Labeled binary planar trees and quasi-Lie algebras

JEROME LEVINE

We study the natural map η between a group of binary planar trees whose leaves are labeled by elements of a free abelian group H and a certain group $\mathsf{D}(H)$ derived from the free Lie algebra over H. Both of these groups arise in several different topological contexts. η is known to be an isomorphism over $\mathbb{Q}$, but not over $\mathbb{Z}$. We determine its cokernel and attack the conjecture that it is injective.

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1 Introduction

Let H be a finitely-generated free abelian group and $\mathsf{L}(H)$ the graded free Lie algebra on H. There is a natural homomorphism $H\otimes\mathsf{L}(H)\to\mathsf{L}(H)$ defined by bracketing, whose kernel is denoted $\mathsf{D}(H)$. If H supports a non-singular symplectic form, eg $H=H_1(\Sigma)$, where Σ is a closed orientable surface, with symplectic basis $\{x_i, y_i\}$, then $\mathsf{D}(H)$ is, in fact, a Lie algebra. It can be identified with the Lie subalgebra of $\mathcal{D}(\mathsf{L}(H))$ (the graded Lie algebra of derivations of $\mathsf{L}(H)$) consisting of those derivations which vanish on the element $\sum_i [x_i, y_i]\in\mathsf{L}_2(H)$.

$\mathsf{D}(H)$ has arisen in several different topological contexts. For example, it was probably first observed by Orr [14] (but see also Habegger and Lin [3]) that it is very natural to regard the Milnor invariants of a link L, or, more precisely, string link, as elements of $\mathsf{D}(H)$, where $H=H_1(S^3-L)$. If Σ is a compact orientable surface with one boundary component and $H=H_1(\Sigma)$ then $\mathsf{D}(H)$ contains, as a Lie subalgebra, the associated graded Lie algebra of the *relative weight* filtration, defined by D. Johnson, of the mapping class group of Σ – see Johnson [8] and Morita [13]. Similarly, if we consider the homology concordance group of homology cylinders over a surface – see Garoufalidis–Levine [2] and Levine [10] – there is also a relative weight filtration and, in this case, the associated graded Lie algebra is actually isomorphic to $\mathsf{D}(H)$.

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$\mathsf{D}(H)$ appears in Kontsevich's work [9] on graph complexes and his computation of the cohomology of the group of outer automorphisms of a free group.

Consider now the abelian group $\mathcal{A}^t(H)$ generated by unitrivalent trees, with cyclic orientations of its trivalent vertices and univalent vertices labelled by elements of H, subject to the anti-symmetry and IHX relations and linearity of the labels. $\mathcal{A}^t(H)$ appears as the indexing of the so-called *tree-level* of the Kontsevich integral of a link or string link. See Habegger and Masbaum [4] where this is related to the Milnor invariants via a natural map $\eta\colon \mathcal{A}^t(H) \to \mathsf{D}(H)$. It is proved there that, rationally, the Milnor invariants of a string link determine the tree-level of its Kontsevich integral. This corresponds to the fact that the map $\eta \otimes \mathbb{Q}\colon \mathcal{A}^t(H) \otimes \mathbb{Q} \to \mathsf{D}(H) \otimes \mathbb{Q}$ is an isomorphism, which is proved by Habegger and Pitsch [5] (see also Garoufalidis–Levine [2] and Levine [11]). $\mathcal{A}^t(H)$ appears in Habiro [6] and the study by Garoufalidis, Goussarov and Polyak [1] of claspers and finite-type invariants of 3-manifolds, and subsequently in Levine [10], mapping onto the associated graded groups of a filtration of the concordance group of homology cylinders defined using *claspers*. In this context the map $\eta\colon \mathcal{A}^t(H) \to \mathsf{D}(H)$ reflects the relation between the clasper filtration and the usual relative weight filtration. Most recently the group $\mathcal{A}^t(H)$ appears in the work of Schneiderman–Teichner [15], where it encodes the obstruction to removing intersection and self-intersection points of immersed connected surfaces in a simply-connected 4-manifold (here H is a free abelian group of rank equal to the number of surfaces) via a tower of Whitney disks. In the special case where the surfaces are disks in the 4-ball bounded by a link in S^3, the obstruction element in $\mathcal{A}^t(H)$ maps to the element in $\mathsf{D}(H)$ corresponding to the Milnor invariants of the link.

In these various situations the study of the homomorphism $\mathcal{A}^t(H) \to \mathsf{D}(H)$ is closely related to the question of whether there are invariants in $\mathcal{A}^t(H)$ which give more information than the analogous, perhaps more easily defined, invariants in $\mathsf{D}(H)$. For example the work of Schneiderman–Teichner may uncover new invariants of link concordance beyond the Milnor invariants.

