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Bottom tangles and universal invariants

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A *bottom tangle* is a tangle in a cube consisting only of arc components, each of which has the two endpoints on the bottom line of the cube, placed next to each other. We introduce a subcategory B of the category of framed, oriented tangles, which acts on the set of bottom tangles. We give a finite set of generators of B, which provides an especially convenient way to generate all the bottom tangles, and hence all the framed, oriented links, via closure. We also define a kind of "braided Hopf algebra action" on the set of bottom tangles.

Using the universal invariant of bottom tangles associated to each ribbon Hopf algebra H, we define a braided functor J from B to the category Mod_H of left H–modules. The functor J, together with the set of generators of B, provides an algebraic method to study the range of quantum invariants of links. The braided Hopf algebra action on bottom tangles is mapped by J to the standard braided Hopf algebra structure for H in Mod_H.

Several notions in knot theory, such as genus, unknotting number, ribbon knots, boundary links, local moves, etc are given algebraic interpretations in the setting involving the category B. The functor J provides a convenient way to study the relationships between these notions and quantum invariants.

57M27; 57M25, 18D10

1 Introduction

The notion of category of tangles (see Yetter [84] and Turaev [80]) plays a crucial role in the study of the quantum link invariants. One can define most quantum link invariants as braided functors from the category of (possibly colored) framed, oriented tangles to other braided categories defined algebraically. An important class of such functorial tangle invariants is introduced by Reshetikhin and Turaev [74]: Given a ribbon Hopf algebra H over a field k, there is a canonically defined functor $F\colon \mathcal{T}_H \to \mathsf{Mod}^f_H$ of the category $\mathcal{T}_H$ of framed, oriented tangles colored by finite-dimensional representations of H into the category Mod^f_H of finite-dimensional left H–modules. The Jones polynomial [29] and many other polynomial link invariants (see Freyd–Yetter–Hoste–Lickorish–Millett–Ocneanu [11], Przytycki–Traczyk [71], Brandt–Lickorish–Millett

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[5], Ho [28] and Kauffman [34]) can be understood in this setting, where H is a quantized enveloping algebra of simple Lie algebra.

One of the fundamental problems in the study of quantum link invariants is *to determine the range of a given invariant over a given class of links*. So far, the situation is far from satisfactory. For example, the range of the Jones polynomial for knots is not completely understood yet.

The purpose of the present paper is to provide a useful algebraic setting for the study of the range of quantum invariants of links and tangles. The main ingredients of this setting are

- a special kind of tangles of arcs, which we call *bottom tangles*,
- a braided subcategory B of the category $\mathcal{T}$ of (uncolored) framed, oriented tangles, which "acts" on the set of bottom tangles by composition, and provides a convenient way to generate all the bottom tangles,
- for each ribbon Hopf algebra H over a commutative, unital ring $\mathbf{k}$, a braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$ from B to the category of left H–modules.

1.1 Bottom tangles

When one studies links in the 3–sphere, it is often useful to represent a link as the *closure* of a tangle consisting only of arc components. In such an approach, one first study tangles, and then obtains results for links from those for tangles, via the closure operation. The advantage of considering tangles of arcs is that one can paste tangles together to obtain another tangle, and such pasting operations produce useful algebraic structures. For example, the set of n–component string links, up to ambient isotopy fixing endpoints, forms a monoid with multiplication induced by vertical pasting.

Bottom tangles, which we study in the present paper, are another kind of tangles of arcs. An n–component bottom tangle $T = T_1 \cup \cdots \cup T_n$ is a framed tangle consisting of n arcs $T_1, \ldots, T_n$ in a cube such that all the endpoints of T are on a line at the bottom square of the cube, and for each $i = 1, \ldots, n$ the component T_i runs from the $2i$ th endpoint on the bottom to the $(2i-1)$ st endpoint on the bottom, where the endpoints are counted from the left. The component T_i is called the i th component of T. For example, see Figure 1 (a).

For $n \geq 0$, let BT_n denote the set of the ambient isotopy classes, relative to endpoints, of n–component bottom tangles. (As usual, we often confuse a tangle and its ambient isotopy class.)

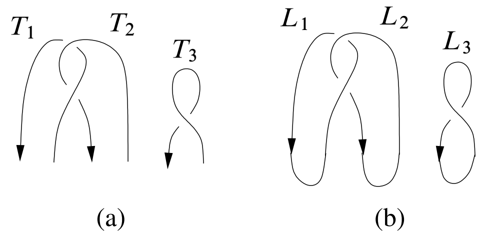


Figure 1: (a) A 3–component bottom tangle $T = T_1 \cup T_2 \cup T_3$. (b) The closure $\mathrm{cl}(T) = L_1 \cup L_2 \cup L_3$ of T.

There is a natural closure operation which transforms an n–component bottom tangle T into an n–component framed, oriented, ordered link $\mathrm{cl}(T)$, see Figure 1 (b). This operation induces a function

$$\mathrm{cl}_n = \mathrm{cl}\colon \mathsf{BT}_n \to \mathsf{L}_n,$$

where L_n denotes the set of the ambient isotopy classes of n–component, framed, oriented, ordered links in the 3–sphere. It is clear that cl_n is surjective, ie, for any link L there is a bottom tangle T such that $\mathrm{cl}(T) = L$. Consequently, one can use bottom tangles to represent links. In many situations, one can divide the study of links into the study of bottom tangles and the study of the effect of closure operation.

Remark 1.1 The notion of bottom tangle has appeared in many places in knot theory, both explicitly and implicitly, and is essentially equivalent to the notion of based links, as is the case with string links. We establish a specific one-to-one correspondence between bottom tangles and string links in Section 13.

1.2 An approach to quantum link invariants using universal invariants of bottom tangles

1.2.1 Universal link invariants associated to ribbon Hopf algebras For each ribbon Hopf algebra H, there is an invariant of links and tangles, which is called the *universal invariant* associated to H, introduced by Lawrence [45; 46] in the case of links and quantized enveloping algebras. Around the same time, Hennings [26] formulated link invariants associated to quasitriangular Hopf algebras which do not involve representations but involve trace functions on the algebras. Reshetikhin [72, Section 4] and Lee [48] considered universal invariants of $(1,1)$–tangles (ie, tangles with one endpoint on the top and one on the bottom) with values in the center of a quantum group, which can be thought of as the $(1,1)$–tangle version of the universal link invariant. Universal invariants are further generalized to more general oriented tangles by Lee [50; 49; 51] and Ohtsuki [65]. Kauffman [35] and Kauffman and Radford [38] defined functorial versions of universal tangle invariant for a generalization of ribbon Hopf algebra which is called "(oriented) quantum algebra".

The universal link invariants are defined at the quantum group level and they do not require any representations. The universal link invariant have a universality property that colored link invariants can be obtained from the universal link invariants by taking traces in the representations attached to components. Thus, in order to study the range of the representation-colored link invariants, it suffices, in theory, to study the range of the universal invariant.

1.2.2 Universal invariant of bottom tangles and their closures Here we briefly describe the relationship between the colored link invariants and the universal invariants, using bottom tangles.

Let H be a ribbon Hopf algebra over a commutative ring $\mathbf{k}$ with unit. For an n–component bottom tangle $T \in \mathsf{BT}_n$, the universal invariant $J_T = J_T^H$ of T associated to H takes value in the n–fold tensor product $H^{\otimes n}$ of H. Roughly speaking, $J_T \in H^{\otimes n}$ is obtained as follows. Choose a diagram D of T. At each crossing of D, place a copy of universal R–matrix $R \in H^{\otimes 2}$, which is modified in a certain rule using the antipode $S\colon H \to H$. Also, place some grouplike elements on the local maxima and minima. Finally, read off the elements placed on each component of H. An example is given in Figure 18 in Section 7.3. For a more precise and more general definition, see Section 7.3.

For each $n \geq 0$, the universal invariant defines a function

$$J\colon \mathsf{BT}_n \to H^{\otimes n}, \quad T \mapsto J_T.$$

In this section we do not give the definition of the universal invariant of links, but it can be obtained from the universal invariant of bottom tangles, as we explain below. Set

$$N = \mathrm{Span}_{\mathbf{k}}\{xy - yS^2(x) \mid x, y \in H\} \subset H. \tag{1–1}$$

The projection

$$\mathrm{tr}_q\colon H \to H/N$$

is called the *universal quantum trace*, since if $\mathbf{k}$ is a field and V is a finite-dimensional left H–module, then the quantum trace $\mathrm{tr}_q^V\colon H \to \mathbf{k}$ in V factors through tr_q.

The universal link invariant $J_L \in (H/N)^{\otimes n}$ for an n–component framed link $L \in \mathsf{L}_n$, which we define in Section 7.3, satisfies

$$J_L = \mathrm{tr}_q^{\otimes n}(J_T),$$

where $T \in \mathsf{BT}_n$ satisfies $\mathrm{cl}(T) = L$.

1.2.3 Reduction to the colored link invariant Let $\mathbf{k}$ be a field, and $V_1, \dots, V_n$ be finite-dimensional left H–modules. Then the quantum invariant $J_{L;V_1,\dots,V_n}$ of an n–component colored link $(L; V_1, \dots, V_n)$ can be obtained from J_L by

$$J_{L;V_1,\dots,V_n} = \big(\overline{\mathrm{tr}}_q^{V_1} \otimes \cdots \otimes \overline{\mathrm{tr}}_q^{V_n}\big)(J_L).$$

where $\overline{\mathrm{tr}}_q^{V_i}\colon H/N \to \mathbf{k}$ is induced by the quantum trace $\mathrm{tr}_q^{V_i}\colon H \to \mathbf{k}$. Hence if $\mathrm{cl}(T) = L$, $T \in \mathsf{BT}_n$, we have

$$J_{L;V_1,\dots,V_n} = \big(\mathrm{tr}_q^{V_1} \otimes \cdots \otimes \mathrm{tr}_q^{V_n}\big)(J_T).$$

These facts can be summarized into a commutative diagram:

$$\begin{tikzcd}
\mathsf{BT}_n \arrow[r, "J"] \arrow[d, "\mathrm{cl}_n"'] & H^{\otimes n} \arrow[d, "\mathrm{tr}_q^{\otimes n}"'] \arrow[dr, "\bigotimes_{i=1}^n \mathrm{tr}_q^{V_i}"] & \\
\mathsf{L}_n \arrow[r, "J"'] & (H/N)^{\otimes n} \arrow[r, "\bigotimes_{i=1}^n \overline{\mathrm{tr}}_q^{V_i}"'] & \mathbf{k}.
\end{tikzcd} \tag{1–2}$$

Given finite-dimensional left H–modules $V_1, \dots, V_n$, we are interested in the range of $J_{L;V_1,\dots,V_n} \in \mathbf{k}$ for $L \in \mathsf{L}_n$. Since cl_n is surjective, it follows from (1–2) that

$$\begin{aligned}
\{J_{L;V_1,\dots,V_n} \mid L \in \mathsf{L}_n\} &= \big(\overline{\mathrm{tr}}_q^{V_1} \otimes \cdots \otimes \overline{\mathrm{tr}}_q^{V_n}\big)(J(\mathsf{L}_n)) \\
&= \big(\mathrm{tr}_q^{V_1} \otimes \cdots \otimes \mathrm{tr}_q^{V_n}\big)(J(\mathsf{BT}_n)).
\end{aligned} \tag{1–3}$$

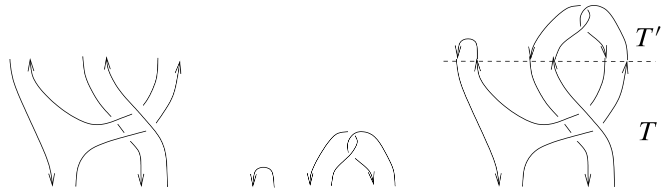
Hence, to determine the range of the representation-colored link invariants, it suffices to determine the images $J(\mathsf{BT}_n) \subset H^{\otimes n}$ for $n \geq 0$ and to study the maps $\mathrm{tr}_q^{V_i}$.

1.3 The category B acting on bottom tangles

To study bottom tangles, and, in particular, to determine the images $J(\mathsf{BT}_n) \subset H^{\otimes n}$, it is useful to introduce a braided subcategory B of the category $\mathcal{T}$ of (uncolored) framed, oriented tangles which *acts on bottom tangles by composition*.

Here we give a brief definition of B, assuming familiarity with the braided category structure of $\mathcal{T}$. For the details, see Section 3.

The objects of B are the expressions $\mathsf{b}^{\otimes n}$ for $n \geq 0$. (Later, b is identified with an object $\mathsf{b} = \downarrow \otimes \uparrow$ in $\mathcal{T}$.) For $m, n \geq 0$, a morphism T from $\mathsf{b}^{\otimes m}$ to $\mathsf{b}^{\otimes n}$ is the ambient isotopy class relative to endpoints of a framed, oriented tangle T satisfying the following.



