbb{Z}) = 0$ implies that A is free.

For those uncomfortable with this assumption, we emphasize that ordinary ZFC set theory is sufficient to prove Lemma 1.3 and Theorem 3.4 when $H^n(X; \mathbb{Z})$ is finitely generated for each n.

1.2 Skeleta

We are concerned with the Lusternik–Schnirelmann category of the CW skeleta of a space. It will simplify some of our later work to use the following slightly abstract notion of skeleton.

Definition 1.1 An *n–skeleton* for a space X is a map $i \colon X_n \to X$, where X_n is a CW complex such that

  (a) X_n is n–dimensional (up to homotopy), and

  (b) i is an n–equivalence.

This definition is justified by the observation that an n–skeleton $i \colon X_n \to X$ can be taken as the nth CW skeleton of a CW replacement for X.

The following result will help us to recognize skeleta. We omit the proof.

Lemma 1.2 *Let $i \colon A \to X$ where A and X are simply-connected. Then i is an n–skeleton for X if and only if*

  (a) *$H^*(A) = 0$ for $* > n$ in all coefficients, and*

  (b) *the induced map $i^* \colon H^*(X) \to H^*(A)$ is an isomorphism for $* < n$ and is injective for $* = n$ in all coefficients.*

When we are working with rational spaces (see Félix, Halperin and Thomas [11]) we will want our skeleta to also be rational spaces. Unfortunately, this won't always happen; for example, the inclusion $\bigvee_{n=1}^{\infty} S^n \hookrightarrow S_{\mathbb{Q}}^n$ is an n–skeleton for the rational n–sphere. We avoid this problem by defining a *rational n–skeleton* of a simply-connected rational space X to be a map $i \colon X_n \to X$ where X_n is a simply-connected rational space such that

(a) $H^*(X_n; \mathbb{Q}) = 0$ for $* > n$, and

(b)   the induced map $i^*: H^*(X; \mathbb{Q}) \to H^*(X_n; \mathbb{Q})$ is an isomorphism for $* < n$ and is injective for $* = n$.

Rational n–skeleta are plentiful: if X is a rational space and X_n is an (integral) n–skeleton of X, then $(X_n)_{\mathbb{Q}}$ is a rational n–skeleton of X. We make a standing convention that if a space X is assumed to be rational, then whenever we refer to an n–skeleton of X, we actually mean a *rational n–skeleton*.

The proof of part (b) of our next result in full generality depends on a positive solution to Whitehead's Problem.

Lemma 1.3   *Let X be a simply-connected space.*

(a)   *If $H_n(X; \mathbb{Z})$ is free abelian, then X has an n–skeleton $i: X_n \to X$ such that $i^*: H^*(X) \to H^*(X_n)$ is an isomorphism for $* \leq n$,*

(b)   *If $H^n(X; A) = 0$ for all coefficient groups A, then X has an $(n-1)$–dimensional n–skeleton.*

The corresponding statements also hold for all simply-connected rational spaces.

Proof   Write $M(G, n)$ for the Moore space with

$$H_n(M(G, n); \mathbb{Z}) \cong G \quad \text{and} \quad H_i(M(G, n); \mathbb{Z}) = 0 \quad \text{for } i \neq 0, n.$$

When G is free abelian, we take $M(G, n) = \bigvee S^n$. According to Brown and Copeland [4], any simply-connected space X admits a homology decomposition, ie, a sequence of CW complexes $X(n)$ which are related to one another by cofiber sequences $M_{n-1} \to X(n-1) \to X(n)$ (where $M_{n-1} = M(H_n(X; \mathbb{Z}), n-1)$) and satisfy $X \simeq \mathrm{hocolim}_n X(n)$. The inclusion map $X(n) \to X$ induces isomorphisms on integral homology through dimension n, and $H_k(X(n); \mathbb{Z}) = 0$ for $k > n$.

With the CW decomposition inherited from the colimit, $X_n \subseteq X(n) \subseteq X_{n+1}$ for each n. If $H_n(X; \mathbb{Z})$ is free abelian then $X(n) = X(n-1) \cup (n\text{–cells})$, so $X(n)$ is n–dimensional and hence $X_n = X(n)$. Thus $X(n)$ is the desired n–skeleton of X.

To prove (b), assume that $H^n(X; A) = 0$ for all A. Using the Universal Coefficient isomorphism (see Switzer [35, Corollary 13.11]), we obtain

$$\mathrm{Hom}(H_n(X; \mathbb{Z}), A) = 0 \quad \text{and} \quad \mathrm{Ext}(H_{n-1}(X; \mathbb{Z}); A) = 0$$

for all A. We claim that (i) $H_n(X; \mathbb{Z}) = 0$ and (ii) $H_{n-1}(X; \mathbb{Z})$ is free abelian. To prove (i), we let $A = H_n(X; \mathbb{Z})$; if A were nonzero, then $\mathrm{Hom}(H_n(X; \mathbb{Z}), A)$ would

be nonzero (since it contains the identity map), thereby contradicting the assumption. For (ii), we set $A = \mathbb{Z}$; now $\text{Ext}(H_{n-1}(X; \mathbb{Z}), \mathbb{Z}) = 0$, and by Whitehead's Problem, we conclude that $H_{n-1}(X; \mathbb{Z})$ is free.

Now apply part (a) to conclude that $X(n-1)$ is an $(n-1)$–dimensional $(n-1)$–skeleton and $X(n)$ is an n–skeleton of X. Since $H_n(M(G, n); \mathbb{Z}) = 0$, we have $M_{n-1} \simeq *$, so $X(n) \simeq X(n-1)$, and hence $X(n)$ is an $(n-1)$–dimensional n–skeleton for X. $\square$

1.3 Lusternik–Schnirelmann category

We make use of three equivalent definitions of the Lusternik–Schnirelmann category of maps and spaces.

Definition 1.4 The Lusternik–Schnirelmann *category* of a map $f \colon X \to Y$ is the least integer k for which X has a cover by open sets

$$X = X_0 \cup X_1 \cup \cdots \cup X_k$$

such that $f|_{X_i} \simeq *$ for each i. When $f = \text{id}_X$, we write $\text{cat}(X) = \text{cat}(\text{id}_X)$ and when $i \colon A \hookrightarrow X$, we write $\text{cat}_X(A) = \text{cat}(i)$.

If X is a CW complex, then it is equivalent to require each X_i to be a subcomplex of X in some CW decomposition.

The category of $f \colon X \to Y$ can also be defined in terms of the *Ganea fibrations* $p_k \colon G_k(Y) \to Y$ with fiber $F_k(Y)$. The inductive definition of these fibrations begins by defining

$$F_0(Y) \longrightarrow G_0(Y) \xrightarrow{\;p_0\;} Y$$

to be the familiar path-loop fibration sequence

$$\Omega(Y) \longrightarrow \mathcal{P}(Y) \longrightarrow Y \,.$$

Given the kth Ganea fibration sequence

$$F_k(Y) \longrightarrow G_k(Y) \xrightarrow{\;p_k\;} Y \,,$$

let $\overline{G}_{k+1}(Y) = G_k(Y) \cup CF_k(Y)$ be the cofiber of p_k and define $\overline{p}_{k+1} \colon \overline{G}_{k+1}(Y) \to Y$ by sending the cone to the base point of Y. The $(k + 1)^{\text{st}}$ Ganea fibration $p_{k+1} \colon G_{k+1}(Y) \to Y$ results from converting the map $\overline{p}_{k+1}$ to a fibration. A result of Ganea [17] implies that $\text{cat}(f) \leq k$ if and only if there is a lift λ of f in the diagram

$$
\begin{array}{ccc}
 & & G_k(Y) \\
 & \overset{\lambda}{\nearrow} & \big\downarrow{\scriptstyle p_k} \\
X & \xrightarrow{\;\;f\;\;} & Y.
\end{array}
$$

Our third definition is due to G W Whitehead. According to [36, page 458], $\mathrm{cat}(f) \le k$ if and only if the composition of f with the diagonal map of pairs

$$(X, *) \xrightarrow{\ f\ } (Y, *) \xrightarrow{\ \Delta_{k+1}\ } (Y, *)^{k+1}$$

factors, up to homotopy of pairs, through the trivial pair (X, X).