Since $\eta \otimes \mathbb{Q}$ is an isomorphism, the kernel and cokernel of η are finite. In [11] some progress was made toward determining them. Toward this end we introduced the notion of a *quasi-Lie algebra* and studied the structure of a free quasi-Lie algebra. In the present work we extend these results. In particular we determine the precise structure of a free quasi-Lie algebra (adapting an argument of Marshall Hall [7]), determine the cokernel of η precisely and show that a "good part" of the kernel of η is trivial. It remains a reasonable conjecture that η is injective.

It has recently come to our attention that some of the results of this note (and of [11]) have been independently obtained by K Habiro – in particular Theorem 2.1, Theorem

4.1 in Section 4 (and Corollary 2.6) by a very similar method, as well as Theorem 2.7 by a different method.

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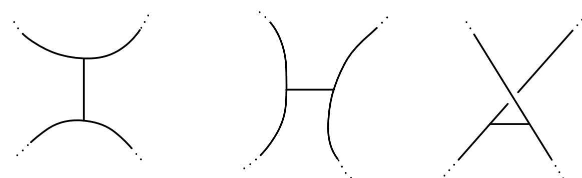
2 Statement of results

We will use the precise definition of the groups $\{\mathsf{D}_n(H)\}$ and $\{\mathcal{A}^t_n(H)\}$ given in [10; 11]. In particular $\mathsf{D}_n(H)$ is the kernel of the bracket map $H \otimes \mathsf{L}_{n+1}(H) \to \mathsf{L}_{n+2}(H)$ and $\mathcal{A}^t_n(H)$ is a quotient of the free abelian group generated by unitrivalent trees with n trivalent vertices, each of which is given an orientation – ie a cyclic ordering of its incident edges – and whose univalent vertices are labeled by elements of H. The "relations" which are divided out are:

- anti-symmetry: $T + T' = 0$, where T' is identical with T except that one trivalent vertex is given the opposite orientation.
- IHX: $T_1 - T_2 + T_3 = 0$, where the T_i are identical except in the neighborhood of two adjacent trivalent vertices, which look as follows:



T_1 T_2 T_3

The orientations of the trivalent vertices are counterclockwise.

- linearity: $T = T_1 + T_2$ where T_1, T_2, T are identical except that one of the univalent vertices has labels $a_1, a_2, a_1 + a_2$, respectively, for some $a_i \in H$.

The graphical representation of free Lie algebras over $\mathbb{Q}$, for example, is well-known. Extending this to Lie algebras over $\mathbb{Z}$ requires some extra considerations. In [11] we introduce the notion of a quasi-Lie algebra, in which the relation $[\alpha, \alpha] = 0$ is replaced by the slightly weaker relation $[\alpha, \beta] + [\beta, \alpha] = 0$. Then the free quasi-Lie algebra $\mathsf{L}'(H)$ over a free abelian group H is isomorphic to a Lie algebra of trees similar to the definition of $\mathcal{A}^t(H)$ except that one univalent vertex (the *root*) is not labeled.

There is an obvious epimorphism $\gamma_n\colon \mathsf{L}'_n(H) \to \mathsf{L}_n(H)$. We can also define a bracketing homomorphism $\beta'_n\colon H \otimes \mathsf{L}'_{n+1}(H) \to \mathsf{L}'_{n+2}(H)$ by $\beta'_n(h \otimes \lambda) = [h, \lambda]$ and then define $\mathsf{D}_n(H) = \operatorname{Ker} \beta'_n$. A map $\eta'_n\colon \mathcal{A}^t_n(H) \to H \otimes \mathsf{L}'_{n+1}(H)$ is defined by

$$\eta'_n(T) = \sum_i h_i \otimes [T_i] \tag{1}$$

where the summation is over all univalent vertices of T. For each univalent vertex h_i is its label and T_i is the rooted tree obtained from T by making that vertex the root – $[T_i]$ is the corresponding element of $\mathsf{L}'_{n+1}(H)$.

In [11] the following theorem is proved.

Theorem 2.1 *The sequence*

$$\mathcal{A}^t_n(H) \xrightarrow{\eta'_n} H \otimes \mathsf{L}'_{n+1}(H) \xrightarrow{\beta'_n} \mathsf{L}'_{n+2}(H) \to 0$$

is exact.

Therefore η'_n defines an epimorphism $\mathcal{A}^t_n(H) \to \mathsf{D}'_n(H)$.