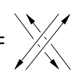
(a) (b) (c)

Figure 2: (a) A morphism $T \in \mathsf{B}(3,2)$. (b) A bottom tangle $T' \in \mathsf{BT}_3$. (c) The composition $TT' \in \mathsf{BT}_2$.

(1) There are $2m$ (resp. $2n$) endpoints on the top (resp. bottom), where the orientations are as $\downarrow\uparrow\cdots\downarrow\uparrow$.

(2) For any m–component bottom tangle T', the composition TT' (obtained by stacking T' on the top of T) is an n–component bottom tangle.

It follows that T consists of $m+n$ arc components and no circle components. For example, see Figure 2 (a). The set $\mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n})$ of morphisms from $\mathsf{b}^{\otimes m}$ to $\mathsf{b}^{\otimes n}$ in B is often denoted simply by $\mathsf{B}(m,n)$. The composition of two morphisms in B is pasting of two tangles vertically, and the identity morphism $1_{\mathsf{b}^{\otimes m}} = \downarrow\uparrow\cdots\downarrow\uparrow$ is a tangle consisting of $2m$ vertical arcs. The monoidal structure is given by pasting two tangles side by side. The braiding is defined in the usual way; ie, the braiding for the generating object $\mathsf{b} \in \mathrm{Ob}(\mathsf{B})$ and itself is given by

$\psi_{\mathsf{b},\mathsf{b}} =$  .

For each $n \geq 0$, we can identify BT_n with $\mathsf{B}(0,n)$. The category B acts on $\mathsf{BT} = \coprod_{n\geq 0} \mathsf{BT}_n$ via the functions

$$\mathsf{B}(m,n) \times \mathsf{BT}_m \to \mathsf{BT}_n, \quad (T,T') \mapsto TT'.$$

In the canonical way, one may regard this action as a functor

$$\mathsf{B}(\mathbf{1}, -)\colon \mathsf{B} \to \mathsf{Sets}$$

from B to the category Sets of sets, which maps an object $\mathsf{b}^{\otimes n}$ into BT_n.

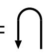
1.4 The braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$

Let Mod_H denote the category of left H–modules, with the standard braided category structure. Unless otherwise stated, we regard H as a left H–module with the left adjoint action.

We study a braided functor

$$\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H,$$

which is roughly described as follows. For the details, see Section 8. For objects, we set $\mathsf{J}(\mathsf{b}^{\otimes n}) = H^{\otimes n}$, where $H^{\otimes n}$ is given the standard tensor product left H–module structure. For $T \in \mathsf{B}(m,n)$, the left H–module homomorphism $\mathsf{J}(T)\colon H^{\otimes m} \to H^{\otimes n}$ maps $x = \sum x_1 \otimes \cdots \otimes x_m \in H^{\otimes m}$ to the element $\mathsf{J}(T)(x)$ in $H^{\otimes n}$ obtained as follows. Set

$\eta_{\mathsf{b}} =$  $\in \mathsf{B}(0,1),$

and set for $m \geq 0$

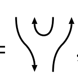
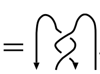
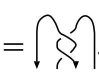
$\eta_m = \eta_{\mathsf{b}}^{\otimes m} =$  $\in \mathsf{B}(0,m).$

Consider a diagram for the composition $T\eta_m$, and put the element x_i on the ith component (from the left) of η_m for $i = 1, \dots, m$, and put the copies of universal R–matrix and the grouplike element on the strings of T as in the definition of J_T. Then the element $\mathsf{J}(T)(x) \in H^{\otimes n}$ is read off from the diagram. (See Figure 21 in Section 8.2 to get a hint for the definition.) We see in Section 8.2 that $\mathsf{J}(T)$ is a left H–module homomorphism, and J is a well-defined braided functor.

1.5 Generators of the braided category B

Recall that a (strict) braided category M is said to be generated by a set $X \subset \mathrm{Ob}(M)$ of objects and a set $Y \subset \mathrm{Mor}(M)$ of morphisms if every object of M is a tensor product of finitely many copies of objects from X, and if every morphism of M is an iterated tensor product and composition of finitely many copies of morphisms from Y and the identity morphisms of the objects from X. In the category $\mathcal{T}$, "tensor product and composition" is "horizontal and vertical pasting of tangles".

Theorem 1 (Theorem 5.16) *As a braided subcategory of $\mathcal{T}$, B is generated by the object b and the morphisms*

$\eta_{\mathsf{b}} =$ , $\mu_{\mathsf{b}} =$ , $v_+ =$ , $v_- =$ , $c_+ =$ , $c_- =$ .

Consequently, any bottom tangle can be obtained by horizontal and vertical pasting from finitely many copies of the above-listed tangles, the braidings $\psi_{\mathsf{b},\mathsf{b}}, \psi_{\mathsf{b},\mathsf{b}}^{-1}$ and the identity $1_{\mathsf{b}} = \downarrow\uparrow$. Theorem 1 implies that, as a category, B is generated by the morphisms

$$f_{(i,j)} = \mathsf{b}^{\otimes i} \otimes f \otimes \mathsf{b}^{\otimes j}$$

for $i, j \geq 0$ and $f \in \{\eta_{\mathsf{b}}, \mu_{\mathsf{b}}, v_\pm, c_\pm, \psi_{\mathsf{b},\mathsf{b}}^{\pm 1}\}$. Theorem 1 provides a convenient way to generate all the bottom tangles, and hence all the links via closure operation.

We can use Theorem 1 to determine the range $J(\mathsf{BT}) = \{J_T \mid T \in \mathsf{BT}\}$ of the universal invariant of bottom tangles as follows. Theorem 1 and functoriality of J implies that any morphism f in B is the composition of finitely many copies of the left H–module homomorphisms

$$\mathsf{J}(f_{(i,j)}) = 1_H^{\otimes i} \otimes \mathsf{J}(f) \otimes 1_H^{\otimes j}, \tag{1–4}$$

for $i, j \geq 0$ and $f \in \{\eta_{\mathsf{b}}, \mu_{\mathsf{b}}, v_{\pm}, c_{\pm}, \psi_{\mathsf{b},\mathsf{b}}^{\pm 1}\}$. Hence we have the following characterization of the range of J for bottom tangles.

Theorem 2 *The set $J(\mathsf{BT})$ is equal to the smallest subset of $\coprod_{n\geq 0} H^{\otimes n}$ containing $1 \in \mathbf{k} = H^{\otimes 0}$ and stable under the functions $\mathsf{J}(f_{(i,j)})$ for $i, j \geq 0$ and $f \in \left\{\eta_{\mathsf{b}}, \mu_{\mathsf{b}}, v_{\pm}, c_{\pm}, \psi_{\mathsf{b},\mathsf{b}}^{\pm 1}\right\}$.*

See Section 9.1 for some variants of Theorem 2, which may be more useful in applications than Theorem 2.

1.6 Hopf algebra action on bottom tangles

We define a kind of "Hopf algebra action" on the set BT, which is formulated as an "exterior Hopf algebra" in the category B. Roughly speaking, this exterior Hopf algebra in B is an "extension" of an algebra structure for the object b in B to a Hopf algebra structure at the level of sets of morphisms in B. This "extension" is formulated as a functor

$$F_{\mathsf{b}} \colon \langle \mathsf{H} \rangle \to \mathsf{Sets},$$

from the free strict braided category $\langle \mathsf{H} \rangle$ generated by a Hopf algebra H to the category Sets of sets, where each object $\mathsf{H}^{\otimes m}$, $m \geq 0$, is mapped to the set $\mathsf{B}(\mathbf{1}, \mathsf{b}^{\otimes m}) = \mathsf{BT}_m$. This functor essentially consists of functions

$$\langle \mathsf{H} \rangle(\mathsf{H}^{\otimes m}, \mathsf{H}^{\otimes n}) \times \mathsf{BT}_m \to \mathsf{BT}_n \quad \text{for } m, n \geq 0, \tag{1–5}$$

which can be regarded as a left action of the category $\langle \mathsf{H} \rangle$ on the graded set BT.

The two functors $\mathsf{B}(\mathbf{1}, -) \colon \mathsf{B} \to \mathsf{Sets}$ and $F_{\mathsf{b}} \colon \langle \mathsf{H} \rangle \to \mathsf{Sets}$ are related as follows. Let $\langle \mathsf{A} \rangle$ denote the braided category freely generated by an algebra A. Note that the morphisms $\mu_{\mathsf{b}} \colon \mathsf{b}^{\otimes 2} \to \mathsf{b}$ and $\eta_{\mathsf{b}} \colon \mathbf{1} \to \mathsf{b}$ in B define an algebra structure for the object b. Consider the following diagram

$$\begin{tikzcd} \langle \mathsf{A} \rangle \arrow[r, "i_{\mathsf{B},\mathsf{b}}"] \arrow[d, "i_{\langle \mathsf{H} \rangle, \mathsf{H}}"'] & \mathsf{B} \arrow[d, "\mathsf{B}(\mathbf{1},-)"] \\ \langle \mathsf{H} \rangle \arrow[r, "F_{\mathsf{b}}"'] & \mathsf{Sets}. \end{tikzcd}$$

Here the two arrows $i_{\mathsf{B},\mathsf{b}}$ and $i_{\langle \mathsf{H}\rangle,\mathsf{H}}$ are the unique braided functors that map the algebra structure for A into those of b and H, respectively. Both $i_{\mathsf{B},\mathsf{b}}$ and $i_{\langle \mathsf{H}\rangle,\mathsf{H}}$ are faithful. The above diagram turns out to be commutative. In other words, the action of the algebra structure in B on BT extends to an action by a Hopf algebra structure on BT.

Remark 1.2 The action of $\langle \mathsf{H}\rangle$ on BT mentioned above extends to an action of a category $\mathcal{B}$ of "bottom tangles in handlebodies" which we shortly explain in Section 14.4. Also, the above Hopf algebra action is closely related to the Hopf algebra structure in the category $\mathcal{C}$ of cobordisms of connected, oriented surfaces with boundary parameterized by a circle (see Crane–Yetter [8] and Kerler [40]), and also to the Hopf algebra properties satisfied by claspers (see Habiro [22]).

Now we explain the relationship between the Hopf algebra action on BT and the braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$. Let H be a ribbon Hopf algebra, hence in particular H is quasitriangular. Recall that the *transmutation* $\underline{H}$ of a quasitriangular Hopf algebra H is a braided Hopf algebra structure in Mod_H defined for the object H with the left adjoint action in the braided category Mod_H, with the same algebra structure and the same counit as H but with a twisted comultiplication $\underline{\Delta}\colon H \to H \otimes H$ and a twisted antipode $\underline{S}\colon H \to H$. For details, see Majid [55; 56]. The transmutation $\underline{H}$ yields a braided functor $F_{\underline{H}}\colon \langle \mathsf{H}\rangle \to \mathsf{Mod}_H$, which maps the Hopf algebra structure of H into that of $\underline{H}$. Via $F_{\underline{H}}$, the category $\langle \mathsf{H}\rangle$ acts on the $H^{\otimes n}$ as

$$\langle \mathsf{H}\rangle(\mathsf{H}^{\otimes m}, \mathsf{H}^{\otimes n}) \times H^{\otimes m} \to H^{\otimes n}, \quad (f, x) \mapsto F_{\underline{H}}(f)(x).$$

We have the following commutative diagram

$$\begin{array}{ccc}
\langle \mathsf{H}\rangle(\mathsf{H}^{\otimes m}, \mathsf{H}^{\otimes n}) \times \mathsf{BT}_m & \longrightarrow & \mathsf{BT}_n \\
{\scriptstyle \mathrm{id}\times J}\big\downarrow & & \big\downarrow{\scriptstyle J} \\
\langle \mathsf{H}\rangle(\mathsf{H}^{\otimes m}, \mathsf{H}^{\otimes n}) \times H^{\otimes m} & \longrightarrow & H^{\otimes n}.
\end{array}$$

This means that the Hopf algebra action on the bottom tangles is mapped by the universal invariant maps $J\colon \mathsf{BT}_n \to H^{\otimes n}$ into the Hopf algebra action on the $H^{\otimes n}$, defined by transmutation. This fact may be considered as a *topological interpretation of transmutation*.

1.7 Local moves

The setting of bottom tangles and the category B is also useful in studying local moves on bottom tangles, and hence on links. Recall that a type of local move can be defined by specifying a pair of tangles (t, t') in a 3–ball with the same set of endpoints and with the same orientations and framings at the endpoints. Two tangles u and u' in another 3–ball B are said to be related by a (t,t')–move if u' is, up to ambient isotopy which fixes endpoints, obtained from u by replacing a "subtangle" t in u with a copy of t'. Here, by a subtangle of a tangle u in B we mean a tangle $u\cap B'$ contained in a 3–ball $B'\subset \operatorname{int} B$.