We will make use of a related invariant, called Qcat, which is defined in terms of the fibrations that result from applying a fiberwise version of the infinite suspension functor Q to the Ganea fibrations. Let $q_k \colon \widetilde{G}_k(Y) \to Y$ denote the fiberwise infinite suspension of the k–th Ganea fibration. Then $\mathrm{Qcat}(f)$ is the least integer k for which f lifts through q_k (see Scheerer, Stanley and Tanré [29]).

1.4 Rational homotopy and Lusternik–Schnirelmann category

We briefly recall some key elements of the rational theory of Lusternik–Schnirelmann category. The reader is encouraged to consult Cornea, Lupton, Oprea and Tanré [7, Chapter 5] or Félix, Halperin and Thomas [11, Part V] for details.

A (simply-connected) *Sullivan algebra* is a commutative differential graded algebra (CDGA) $\mathcal{A}$ over $\mathbb{Q}$ such that: (a) $\mathcal{A}^0 \cong \mathbb{Q}$ and $\mathcal{A}^1 = 0$; (b) as a $\mathbb{Q}$–algebra, $\mathcal{A} \cong \Lambda(V)$ where V is a graded vector space; and (c) the differential d is decomposable in the sense that $d(\mathcal{A}) \subseteq \bar{\mathcal{A}}^2$, where $\bar{\mathcal{A}}$ is the augmentation ideal of $\mathcal{A}$. Every simply-connected space X has a *Sullivan minimal model*, $\mathcal{M}(X)$, which is a Sullivan algebra such that $H^*(\mathcal{M}(X)) \cong H^*(X; \mathbb{Q})$. A *model* for X is any CDGA for which there is a map $\phi \colon \mathcal{M}(X) \to \mathcal{A}$ which induces an isomorphism in cohomology (ϕ is a *quasi-isomorphism*).

Definition 1.5 Let $\mathcal{A}$ be an augmented CDGA and write $\bar{\mathcal{A}}$ for the augmentation ideal. The *nilpotency* of A, denoted $\mathrm{nil}(\mathcal{A})$, is the greatest integer k such that $(\bar{\mathcal{A}})^k \ne 0$.

The algebraic study of the Lusternik–Schnirelmann category of rational spaces can be developed from the following result, which can be found in [7, Corollary 5.16] (though, historically, it was not [7, Remark 5.15]).

Theorem 1.6 *If X is a rational space, then the following are equivalent*

(a) $\mathrm{cat}(X) \le k$, *and*

(b) $\mathcal{M}(X)$ *is a retract (up to chain homotopy) of a Sullivan algebra $\mathcal{B}$ which is quasi-isomorphic with another CDGA $\mathcal{A}$ with $\mathrm{nil}(\mathcal{A}) \le k$.*

It follows immediately from Theorem 1.6 that if $u \in H^*(Y) = H^*(\mathcal{M}(Y))$ can be represented by a cocycle which is a k–fold product, then $f^*(u) = 0$ for any map $f \colon X \to Y$ with $\mathrm{cat}(X) < k$. In this case, the *(rational) category weight* of u is at least k. We write $\mathrm{wgt}(u) \geq k$ and observe that $\mathrm{cat}(X) \geq \mathrm{wgt}(u)$ whenever $u \neq 0 \in H^*(X)$. The maximum value of $\mathrm{wgt}(u)$ for $u \in H^*(X)$ is known as the *Toomer invariant* of X, and is denoted $e_0(X)$.

There is a related invariant, denoted Mcat (see Félix [9]). It is known that $\mathrm{Mcat}(X) = \mathrm{cat}(X)$ for simply-connected rational spaces (see Hess [21, Theorem 0]). The equality of Mcat and cat is known to fail for maps: according to Parent [26, Theorems 2 and 11] $\mathrm{Mcat}(f \times g) = \mathrm{Mcat}(f) + \mathrm{Mcat}(g)$; on the other hand, Stanley [33] has produced examples of maps f and g between simply-connected rational spaces such that $\mathrm{cat}(f \times g) < \mathrm{cat}(f) + \mathrm{cat}(g)$. It is also known that $\mathrm{Mcat}(X) = \mathrm{Qcat}(X)$ when X is a simply-connected rational space (see Scheerer and Stelzer [30], but see also [7, Theorem 5.49]). A simple adaptation of the proof of [7, Theorem 5.49] yields the following generalization to maps; we omit the proof.

Proposition 1.7 *If $f \colon X \to Y$ is a map between simply-connected rational spaces, then* $\mathrm{Qcat}(f) = \mathrm{Mcat}(f)$.

2 Categorical sequences

In this section we will define our object of study, the categorical sequence associated to a space X. To ensure that our sequences are well-defined, we must first prove some results concerning the relative category of an n–skeleton.

2.1 Relative category of skeleta

Since we usually think of an n–skeleton as a subspace of X, we will sometimes write $\mathrm{cat}_X(X_n)$ instead of $\mathrm{cat}(i)$ when $i \colon X_n \to X$ is an n–skeleton.

Proposition 2.1 *For fixed n, the integer $\mathrm{cat}_X(X_n)$ depends only on the homotopy type of X, and not on the choice of n–skeleton.*

Proof Let $i \colon A \to X$ and $j \colon B \to X$ be two n–skeleta of X and consider the diagram

$$
\begin{array}{ccc}
 & & B \\
 & \nearrow\raisebox{0.5ex}{\scriptsize l} & \downarrow{\scriptstyle j} \\
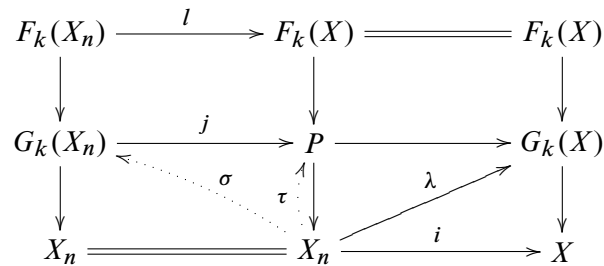A & \xrightarrow[i]{} & X.
\end{array}
$$

Since j is an n–equivalence and A is n–dimensional, there is a lift $l\colon A \to B$ such that $j \circ l \simeq i$ (see Switzer [35, Theorem 6.31]). It follows that $\mathrm{cat}_X(A) = \mathrm{cat}(i) \leq \mathrm{cat}(j)$ (see Berstein and Ganea [2, 1.4]). Since the situation is symmetrical, we also have $\mathrm{cat}(j) \leq \mathrm{cat}(i)$. □

It can be conceptually easier to work with the Lusternik–Schnirelmann category of spaces rather than of maps. Happily, there is no difference between the two for skeleta.