In order to completely understand the original map $\mathcal{A}^t_n(H) \to \mathsf{D}_n(H)$ we need to resolve the following two problems:

(1) Determine the map $\gamma_n\colon \mathsf{D}'_n(H) \to \mathsf{D}_n(H)$, ie determine its kernel and cokernel.

(2) Determine the kernel of η'_n.

Problem (1) essentially reduces to determining $\mathsf{L}'_n(H)$, since $\mathsf{L}_n(H)$ is well-understood.

In [11] it is shown that γ_n is an isomorphism if n is odd and, for n even there is an exact sequence

$$\mathsf{L}_k(H) \otimes \mathbb{Z}/2 \xrightarrow{\theta_k} \mathsf{L}'_{2k}(H) \xrightarrow{\gamma_{2k}} \mathsf{L}_{2k}(H) \to 0$$

where θ_k is defined by $\theta_k(\alpha) = [\alpha, \alpha]$ (where $\alpha \in \mathsf{L}_k(H)$ is lifted into $\mathsf{L}'_k(H)$). It was conjectured in [11] that θ_k is injective. Our first result proves this conjecture.

Theorem 2.2 *The sequence*

$$0 \to \mathsf{L}_k(H) \otimes \mathbb{Z}/2 \xrightarrow{\theta_k} \mathsf{L}'_{2k}(H) \xrightarrow{\gamma_{2k}} \mathsf{L}_{2k}(H) \to 0$$

is exact (and therefore split exact).

As a consequence of this theorem, we have the following relations between $\mathsf{D}'_n(H)$ and $\mathsf{D}_n(H)$.

Corollary 2.3 *There exist exact sequences:*

$$0 \to \mathsf{D}'_{2k}(H) \to \mathsf{D}_{2k}(H) \to \mathsf{L}_{k+1}(H) \otimes \mathbb{Z}/2 \to 0$$

$$0 \to H \otimes \mathsf{L}_k(H) \otimes \mathbb{Z}/2 \to \mathsf{D}'_{2k-1}(H) \to \mathsf{D}_{2k-1}(H) \to 0$$

These exact sequences are derived in [11].

Remark 2.4 We can describe the elements of $\mathsf{D}_{2k}(H)$ which do not come from $\mathsf{D}'_{2k}(H)$ in the following graphical manner. Let $\alpha \in \mathsf{L}_{k+1}(H)$ be represented by a labeled rooted tree T of degree k. Let T' be another copy of T and let $T \odot T'$ be the labeled tree (representing an element of $\mathcal{A}^t_{2k}(H)$) obtained by welding the roots of T and T' together. For each labeled univalent vertex v_i of T let $T_i \odot T'$ be the rooted labeled tree obtained from $T \odot T'$ by making v_i the root. If h_i is the label of v_i in T then consider the element $\sum_i h_i \otimes (T_i \odot T') \in H \otimes \mathsf{L}'_{2k+1}(H)$. This does not lie in $\mathsf{D}'_{2k}(H)$ but its projection into $H \otimes \mathsf{L}_{2k+1}(H)$ does lie in $\mathsf{D}_{2k}(H)$ and maps to $\alpha \otimes 1 \in \mathsf{L}_{k+1}(H) \otimes \mathbb{Z}/2$.

We now turn to Problem (2). In [11] the following is proved.

Theorem 2.5 *η'_n is a split surjection.* $\operatorname{Ker} \eta'_n$ *is the torsion subgroup of $\mathcal{A}^t_n(H)$ if n is even, and is the odd torsion subgroup of $\mathcal{A}^t_n(H)$ if n is odd.*

In both cases $(n+2)\operatorname{Ker} \eta'_n = 0$.

One immediate consequence is the known result that $\mathcal{A}^t_n(H) \otimes \mathbb{Q} \cong \mathsf{D}_n(H) \otimes \mathbb{Q}$.

We will improve on this result by constructing a splitting of $\mathcal{A}^t(H)$ and $\mathsf{D}'(H)$ such that η' preserves components, and give a better estimate on the order of the kernel of each factor. In particular we will show:

Corollary 2.6 *If $n+2$ is a prime power p^k, then $p^{k-1}\operatorname{Ker} \eta'_n = 0$. For example, if $n+2$ is prime, then η'_n is an isomorphism.*

Finally, by a direct computation of ranks we will show:

Theorem 2.7 *η'_2 is an isomorphism.*

Since it is obvious that η'_1 is an isomorphism, the first unsettled case is $n = 4$.