In the following, we restrict our attention to the case where both the tangles t and t' consist only of arcs. In this case, we can choose two bottom tangles $T,T'\in \mathsf{BT}_n$ such that the notion of (t,t')–move is the same as that of (T,T')–move.

The following result implies that the notion of (T,T')–move for bottom tangles, where $T,T'\in \mathsf{BT}_n$, can be formulated in a totally algebraic way within the category B.

Theorem 3 (Consequence of Proposition 5.5 and Theorem 5.8) *Let $T,T'\in \mathsf{BT}_m$ and $U,U'\in \mathsf{BT}_n$ with $m,n\geq 0$. Then U and U' are related by a (T,T')–move if and only if there is $W\in \mathsf{B}(m,n)$ such that $U=WT$ and $U'=WT'$.*

The setting of bottom tangles and the category B is useful in studying local moves on bottom tangles. In Section 5, we formulate in algebraic terms the following typical questions in the theory of local moves, under some mild conditions.

- When are two bottom tangles related by a sequence of a given set of local moves? (Proposition 5.11.)
- When are two bottom tangles equivalent under the equivalence relation generated by a given set of local moves? (Proposition 5.12.)
- When is a bottom tangle related to the trivial bottom tangle η_n by just one local move of a given type? (Theorem 5.14.)
- When is a bottom tangle equivalent to η_n under the equivalence relation generated by a given set of local moves? (Corollary 5.15.)

It is easy to modify the answers to the above questions for bottom tangles into those for links, via closure. Some of the algebraic formulations of the above questions are combined with the functor J in Section 9.1.

A remarkable application of the algebraic formulation of local moves is the following. A *delta move* (see Murakami and Nakanishi [59]) or a *Borromean transformation* (see Matveev [57]) can be defined as a (η_3,B)–move, where $B\in \mathsf{BT}_3$ is the *Borromean tangle* depicted in Figure 3. An n–component link has a zero linking matrix if and only if it is related to the n–component unlink by a sequence of delta moves [59]. Using the obvious variation of this fact for bottom tangles, we obtain the following.

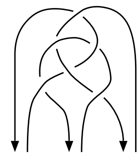
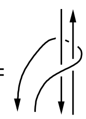
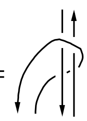


Figure 3: The Borromean tangle B

Theorem 4 (Part of Corollary 9.13) *A bottom tangle with zero linking matrix is obtained by pasting finitely many copies of* $1_{\mathsf{b}}, \psi_{\mathsf{b},\mathsf{b}}, \psi^{-1}_{\mathsf{b},\mathsf{b}}, \mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-, B$, *where*

$\gamma_+ =$  $,\quad \gamma_- =$  $\in \mathsf{B}(1,2).$

For applications of Theorem 4 to the universal invariant of bottom tangles with zero linking matrix, see Corollary 9.15, which states that the universal invariant associated to a ribbon Hopf algebra H of n–component bottom tangles with zero linking matrix is contained in a certain subset of $H^{\otimes n}$. The case of the quantized enveloping algebra $U_h(\mathfrak{g})$ of simple Lie algebra $\mathfrak{g}$ will be used in future publications [20] (for $\mathfrak{g} = sl_2$) and [25] (for general $\mathfrak{g}$) to show that there is an invariant of integral homology spheres with values in the completion $\varprojlim_n \mathbb{Z}[q]/((1-q)(1-q^2)\cdots(1-q^n))$ studied in [24], which unifies the quantum $\mathfrak{g}$ invariants at all roots of unity, as announced in [23], [67, Conjecture 7.29].

1.8 Other applications

The setting of bottom tangles, the category B and the functor J can be applied to the following notions in knot theory: unknotting number (Section 9.2), Seifert surface, knot genus and boundary links (Section 9.3), unoriented spanning surface, crosscap number and $\mathbb{Z}_2$–boundary links (Section 9.4), clasper moves (Section 9.6), Goussarov–Vassiliev finite type invariants (Section 9.7), twist moves and Fox's congruence (Section 9.8), ribbon knots (Section 11.2), and the Hennings 3–manifold invariants (Section 12).

1.9 Organization of the paper

Here we briefly explain the organization of the rest of the paper. Section 2 provides some preliminary facts and notations about braided categories and Hopf algebras. In Section 3, we recall the definition of the category $\mathcal{T}$ of framed oriented tangles, and define the subcategory B of $\mathcal{T}$. In Section 4, we study a subcategory B_0 of B, which is used in Section 5, where we study local moves on tangles and give a set of generators of B. Section 6 deals with the Hopf algebra action on bottom tangles. In Section 7, we recall the definition of the universal tangle invariant and provide some necessary facts. In Section 8, we define and study the braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$. In Section 9, we give several applications of the results in the previous sections to the values of the universal invariant. In Section 10, we modify the functor J into a braided functor $\tilde{\mathsf{J}}\colon \mathsf{B} \to \mathsf{Mod}_H$, which is useful in the study of the universal invariants of bottom knots. In Section 11, we define a refined version of the universal link invariant by giving a necessary and sufficient condition that two bottom tangles have the same closures. In Section 12, we reformulate the Hennings invariant of 3–manifolds in our setting, using an algebraic version of Kirby calculus. In Section 13, we relate the structure of the sets of bottom tangles to the sets of string links. In Section 14, we give some remarks.

Remark 1.3 Around the same time as the first preprint version of the present paper is completed, a paper by Bruguières and Virelizier [6] appeared on the arXiv. Part of [6] is closely related to part of the present paper. The present paper is independent of [6] (except this Remark 1.3).

The notion of bottom tangle is equivalent to that of "ribbon handle" in [6]. Theorem 1 is related to [6, Theorems 1.1 and 3.1]. Section 12 is related to [6, Section 5]. Section 13 is closely related to [6].

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2 Braided categories and Hopf algebras

In this section, we fix some notations concerning monoidal and braided categories, and braided Hopf algebras.

If C is a category, then the set (or class) of objects in C is denoted by $\mathrm{Ob}(C)$, and the set (or class) of morphisms in C is denoted by $\mathrm{Mor}(C)$. For $a, b \in \mathrm{Ob}(C)$, the set of morphisms from a to b is denoted by $C(a, b)$. For $a \in \mathrm{Ob}(C)$, the identity morphism $1_a \in C(a, a)$ of a is sometimes denoted by a.

2.1 Monoidal and braided categories

We use the following notation for monoidal and braided categories. See Mac Lane [54] for the definitions of monoidal and braided categories. If $\mathcal{M}$ is a monoidal category (also called tensor category), then the tensor functor is denoted by $\otimes_{\mathcal{M}}$ and the unit object by $\mathbf{1}_{\mathcal{M}}$. We omit the subscript $\mathcal{M}$ and write $\otimes_{\mathcal{M}} = \otimes$ and $\mathbf{1}_{\mathcal{M}} = \mathbf{1}$ if there is no fear of confusion. If $\mathcal{M}$ is a braided category, then the braiding of $a, b \in \mathrm{Ob}(\mathcal{M})$ is denoted by

$$\psi^{\mathcal{M}}_{a,b} = \psi_{a,b}\colon a \otimes b \to b \otimes a$$

for $a, b \in \mathrm{Ob}(\mathcal{M})$.

Definition of monoidal category involves also the associativity and the unitality constraints, which are functorial isomorphisms

$$a \otimes (b \otimes c) \overset{\cong}{\to} (a \otimes b) \otimes c, \quad \mathbf{1} \otimes a \overset{\cong}{\to} a, \quad a \otimes \mathbf{1} \overset{\cong}{\to} a.$$

A monoidal category is called *strict* if these constraints are identity morphisms. In what follows, a strict monoidal category is simply called a "monoidal category". Similarly, a strict braided category is called a "braided category". Also, we sometimes need non-strict monoidal or braided categories, such as the category of left modules over a Hopf algebra. When this is the case, we usually suppress the associativity and the unitality constraints, and we argue as if they are strict monoidal or braided categories. This should not cause confusion.

A *strict monoidal functor* from a (strict) monoidal category $\mathcal{M}$ to a (strict) monoidal category $\mathcal{M}'$ is a functor $F\colon \mathcal{M} \to \mathcal{M}'$ such that

$$\begin{aligned} F\otimes &= \otimes(F \times F)\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}', \\ F(\mathbf{1}_{\mathcal{M}}) &= 1_{\mathcal{M}'}, \end{aligned}$$

ie, a strict monoidal functor is a functor which (strictly) preserves $\otimes$ and $\mathbf{1}$. Unless otherwise stated, by a "monoidal functor" we mean a strict monoidal functor. A *braided functor* from a braided category $\mathcal{M}$ to a braided category $\mathcal{M}'$ is a monoidal functor $F\colon \mathcal{M} \to \mathcal{M}'$ such that $F(\psi^{\mathcal{M}}_{a,b}) = \psi^{\mathcal{M}'}_{F(a),F(b)}$ for all $a, b \in \mathrm{Ob}(\mathcal{M})$.

A monoidal category M is said to be *generated* by a set $X \subset \mathrm{Ob}(M)$ of objects and a set $Y \subset \mathrm{Mor}(M)$ of morphisms if every object of M is a tensor product of finitely many copies of objects from X, and every morphism of M is an iterated tensor product and composition of finitely many copies of morphisms from Y and the identity morphisms of the objects from X. A braided category M is said to be generated by $X \subset \mathrm{Ob}(M)$ and $Y \subset \mathrm{Mor}(M)$ if M is generated as a monoidal category by X and $Y \cup \{\psi^{\pm 1}_{x,y} \mid x, y \in X\}$.

2.2 Algebras and Hopf algebras in braided categories

Here we fix the notations for algebras and Hopf algebras in a monoidal or braided category. We refer the reader to Majid [55; 56] for the details.

An *algebra* (also called *monoid*) in a monoidal category $\mathcal{M}$ is an object A equipped with morphisms $\mu\colon A \otimes A \to A$ (*multiplication*) and $\eta\colon \mathbf{1} \to A$ (*unit*) satisfying

$$\mu(\mu \otimes A) = \mu(A \otimes \mu), \qquad \mu(\eta \otimes A) = 1_A = \mu(A \otimes \eta).$$

A *coalgebra* C in $\mathcal{M}$ is an object C equipped with morphisms $\Delta\colon A \to A \otimes A$ (*comultiplication*) and $\epsilon\colon A \to \mathbf{1}$ (*counit*) satisfying

$$(\Delta \otimes A)\Delta = (A \otimes \Delta)\Delta, \qquad (\epsilon \otimes A)\Delta = 1_A = (A \otimes \epsilon)\Delta.$$

A *Hopf algebra* in a braided category $\mathcal{M}$ is an object H in $\mathcal{M}$ equipped with an algebra structure μ, η, a coalgebra structure Δ, ϵ and a morphism $S\colon A \to A$ (*antipode*) satisfying

$$\epsilon\eta = 1_{\mathbf{1}}, \qquad \Delta\eta = \eta \otimes \eta, \qquad \epsilon\mu = \epsilon \otimes \epsilon,$$
$$\Delta\mu = (\mu \otimes \mu)(A \otimes \psi_{A,A} \otimes A)(\Delta \otimes \Delta),$$
$$\mu(A \otimes S)\Delta = \mu(S \otimes A)\Delta = \eta\epsilon.$$

Later, we sometimes use the notations $\mu_A = \mu$, $\eta_A = \eta$, $\Delta_A = \Delta$, $\epsilon_A = \epsilon$, $S_A = S$, to distinguish structure morphisms from different Hopf algebras.

A Hopf algebra (in the usual sense) over a commutative, unital ring $\mathbf{k}$ can be regarded as a Hopf algebra in the symmetric monoidal category of $\mathbf{k}$–modules.

If A is an algebra in $\mathcal{M}$, then let $\mu_A^{[n]} = \mu^{[n]}\colon A^{\otimes n} \to A$ $(n \geq 0)$ denote the *n–input multiplication* defined by $\mu^{[0]} = \eta$, $\mu^{[1]} = 1_A$, and

$$\mu^{[n]} = \mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes(n-2)})$$

for $n \geq 2$. Similarly, if C is an coalgebra, then let $\Delta_C^{[n]} = \Delta^{[n]}\colon C \to C^{\otimes n}$ denote the *n–output comultiplication* defined by $\Delta^{[0]} = \epsilon$, $\Delta^{[1]} = 1_C$ and

$$\Delta^{[n]} = (\Delta \otimes 1^{\otimes(n-2)}) \cdots (\Delta \otimes 1)\Delta$$

for $n \geq 2$.

3 The category $\mathcal{T}$ of tangles and the subcategory B

In this section, we first recall the definition of the category $\mathcal{T}$ of framed, oriented tangles. Then we give a precise definition of the braided subcategory B of $\mathcal{T}$.