Proposition 2.2 *If X is $(c-1)$–connected and $i\colon X_n \to X$ is an n–skeleton with $n \geq c$, then*

   (a) $\mathrm{cat}(X_n) = \mathrm{cat}(i)$,

   (b) $\mathrm{Qcat}(X_n) = \mathrm{Qcat}(i)$, *and*

   (c) *if X is a rational space and $i\colon X_n \to X$ is a rational n–skeleton for X, then $\mathrm{cat}(X_n) = \mathrm{Mcat}(i)$.*

Proof We begin by proving (a). It is trivial that $\mathrm{cat}_X(X_n) \leq \mathrm{cat}(X_n)$; we wish to prove the reverse inequality. Assume that $\mathrm{cat}_X(X_n) = k$; we will show that $\mathrm{cat}(X_n) \leq k$. Since $n \geq c$, the map $i_*\colon \pi_c(X_n) \to \pi_c(X)$ is nontrivial, and hence $k \geq 1$. Now consider the diagram

$$
\begin{array}{ccccc}
F_k(X_n) & \xrightarrow{\;\;l\;\;} & F_k(X) & =\!=\!= & F_k(X) \\
\downarrow & & \downarrow & & \downarrow \\
G_k(X_n) & \xrightarrow{\;\;j\;\;} & P & \longrightarrow & G_k(X) \\
\downarrow & {}^{\sigma}\!\nwarrow\;\;{}_{\tau}\!\uparrow & \uparrow & {}^{\lambda}\!\nearrow & \downarrow \\
X_n & =\!=\!= & X_n & \xrightarrow{\;\;i\;\;} & X
\end{array}
$$

in which the bottom right square is a pullback. Since $\mathrm{cat}_X(X_n) = k$ there is a lift λ of i. By the pullback property, there is a section $\tau\colon X_n \to P$.

According to [7, Lemma 6.26], the map $l\colon F_k(X_n) \to F_k(X)$ is an $(n+kc-1)$–equivalence since $k \geq 1$, and it follows that j is also a $(n+kc-1)$–equivalence. Since $n \leq n + kc - 1$, it follows that there is a (unique) map $\sigma\colon X_n \to G_k(X_n)$ with $j \circ \sigma = \tau$ [35, Theorem 6.31]. This σ is a section (up to homotopy) of the fibration $G_k(X_n) \to X_n$, and so $\mathrm{cat}(X_n) \leq k$.

The key to the proof of part (a) is the fact that $l\colon F_k(X_n) \to F_k(X)$ is an $(n+kc-1)$–equivalence. But this implies that $Ql\colon QF_k(X_n) \to QF_k(X)$ is also an $(n+kc-1)$–equivalence, and so the proof of (a) can be used again to show $\mathrm{Qcat}(i) = \mathrm{Qcat}(X_n)$.

It remains to prove (c). For this we simply compute

$$
\begin{aligned}
\mathrm{cat}(X_n) &= \mathrm{Mcat}(X_n) & \text{by Hess [21, Theorem 0]} \\
&= \mathrm{Qcat}(X_n) & \text{by Scheerer and Stelzer [30]} \\
&= \mathrm{Qcat}(i) & \text{by part (b)} \\
&= \mathrm{Mcat}(i) & \text{by Proposition 1.7.}
\end{aligned}
$$

This completes the proof.   □

Remark 2.3   The proof of Proposition 2.2(a) is an adaptation of the proof of [12, Theorem 1]. The argument actually works equally well with $i\colon X_n \to X$ replaced by any n–equivalence $f\colon Z \to X$ with $\dim(Z) \leq n + kc - 1$. The conclusion in this case is that $\mathrm{cat}(f) = \mathrm{cat}(Z) = k$.

2.2   Sequences from topology and algebra

We will be concerned with sequences whose values are either nonnegative integers or ∞; thus a sequence is a function $\sigma\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. We say that $\sigma \leq \tau$ if $\sigma(k) \leq \tau(k)$ for each $k \geq 0$. We write $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$ ($\sigma < \tau$ does *not* mean that $\sigma(k) < \tau(k)$ for every k). If σ is increasing, then the *length* of σ is $\sup\{k \mid \sigma(k) < \infty\}$.

In view of Propositions 2.1 and 2.2, we may make the following definition.

Definition 2.4   The *categorical sequence* of a CW complex X is the sequence $\sigma_X\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by

$$
\sigma_X(k) = \inf\{n \mid \mathrm{cat}_X(X_n) \geq k\}.
$$

Remark 2.5   The following elementary observations about categorical sequences will be used frequently in what follows.

(a) σ_X is an invariant of the weak homotopy type of X.

(b)   If X is $(c-1)$–connected but not c–connected, then $\sigma_X(0) = 0$ and $\sigma_X(1) = c$.

(c)   The finite values of σ_X are strictly increasing.

(d)   If $\sigma_X(k) = n$, then $X_n \neq X_{n-1}$ in every cellular decomposition of X. In particular, if X is simply-connected and $\sigma_X(k) = n$, then $H^n(X) \neq 0$ for some coefficients (see Theorem 3.4(b) below).

(e)   If X is finite-dimensional, then $\mathrm{cat}(X) = \mathrm{length}(\sigma_X)$; if X is infinite-dimensional, then $\mathrm{length}(\sigma_X) \leq \mathrm{cat}(X) \leq 2 \cdot \mathrm{length}(\sigma_X)$ (see Hardie [20]).

(f)   In particular, $\mathrm{cat}(X) = \infty$ if and only if $\mathrm{length}(\sigma_X) = \infty$.

(g) If $\sigma_X \leq \sigma_Y$ and Y is finite-dimensional, then $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$.

Before proceeding further, we give some examples.

Example 2.6 (a) As is well-known, the integral cohomology of the symplectic group $Sp(2)$ is $H^*(Sp(2)) = \Lambda(x_3, x_7)$, an exterior algebra on generators in dimensions 3 and 7. It follows from Theorem 3.4(b) that the only possible finite values for $\sigma_X(k)$ are $0, 3, 7$ and 10. Since it is known (see Schweitzer [31, Example 4.4]) that $\mathrm{cat}(Sp(2)) = 3$, $\sigma_{Sp(2)}(3) < \infty$, and hence

$$\sigma_{Sp(2)} = (0, 3, 7, 10, \infty, \infty, \ldots).$$

(b) Define a Sullivan algebra $\mathcal{M} = \Lambda(x_3, y_3, z_5)$ with $d(z_5) = x_3 y_3$, and let X be a rational space whose minimal model is isomorphic to $\mathcal{M}$ (this algebra and space appear in [11, page 387]). The nontrivial cohomology of X is

$$\begin{aligned}
H^3(X) &= \mathbb{Q}[x] \oplus \mathbb{Q}[y] \\
H^8(X) &= \mathbb{Q}[xz] \oplus \mathbb{Q}[yz] \\
H^{11}(X) &= \mathbb{Q}[xyz]
\end{aligned}$$

where brackets indicate cohomology classes. Thus $\mathrm{cat}(X) \leq 3$, and since $\mathrm{cat}(X) \geq \mathrm{wgt}([xyz]) = 3$, we have $\mathrm{cat}(X) = 3$. This forces $\sigma_X = (0, 3, 8, 11, \infty, \infty, \ldots)$.

(c) The 'finite-dimensional' hypothesis in Remark 2.5(g) cannot be removed. Roitberg has shown that the cofibers C of certain (phantom) maps $f: \Sigma K(\mathbb{Z}, 5) \to S^4$ have the property that C_n is a suspension for all n, but $\mathrm{cat}(C) = 2$. Thus $\sigma_C = (0, 4, \infty, \infty, \ldots) = \sigma_{S^4}$, but $\mathrm{cat}(C) = 2 > 1 = \mathrm{cat}(S^4)$.

We will often abbreviate a sequence by deleting any terms known to be infinite. Thus, for example, we could summarize the results of Example 2.6(a,b) by writing $\sigma_{Sp(2)} = (0, 3, 7, 10)$ and $\sigma_X = (0, 3, 8, 11)$. If we were unsure of the later values of the sequence, we would write, for example, $\sigma_{Sp(2)} = (0, 3, 7, \ldots)$; knowing that $\mathrm{cat}(Sp(2)) \leq 3$, we might write $\sigma_{Sp(2)} = (0, 3, 7, a)$, where $a = 10$ or $a = \infty$.

We will also make use of the algebraic *product length sequence* of a nonnegatively graded augmented CGA $\mathcal{A}$, defined by

$$\sigma_{\mathcal{A}}(k) = \inf\{n \mid \exists \text{ nontrivial } k-\text{fold products in } \mathcal{A}^n\}.$$

If each of P and Q is either a space or a graded algebra, then it may happen that $\sigma_P = \sigma_Q$. If so, then we say that P and Q are *isosequential*.