3 Structure of the quasi-Lie algebra: Proof of Theorem 2.2

Choose a basis $\{a_1, \dots, a_m\}$ of H. Let $\mathcal{C}_n$ denote the set of formal commutators of degree n in the a_i and $\mathcal{C} = \cup_n \mathcal{C}_n$. Recall the definition of a *Hall basis* (see, for example, Hall [7]). Choose a linear ordering of the elements of $\mathcal{C}$ satisfying only that if $d(x) > d(y)$ (where d denotes degree), then $x > y$. Let $\mathcal{H}$ be the subset of $\mathcal{C}$ defined recursively by the following properties:

(1) Each $a_i \in \mathcal{H}$

(2) If $u, v \in \mathcal{C}$, then $[u, v] \in \mathcal{H}$ if and only if:

(a) $u, v \in \mathcal{H}$

(b) $u > v$

(c) If $u = [x, y]$ (and so $x, y \in \mathcal{H}$ and $x > y$), then $v \geq y$.

Note that $\mathcal{H}$ depends on the choice of ordering.

It is a well-known result (see, for example, [7]) that any Hall basis is a basis of the free Lie algebra $\mathsf{L}(H)$).

Let $\widetilde{\mathcal{H}}$ denote the subset of $\mathcal{C}$ consisting of all elements of the form $[h, h]$ for some $h \in \mathcal{H}$. It is clear that Theorem 2.2 will follow from:

Lemma 3.1 $\mathsf{L}'(H) \otimes \mathbb{Z}/2$ *has, as basis,* $\mathcal{H}' = \mathcal{H} \cup \widetilde{\mathcal{H}}$.

Proof We will follow closely the proof in [7], making a few necessary modifications to apply to our situation.

Let V_n be the $\mathbb{Z}/2$–vector space with basis $\mathcal{C}_n$, $V = \oplus_n V_n$, and W_n the $\mathbb{Z}/2$–vector space with basis $\mathcal{H}'_n$ and $W = \oplus_n W_n$. There are obvious maps:

$$W_n \subseteq V_n \to \mathsf{L}'_n(H)$$

We will define a retraction $r\colon V_n \to W_n$ recursively on n, satisfying

(1) If $h \in \mathcal{H}'$ then $r(h) = h$.

(2) For any $c \in \mathcal{C}$, $r(c) = c$ in $\mathsf{L}'(H)$.

(3) For any $c_1, c_2 \in \mathcal{C}$, $r[c_1, c_2] = r[r(c_1), r(c_2)]$.

For $n = 1$ we define $r(a_i) = a_i$.

Now suppose r is defined on V_k for all $k < n$ satisfying (1)–(3). We will define a sequence of additive moves $V_n \to V_n$ which will define r when it stops.

Step 1 If $c = [c_1, c_2]$, then $c \to [r(c_1), r(c_2)]$.

Now apply Step 2 to each term of the sum.

Step 2 If $c = [h_1, h_2]$, where $h_1, h_2 \in \mathcal{H}'$, then

$$c \to \begin{cases} 0 & \text{if } h_1 \text{ or } h_2 \text{ belongs to } \widetilde{\mathcal{H}} \text{ (case 1)} \\ c & \text{if } h_1, h_2 \in \mathcal{H} \text{ and } h_1 \geq h_2 \text{ (case 2)} \\ [h_2, h_1] & \text{if } h_1, h_2 \in \mathcal{H} \text{ and } h_1 < h_2 \text{ (case 3)} \end{cases}$$

In case 1 stop. In case 2 or 3 go on to Step 3.

Step 3 If $c = [h_1, h_2]$, with $h_i \in \mathcal{H}$ and $h_1 \geq h_2$, write $h_1 = [h_3, h_4]$ (note $h_3 > h_4$). Then

$$c \to \begin{cases} c & \text{if } h_2 \geq h_4 \text{ (case 1)} \\ [[h_3, h_2], h_4] + [[h_2, h_4], h_3] & \text{if } h_2 < h_4 \text{ (case 2)} \end{cases}$$

In case 1 stop. In case 2, apply Step 1 to each of the terms in the sum.

It is clear that if $c \in \mathcal{H}'$, then the process will stop at Step 2 or 3 at c. In general we need to show that this process will stop after a finite number of steps. It is clear then that properties (1)–(3) will be satisfied.