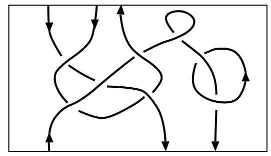


Figure 4: A tangle $T\colon \downarrow\otimes\downarrow\otimes\uparrow\to\uparrow\otimes\downarrow\otimes\downarrow$

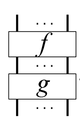
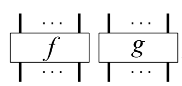
In the rest of the paper, by an "isotopy" between two tangles we mean "ambient isotopy fixing endpoints". Thus, two tangles are said to be "isotopic" if they are ambient isotopic relative to endpoints.

3.1 The category $\mathcal{T}$ of tangles

Here we recall the definition of the braided category $\mathcal{T}$ of framed, oriented tangles, and fix the notations. For details, see Yetter [84; 86], Turaev [80; 81], Freyd and Yetter [10], Shum [78] and Kassel [31].

The objects in the category $\mathcal{T}$ are the tensor words of symbols $\downarrow$ and $\uparrow$, ie, the expressions $x_1\otimes\cdots\otimes x_n$ with $x_1,\dots,x_n\in\{\downarrow,\uparrow\}$, $n\geq 0$. The tensor word of length 0 is denoted by $\mathbf{1}=\mathbf{1}_{\mathcal{T}}$. The morphisms $T\colon w\to w'$ between $w,w'\in\mathrm{Ob}(\mathcal{T})$ are the isotopy classes of framed, oriented tangles in a cube $[0,1]^3$ such that the endpoints at the top are described by w and those at the bottom by w', see Figure 4 for example. We use the blackboard framing convention in the figures. In what follows, by "tangles" we mean framed, oriented tangles unless otherwise stated. As usual, we systematically confuse a morphism in $\mathcal{T}$ with a tangle representing it.

The composition gf of a composable pair (f,g) of morphisms in $\mathcal{T}$ is obtained by placing g below f, and the tensor product $f\otimes g$ of two morphisms f and g is obtained by placing g on the right of f. Graphically,

$gf =$ , $\quad f\otimes g =$ .

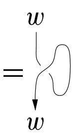
The braiding $\psi_{w,w'}\colon w\otimes w'\to w'\otimes w$ for $w,w'\in\mathrm{Ob}(\mathcal{T})$ is the positive braiding of parallel families of strings. For $w\in\mathrm{Ob}(\mathcal{T})$, the dual $w^*\in\mathrm{Ob}(\mathcal{T})$ of w is defined by $\mathbf{1}^*=\mathbf{1}$, $\downarrow^*=\uparrow$, $\uparrow^*=\downarrow$, and

$$(x_1\otimes\cdots\otimes x_n)^* = x_n^*\otimes\cdots\otimes x_1^* \quad (x_1,\dots,x_n\in\{\downarrow,\uparrow\}, n\geq 2).$$

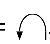
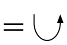
For $w \in \mathrm{Ob}(\mathcal{T})$, let

$$\mathrm{ev}_w\colon w^* \otimes w \to \mathbf{1}, \quad \mathrm{coev}_w\colon \mathbf{1} \to w \otimes w^*$$

denote the duality morphisms. For each object w in $\mathcal{T}$, let $t_w\colon w \to w$ denote the positive full twist defined by

$$t_w = (w \otimes \mathrm{ev}_{w^*})(\psi_{w,w} \otimes w^*)(w \otimes \mathrm{coev}_w) =$$ .

It is well known that $\mathcal{T}$ is generated as a monoidal category by the objects $\downarrow, \uparrow$ and the morphisms

$\psi_{\downarrow,\downarrow} =$ , $\psi_{\downarrow,\downarrow}^{-1} =$ , $\mathrm{ev}_\downarrow =$ , $\mathrm{ev}_\uparrow =$ , $\mathrm{coev}_\downarrow =$ , $\mathrm{coev}_\uparrow =$ .

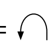
3.2 The braided subcategory B of $\mathcal{T}$

Two tangles $T, T' \in \mathcal{T}(w, w')$, $w, w' \in \mathrm{Ob}(\mathcal{T})$, are said to be *homotopic* (to each other) if there is a homotopy between T and T' which fixes the endpoints, where the framings are ignored. We write $T \sim_h T'$ if T and T' are homotopic. Note that two tangles are homotopic if and only if they are related by a finite sequence of isotopies, crossing changes, and framing changes.

Now we define a braided subcategory B of $\mathcal{T}$. Set

$$\mathrm{Ob}(\mathsf{B}) = \{\mathsf{b}^{\otimes m} \mid m \geq 0\} \subset \mathrm{Ob}(\mathcal{T}),$$

where $\mathsf{b} = \downarrow \otimes \uparrow \in \mathrm{Ob}(\mathcal{T})$. Set

$\eta_{\mathsf{b}} = \mathrm{coev}_\downarrow =$  $\in \mathcal{T}(\mathbf{1}, \mathsf{b})$,

$$\eta_n = \eta_{\mathsf{b}}^{\otimes n} \in \mathcal{T}(\mathbf{1}, \mathsf{b}^{\otimes n}) \quad \text{for } n \geq 0.$$

For $m, n \geq 0$, set

$$\mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n}) = \{T \in \mathcal{T}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n}) \mid T\eta_m \sim_h \eta_n\}.$$

If $T \in \mathsf{B}(\mathsf{b}^{\otimes l}, \mathsf{b}^{\otimes m})$ and $T' \in \mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n})$, then we have $T'T\eta_l \sim_h T'\eta_m \sim_h \eta_n$, hence $T'T \in \mathsf{B}(\mathsf{b}^{\otimes l}, \mathsf{b}^{\otimes n})$. Clearly, we have $1_{\mathsf{b}^{\otimes n}} \in \mathsf{B}(\mathsf{b}^{\otimes n}, \mathsf{b}^{\otimes n})$. Hence $\mathrm{Ob}(\mathsf{B})$ and the $\mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n})$ form a subcategory of $\mathcal{T}$.

If $T \in \mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n})$ and $T' \in \mathsf{B}(\mathsf{b}^{\otimes m'}, \mathsf{b}^{\otimes n'})$, then we have

$$(T \otimes T')\eta_{m+m'} = (T\eta_m) \otimes (T'\eta_{m'}) \sim_h \eta_n \otimes \eta_{n'} = \eta_{n+n'}.$$

We also have $\mathbf{1} \in \mathrm{Ob}(\mathsf{B})$. Hence B is a monoidal subcategory of $\mathcal{T}$.

We have $\psi_{\mathsf{b},\mathsf{b}}^{\pm1}\eta_2 = \eta_2$, hence $\psi_{\mathsf{b},\mathsf{b}}^{\pm1} \in \mathsf{B}(\mathsf{b}^{\otimes 2}, \mathsf{b}^{\otimes 2})$. Since the object b generates the monoid $\mathrm{Ob}(\mathsf{B})$, it follows that $\psi_{\mathsf{b}^{\otimes m},\mathsf{b}^{\otimes n}} \in \mathsf{B}(\mathsf{b}^{\otimes(m+n)}, \mathsf{b}^{\otimes(m+n)})$. Hence B is a braided subcategory of $\mathcal{T}$.

For simplicity of notation, we set

$$\mathsf{B}(m,n) = \mathsf{B}(\mathsf{b}^{\otimes m}, \mathsf{b}^{\otimes n})$$

for $m,n \geq 0$. We use the similar notation for subcategories of B defined later.

For $n \geq 0$, we have

$$\mathsf{B}(0,n) = \{T \in \mathcal{T}(\mathbf{1}, \mathsf{b}^{\otimes n}) \mid T \sim_h \eta_n\}.$$

Hence we can naturally identify $\mathsf{B}(0,n)$ with the set BT_n of isotopy classes of n–component bottom tangles.

4 The subcategory B_0 of B

In this section, we introduce a braided subcategory B_0 of B, and give a set of generators of B_0. The category B_0 is used in Section 5.

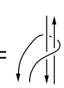
4.1 Definition of B_0

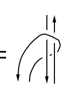
Let B_0 denote the subcategory of B with $\mathrm{Ob}(\mathsf{B}_0) = \mathrm{Ob}(\mathsf{B})$ and

$$\mathsf{B}_0(m,n) = \{T \in \mathsf{B}(m,n) \mid T\eta_m = \eta_n\}$$

for $m,n \geq 0$. It is straightforward to check that the category B_0 is well-defined and it is a braided subcategory of B. Note that we have $\mathsf{B}_0(0,n) = \{\eta_n\}$.

Set

$$\gamma_+ = \big(\downarrow \otimes \psi_{\mathsf{b},\uparrow}\psi_{\uparrow,\mathsf{b}}\big)(\mathrm{coev}_\downarrow \otimes \mathsf{b}) =$$  $$\in \mathsf{B}_0(1,2),$$

$$\gamma_- = \big(\downarrow \otimes \psi_{\uparrow,\mathsf{b}}^{-1}\psi_{\mathsf{b},\uparrow}^{-1}\big)(\mathrm{coev}_\downarrow \otimes \mathsf{b}) =$$  $$\in \mathsf{B}_0(1,2),$$

$$t_{+-} = t_\downarrow \otimes t_\uparrow^{-1} \in \mathsf{B}_0(1,1).$$

Note that t_{+-} is an isomorphism.

The purpose of this section is to prove the following theorem, which is used in Section 5.

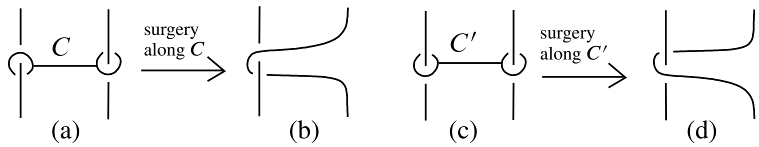


Figure 5: Here each string may be replaced with parallel strings.

Theorem 4.1 *As a braided subcategory of* B, B_0 *is generated by the object* b *and the morphisms* $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-, t_{+-}, t_{+-}^{-1}$.

The proof of Theorem 4.1 is given in Section 4.3, after giving a lemma on string links in Section 4.2.

4.2 Clasper presentations for string links

To prove the case of "doubled string links" of Theorem 4.1 (see Section 4.3.1), we need a lemma which presents an n–component string link as the result of surgery on $1_{\downarrow^{\otimes n}}$ along some claspers (see Goussarov [14] and Habiro [22]). In this and the next subsections (but not elsewhere in this paper), a "clasper" means a "strict tree clasper of degree 1" in the sense of [22], ie, a clasper consisting of two disc-leaves and one edge which looks as depicted in Figure 5 (a). One can perform surgery on a clasper as depicted in Figure 5 (b), see [22, Remark 2.4]. We also use the fact that the result of surgery on another clasper C' depicted in Figure 5 (c) is as depicted in Figure 5 (d).

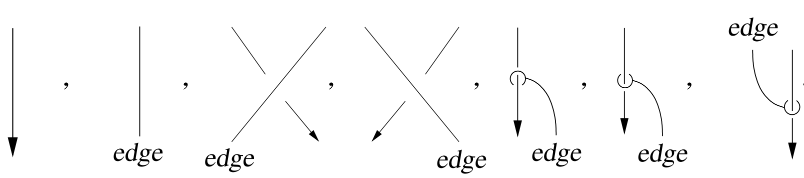
For $n \geq 0$, let SL_n denote the submonoid of $\mathcal{T}(\downarrow^{\otimes n}, \downarrow^{\otimes n})$ consisting of the isotopy classes of the n–component framed string links. Thus we have

$$\mathsf{SL}_n = \{T \in \mathcal{T}(\downarrow^{\otimes n}, \downarrow^{\otimes n}) \mid T \sim_h \downarrow^{\otimes n}\}.$$

Lemma 4.2 *If* $T \in \mathsf{SL}_n$, *then there are mutually disjoint claspers* $C_1, \dots, C_r$ $(r \geq 0)$ *for* $1_{\downarrow^{\otimes n}}$ *satisfying the following properties.*

(1) *The tangle* T *is obtained from* $1_{\downarrow^{\otimes n}}$ *by surgery along* $C_1, \dots, C_r$ *and framing change.*

(2) $1_{\downarrow^{\otimes n}}$ *and* $C_1, \dots, C_r$ *is obtained by pasting horizontally and vertically finitely many copies of the following:*



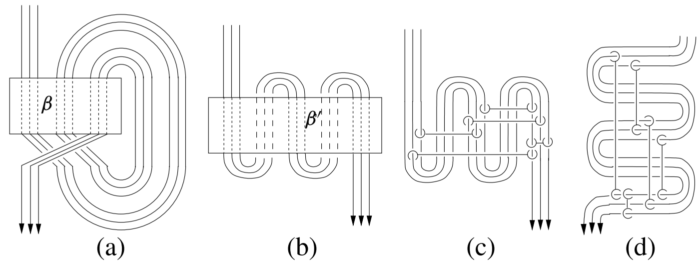


Figure 6

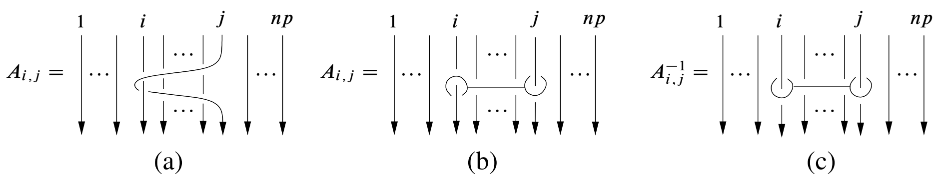


Figure 7

Proof In this proof, we can ignore the framings.