Example 2.7 (a) The spaces

$$\overbrace{S^2 \times \cdots \times S^2}^{n \text{ factors}} \quad \text{and} \quad \mathbb{C}\mathrm{P}^n$$

are easily seen to be isosequential.

(b) It is easy to verify that $\sigma_{\mathbb{C}\mathrm{P}^\infty} = (0, 2, 4, 6, \ldots)$; it is even easier to check that if $\mathcal{A} = H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q})$, $\sigma_{\mathcal{A}} = (0, 2, 4, 6, \ldots)$. Thus the space $\mathbb{C}\mathrm{P}^\infty$ and the graded algebra $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Q})$ are isosequential.

(c) Let X be the space of Example 2.6(b). Then $\sigma_X = (0, 3, 8, 11)$, but $\sigma_{H^*(X)} = (0, 3, 11)$, and $\sigma_{\mathcal{M}(X)} = (0, 3, 6, 11)$, so X is not isosequential with either $H^*(X)$ or $\mathcal{M}(X)$. Instead, these sequences are related by the string of strict inequalities $\sigma_{\mathcal{M}(X)} < \sigma_X < \sigma_{H^*(X)}$.

3 Inequalities between sequences

One of our goals is to develop techniques for computing categorical sequences σ_X. As with formulas for the calculation of $\mathrm{cat}(X)$, many of our results for sequences come in the form of inequalities.

3.1 Inequalities for general spaces

We begin by dispensing with wedges and retracts.

Proposition 3.1 *Let X and Y be any two spaces. Then*

   (a) $\sigma_{X \vee Y}(k) = \min\{\sigma_X(k), \sigma_Y(k)\}$, *and*

   (b)   *if X is a homotopy retract of Y, then $\sigma_X \geq \sigma_Y$.*

Proof Part (a) follows from the formula $\mathrm{cat}(f \vee g) = \max\{\mathrm{cat}(f), \mathrm{cat}(g)\}$. For (b), we consider the homotopy commutative diagram

in which the map s exists by cellular approximation. It follows from the commutativity of the diagram that $\mathrm{cat}_X(X_{n-1}) = \mathrm{cat}(i) \leq \mathrm{cat}(j) = \mathrm{cat}_Y(Y_{n-1})$. Now $\sigma_Y(k) = n$

implies that $\mathrm{cat}_Y(Y_{n-1}) < k$ and hence that $\mathrm{cat}_X(X_{n-1}) < k$. Therefore $\sigma_X(k) \geq n = \sigma_Y(k)$. $\qquad\square$

Our next result recasts the classical cup length lower bound for Lusternik–Schnirelmann category in terms of sequences.

Proposition 3.2 *For any space X and any ring R, $\sigma_X \leq \sigma_{H^*(X;R)}$.*

Proof Suppose that $\sigma_{H^*(X)}(k) = n$, so there is a nontrivial k-fold cup product $u_1 \cdots u_k \in H^n(X)$. Let $i\colon X_n \to X$ be an n-skeleton. Then i induces an injection $i^*\colon H^n(X) \to H^n(X_n)$, so $i^*(u_1 \cdots u_k) = i^*(u_1) \cdots i^*(u_k) \neq 0 \in H^n(X_n)$. Therefore $\mathrm{cat}(X_n) \geq k$ (see Cornea, Lupton, Oprea and Tanré [7, Proposition 1.5]) and so $\sigma_X(k) \leq n$. $\qquad\square$

Proposition 3.2 can be used to determine the categorical sequence of a product of spheres. This simple corollary will play an important role in our characterization of the categorical sequences of formal rational spaces (§5).

Corollary 3.3 *If $X = S^{n_1} \times \cdots \times S^{n_r}$ with $n_1 \leq n_2 \leq \cdots \leq n_r$, then σ_X is given by the formula $\sigma_X(k) = \sigma_{H^*(X)}(k) = n_1 + n_2 + \cdots + n_k$ for $1 \leq k \leq r$ and $\sigma_X(k) = \infty$ for $k > r$.*

Proof Clearly $\sigma_{H^*(X)}(k) = n_1 + n_2 + \cdots + n_k$, and Proposition 3.2 implies that $\sigma_X \leq \sigma_{H^*(X)}$. For the reverse inequality, let

$$X(k) = \{(x_1, \ldots, x_r) \mid \text{at least } r - k \text{ entries are } *\} \subseteq X.$$

It is well-known that $X(0), X(1), \ldots, X(r)$ constitute a (spherical) cone decomposition of X. Furthermore, $X(k-1)$ contains the cellular $(n_1 + n_2 + \cdots + n_k - 1)$–skeleton of X, and so

$$\mathrm{cat}(X_{n_1 + n_2 + \cdots + n_k - 1}) \leq \mathrm{cat}(X(k-1)) < k.$$

Therefore $\sigma_X(k) \geq n_1 + n_2 + \cdots + n_k = \sigma_{H^*(X)}(k)$. $\qquad\square$

The following theorem gives surprisingly strong algebraic control over categorical sequences. The proofs of parts (b) and (c) in full generality depend on the positive solution to Whitehead's Problem; but they are valid in ordinary ZFC set theory if X is of finite type.

Theorem 3.4 *For any space X,*

(a) $\sigma_X(k + l) \geq \sigma_X(k) + \sigma_X(l)$,

(b) *if X is simply-connected and $\sigma_X(k) = n$, then $H^n(X; A) \neq 0$ for some coefficient group A, and*

(c) *if equality occurs in (a) and X is simply-connected, then the cup product*

$$H^k(X; A) \otimes H^l(X; B) \to H^{k+l}(X; A \otimes B)$$

is nontrivial for some choice of coefficients.

Proof Write $\sigma_X(k) = a$ and $\sigma_X(l) = b$. Then $\mathrm{cat}(X_{a-1}) = k - 1$ and $\mathrm{cat}(X_{b-1}) = l - 1$, which means that there are factorizations

$$(X, *) \to (X, X_{a-1}) \to (X, *)^k \qquad \text{and} \qquad (X, *) \to (X, X_{b-1}) \to (X, *)^l$$

of Δ_k and Δ_l, up to homotopy of pairs. Putting these together using cellular approximation and the factorization $\Delta_{k+l} = (\Delta_k \times \Delta_l) \circ \Delta_2$, we obtain the homotopy-commutative diagram of pairs

Taking $n = a + b - 1$ we see that $\Delta_{k+l}|_{X_{a+b-1}}$ factors, up to homotopy of pairs, through (X_{a+b-1}, X_{a+b-1}), and so $\mathrm{cat}_X(X_{a+b-1}) < k + l$ by the Whitehead definition and Proposition 2.2. Therefore $\sigma_X(k + l) \geq a + b$, proving (a).

Now we prove part (b). If $\sigma_X(k) = n$, then $\mathrm{cat}_X(X_n) > \mathrm{cat}_X(X_{n-1})$, so X does not have an $(n-1)$–dimensional n–skeleton. By Lemma 1.3(c), then, it cannot be that $H^n(X; A) = 0$ for all A.

To prove the statement (c) about cup products, we first recall that by Theorem 3.4(b), if $\sigma_X(i) = m$, then $H^m(X; A) \neq 0$ for some coefficient group A. Let $u \in H^m(X; A)$ be nonzero, and interpret it as a map $u\colon X \to K(A, m)$. This map factors

where $\pi_m = \pi_m(X/X_{m-1})$. Since $u \not\simeq *$, $\mu_m \not\simeq *$ as well. Note also that $K(\pi_m, m)$ may be constructed from X/X_{m-1} by attaching cells of dimension $m + 2$ and higher, so κ_m is an $(m+1)$–equivalence.