Define a new relation among the elements of $\mathcal{C}_n$, for $n \geq 2$. Let $c = [c_1, c_2]$ and $c' = [c_1', c_2']$. We will say

$$c \succ c' \text{ if } \min(c_1, c_2) > \min(c_1', c_2')$$

Now if b is one of the terms in $[r(c_1), r(c_2)]$ obtained after Step 1, then applying Steps 2 and 3 to b will either stop, resulting in an element of $\mathcal{H}'$ or lead us to case 2 of Step 3. Take $[c_1', c_2']$ to be either of the resulting terms in case 2 of Step 3. When we then apply Step 1 we have $[r(c_1'), r(c_2')] \succ b$, since $r(c_2') = c_2' > h_2$ and $d(r(c_1)) = d(c_1) > d(h_2)$. Thus iterating the process results in a sum of terms each of which stabilizes or leads to a sum of terms which are greater under the relation $\succ$. Since there are only a finite number of elements in $\mathcal{C}_n$ the process must eventually stop. In fact it must stop whenever the element $[h_1, h_2]$ to which we apply Step 3 satisfies $d(h_2) > \frac{n}{3}$ since this will force $d(h_4) < \frac{n}{3}$ from which it follows that $h_2 > h_4$.

We now have defined a retraction $r\colon V_n \to W_n$ satisfying properties (1)–(3). To complete the proof we need to show that r induces a map $L_n'(H) \to W_n$. Notice that we can regard V as the *free* $\mathbb{Z}/2$*–magma over* H and that $L'(H) \otimes \mathbb{Z}/2$ is the quotient of V by the ideal I generated by elements of the form

$$\xi = [x, y] + [y, x]] \quad \text{or} \quad [[x, y], z] + [[y, z], x] + [[z, x], y]$$

where $x, y, z \in \mathcal{C}$. So we need to show that $r(I) = 0$.

Now I is generated additively by formal brackets of elements of V, one of which is of the form ξ above. By property (3) of r it is only necessary then to show that $r(\xi) = 0$. In fact, again by property (3), we may assume that $x, y, z \in \mathcal{H}'$.

$\xi = [x, y] + [y, x]$: We may assume $x \geq y$. In the definition of r, we see that Step 2 will change ξ either to 0, if x or y belongs to $\widetilde{\mathcal{H}}$, or to $2[x, y] = 0$ otherwise.

$\xi = [[x, y], z] + [[y, z], x] + [[z, x], y]$: We may assume $x \geq y \geq z$ and proceed by induction on $d([[x, y], z])$.

Case 1 $x = y$: So $\xi = [[x, x], z] + [[x, z] + [z, x], x]$. Now $[x, x]$ either belongs to $\widetilde{\mathcal{H}}$ (if $x \in \mathcal{H}$) or $r([x, x]) = 0$ if $x \in \widetilde{\mathcal{H}}$. In either case $r([[x, x], z]) = 0$.

For the remaining terms note that $[x, z] + [z, x] \to 2[x, z] = 0$.

Case 2 $x \in \widetilde{\mathcal{H}}$: Then $r([x, y]) = r([[y, z], x]) = r([z, x]) = 0$ by Step 2.

Case 3 $[x, y] \in \mathcal{H}$: In evaluating $r([[x, y], z])$ we proceed to Step 3 and apply case 2:

$$[[x, y], z] \to [[x, z], y] + [[z, y]x]$$

At this point ξ has been reduced to 0.

Case 4 $[x, y] \notin \mathcal{H}'$ and $x \in \mathcal{H}$: We proceed by a downward lexicographical induction. Assume that $r(\xi) = 0$ when

$$\xi = [[x', y'], z'] + [[y', z'], x'] + [[z', x'], y']$$

and $x' \geq y' \geq z'$ and either $z' > z$ or $z' = z$ and $y' > y$.

Write $x = [x_1, x_2]$. Since $x \in \mathcal{H}$, then $x_1, x_2 \in \mathcal{H}$ and $x_1 > x_2$. Since $[x, y] \notin \mathcal{H}$, then $x_2 > y$.

We therefore have:

$$\begin{aligned} r([[x, y], z]) &= r([[[x_1, x_2], y], z]) \\ &= r([([x_1, y], x_2] + [[x_2, y], x_1]), z]) \end{aligned}$$

by Step 3, case 2, since $x_2 > y$, and Step 2, case 3, for the first term, since $x_1 > y$.

We can apply our downward induction to both terms, since $[x_1, y] > x_2 > y$ and both $[x_2, y]$ and x_1 are $> y$ (using Step 2 case 3, if $x_1 > [x_2, y]$) to obtain:

$$\begin{aligned} (2) \quad r([[x, y], z]) = r([[x_2, z], [x_1, y]] &+ [[[x_1, y], z], x_2] \\ &+ [[x_1, z], [x_2, y]] + [[[x_2, y], z], x_1]) \end{aligned}$$

By our ongoing induction on $\deg([[x, y], z])$ we have

$$\begin{aligned} r([[x_1, y], z]) &= r([[x_1, z], y] + [[y, z], x_1]) \\ r([[x_2, y], z]) &= r([[x_2, z], y] + [[y, z], x_2]) \end{aligned}$$