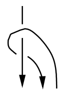
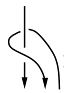
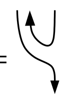
As is well known, we can express T as a "partially closed braid" in the sense that there is an integer $p \geq 1$ and a pure braid $\beta \in \mathcal{T}(\downarrow^{\otimes np}, \downarrow^{\otimes np})$ of np strings such that

$$T = \big(\downarrow^{\otimes n} \otimes \mathrm{ev}_{\uparrow^{\otimes n(p-1)}}\big)\big(\psi_{\downarrow^{\otimes n(p-1)},\downarrow^{\otimes n}}\beta \otimes \uparrow^{\otimes n(p-1)}\big)\big(\downarrow^{\otimes n} \otimes \mathrm{coev}_{\downarrow^{\otimes n(p-1)}}\big),$$

see Figure 6 (a). By isotopy, T can be expressed as in Figure 6 (b), where $\beta' = ((t_{\downarrow^{\otimes n}})^{\otimes(p-1)} \otimes \downarrow^{\otimes n})\beta$ is a pure braid, and where the upward parts of the strings run under, and are not involved in, the pure braid β'. We express β' as the product of copies of generators $A_{i,j}$ $(1 \leq i < j \leq np)$ of the np–string pure braid group and their inverses. Here $A_{i,j}$ is as depicted in Figure 7 (a). (See Birman [2] for the generators of the pure braid group.) Using claspers, we can express $A_{i,j}^{\pm 1}$ as depicted in Figure 7 (b), (c). Let T_0 denote the string link obtained from the tangle depicted in Figure 6 (b) by replacing the pure braid β' with $1_{\downarrow^{\otimes np}}$. There are claspers $C'_1, \dots, C'_r$ $(r \geq 0)$ for T_0 corresponding to the generators and inverses involved in β' such that surgery on T_0 along $C'_1, \dots, C'_r$ yields T, see eg Figure 6 (c). We can isotop $T_0, C'_1, \dots, C'_r$ to the identity braid $1_{\downarrow^{\otimes n}}$ and claspers $C_1, \dots, C_r$ satisfying the desired properties, as is easily seen from Figure 6 (d). □

The rest of this subsection is not necessary in the rest of the paper, but seems worth mentioned. Let $\mathcal{S}$ denote the monoidal subcategory of $\mathcal{T}$ generated by the objects $\downarrow$, b and the following morphisms

$$t_\downarrow, t_\downarrow^{-1} \colon \downarrow \to \downarrow, \quad \psi_{\downarrow,\mathsf{b}} \colon \downarrow \otimes \mathsf{b} \to \mathsf{b} \otimes \downarrow, \quad \psi_{\downarrow,\mathsf{b}}^{-1} \colon \mathsf{b} \otimes \downarrow \to \downarrow \otimes \mathsf{b},$$

$\delta_+ =$ , $\delta_- =$ $\colon \downarrow \to \downarrow \otimes \mathsf{b}$, $\alpha =$ $\colon \mathsf{b} \otimes \downarrow \to \downarrow$.

Proposition 4.3 *For $n \geq 0$, we have $\mathsf{SL}_n = \mathcal{S}(\downarrow^{\otimes n}, \downarrow^{\otimes n})$.*

Proof The inclusion $\mathsf{SL}_n \subset \mathcal{S}(\downarrow^{\otimes n}, \downarrow^{\otimes n})$ easily follows from Lemma 4.2. We prove the other inclusion $\mathcal{S}(\downarrow^{\otimes n}, \downarrow^{\otimes n}) \subset \mathsf{SL}_n$. Let $\mathcal{S}_0$ denote the monoidal subcategory of $\mathcal{T}$ generated by the objects $\downarrow$, b and the morphisms $\psi_{\downarrow,\mathsf{b}}$, $\psi_{\downarrow,\mathsf{b}}^{-1}$, $\downarrow \otimes \eta_{\mathsf{b}}$, α. We can prove that $\mathcal{S}_0(\downarrow^{\otimes n}, \downarrow^{\otimes n}) = \{1_{\downarrow^{\otimes n}}\}$. Since any morphism in $\mathcal{S}$ is homotopic to a morphism in $\mathcal{S}_0$, the assertion follows. $\square$

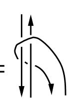
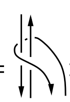
Proposition 4.3 may be useful in studying quantum invariants of string links. For another approach to string links, see Section 13.

4.3 Proof of Theorem 4.1

In this subsection we prove Theorem 4.1. Let B_0' denote the braided subcategory of B generated by the object b and the morphisms

$$\eta_{\mathsf{b}}, \mu_{\mathsf{b}}, \gamma_+, \gamma_-, t_{+-}, t_{+-}^{-1}, t_{\mathsf{b}}, t_{\mathsf{b}}^{-1}, \gamma_+', \gamma_-',$$

where

$\gamma_+' =$ , $\gamma_-' =$ $\colon \mathsf{b} \to \mathsf{b}^{\otimes 2}$.

Since these morphisms are in B_0, it follows that B_0' is a subcategory of B_0. Since

$$t_{\mathsf{b}}^{\pm 1} = \mu_{\mathsf{b}} \gamma_\mp t_{+-}^{\mp 1}, \quad \gamma_\pm' = \psi_{\mathsf{b},\mathsf{b}}^{\pm 1} \gamma_\mp,$$

it follows that B_0' is generated as a braided subcategory of B by the object b and the morphisms $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-, t_{+-}, t_{+-}^{-1}$. Hence it suffices to prove that any morphism in B_0 is in B_0'.

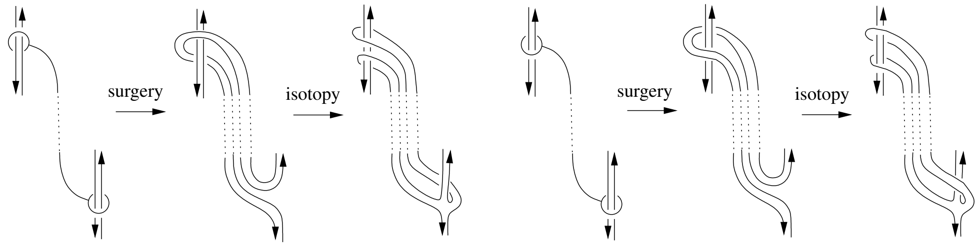


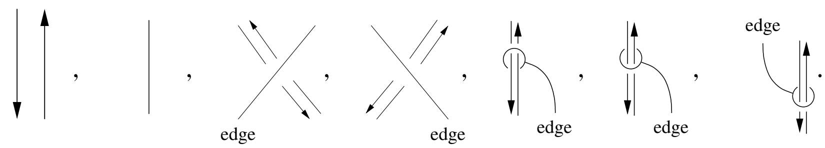
Figure 8

4.3.1 The case of doubled string links We here prove that if $T \in \mathsf{B}_0(n,n)$ is obtained from a framed string link $T' \in \mathsf{SL}_n$ by doubling each component, then we have $T \in \mathsf{B}'_0(n,n)$.

By Lemma 4.2, there are mutually disjoint claspers $C_1, \dots, C_r$ $(r \geq 0)$ for $1_{\mathsf{b}^{\otimes n}}$ and integers $l_1, \dots, l_n \in \mathbb{Z}$ satisfying the following properties.

(1) $\widetilde{T} = T\big(t_{\mathsf{b}}^{l_1} \otimes \cdots \otimes t_{\mathsf{b}}^{l_n}\big)$ is obtained from $1_{\downarrow^{\otimes n}}$ by surgery along $C_1, \dots, C_r$.

(2) $1_{\downarrow^{\otimes n}}$ and $C_1, \dots, C_r$ is obtained by pasting horizontally and vertically finitely many copies of the following:



Surgery on each C_i moves the band intersecting the lower leaf of C_i and let it clasp with the band intersecting the upper leaf of C_i, and we can isotop the result of surgery to the tangle representing a morphism in B'_0 as depicted in Figure 8. Hence it follows that $\widetilde{T}$ is in B'_0. Since $t_{\mathsf{b}}^{l_1} \otimes \cdots \otimes t_{\mathsf{b}}^{l_n} \in \mathsf{B}'_0$, we have $T \in \mathsf{B}'_0$.

4.3.2 The general case Suppose that a tangle T in $[0,1]^3$ represents a morphism $T \in \mathsf{B}_0(m,n)$. We also assume that the endpoints of T are contained in the two intervals $\{\frac{1}{2}\} \times [0,1] \times \{\xi\}$, $\xi = 0, 1$.

For $i = 1, \dots, m$, let c_i denote the interval in $\{\frac{1}{2}\} \times [0,1] \times \{1\}$ bounded by the $(2i-1)$st and the $2i$th upper endpoints of T. Set $c = c_1 \cup \cdots \cup c_m$. Similarly, for $j = 1, \dots, n$, let d_j denote the interval in $\{\frac{1}{2}\} \times [0,1] \times \{0\}$ bounded by the $(2j-1)$st and the $2j$th lower endpoints of T. Set $d = d_1 \cup \cdots \cup d_n$. Note that $T \cup c$ consists of n mutually disjoint arcs $e_1, \dots, e_n$, such that $\partial e_j = \partial d_j$ for $j = 1, \dots, n$. Set $e = e_1 \cup \cdots \cup e_n$.

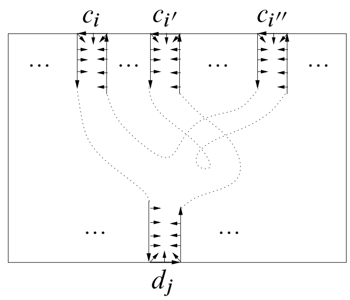


Figure 9: Here the small arrows determines the framing of $e \cup d$ near $c \cup d$.

Consider $T\eta_m$, which is regarded as a tangle in $[0,1]^2 \times [0,2]$, where the lower cube $[0,1]^2 \times [0,1]$ contains T and the upper cube $[0,1]^2 \times [1,2]$ contains η_m. Note that $e \subset [0,1]^2 \times [0,2]$ can be regarded as a tangle, and is equivalent to $T\eta_m$, and hence, by the assumption, equivalent to η_n (after identifying $[0,1]^3$ and $[0,1]^2 \times [0,2]$ in a natural way). Hence for $j = 1, \dots, n$, $e_j \cup d_j$ bounds a disc D_j in $[0,1]^2 \times [0,2]$, where $D_1, \dots, D_n$ are mutually disjoint. Here each D_i is chosen so that the framing of $e_i \cup d_i$ induced by D_i is the same as the framing of $e_i \cup d_i$ induced by that of T. (Here we use the convention that the framing of oriented tangle component is given by the blackboard framing convention, see Figure 9.) Set $D = D_1 \cup \dots \cup D_n$.

Let $\pi\colon [0,1]^2 \times \{1\} \to [0,1]^2$ denote the projection. Using a small isotopy if necessary, we may assume that for small $\epsilon > 0$ we have the following.

- $N = \pi(c) \times [1-\epsilon, 1]$ is a regular neighborhood of c in D.
- $e \setminus N \subset [0,1]^2 \times [0, 1-\epsilon)$.

For $i = 1, \dots, m$, let U_i denote a small regular neighborhood of c_i in $[0,1]^2 \times \{1\}$. Using an isotopy of $[0,1]^2 \times [0,2]$ fixing $[0,1]^3$, we can assume that for each $i = 1, \dots, m$, we have

$$\big(\pi(U_i) \times (1,2]\big) \cap D = \pi(U_i) \times \{p_{i,1}, \dots, p_{i,l_i}\},$$

where $1 < p_{i,1} < \dots < p_{i,l_i} < 2$, $l_i \geq 0$. Define a piecewise-linear homeomorphism $f\colon [0,2] \to [0,1]$ by

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1-\epsilon, \\ \frac{\epsilon t + (1-\epsilon)}{1+\epsilon}, & \text{if } 1-\epsilon \leq t \leq 2. \end{cases}$$

Define $\widetilde{f}\colon [0,1]^2\times[0,2]\to[0,1]^3$ by $\widetilde{f}(x,y,t)=(x,y,f(t))$ for $x,y\in[0,1]$, $t\in[0,2]$. (Thus $\widetilde{f}$ fixes $[0,1]^2\times[0,1-\epsilon]$, and maps $[0,1]^2\times[1-\epsilon,2]$ onto $[0,1]^2\times[1-\epsilon,1]$ linearly.) Set

$$D'=\widetilde{f}(D\setminus N)\cup N.$$

Note that D' is a union of n immersed discs whose only singularities are ribbon singularities

$$\pi(c_i)\times\{f(p_{i,k})\}\quad \text{for } 1\le i\le m,\ 1\le k\le l_i.$$

For $j=1,\dots,n$, let $D'_j\subset D'$ denote the unique immersed disc containing d_j. Note that $\partial D'_j=e_j\cup d_j$.