Since $(X, X_{a-1}) \times (X, X_{b-1}) = (X \times X, X \times X_{b-1} \cup X_{a-1} \times X)$ is an $(a+b-1)$–connected pair and $X \times X_{b-1} \cup X_{a-1} \times X$ is 1–connected, we apply the Blakers–Massey theorem (see Switzer [35, Corollary 6.22]) to conclude that the collapse map

$$(X, X_{a-1}) \times (X, X_{b-1}) \to (X/X_{a-1} \wedge X/X_{b-1}, *)$$

is an $(a+b+1)$–equivalence.

Assuming $\sigma_X(k + l) = \sigma_X(k) + \sigma_X(l) = a + b$, we may set $n = a + b$ in the diagram of part (a) and conclude that the composite map $(X_{a+b}, *) \to (X, X_{a-1}) \times (X, X_{b-1})$ is nontrivial. Because the collapse map is an $(a+b+1)$–equivalence and X_{a+b} is $(a+b)$–dimensional, we see that the composition

$$(X_{a+b}, *) \to (X, X_{a-1}) \times (X, X_{b-1}) \to (X/X_{a-1} \wedge X/X_{b-1}, *)$$

is also nontrivial. Now the desired cup product is

and it is nontrivial because the horizontal maps are all $(a+b+1)$–equivalences and X_{a+b} is $(a+b)$–dimensional. $\square$

The following elementary computation illustrates the use of Theorem 3.4.

Example 3.5 Let us consider the exceptional Lie group G_2. It is known (see Mimura and Toda [25]) that $H^*(G_2; \mathbb{Z}/2) \cong \left(\mathbb{Z}/2[x_3]/(x_3^4)\right) \otimes \Lambda(x_5)$. Therefore

$$\sigma_{G_2} \leq \sigma_{H^*(G_2; \mathbb{Z}/2)} = (0, 3, 6, 9, 14, \infty, \ldots)$$

by Proposition 3.2. On the other hand, we know $\sigma_{G_2}(1) = 3$ by Remark 2.5(b), so $\sigma_{G_2} \geq (0, 3, 6, 9, 12, \infty, \ldots)$ by Theorem 3.4(b,c); this determines σ_{G_2} except for $\sigma_{G_2}(4)$. However, $H^*(G_2; A) = 0$ for $* = 12, 13$ and any abelian group A, so $\sigma_{G_2}(4) \neq 12, 13$ by Theorem 3.4(b). We conclude that $\sigma_{G_2} = (0, 3, 6, 9, 14)$.

Theorem 3.4 implies the well-known result:

$$\mathrm{cat}(X) \leq \frac{\mathrm{dimension}(X)}{\mathrm{connectivity}(X)}$$

In [16], Ganea generalized this familiar upper bound to obtain an upper bound for the category of X in terms of the set of dimensions in which $H^*(X)$ is nontrivial. We

now prove a further generalization by a completely different method. For a space X, let

$$h(X) = \{n \mid \widetilde{H}^n(X;G) \neq 0 \text{ for some } G\}.$$

Corollary 3.6 *Let X be simply-connected and of finite type with $\sigma_X(k) = n$. If there are integers $0 < a_1 < a_2 < \cdots < a_l$ such that*

$$h(X) \subseteq I_1 \cup I_2 \cup \cdots \cup I_l$$

where $I_j = [a_j, a_j + (n-1)]$ (brackets denote closed intervals in $\mathbb{R}$), then $\operatorname{cat}(X) < k(l+1)$.

Proof Consider the integers $\sigma_X(kj)$, $j = 1, 2, \ldots$. We show by induction that $\sigma_X(kj) \geq a_j$. If $\sigma_X(kj) = \infty$ we are done, so we assume that this value is finite, and hence is an element of $h(X)$ by Theorem 3.4(b). Since $n \in h(X)$, $a_1 \leq n \leq a_1 + (n-1)$. Now assume that $\sigma_X(k(j-1)) \geq a_{j-1}$. By Theorem 3.4(a),

$$\sigma_X(kj) \geq \sigma_X(k(j-1)) + \sigma_X(k) \geq a_{j-1} + n,$$

which implies that $\sigma_X(kj) \notin \bigcup_{t<j} I_t$ and forces $\sigma_X(kj) \in \bigcup_{t \geq j} I_t \subseteq [a_j, \infty)$.

In particular, $\sigma_X(kl) \geq a_l$, and so $\sigma_X(k(l+1)) > \sigma_X(kl) + \sigma_X(k) = a_l + n$ by Theorem 3.4(a). Thus $\sigma_X(k(l+1)) \notin h(X)$, and so $\sigma_X(k(l+1)) = \infty$. Therefore $\operatorname{cat}(X) < k(l+1)$ by Remark 2.5(e), since the hypotheses imply that X is weakly equivalent to a finite dimensional CW complex. $\square$

Ganea's theorem is the special case $k = 1$ when X is $(n-1)$–connected. It should be noted, though, that Ganea's result applies for *strong category* (ie, cone length), where ours only applies for ordinary Lusternik–Schnirelmann category. It would be interesting to know whether our generalization holds with cone length in place of category.

3.2 Rational spaces

The categorical sequence σ_X for a rational space X can be easily bounded above in terms of *any one* of its models.

Proposition 3.7 *For any simply-connected rational space X, and any model $\mathcal{A}$ for X, $\sigma_X \geq \sigma_{\mathcal{A}}$.*

Proof Write $\sigma_{\mathcal{A}}(k) = n$ and let $\mathcal{B}$ be the quotient of $\mathcal{A}$ by the differential ideal consisting of all elements of dimension n or greater. Then $\operatorname{nil}(\mathcal{B}) < k$ and the quotient

$q\colon \mathcal{A} \to \mathcal{B}$ induces an isomorphism on cohomology in dimensions $< n - 1$ and an injection in dimension $n - 1$.

Let $\mathcal{N}$ be the Sullivan minimal model for $\mathcal{B}$ and let $r\colon \mathcal{M}(X) \to \mathcal{N}$ cover the map q. Then r has a *spatial realization* $i\colon Z \to X$ such that $q^* = i^*\colon H^*(X) \to H^*(Z)$ (see Félix, Halperin and Thomas [11, Chapter 17]). It follows that $i\colon Z \to X$ is a rational $(n-1)$–skeleton. Since $\mathcal{M}(Z) \sim \mathcal{B}$ by construction and $\mathrm{nil}(\mathcal{B}) < k$, we conclude using Theorem 1.6 that $\mathrm{cat}(Z) < k$. It follows that $\sigma_X(k) \geq n$. $\qquad\square$

Example 3.8 Let $(\mathcal{A}, d)$ be the CDGA with generators x_n, y_m and w_{n+m-1} (subscripts indicate dimension; $2 \leq n \leq m$) subject to the relations $x^2 = y^2 = w^2 = 0$ and with differential determined by $dx = dy = 0$ and $dw = xy$. This is not a Sullivan algebra, but it does have a Sullivan model, $\mathcal{M}$, and $\mathcal{M}$ has a spatial realization, X. Then $\mathcal{A}$ is a model for X, and according to Proposition 3.7,

$$\sigma_X \geq \sigma_\mathcal{A} = (0, n, n + m, 2(n + m) - 1, \infty, \infty, \ldots).$$

But we can say even more, because the nonzero cohomology of X occurs in dimensions $n, m, 2n + m - 1, n + 2m - 1$ and $2(n + m) - 1$. Since X is indistinguishable from $S^n \vee S^m$ through dimension $n + m$, we know that $\sigma_X(2) > n + m$, and therefore $\sigma_X(2) \geq 2n + m - 1$. Thus

$$\sigma_X \geq (0, n, 2n + m - 1, 2(n + m) - 1, \infty, \infty, \ldots).$$

Since $\mathcal{A}$ is finite-dimensional, so is $H^*(X)$, and we conclude that $\mathrm{cat}(X) \leq 3$.