Substituting these equalities into equation (2) gives

$$\begin{aligned}(3)\quad r([[x,y],z]) = r(&[[x_2,z],[x_1,y]] + [[[x_1,z],y],x_2] + [[[y,z],x_1],x_2]\\ &+ [[x_1,z],[x_2,y]] + [[[x_2,z],y],x_1] + [[[y,z],x_2],x_1])\end{aligned}$$

Now we write

$$\begin{aligned} r([[z,x],y]) &= r([[z,[x_1,x_2]],y]) = r([[[x_1,x_2],z],y]) \\ &= r([[[x_1,z],x_2],y] + [[[x_2,z],x_1],y]) \end{aligned}$$

using the Jacobi identity on elements of degree $< \deg([[x,y],z])$.

We can now use our downward induction on each of the two terms on the right to get:

$$\begin{aligned}(4)\quad r([[z,x],y]) = r(&[[x_1,z],[x_2,y]] + [[[x_1,z],y],x_2] + [[x_2,z],[x_1,y]]\\ &+ [[[x_2,z],y],x_1])\end{aligned}$$

We can now add equations (3) and (4), cancelling out many of the terms, to get

$$\begin{aligned}(5)\quad r([[x,y],z] + [[y,z],x] + [[z,x],y]) = r(&[[[y,z],x_1],x_2]\\ &+ [[[y,z],x_2],x_1] + [[y,z],[x_1,x_2]])\end{aligned}$$

If $[y,z] > x_2$, then our downward induction, applied to $[[[y,z],x_1],x_2]$, will tell us that $r([[[y,z],x_1],x_2] + [[[y,z],x_2],x_1] + [[y,z],[x_1,x_2]]) = 0$. (In case $x_1 < [y,z]$, we use Step 2.) If $[y,z] < x_2$ then we apply downward induction on $[[x_1,x_2],[y,z]]$.

Finally note that, since $x_1 > x_2 > y > z$, we conclude that

$$\deg z \le \tfrac{1}{4}\deg([[x,y],z]) = \tfrac{n}{4},$$

which ensures the beginning of the induction.

This completes the proof of Lemma 3.1 and Theorem 2.2. □

4 Study of η'

4.1 A splitting of $\mathcal{A}^t_n(H)$

We now consider the maps $\eta'_n\colon \mathcal{A}^t_n(H) \to \mathsf{D}'_n(H)$. In [11] it is proved that

$$(n+2)\operatorname{Ker}\eta'_n = 0.$$

We will construct a splitting of the various groups

$$\mathcal{A}^t(H), \mathsf{L}'(H), H \otimes \mathsf{L}'(H)$$

so that η' will respect the summands of these splittings and then give better estimates of the order of $\operatorname{Ker}\eta'$ on each summand. The splitting will depend on the choice of a basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_d\}$ of H. We adopt a slightly different, but equivalent, view of $\mathcal{A}^t(H)$ as generated by vertex-oriented unitrivalent trees with univalent vertices labeled by elements of $\mathcal{B}$, subject to the anti-symmetry and IHX relations (but now the linearity relation is not needed). Similarly $\mathsf{L}'(H)$ is generated by formal brackets in the elements of $\mathcal{B}$, subject to anti-symmetry and Jacobi relations. Thus $\mathsf{L}'(H)$ is graphically described as generated by vertex-oriented unitrivalent trees with univalent vertices labeled by elements of $\mathcal{B}$ and one unlabeled univalent vertex chosen as a "root", subject to anti-symmetry and IHX.

Let $\omega = (n_1, \dots, n_d)$ be a sequence of non-negative integers. We will say that a labeled vertex-oriented unitrivalent tree has *signature* ω if exactly n_i of the vertices are labeled by α_i. A formal bracket has signature ω if exactly n_i of the entries in the bracket are α_i. Notice that each anti-symmetry, IHX or Jacobi relation is defined by a sum of trees or brackets which all have the same signature.

We now define $\mathcal{A}^t_\omega(H)$ to be the abelian group generated by labeled vertex-oriented unitrivalent trees of signature ω, subject to the anti-symmetry and IHX relations, and $\mathsf{L}'_\omega(H)$ to be the group generated by brackets of signature ω, subject to anti-symmetry and Jacobi relations. It is clear that

$$\mathcal{A}^t_n(H) = \oplus_\omega \mathcal{A}^t_\omega(H) \qquad \mathsf{L}'_n(H) = \oplus_\omega \mathsf{L}'_\omega(H)$$

where the sums range over all ω with $\sum_i n_i = n+2$, for $\mathcal{A}^t_n(H)$ and with $\sum_i n_i = n$ for $\mathsf{L}'_n(H)$. Note that the Lie bracket defines a pairing

$$\mathsf{L}'_\omega(H) \otimes \mathsf{L}'_{\omega'}(H) \to \mathsf{L}'_{\omega+\omega'}(H)$$

where, if $\omega = (n_1, \dots, n_d)$ and $\omega' = (n'_1, \dots, n'_d)$, then $\omega + \omega' = (n_1 + n'_1, \dots, n_d + n'_d)$.