We prove the assertion by induction on the number $l=l_1+\cdots+l_m$ of ribbon singularities. There are two cases.

Case 1 $l=0$, ie, there are no ribbon singularities in D'. In this case, we claim that T is equivalent to

$$(\mu_{\mathsf{b}}^{[m_1]}\otimes\cdots\otimes\mu_{\mathsf{b}}^{[m_n]})\sigma\beta,$$

where m_j is the number of arcs c_i contained in D'_j for $j=1,\dots,n$, σ is a doubled braid, and β is a doubled string link. This claim can be proved as follows. Choose points $x_i\in\operatorname{int}c_i$ for $i=1,\dots,m$, and $y_j\in\operatorname{int}d_j$ for $j=1,\dots,n$. For each $i=1,\dots,m$, let b_i denote a proper arc in D' which connects x_i and $y_{j(i)}$, where $j(i)$ is such that x_i and y_j are in the same component of D'. Set $b=b_1\cup\cdots\cup b_m$. We may assume that for $i\ne i'$ the intersection $b_i\cap b_{i'}$ is either empty if $y(i)\ne y(i')$, or the common endpoint y_j if $y(i)=y(i')$. By small isotopy, we may assume that for sufficiently small $\epsilon'>0$ the intersection $([0,1]^2\times[0,\epsilon'])\cap b$ consists of "stars" rooted at $y_1,\dots,y_n$. Here each star at y_j consists of m_j line segments, and each $b'_i=([0,1]^2\times[\epsilon',1])\cap b_i$ is an arc. Let D'' be a sufficiently small regular neighborhood of $c\cup b\cup d$ in D' so that $D''\cap([0,1]^2\times[\epsilon',1])$ consists of mutually disjoint bicollar neighborhoods of $b'_1,\dots,b'_m$. There is an isotopy of $[0,1]^3$ which fixes $c\cup b\cup d$ and deforms D' to D''. Set $T'=\partial D''\setminus(\operatorname{int}c\cup\operatorname{int}d)$, which is a tangle isotopic to T. By an isotopy fixing $[0,1]^2\times\{\epsilon'\}$ as a set, we may assume that $([0,1]^2\times[0,\epsilon'])\cap T'$ is a tangle of the form $\mu_{\mathsf{b}}^{[m_1]}\otimes\cdots\otimes\mu_{\mathsf{b}}^{[m_n]}$, and $([0,1]^2\times[\epsilon',1])\cap T'$ is a a tangle of the form $\sigma\beta$, as desired. Hence we have the claim.

It follows from Section 4.3.1 that β is a morphism in B'_0. Obviously, σ and $\mu_{\mathsf{b}}^{[m_1]}\otimes\cdots\otimes\mu_{\mathsf{b}}^{[m_n]}$ are morphisms in B'_0. Hence T is in B'_0.

Case 2 $l\ge1$, ie, there is at least one ribbon singularity in D'. Suppose $i\in\{1,\dots,m\}$ with $l_i\ge1$. Using isotopy of $[0,1]^3$ fixing $(\pi(c)\times[1-\epsilon,1])\cup d$, we see that T is equivalent to

$$T'(\mathsf{b}^{\otimes(i-1)}\otimes\gamma_{\pm}\otimes\mathsf{b}^{\otimes(n-i)}),\tag{4–1}$$

where $T' \in \mathcal{T}(\mathsf{b}^{\otimes(m+1)}, \mathsf{b}^{\otimes n})$. Here the ribbon singularity $\pi(c_i) \times \{f(p_{i,l_i})\}$ of D is isotoped to the obvious ribbon singularity involved in the copy of $\gamma_\pm$ in (4–1). Since we have

$$T'\eta_{m+1} = T'(\mathsf{b}^{\otimes(i-1)} \otimes \gamma_\pm \otimes \mathsf{b}^{\otimes(n-i)})\eta_m = T\eta_m = \eta_n,$$

it follows that $T' \in \mathsf{B}_0(m+1, n)$. Note that T' bounds ribbon discs with less singularities than T by 1. By the induction assumption, it follows that T' is in B'_0. Hence we have $T \in \mathsf{B}'_0(m, n)$.

This completes the proof of Theorem 4.1.

5 Local moves

In this section, we explain how the category B can be used in the study of local moves on links and tangles.

5.1 Monoidal relations and monoidal congruences

In this subsection, we recall the notions of monoidal relations and monoidal congruences in monoidal categories.

Two morphisms in a monoidal category $\mathcal{M}$ are said to be *compatible* if they have the same source and the same target.

A *monoidal relation* in a monoidal category $\mathcal{M}$ is a binary relation $R \subset \mathrm{Mor}(\mathcal{M}) \times \mathrm{Mor}(\mathcal{M})$ on $\mathrm{Mor}(\mathcal{M})$ satisfying the following conditions.

(1) If $(f, f') \in R$, then f and f' are compatible.

(2) If $(f, f') \in R$, $a \in \mathrm{Ob}(\mathcal{M})$, then $(a \otimes f, a \otimes f') \in R$ and $(f \otimes a, f' \otimes a) \in R$.

(3) If $(f, f') \in R$, $g \in \mathrm{Mor}(\mathcal{M})$ and $\mathrm{target}(f) = \mathrm{source}(g)$ (resp. $\mathrm{source}(f) = \mathrm{target}(g)$), then $(gf, gf') \in R$ (resp. $(fg, f'g) \in R$).

For any relation $X \subset \mathrm{Mor}(\mathcal{M}) \times \mathrm{Mor}(\mathcal{M})$ satisfying the condition (1) above, there is the smallest monoidal relation R_X containing X, which is called the monoidal relation in $\mathcal{M}$ generated by X. If $X = \{(f, f')\}$, then R_X is also said to be generated by the pair (f, f').

Suppose $f, f' \in \mathcal{M}(a, b)$ and $g, g' \in \mathcal{M}(c, d)$ with $a, b, c, d \in \mathrm{Ob}(\mathcal{M})$. Then g and g' are related by the monoidal relation generated by (f, f'), if and only if there are $z, z' \in \mathrm{Ob}(\mathcal{M})$ and morphisms $h_1 \in \mathcal{M}(c, z \otimes a \otimes z')$, $h_2 \in \mathcal{M}(z \otimes b \otimes z', d)$ such that

$$g = h_2(z \otimes f \otimes z')h_1, \quad g' = h_2(z \otimes f' \otimes z')h_1. \tag{5–1}$$

Lemma 5.1 *Let $\mathcal{M}$ be a braided category and let $(f, f') \in \mathcal{M}(\mathbf{1}, b)$ with $b \in \mathrm{Ob}(\mathcal{M})$. Then $g, g' \in \mathcal{M}(c, d)$ $(c, d \in \mathrm{Ob}(\mathcal{T}))$ are related by the monoidal relation generated by (f, f') if and only if there is $h \in \mathcal{M}(c \otimes b, d)$ such that*

$$g = h(c \otimes f), \quad g' = h(c \otimes f').$$

Proof The "if" part is obvious. We prove the "only if" part. By assumption, there are $z, z' \in \mathrm{Ob}(\mathcal{M})$ and $h_1 \in \mathcal{M}(c, z \otimes z')$, $h_2 \in \mathcal{M}(z \otimes b \otimes z', d)$ satisfying (5–1). Set $h = h_2(z \otimes \psi_{z',b})(h_1 \otimes b)$. Then we have the assertion. □

A *monoidal congruence*, also called *four-sided congruence*, in a monoidal category $\mathcal{M}$ is a monoidal relation in $\mathcal{M}$ which is an equivalence relation. If $\sim$ is a monoidal congruence in a monoidal (resp. braided) category $\mathcal{M}$, then the quotient category $\mathcal{M}/\sim$ is equipped with a monoidal (resp. braided) category structure induced by that of $\mathcal{M}$.

Example 5.2 The notion of homotopy (see Section 3.2) for morphisms in $\mathcal{T}$ is a monoidal congruence in $\mathcal{T}$ generated by $\{(\psi_{\downarrow,\downarrow}, \psi_{\downarrow,\downarrow}^{-1}), (1_{\downarrow}, t_{\downarrow})\}$.

5.2 Topological and algebraic definitions of local moves

In this subsection, we first recall a formulation of local moves on links and tangles, and then we reformulate it in the setting of the category $\mathcal{T}$.

Informally, a "local move" is an operation on a link (or a tangle) which replaces a tangle in a link (or a tangle) contained in a 3–ball B with another tangle. The following is a precise definition of the notion of local moves.

Definition 5.3 Two tangles t and t' in a 3–ball B are said to be *compatible* if we have $\partial t = \partial t'$ and t and t' have the same framings and the same orientations at the endpoints. Let (t, t') be a compatible pair of tangles in B. For two compatible tangles u and u' in another 3–ball D, we say that u and u' are (t, t')*–related*, or u' is obtained from u by a (t, t')*–move*, if there is an orientation-preserving embedding $f\colon B \hookrightarrow \mathrm{int}\, D$ and a tangle u'' in D isotopic to u' such that

$$f(t) = u \cap f(B), \quad f(t') = u'' \cap f(B), \quad u \setminus \mathrm{int}\, f(B) = u'' \setminus \mathrm{int}\, f(B), \tag{5–2}$$

where the orientations and framings are the same in the two 1–submanifolds in each side of these three identities.

Now we give an algebraic formulation of local moves.

Definition 5.4 Let (T, T') be a compatible pair of morphisms in $\mathcal{T}$. Let $R_{(T,T')}$ be the monoidal relation generated by the pair (T, T'). For two morphisms U and U' in $\mathcal{T}$, we say that U and U' are (T, T')*–related*, or U' is obtained from U by a (T, T')*–move*, if $(U, U') \in R_{(T,T')}$.

The following shows that we can reduce the study of local moves defined by compatible pairs of tangles in a 3–ball B to the study of local moves defined by compatible pairs of morphisms in $\mathcal{T}$.

Proposition 5.5 *Let (t, t') be a compatible pair of tangles in a 3–ball B, and let C be a compatibility class of tangles in another 3–ball D. Then there is a compatible pair (T, T') of morphisms in $\mathcal{T}$ and an orientation-preserving homeomorphism $g\colon D \cong [0,1]^3$ such that two tangles $u, u' \in C$ are (t, t')–related if and only if the morphisms in $\mathcal{T}$ represented by $g(u)$ and $g(u')$ are (T, T')–related as morphisms in $\mathcal{T}$.*

Proof We choose an orientation-preserving homeomorphism $h\colon B \cong [0,1]^3$ such that we have $h(\partial t) = h(\partial t') \subset \{\frac{1}{2}\} \times (0,1) \times \{0,1\}$ so that the tangles $h(\partial t), h(\partial t') \subset [0,1]^3$ represents (compatible) morphisms in $\mathcal{T}$. Similarly, we choose an orientation-preserving homeomorphism $g\colon D \cong [0,1]^3$ such that for any $u \in C$ we have $g(\partial u) \subset \{\frac{1}{2}\} \times (0,1) \times \{0,1\}$ so that for $u \in C$ the tangle $g(u) \subset [0,1]^3$ represents a morphism in $\mathcal{T}$. Set $T = h(t), T' = h(t') \subset [0,1]^3$, which are regarded as morphisms in $\mathcal{T}$.

It is easy to verify the "if" part. We prove the "only if" part below. Suppose $u, u' \in C$ are (t, t')–related. By the definition, there is an orientation-preserving embedding $f\colon B \hookrightarrow D$ and a tangle $u'' \in C$ isotopic to u' satisfying (5–2). We can assume without loss of generality that $u'' = u'$. Set

$$f' = gfh^{-1}\colon [0,1]^3 \hookrightarrow [0,1]^3.$$

Then $g(u)$ and $g(u')$, as tangles in $[0,1]^3$, are related by a (T, T')–move via f'. By applying an appropriate self-homeomorphism of $[0,1]^3$ fixing boundary to both $g(u)$ and $g(u')$, we have in $\mathcal{T}$

$$g(u) = W_2(z \otimes T \otimes z')W_1, \quad g(u') = W_2(z \otimes T' \otimes z')W_1,$$

where $z, z' \in \mathrm{Ob}(\mathcal{T})$ and

$$W_1 \in \mathcal{T}(\mathrm{source}(g(u)), z \otimes \mathrm{source}(T) \otimes z'),$$
$$W_2 \in \mathcal{T}(z \otimes \mathrm{target}(T) \otimes z', \mathrm{target}(g(u))).$$

This shows that $g(u)$ and $g(u')$ are (T, T')–related as morphisms in $\mathcal{T}$. □

Definition 5.6 Two compatible pairs (T_1, T_1') and (T_2, T_2') of morphisms in $\mathcal{T}$ are said to be *equivalent* to each other if there is an orientation-preserving self-homeomorphism ϕ (not necessarily fixing the boundary) of the cube $[0,1]^3$ such that $\phi(T_1) = T_2$ and $\phi(T_1') = T_2'$.