4 Sequences and fibrations

In this section we study the relationship between the sequences σ_F, σ_E and σ_B when $F \to E \to B$ is a fibration sequence. Our general result is the key to a mapping theorem for categorical sequences of rational spaces.

4.1 General spaces

Our first result is proved by a slight generalization of the method Hardie used to prove the main result of [19].

Proposition 4.1 *Consider the diagram*

$$
\begin{array}{ccccc}
 & & X & & \\
 & {\scriptstyle f}\downarrow & & \searrow^{p \circ f} & \\
F & \xrightarrow{\ q\ } & E & \xrightarrow{\ p\ } & B
\end{array}
$$

in which the bottom row is a fibration sequence. Then

$$\mathrm{cat}(f) + 1 \le (\mathrm{cat}(p \circ f) + 1) \cdot (\mathrm{cat}(q) + 1).$$

Proof Suppose $\mathrm{cat}(p \circ f) = k$ and that $\mathrm{cat}(q) = l$. Then X has a cover $X = A_0 \cup A_1 \cup \cdots \cup A_k$ by subcomplexes such that $(p \circ f)|_{A_i} \simeq *$ for each i. Since p is a fibration with fiber F, $f|_{A_i}$ factors (up to homotopy) as $j \circ g_i$, where $g_i \colon A_i \to F$. Therefore $\mathrm{cat}(f|_{A_i}) \le \mathrm{cat}(q) = l$ and so we can write $A_i = A_{i0} \cup A_{i1} \cup \ldots \cup A_{il}$ where $(q \circ g_i)|_{A_{ij}} \simeq *$. Thus $X = \bigcup_{i,j} A_{ij}$ where $0 \le i \le k$ and $0 \le j \le l$ and $f|_{A_{ij}} \simeq *$ for all i and j. Therefore $\mathrm{cat}(f) + 1 \le (k+1)(l+1)$. $\square$

Hardie's result is the special case in which $f = \mathrm{id}_E$. We are interested in the more general situation in which $f \colon X \to E$ is an n–skeleton.

Theorem 4.2 *Let*

$$F \xrightarrow{\ q\ } E \xrightarrow{\ p\ } B$$

be a fibration sequence and write $a = \mathrm{cat}(q) \le \mathrm{cat}(F)$ and $b = \mathrm{cat}(p) \le \mathrm{cat}(B)$. Then

(a) $\sigma_E(k(a+1)) \ge \sigma_B(k)$, *and*

(b) $\sigma_E(k(b+1)) \ge \sigma_F(k)$.

Proof Let $\sigma_B(k) = n$. Thus $\mathrm{cat}(B_{n-1}) < k$ and we have to show that $\mathrm{cat}_E(E_{n-1}) < k(a+1)$. Consider the homotopy-commutative diagram

$$
\begin{array}{ccc}
E_{n-1} & \dashrightarrow & B_{n-1} \\
\big\downarrow{\scriptstyle i} & & \big\downarrow \\
F \xrightarrow{\ q\ } E & \xrightarrow{\ p\ } & B,
\end{array}
$$

in which the dotted arrow exists by cellular approximation. According to Proposition 4.1,

$$
\begin{aligned}
\mathrm{cat}(i) &\le (\mathrm{cat}(p \circ i) + 1) \cdot (\mathrm{cat}(q) + 1) - 1 \\
&< (\mathrm{cat}(B_{n-1}) + 1) \cdot (a+1) \\
&< k(a+1),
\end{aligned}
$$

proving (a).

For part (b), we let $\sigma_F(k) = n$, so $\mathrm{cat}(F_{n-1}) = k - 1$. Choose an $(n-1)$–skeleton $i \colon E_{n-1} \to E$. Since $\mathrm{cat}(p \circ i) \le \mathrm{cat}(p) = b$, we can write $E_{n-1} = A_0 \cup A_1 \cup \cdots \cup A_b$

where A_j is a subcomplex of E_{n-1} (so $\dim(A_j) < n$) and $(p \circ i)|_{A_j} \simeq *$ for each j. Thus $i|_{A_j}$ factors (up to homotopy) through the $F \to E$, and so we have the diagram

$$
\begin{array}{ccc}
F_{n-1} & \xleftarrow{\;\;\;\;\;\;} & A_j \\
\downarrow & & \downarrow{\scriptstyle i|_{A_j}} \\
F & \xrightarrow[q]{} E \xrightarrow[p]{} & B,
\end{array}
$$

in which the dotted arrow exists by cellular approximation. This proves that $\mathrm{cat}(i|_{A_j}) \le \mathrm{cat}(F_{n-1}) = k - 1$, and so $\mathrm{cat}(i) < (b+1)k$, which implies the desired inequality $\sigma_E((b+1)k) \ge n$. $\qquad\square$

Remark 4.3 These inequalities are not the best possible. A quick look at the proof of Theorem 4.2 shows that, in studying the category of E_n, for example, the estimate $\mathrm{cat}(p \circ i) \le b$ can be improved to $\mathrm{cat}(p \circ i) \le \mathrm{cat}(B_n)$, and similarly for the second formula. We leave the cumbersome formulation of the sharper results to the reader.

Since the reverse formulas expressing σ_E in terms of σ_B and σ_F are not entirely obvious, we record them here.

Corollary 4.4 *In the situation of Theorem 4.2,*

(a) $\sigma_E(k) \le \sigma_B\left(\left\lceil \frac{k-a}{a+1} \right\rceil\right)$, *and*

(b) $\sigma_E(k) \le \sigma_F\left(\left\lceil \frac{k-b}{b+1} \right\rceil\right)$.

In [8], Fadell and Husseini studied the Lusternik–Schnirelmann category of free loop spaces using a general result that relates the category of the fiber and the total space in a fibration sequence with a section. This result generalizes to a statement about categorical sequences.

Corollary 4.5 *Let*

$$
F \longrightarrow E \xrightarrow{\;p\;} B
$$

be a fibration sequence. If Ωp has a section s, then $\sigma_E \le \sigma_F$.

Proof Extend the given fibration sequence to the left to obtain

$$
\Omega E \underset{\Omega p}{\overset{s}{\rightleftarrows}} \Omega B \xrightarrow{\;\partial\;} F \longrightarrow E.
$$

Since Ωp has a section, the map $\partial : \Omega B \to F$ is trivial. Thus $\mathrm{cat}(\partial) = 0$, and Theorem 4.2(a) implies $\sigma_F(k) = \sigma_F((\mathrm{cat}(\partial)+1)k) \ge \sigma_E(k)$. $\qquad\square$

We can now expand upon the main homotopy-theoretical result of [8].

Example 4.6 Let $L(X) = \mathrm{map}(S^1, X)$ denote the free loop space on X. Evaluation at the basepoint determines a fibration $p\colon L(X) \to X$ with fiber ΩX, and the map $s\colon x \mapsto l_x$, where l_x is the constant map $l_x(S^1) = x$, is a section of p; thus Ωs is a section of Ωp. Therefore Corollary 4.5 shows that

$$\sigma_{L(X)} \leq \sigma_{\Omega X}.$$

In particular, $\mathrm{cat}(L(X)) = \infty$ if $\mathrm{cat}(\Omega X) = \infty$.

4.2 A mapping theorem for sequences

One of the most powerful early results concerning the Lusternik–Schnirelmann category of rational spaces is the Mapping Theorem (see Félix and Halperin [10]); the nice 'book proof' of this result (see Félix and Lemaire [13]) uses Proposition 4.1 in the special case $\mathrm{cat}(j) = 0$. We use exactly the same argument to get an inequality for categorical sequences.

Proposition 4.7 *Let* $f\colon X \to Y$ *be a map between rational spaces which induces an injective map* $f_*\colon \pi_*(X) \to \pi_*(Y)$. *Then* $\sigma_X \geq \sigma_Y$.