The map $\eta'_n\colon \mathcal{A}^t_n(H) \to H \otimes \mathsf{L}'_{n+1}(H)$ is defined in equation (1). If the tree T in that formula has signature $\omega = (n_1, \dots, n_d)$, then each term on the right side will be of the form $\alpha_j \otimes [T_i]$, for some j, and where $[T_i] \in \mathsf{L}'_{n+1}(H)$ has signature $\omega_j = (n_1, \dots, n_j - 1, \dots, n_d)$. Therefore we can write

$$\eta'_n(T) = \sum_j \alpha_j \otimes \lambda_j \qquad \text{where } \lambda_j \in \mathsf{L}'_{\omega_j}(H)$$

Now define $(H \otimes \mathsf{L}'(H))_\omega = \sum_j \alpha_j \otimes \mathsf{L}'_{\omega_j}(H)$. It is clear that

$$H \otimes \mathsf{L}'_{n+1}(H) = \oplus_\omega (H \otimes \mathsf{L}'(H))_\omega,$$

where ω ranges over all ω with $\sum_i n_i = n+2$ and $\eta'_n(\mathcal{A}^t_\omega(H)) \subseteq (H \otimes \mathsf{L}'(H))_\omega$. If $\omega = (n_1, \ldots, n_d)$ we define $\delta(\omega) =$greatest common divisor of $n_1, \ldots, n_d$.

Theorem 4.1

$$\delta(\omega)\left(\operatorname{Ker} \eta'_n \,\middle|\, \mathcal{A}^t_\omega(H)\right) = 0$$

Proof In the proof of Theorem 2.1 in [11] we use a map $\rho_n\colon H \otimes \mathsf{L}'_{n+1}(H) \to \mathcal{A}^t_n(H)$ which sends any labeled tree with a root to the same tree, forgetting which vertex is the root. The observation that $\rho_n \circ \eta'_n =$multiplication by $n+2$ shows that $(n+2)\operatorname{Ker}\eta'_n = 0$.

Now it is clear that $\rho_n(H \otimes \mathsf{L}'(H))_\omega \subseteq \mathcal{A}^t_\omega(H)$. But the restriction of ρ_n to $(H \otimes \mathsf{L}'(H))_\omega$ can be decomposed into a sum of maps $\rho_n^{\omega,i}\colon (H \otimes \mathsf{L}'(H))_\omega \to \mathcal{A}^t_\omega(H)$ defined by

$$\rho_n^{\omega,i}(\alpha_j \otimes \lambda) = \begin{cases} \rho_n(\alpha_j \otimes \lambda) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It is clear that $\rho_n^{\omega,i} \circ \eta'_n \,|\mathcal{A}^t_\omega(H)$ is just multiplication by n_i, and so

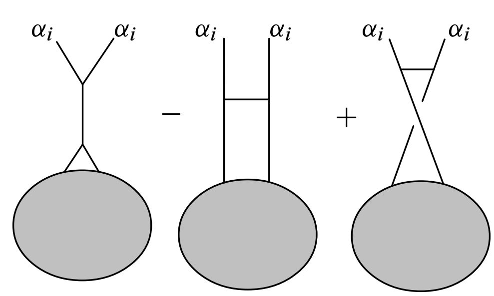
$$n_i \operatorname{Ker} \eta'_n \,|\mathcal{A}^t_\omega(H) = 0$$

for $i = 1, \ldots, d$.

This completes the proof. $\square$

4.2 Proof of Corollary 2.6

We only need show that, for any $\omega = (n_1, \ldots, n_d)$ with $\sum_i n_i = p^k$, $\delta(\omega) \,\big|\, p^{k-1}$. Clearly $\delta(\omega) \,\big|\, p^k$, so suppose $\delta(\omega) = p^k$. This can only happen if some $n_i = p^k$ and the remaining $n_j = 0$, which means that every tree T in the generating set of $\mathcal{A}^t_\omega(H)$ has all its univalent vertices labeled by α_i. Choose two univalent vertices which are each connected by an edge to the same trivalent vertex. Unless $n = 1$ this trivalent vertex is connected by its third edge to another trivalent vertex. If we apply the IHX relation here, we see that $T = 0$ in $\mathcal{A}^t(H)$.