Note that if (T_1, T_1') and (T_2, T_2') are equivalent pairs of mutually compatible morphisms in $\mathcal{T}$, then the notions of (T_1, T_1')–move and (T_2, T_2')–move are the same, ie, two tangles U and U' are (T_1, T_1')–related if and only if U and U' are (T_2, T_2')–related.

5.3 Local moves defined by pairs of bottom tangles

In the following we restrict our attention to local moves defined by pairs of mutually homotopic tangles T and T' consisting only of arc components. Let us call such a local move an *arc local move*. Arc local moves fit nicely into the setting of the category B.

The following implies that, to study the arc local moves, it suffices to study the local moves defined by pairs of bottom tangles.

Proposition 5.7 *Let (T, T') be a pair of mutually homotopic morphisms in $\mathcal{T}$, each consisting of n arc components and no circle components. Then there is a pair (T_1, T_1') of mutually homotopic n–component bottom tangles which is equivalent to (T, T').*

Proof Set $a = \mathrm{source}(T)$ and $b = \mathrm{target}(T)$. There is a (not unique) framed braid $\beta \in \mathcal{T}(b \otimes a^*, \mathsf{b}^{\otimes n})$ such that

$$T_1 = \beta(T \otimes a^*)\mathrm{coev}_a \quad \text{and} \quad T_1' = \beta(T' \otimes a^*)\mathrm{coev}_a$$

are bottom tangles. Clearly, the two pairs (T, T') and (T_1, T_1') are equivalent, and T_1 and T_1' are homotopic to each other. Hence we have the assertion. □

We are in particular interested in arc local moves on bottom tangles. The following theorem implies that the study of arc local moves on tangles in B is reduced to the study of monoidal relations in B generated by pairs of bottom tangles.

Theorem 5.8 *For $T, T' \in \mathsf{BT}_n$ and $U, U' \in \mathsf{B}(k, l)$, the following conditions are equivalent.*

(1) *U and U' are (T, T')–related.*

(2) *U and U' are "(T, T')–related in B", ie, related by the monoidal relation in B generated by (T, T').*

(3) *There is a morphism* $W \in \mathsf{B}(k+n, l)$ *such that*

$$U = W(\mathsf{b}^{\otimes k} \otimes T), \quad U' = W(\mathsf{b}^{\otimes k} \otimes T'). \tag{5–3}$$

Proof By Lemma 5.1, (2) and (3) are equivalent. Obviously, (2) implies (1). We show that (1) implies (3). By assumption, U and U' are (T, T')–related. By Lemma 5.1, there is $W \in \mathcal{T}(\mathsf{b}^{\otimes(k+n)}, \mathsf{b}^{\otimes l})$ satisfying (5–3). We have

$$W\eta_{k+n} = W(\mathsf{b}^{\otimes k} \otimes \eta_n)\eta_k \sim_h W(\mathsf{b}^{\otimes k} \otimes T)\eta_k = U\eta_k \sim_h \eta_l,$$

where we used the fact that T is a bottom tangle and U is a morphism in B. Hence we have $W \in \mathsf{B}(k+n, l)$. □

5.4 Admissible local moves

An n–component tangle t in a 3–ball B is said to be *admissible* if the pair (B, t) is homeomorphic to the pair $([0,1]^3, \eta_n)$. (In the literature, such a tangle is sometimes called "trivial tangle", but here we do not use this terminology, since it may give an impression that a tangle is equivalent to a "standard" tangle such as η_n.)

A compatible pair (t, t') of tangles in a 3–ball B is called *admissible* if both t and t' are admissible. A local move defined by admissible pair is called an admissible local move. In this subsection, we translate some well-known properties of admissible local moves into our category-theoretical setting.

It follows from the previous subsections that, to study admissible local moves on morphisms in B, it suffices to study the monoidal relations in B generated by pairs of admissible bottom tangles with the same number of components.

For $n \geq 0$, let ABT_n denote the subset of BT_n consisting of admissible bottom tangles. We set

$$\mathsf{ABT} = \bigcup_{n \geq 0} \mathsf{ABT}_n \subset \mathsf{BT}.$$

Lemma 5.9 *If* $T, T' \in \mathsf{ABT}_n$, *then there is* $V \in \mathsf{ABT}_n$ *such that the two pairs* (T, T') *and* (η_n, V) *are equivalent.*

Proof Since $T \in \mathsf{ABT}_n$, there is a framed pure braid $\beta \in \mathcal{T}(\mathsf{b}^{\otimes n}, \mathsf{b}^{\otimes n})$ of $2n$–strings such that $\beta T = \eta_n$. Setting $V = \beta T'$, we can easily verify the assertion. □

Lemma 5.9 above implies that, to study admissible local moves on morphisms in B, it suffices to study the admissible local moves defined by pairs (η_n, T) for $T \in \mathsf{ABT}_n$. Hence it is useful to make the following definition. If two tangles U and U' in B are

(η_n, T)–related, then we simply say that U and U' are T*–related*, or U' is obtained from U by a T*–move*.

Now we consider sequences of admissible local moves.

Proposition 5.10 *Let $T_1, \dots, T_r \in \mathsf{ABT}$, $r \geq 0$, and let $U, U' \in \mathsf{B}(k,l)$ be two morphisms in B. Then the following conditions are equivalent.*

(1) *There is a sequence $U_0 = U, U_1, \dots, U_r = U'$ of morphisms in B from U to U' such that, for $i = 1, \dots, r$, the tangles U_{i-1} and U_i are T_i–related.*

(2) *U and U' are $(T_1 \otimes \cdots \otimes T_r)$–related.*

Proof Obviously, (2) implies (1). We show that (1) implies (2). It is well known (see, for example, [22, Lemma 3.21]) that if there is a sequence from a tangle U to another tangle U' of admissible local moves, then T' can be obtained from T by simultaneous application of admissible local moves of the same types as those appearing in the sequence. Hence, after suitable isotopy of $[0,1]^3$ fixing the boundary, there are mutually disjoint small cubes $C_1, \dots, C_r$ in $[0,1]^3$ such that

- for $i = 1, \dots, r$, the tangle $C_i \cap U$ in C_i is equivalent to η_{n_i}, where we set $n_i = |T_i|$,
- the tangle obtained from U by replacing the copy of η_{n_i} in C_i with a copy of T_i for all $i = 1, \dots, r$ is equivalent to U'.

Using an isotopy of $[0,1]^3$ fixing the boundary which move the cubes $C_1, \dots, C_r$ to the upper right part of $[0,1]^3$, we can express U and U' as

$$U = W(\mathsf{b}^{\otimes k} \otimes \eta_{n_1 + \cdots + n_r}), \quad U' = W(\mathsf{b}^{\otimes k} \otimes T_1 \otimes \cdots \otimes T_r),$$

where $W \in \mathcal{T}(\mathsf{b}^{\otimes(k+n_1+\cdots+n_r)}, \mathsf{b}^{\otimes l})$. One can easily verify $W\eta_{k+n_1+\cdots+n_r} \sim_h \eta_l$, hence $W \in \mathsf{B}(k+n_1+\cdots+n_r, l)$. Hence we have the assertion. □

In the study of local moves, it is often useful to consider the relations on tangles defined by several types of moves. Let $M \subset \mathsf{ABT}$ be a subset. For two tangles U and U' in B, we say that U and U' are M*–related*, or U' is obtained from U by an M*–move*, if there is $T \in M$ such that U and U' are T–related.

For $M \subset \mathsf{ABT}$, let M^* denote the subset of ABT of the form $T_1 \otimes \cdots \otimes T_r$ with $T_i \in M$ for $i = 1, \dots, r$, $r \geq 0$. Note that $M^* \subset \mathsf{ABT}$. The following immediately follows from Proposition 5.10.

Proposition 5.11 *Let $M \subset \mathsf{ABT}$. Then $U, U' \in \mathsf{B}(k,l)$ are related by a finite sequence of M–moves if and only if U and U' are M^*–related.*

For $M \subset \mathsf{ABT}$, the *M–equivalence* is the equivalence relation on tangles generated by the M–moves. Note that, for morphisms in B, the M–equivalence is the same as the monoidal congruence in B generated by the set $\{(\eta_{|T|}, T) \mid T \in M\}$.

A subset $M \subset \mathsf{ABT}$ is said to be *inversion-closed* if for each $T \in M$, there is a sequence of M–moves from T to $\eta_{|T|}$. In this case, two tangles U and U' are M–equivalent if and only if there is a sequence of M–moves from U to U'.

Given any subset $M \subset \mathsf{ABT}$, one can construct an inversion-closed subset $M' \subset \mathsf{ABT}$ such that any two morphisms U and U' in B are M–equivalent if and only if there is a sequence from U to U' of M'–moves. For example, $M' = M \cup \{\overline{T} \mid T \in M\}$ satisfies this condition, where $\overline{T} = \beta^{-1}\eta_n$ with $n = |T|$ and $\beta \in \mathsf{B}(n,n)$ a (not unique) framed pure braid such that $T = \beta\eta_n$. (Note that the pair $(\overline{T}, \eta_n)$ is equivalent to (η_{b}, T).)

By Proposition 5.11, we have the following.

Proposition 5.12 *Let $M \subset \mathsf{ABT}$ be inversion-closed. Then $U, U' \in \mathsf{B}(k,l)$ are M–equivalent if and only if they are M^*–related.*

5.5 Tangles obtained from η_n by an admissible local move

Let $\check{\mathsf{B}}_0$ denote the braided subcategory of B_0 generated by the object b and the morphisms $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-$, and let T denote the monoidal subcategory of B_0 generated by the object b and the morphisms t_{+-} and t_{+-}^{-1}. It is easy to see that for $n \geq 0$ we have

$$\mathsf{T}(n,n) = \{t_{+-}^{k_1} \otimes \cdots \otimes t_{+-}^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\},$$

and $\mathsf{T}(m,n)$ is empty if $m \neq n$.

Lemma 5.13 *For any morphism $T \in \mathsf{B}_0(m,n)$ with $m, n \geq 0$, there are $T' \in \check{\mathsf{B}}_0(m,n)$ and $T'' \in \mathsf{T}(m,m)$ such that $T = T'T''$. (The decomposition $T = T'T''$ is not unique.)*

Proof Using Theorem 4.1 and the identities

$$t_{+-}^{k}\mu_{\mathsf{b}} = \mu_{\mathsf{b}}(t_{+-}^{k} \otimes t_{+-}^{k}), \quad t_{+-}^{k}\eta_{\mathsf{b}} = \eta_{\mathsf{b}}, \quad (t_{+-}^{k} \otimes t_{+-}^{l})\gamma_{\pm} = \gamma_{\pm}t_{+-}^{l},$$

for $k, l \in \mathbb{Z}$, we can easily prove the assertion. □

Theorem 5.14 *Let $T \in \mathsf{ABT}_m$ and $U \in \mathsf{BT}_n$. Then U is obtained from η_n by one T–move if and only if there is $W \in \check{\mathsf{B}}_0(m,n)$ such that $U = WT$.*

Proof The "if" part is obvious. We prove the "only if" part. Suppose that η_n and $U \in \mathsf{BT}_n$ are T–related. By Theorem 5.8, there is $W' \in \mathsf{B}(m,n)$ such that $\eta_n = W'\eta_m$ and $U = W'T$. The first identity means that $W' \in \mathsf{B}_0(m,n)$. By Lemma 5.13, we have $W' = WV$, where $W \in \check{\mathsf{B}}_0(m,n)$ and $V \in \mathsf{T}(m,m)$. It is easy to see that $VT = T$. Hence we have $U = W'T = WVT = WT$. □

For $M \subset \mathsf{ABT}$, a bottom tangle $T \in \mathsf{BT}_n$ is said to be M*–trivial* if T is M–equivalent to η_n. The following immediately follows from Proposition 5.12 and Theorem 5.14.