Proof Let $q\colon F \to X$ be the homotopy fiber of f. According to the proof of the standard Mapping Theorem, the injectivity hypothesis on f_* implies that $q \simeq *$ and so $\mathrm{cat}(q) = 0$ [7, Theorem 4.11]. It now follows from Theorem 4.2 that $\sigma_X(k) \geq \sigma_Y(k)$ for all k. □

5 Formal sequences

A simply-connected space X is *formal* if its cohomology algebra, with trivial differential, is a model for X [11, page 156]. In this section we characterize the categorical sequences of formal rational spaces in several ways.

First we show that formal rational spaces and their cohomology algebras are isosequential.

Proposition 5.1 *If* X *is a simply-connected formal rational space, then* $\sigma_X = \sigma_{H^*(X)}$.

Proof By assumption, $H^*(X)$ is a model for X. Propositions 3.7 and 3.2 show that $\sigma_{H^*(X)} \leq \sigma_X \leq \sigma_{H^*(X)}$, which proves the result. □

Our main result in this section completely characterizes the sequences which can occur as categorical sequences of simply-connected rational formal spaces.

Theorem 5.2 *The following conditions on a sequence σ with $\sigma(1) > 1$ are equivalent:*

(a) $\sigma = \sigma_{\mathcal{A}}$ *for some CGA $\mathcal{A}$,*

(b) $\sigma(k+1) \geq \frac{k+1}{k}\sigma(k)$ *for each k,*

(c) $\sigma = \sigma_W$ *where $W = \bigvee P_i$ and $P_i = \prod S^{n_j}$ is a product of spheres, and*

(d) $\sigma = \sigma_X$ *for some formal space X.*

Before proceeding to the proof of Theorem 5.2 we need to establish a technical result about sequences. Let $0 < k \leq n$ be integers, write $n = kx + r$ with $0 \leq r < k$ and let $r + s = k$. Define τ to be the sequence whose finite values are

$$\tau = (0, x, 2x, \ldots, sx, sx + (x+1), sx + 2(x+1), \ldots, \underbrace{sx + r(x+1)}_{n}).$$

We call τ the *optimal k–term sequence with $\tau(k) = n$.*

Lemma 5.3 *Assume that σ is a sequence satisfying condition (b) of Theorem 5.2, and that $\sigma(k) < \infty$. Let τ be the optimal k–term sequence with $\tau(k) = \sigma(k)$. Then $\sigma \leq \tau$.*

Proof This is clearly true for $j > k$, because $\tau(j) = \infty$ for such j. If $\sigma(j) > \tau(j)$ for some $j \leq k$, then $\sigma(j) \geq \tau(j) + 1$, and so

$$\sigma(j+1) \geq \tfrac{1}{j}\sigma(j) + \sigma(j) \geq \tfrac{1}{j}\sigma(j) + (\tau(j) + 1)$$

Now $\sigma(j) > \tau(j) \geq jx$, so $\tfrac{1}{j}\sigma(j) > x$. Therefore

$$\sigma(j+1) > \tau(j) + (x+1) \geq \tau(j+1).$$

Inductively, we see that $\sigma(l) > \tau(l)$ for all $i \leq l \leq k$, which contradicts the hypothesis $\sigma(k) = \tau(k)$. $\square$

Proof of Theorem 5.2 We begin by proving that (a) implies (b). Let $\mathcal{A}$ be a CGA such that $\sigma = \sigma_{\mathcal{A}}$. If $\sigma = (0, n)$ has length 1, then there is nothing to prove, so we proceed by induction, assuming that the implication is valid for sequences of length $\leq k$. Write $n = \sigma(k+1) = \sigma_{\mathcal{A}}(k+1)$. Then there is a nontrivial product $x_1 x_2 \cdots x_{k+1} \in \mathcal{A}^n$, where we write the terms in order so that $|x_1| \leq |x_2| \leq \cdots \leq |x_{k+1}|$. For $j \leq k+1$ we have

$$x_1 x_2 \cdots x_j \neq 0 \in \mathcal{A}^{|x_1| + |x_2| + \cdots + |x_j|},$$

so $\sigma_{\mathcal{A}}(j) \leq |x_1| + |x_2| + \cdots + |x_j|$ for each j. Since $\sigma_{\mathcal{A}}(k+1) = |x_1| + |x_2| + \cdots + |x_{k+1}|$ by construction, we have

$$
\begin{aligned}
\sigma_{\mathcal{A}}(k+1) - \sigma_{\mathcal{A}}(k) &\geq (|x_1| + \cdots + |x_{k+1}|) - (|x_1| + \cdots + |x_k|) \\
&= |x_{k+1}| \\
&= \tfrac{1}{k} \overbrace{(|x_{k+1}| + |x_{k+1}| + \cdots + |x_{k+1}|)}^{k \text{ terms}} \\
&\geq \tfrac{1}{k}(|x_1| + |x_2| + \cdots + |x_k|) \\
&\geq \tfrac{1}{k} \sigma_{\mathcal{A}}(k),
\end{aligned}
$$

which proves the result.

Next we prove that (b) implies (c) by induction on the length k of the sequence σ. If $\sigma = (0, n)$, then $\sigma = \sigma_{S^n}$ and the result holds. Suppose now that the result is known for all sequences with length $\leq k$, and let σ be a sequence with length $k + 1$. Write $\bar{\sigma}$ for the sequence

$$
\bar{\sigma}(j) = \begin{cases} \sigma(j) & \text{if } j \leq k \\ \infty & \text{if } j > k. \end{cases}
$$

Since $\text{length}(\bar{\sigma}) \leq k$, we can apply the inductive hypothesis, to find a wedge of products of spheres W such that $\sigma_W = \bar{\sigma}$. Let τ be the optimal $(k+1)$–term sequence with $\tau(k+1) = \sigma(k+1)$, and define

$$
P = \overbrace{S^x \times S^x \times \cdots \times S^x}^{s \text{ factors}} \times \overbrace{S^{x+1} \times S^{x+1} \times \cdots \times S^{x+1}}^{r \text{ factors}}.
$$

Then $\sigma_P = \tau$ by Corollary 3.3, and Proposition 3.1 shows that

$$
\sigma_{W \vee P}(j) = \min\{\sigma_W(j), \sigma_P(j)\}
$$

for all j. For $j \leq k$, we have $\sigma_W(j) = \sigma(j) \leq \tau(j) = \sigma_P(j)$ by Lemma 5.3, so $\sigma_{W \vee P} = \sigma(j)$ for $j < k$ by Proposition 3.1(a). Also $\sigma_P(k+1) = \sigma(k+1) < \infty = \sigma_W(k+1)$, so $\sigma_{W \vee P}(k+1) = \sigma(k+1)$.

The implication (c) $\Rightarrow$ (d) follows from the fact that the rationalization of a wedge of products of spheres is formal.

According to Proposition 5.1, if X is a formal rational space, then $\sigma_X = \sigma_{H^*(X)}$. Thus (d) implies (a). $\square$

In view of Theorem 5.2, we define a *formal sequence* to be any sequence σ which satisfies the condition

$$
\sigma(k+1) \geq \frac{k+1}{k} \sigma(k) \qquad \text{for all } k.
$$

It is *not* true that every formal space is isosequential with its minimal model. For example, the minimal model of S^4 is $\Lambda(x_4, x_7)$, so

$$\sigma_{S^4} = (0, 4) > (0, 4, 8, 12, \ldots) = \sigma_{\mathcal{M}(S^4)}.$$

Our study of formal sequences grew out of a simple question: is every simply-connected rational space isosequential with a product of spheres, or a wedge of products of spheres, or a product of wedges of products of spheres, etc?