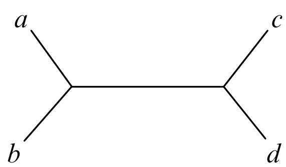
$= 0$

In case $n=1$, we see, by anti-symmetry, that $2T=0$. But since $\delta(\omega)=3$ we also have $3T=0$.

4.3 Proof of Theorem 2.7

If follows from Theorem 2.1 and Corollary 2.3 that $\mathsf{D}'_2(H)=\operatorname{Im}\eta'_2$ is a free abelian group of rank dd_3-d_4, where d_n denotes the rank of $\mathsf{L}_n(H)$. Therefore in order to prove that η'_2 is injective it suffices to show that $\mathcal{A}^t_2(H)$ is generated by dd_3-d_4 elements. Witt's formula (see, for example, Magnus, Karass and Solitar [12, Theorem 5.11]) gives a general formula for d_n – in particular $d_3=2\binom{d+1}{3}$ and $d_4=\frac{1}{4}(d^4-d^2)$. Therefore $dd_3-d_4=\frac{1}{12}(d^4-d^2)$.

Now $\mathcal{A}^t_2(H)$ is generated by trees



which we denote $T(a,b,c,d)$, where a,b,c,d are elements of a basis $\mathcal{B}$ of H. Choose an ordering of $\mathcal{B}$.

Lemma 4.2 *$\mathcal{A}^t_2(H)$ is generated by $\{T(a,b,c,d)\}$ with $a>b, c>d, a\geq c\geq b$ and, if $a=c$, $b\geq d$.*

Assuming the lemma we can count the number of $T(a,b,c,d)$ satisfying the conditions of the lemma:

$\binom{d}{4}$	for $a>c>b>d$	$\binom{d}{4}$	for $a>c>d>b$
$\binom{d}{3}$	for $a=c>b>d$	$\binom{d}{3}$	for $a>c=b>d$
$\binom{d}{3}$	for $a>c>b=d$	$\binom{d}{2}$	for $a=c>b=d$

The sum of these six cases is exactly $\frac{1}{12}(d^4-d^2)$, which proves the Theorem.

Proof of Lemma 4.2 We first list some equalities:

(6) $\quad T(a,b,c,d)=T(d,c,b,a)$

(7) $\quad T(a,b,c,d)=-T(b,a,c,d)=-T(a,b,d,c)=T(b,a,d,c)$

(8) $\quad T(a,b,c,d)=T(a,c,b,d)-T(a,d,b,c)$

(9) $\quad T(a,a,c,d)=T(a,b,c,c)=0$

Equation (6) follows by rotating the tree. Equation (7) is anti-symmetry and equation (8) is the IHX relation. Equation (9) follows from IHX (equation (8)) for $c = d$, and then from (6) for $a = b$.

We now prove the Lemma.

It follows from equations (7) and (9) that $\mathcal{A}^t_2(H)$ is generated by $T(a,b,c,d)$ with $a > b$ and $c > d$. Next let $\mathcal{T}$ consist of all $\{T(a,b,c,d)\}$ satisfying $a > b, c > d, a \geq c$ – we show that $\mathcal{T}$ generates $\mathcal{A}^t_2(H)$.

Suppose $a > b, c > d$ but $c > a$. Then we can use equations (6) and (7) to write $T(a,b,c,d) = T(c,d,a,b)$, which belong to $\mathcal{T}$.

Now define $\mathcal{T}'$ to consist of all $\{T(a,b,c,d)\}$ satisfying $a > b, c > d, a \geq c \geq b$. Suppose $T(a,b,c,d) \in \mathcal{T}$ but $b > c$. Then apply equation (8) to $T(a,b,c,d)$ and note that $T(a,c,b,d)$ and $T(a,d,b,c)$ both belong to $\mathcal{T}'$.

It remains to eliminate those $T(a,b,c,d) \in \mathcal{T}'$ for which $a = c$ and $b < d$. Applying equations (7), (8) and (9) we have

$$\begin{aligned} T(a,b,a,d) &= T(a,a,b,d) - T(a,d,b,a) \\ &= T(a,d,a,b) \end{aligned}$$

This completes the proof. □

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Department of Mathematics, Brandeis University
Waltham MA 02454-9110, USA

Correspondence to: K Orr
Department of Mathematics, Indiana University
831 East 3rd St, Bloomington IN 47405-7106, USA

korr@indiana.edu

http://php.indiana.edu/~korr/

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