Any space constructed from spheres by repeatedly taking products and wedges is automatically formal [7, Example 5.4]. Using Theorem 5.2, we see that any such space is isosequential with a simple wedge of products of spheres. Furthermore, Theorem 5.2 reveals that our original question reduces to asking whether or not σ_X is a formal sequence whenever X is a rational space. We have already seen that this is not the case!

Example 5.4 The space X of Example 2.6(b) is a rational space whose categorical sequence is $\sigma_X = (0, 3, 8, 11)$. Since $11 < \frac{3}{2} \cdot 8$, σ_X is not a formal sequence. By Theorem 5.2, X is not isosequential with any wedge of products of spheres.

6 Products

For two sequences σ and τ, we define a new sequence $\sigma * \tau$ by

$$\sigma * \tau(k) = \min\{\sigma(i) + \sigma(j) \mid i + j = k\}.$$

Our goal in this section is to prove a result linking the sequences $\sigma_{X \times Y}$ and $\sigma_X * \sigma_Y$. When the spaces in question are formal, this is not hard to do.

Proposition 6.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be simply-connected CGAs and let X and Y be simply-connected formal rational spaces. Then*

  (a) $\sigma_{\mathcal{A} \otimes \mathcal{B}} = \sigma_{\mathcal{A}} * \sigma_{\mathcal{B}}$, *and*

  (b) $\sigma_{X \times Y} = \sigma_X * \sigma_Y$.

Proof We omit the easy proof of (a), and use it to prove (b) as follows: since X, Y and $X \times Y$ are each formal and rational,

$$\sigma_{X \times Y} = \sigma_{H^*(X \times Y)} = \sigma_{H^*(X) \otimes H^*(Y)} = \sigma_{H^*(X)} * \sigma_{H^*(Y)} = \sigma_X * \sigma_Y$$

by Proposition 5.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following conjecture seems quite plausible.

Conjecture A   *For simply-connected rational X and Y, $\sigma_{X \times Y} = \sigma_X * \sigma_Y$.*

Unfortunately, we have been unable to prove this. However, we can prove that there is an inequality relating these sequences.

Theorem 6.2   *For simply-connected rational X and Y, $\sigma_{X \times Y} \leq \sigma_X * \sigma_Y$.*

Proof   Let $\sigma_X * \sigma_Y(k) = n$. Thus there are i and j with $i + j = k$, $\sigma_X(i) = a$, $\sigma_Y(j) = b$, and $a + b = n$. Now let $i_a \times i_b \colon X_a \times Y_b \to X \times Y$ and compute

$$
\begin{aligned}
\mathrm{cat}((X \times Y)_n) &\geq \mathrm{cat}_{X \times Y}(X_a \times Y_b) \\
&\geq \mathrm{Mcat}(i_a \times i_b) && \text{by Proposition 2.2(c)} \\
&= \mathrm{Mcat}(i_a) + \mathrm{Mcat}(i_b) && \text{by Parent [26, Theorem 2]} \\
&= \mathrm{cat}(X_a) + \mathrm{cat}(X_b) && \text{by Proposition 2.2(c)} \\
&= k,
\end{aligned}
$$

which means that $\sigma_{X \times Y}(k) \leq n = \sigma_X * \sigma_Y(k)$, $\square$

The inequality of Theorem 6.2 fails when the spaces are not rational, as the following example demonstrates.

Example 6.3   Iwase [22] has constructed a space $X = S^2 \cup D^{10}$ with the property that $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) = 2$ for all $k \geq 2$. The categorical sequences for X and S^2 are $\sigma_X = (0, 2, 10, \infty, \ldots)$ and $\sigma_{S^2} = (0, 2, \infty, \ldots)$, respectively. Now we have $\sigma_X * \sigma_{S^2} = (0, 2, 4, 12, \infty, \ldots) < (0, 2, 4, \infty, \ldots) = \sigma_{X \times S^2}$.

Nevertheless, the following conjecture seems reasonable.

Conjecture B   *For general spaces X and Y, $\sigma_{X \times Y} \geq \sigma_X * \sigma_Y$.*

Conjecture B, together with Theorem 6.2, implies Conjecture A.

7   The Mislin genus of $Sp(3)$

In this final section we use categorical sequences to give a simple proof of a theorem of Ghienne [18].

The *Mislin genus* of a nilpotent space X is the set $\mathcal{G}(X)$ of homotopy types of nilpotent spaces Y such that the p–localizations $X_{(p)}$ and $Y_{(p)}$ are homotopy equivalent for

every prime p. McGibbon [24, Section 8] asked whether Lusternik–Schnirelmann category is an invariant of Mislin genus; that is, if $X \in \mathcal{G}(Y)$, does it follow that $\operatorname{cat}(X) = \operatorname{cat}(Y)$? This is known to be false for certain infinite-dimensional spaces (see Roitberg [28]), but the question remains open for finite complexes Y.

In [18], Ghienne proved that McGibbon's conjecture holds in the special case $Y = Sp(3)$. We use a sequence computation to give a simple alternative proof of this result.

Theorem 7.1 (Ghienne) *If $X \in \mathcal{G}(Sp(3))$, then* $\operatorname{cat}(X) = 5$.

Proof According to Fernández-Suárez, Gómez-Tato, Strom and Tanré [14], and Iwase and Mimura [23], $\operatorname{wcat}(Sp(3)) = \operatorname{cat}(Sp(3)) = 5$. Since weak category is a genus invariant, we have

$$\operatorname{cat}(X) \geq \operatorname{wcat}(X) = \operatorname{wcat}(Sp(3)) = 5$$

for any space $X \in \mathcal{G}(Sp(3))$. It remains to show that $\operatorname{cat}(X) \leq 5$ for every $X \in \mathcal{G}(Sp(3))$. In fact, we prove the following stronger statement: any simply-connected space X whose cohomology ring $H^*(X; \mathbb{Z})$ is isomorphic to $H^*(Sp(3); \mathbb{Z})$ must have $\operatorname{cat}(X) \leq 5$.

The categorical sequence σ_X clearly has $\sigma_X(1) = 3$ and $\sigma_X(2) \geq 7$ by Theorem 3.4(b). By Theorem 3.4(a), $\sigma_X(4) \geq \sigma_X(2) + \sigma_X(2) \geq 14$. Furthermore, $\sigma_X(4) > 14$ by Theorem 3.4(c), because the cup product $H^7(X) \otimes H^7(X) \to H^{14}(X)$ is trivial. Now we have $\sigma_X(4) \geq 18$ by Theorem 3.4(b), and hence $\sigma_X(5) \geq \sigma_X(4) + \sigma_X(1) = 21$. From this we immediately conclude that $\operatorname{cat}(X) = \operatorname{cat}(X_{21}) \leq 5$. $\qquad\square$

McGibbon's conjecture for finite complexes is equivalent to the following conjecture for finite type spaces.

Conjecture C *If X is a nilpotent space of finite type, then $\sigma_Y = \sigma_X$ for every $Y \in \mathcal{G}(X)$.*

Conjecture C is easily seen to be valid for $X = Sp(2)$. We can also verify the conjecture for $X = Sp(3)$.

Corollary 7.2 *If $X \in \mathcal{G}(Sp(3))$, then* $\sigma_X = \sigma_{Sp(3)} = (0, 3, 7, 10, 18, 21)$.

Proof The proof of Theorem 7.1 shows that if $H^*(X) \cong H^*(Sp(3))$ then $\sigma_X \geq (0, 3, 7, 10, 18, 21)$. If $\operatorname{cat}(X) = 5$, then $\sigma_X(5) \leq 21$, and this implies $\sigma_X(2) = 7$. Now [34, Theorem 8] implies that $\operatorname{cat}(X_{10}) = 3$, and hence $\sigma_X(3) = 10$. The analysis used in the proof of Theorem 7.1 shows that $\sigma_X = (0, 3, 7, 10, 18, 21)$. $\qquad\square$

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