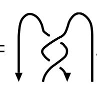
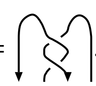
Corollary 5.15 *Let $M \subset \mathsf{ABT}$ be inversion-closed, and let $U \in \mathsf{BT}_n$. Then U is M–trivial if and only if there are $T \in M^*$ and $W \in \check{\mathsf{B}}_0(|T|,n)$ such that $U = WT$.*

5.6 Generators of B

Here we use the results in the previous subsections to obtain a simple set of generators of B. Define morphisms $v_\pm \in \mathsf{BT}_1$ and $c_\pm \in \mathsf{BT}_2$ by

$$v_\pm = (t_\downarrow^{\mp 1} \otimes \uparrow)\eta_{\mathsf{b}}, \quad c_\pm = (\downarrow \otimes (\psi_{\downarrow,\uparrow}\psi_{\uparrow,\downarrow})^{\pm 1} \otimes \uparrow)(\eta_{\mathsf{b}} \otimes \eta_{\mathsf{b}}).$$

Graphically, we have

$v_+ =$ , $\quad v_- =$ , $\quad c_+ =$ , $\quad c_- =$ .

Note that a $v_\pm$–move is change of framing by 1, and a $c_\pm$–move is a crossing change. Hence two morphisms in B are homotopic if and only if they are $\{v_+, v_-, c_+, c_-\}$–equivalent.

Theorem 5.16 *As a braided subcategory of $\mathcal{T}$, B is generated by the object b and the morphisms $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, v_+, v_-, c_+, c_-$.*

Proof Let B' denote the braided subcategory of $\mathcal{T}$ generated by the object b and the morphisms $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, v_+, v_-, c_+, c_-$. It suffices to show that any tangle $U' \in \mathsf{B}(m,n)$ is a morphism in B'.

Choose a tangle $U \in \mathsf{B}_0(m,n)$ which is homotopic to U', ie, $\{v_+, v_-, c_+, c_-\}$–equivalent to U'. Since $\{v_+, v_-, c_+, c_-\}$ is an inversion-closed subset of ABT, Theorem 5.8 and Proposition 5.12 imply that there are $T \in \{v_+, v_-, c_+, c_-\}^*$ and $W \in \mathsf{B}(m+|T|,n)$ such that

$$U = W(\mathsf{b}^{\otimes m} \otimes \eta_{|T|}), \tag{5–4}$$

$$U' = W(\mathsf{b}^{\otimes m} \otimes T). \tag{5–5}$$

Since U is a morphism in B_0, (5–4) implies that W is a morphism in B_0. The generators of B_0 given in Theorem 4.1 are in B', since we have

$$\gamma_\pm = (\mu_{\mathsf{b}} \otimes \mathsf{b})(\mathsf{b} \otimes \psi_{\mathsf{b},\mathsf{b}}^{\pm 1})(\mathsf{b} \otimes \mu_{\mathsf{b}}^{[3]} \otimes \mathsf{b})(c_\pm \otimes \mathsf{b} \otimes c_\mp),$$
$$t_{+-}^{\pm 1} = \mu_{\mathsf{b}}^{[3]}(v_\mp \otimes \mathsf{b} \otimes v_\pm).$$

Hence W is in B'. Since $\mathsf{b}^{\otimes p} \otimes T$ is in B', it follows from (5–5) that U' is in B'. This completes the proof. □

Remark 5.17 The set of generators of B given in Theorem 5.16 is not minimal. One can show, for example, that B is minimally generated as a braided subcategory of $\mathcal{T}$ by the object b and the morphisms μ_{b}, v_+, c_+ and c_-.

Theorem 5.16 implies that each bottom tangle can be obtained as a result of horizontal and vertical pasting of finitely many copies of the tangles 1_{b}, $\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$, μ_{b}, η_{b}, v_+, v_-, c_+, c_-.

In the following we give several corollaries to Theorem 5.16.

The following notation is useful in the rest of the paper. For $f \in \mathsf{B}(m,n)$ and $i, j \geq 0$, set

$$f_{(i,j)} = \mathsf{b}^{\otimes i} \otimes f \otimes \mathsf{b}^{\otimes j} \in \mathsf{B}(i+m+j, i+n+j).$$

The following corollary to Theorem 5.16 is sometimes useful.

Corollary 5.18 *As a subcategory of $\mathcal{T}$, B is generated by the objects $\mathsf{b}^{\otimes n}$, $n \geq 0$, and the morphisms*

$$\begin{aligned} (\psi_{\mathsf{b},\mathsf{b}})_{(i,j)}, (\psi_{\mathsf{b},\mathsf{b}}^{-1})_{(i,j)} \quad & \text{for } i, j \geq 0,\\ f_{(i,0)} \quad & \text{for } f \in \{\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, v_\pm, c_\pm\}, i \geq 0. \end{aligned}$$

Proof By Theorem 5.16, B is generated as a subcategory of $\mathcal{T}$ by the morphisms $f_{(i,j)}$ with $f \in \{\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}, \mu_{\mathsf{b}}, \eta_{\mathsf{b}}, v_\pm, c_\pm\}$, $i, j \geq 0$. For $f \neq \psi_{\mathsf{b},\mathsf{b}}^{\pm 1}$, we can express $f_{(i,j)}$ as a conjugate of $f_{(i+j,0)}$ by a doubled braid. (Here a *doubled braid* means a morphism in the braided subcategory of B generated by the object b.) This implies the assertion. □

Corollary 5.19 *(1) Each $T \in \mathsf{BT}_n$ can be expressed as*

$$T = \left(t_{+0}^{p_1} \otimes \cdots \otimes t_{+0}^{p_n}\right)\left(\mu_{\mathsf{b}}^{[j_1]} \otimes \cdots \otimes \mu_{\mathsf{b}}^{[j_n]}\right)\beta\left(c_+^{\otimes l_+} \otimes c_-^{\otimes l_-}\right) \tag{5–6}$$

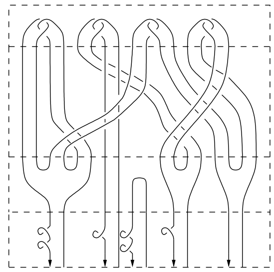


Figure 10: An example of T with $n=5$, $(p_1,\dots,p_5)=(-2,1,2,-1,0)$, $(j_1,\dots,j_5)=(3,1,0,2,2)$, and $l_+=l_-=2$

with $t_{+0}=t_{\downarrow}\otimes\uparrow(=\mu_{\mathsf{b}}(v_-\otimes\mathsf{b}))\in\mathsf{B}(1,1)$, $p_1,\dots,p_n\in\mathbb{Z}$, $j_1,\dots,j_n\geq 0$, $l_+,l_-\geq 0$, $2(l_++l_-)=j_1+\cdots+j_n$, *and* $\beta\in\mathsf{B}(2(l_++l_-),2(l_++l_-))$ *a doubled braid. (For example, see Figure 10.)*

(2) Each $T\in\mathsf{BT}_n$ *can be expressed up to framing change as*

$$T=\big(\mu_{\mathsf{b}}^{[j_1]}\otimes\cdots\otimes\mu_{\mathsf{b}}^{[j_n]}\big)\beta\big(c_+^{\otimes l_+}\otimes c_-^{\otimes l_-}\big) \tag{5–7}$$

where $j_1,\dots,j_n\geq 0$, $l_+,l_-\geq 0$ *with* $2(l_++l_-)=j_1+\cdots+j_n$, *and where* $\beta\in\mathsf{B}(2(l_++l_-),2(l_++l_-))$ *is a doubled braid.*

Proof Note that composing $\mathsf{b}^{\otimes(i-1)}\otimes t_{+0}^{p_i}\otimes\mathsf{b}^{\otimes(n-i)}$ from the left to $T\in\mathsf{BT}_n$ just changes the framing of the ith component of T by p_i. Hence we have only to prove (2). In the following we ignore the framing. It suffices to prove that if T is as in (5–7), and $U=f_{(r,s)}$ with $f\in\{\psi_{\mathsf{b},\mathsf{b}}^{\pm1},\mu_{\mathsf{b}},\eta_{\mathsf{b}},c_{\pm}\}$ and $r,s\geq 0$ such that UT is well defined, then UT has a decomposition similar to (5–7). In the following we use the notation

$$\mu_{\mathsf{b}}^{[a_1,a_2,\dots,a_k]}=\mu_{\mathsf{b}}^{[a_1]}\otimes\mu_{\mathsf{b}}^{[a_2]}\otimes\cdots\otimes\mu_{\mathsf{b}}^{[a_k]}$$

for $a_1,\dots,a_k\geq 0$.

The case $f=\psi_{\mathsf{b},\mathsf{b}}$ follows from

$$(\psi_{\mathsf{b},\mathsf{b}})_{(r,s)}\mu_{\mathsf{b}}^{[j_1,\dots,j_n]}=\mu_{\mathsf{b}}^{[j_1,\dots,j_r,j_{r+2},j_{r+1},j_{r+3},\dots,j_n]}(\psi_{\mathsf{b}^{\otimes j_{r+1}},\mathsf{b}^{\otimes j_{r+2}}})_{(j_1+\cdots+j_r,\,j_{r+3}+\cdots+j_n)}.$$

The case $f=\psi_{\mathsf{b},\mathsf{b}}^{-1}$ is similar. The cases $f=\mu_{\mathsf{b}},\eta_{\mathsf{b}}$ follow from

$$\mu_{(r,s)}\mu_{\mathsf{b}}^{[j_1,\dots,j_n]}=\mu_{\mathsf{b}}^{[j_1,\dots,j_r,j_{r+1}+j_{r+2},j_{r+3},\dots,j_n]},$$
$$\eta_{(r,s)}\mu_{\mathsf{b}}^{[j_1,\dots,j_n]}=\mu_{\mathsf{b}}^{[j_1,\dots,j_r,0,j_{r+1},\dots,j_n]}.$$

For $f = c_\pm$, we have

$$f_{(r,s)}\mu_{\mathsf{b}}^{[j_1,\dots,j_n]}\beta(c_+^{\otimes l_+}\otimes c_-^{\otimes l_-}) = \mu_{\mathsf{b}}^{[j_1,\dots,j_r,1,1,j_{r+1},\dots,j_n]}(1^{\otimes(j_1+\dots+j_r)}\otimes\psi_{\mathsf{b}^{\otimes(j_{r+1}+\dots+j_n)},\mathsf{b}^{\otimes 2}}) (\beta\otimes\mathsf{b}^{\otimes 2})(c_+^{\otimes l_+}\otimes c_-^{\otimes l_-}\otimes f).$$

If $f = c_-$, then we are done. The other case $f = c_+$ follows from

$$c_+^{\otimes l_+}\otimes c_-^{\otimes l_-}\otimes c_+ = (\mathsf{b}^{\otimes l_+}\otimes\psi_{\mathsf{b}^{\otimes 2},\mathsf{b}^{\otimes 2l_-}})(c_+^{\otimes(l_++1)}\otimes c_-^{\otimes l_-}). \qquad \square$$

Remark 5.20 In Corollary 5.19 (1), we may assume that $p_1,\dots,p_n\in\{0,1\}$. This follows from the identity $t_{+0}^{\pm 2} = \mu_{\mathsf{b}}^{[3]}(c_\mp\otimes\mathsf{b})$. In particular, it follows that if each component of $T\in\mathsf{BT}_n$ is of even framing, then T can be expressed as in Corollary 5.19 (1) with $p_1=\dots=p_n=0$. This fact is used in Section 14.2.5.

Let $\mathcal{A}$ denote the braided subcategory of B generated by the object b and the morphisms μ_{b} and η_{b}. ($\mathcal{A}$ is naturally isomorphic to the braided category $\langle\mathsf{A}\rangle$ freely generated by an algebra A, defined later in Section 6.2, but we do not need this fact.) Clearly, $\mathcal{A}$ is a subcategory of $\check{\mathsf{B}}_0$ (and hence of B_0). We need the following corollary later.

Corollary 5.21 *Any $T\in\mathsf{BT}_n$ can be expressed as a composition $T = T'T''$ with $T'\in\mathcal{A}(m,n)$ and $T''\in\{v_\pm, c_\pm\}^*\cap\mathsf{BT}_m$, $m\ge 0$.*

Proof This easily follows from Theorem 5.16, similarly to Corollary 5.19. $\square$

6 Hopf algebra action on bottom tangles

6.1 The braided category $\langle\mathsf{H}\rangle$ freely generated by a Hopf algebra H

Let $\langle\mathsf{H}\rangle$ denote the braided category freely generated by a Hopf algebra H. In other words, $\langle\mathsf{H}\rangle$ is a braided category with a Hopf algebra H such that if $\mathcal{M}$ is a braided category and H is a Hopf algebra in $\mathcal{M}$, then there is a unique braided functor $F_H\colon\langle\mathsf{H}\rangle\to\mathcal{M}$ that maps the Hopf algebra structure of H into that of H. Such $\langle\mathsf{H}\rangle$ is unique up to isomorphism.

A more concrete definition of $\langle\mathsf{H}\rangle$ (up to isomorphism) is sketched as follows. Set $\mathrm{Ob}(\langle\mathsf{H}\rangle)=\{\mathsf{H}^{\otimes n}\mid n\ge 0\}$. Consider the expressions obtained by compositions and tensor products from copies of the morphisms

$$1_{\mathbf{1}}\colon\mathbf{1}\to\mathbf{1},\quad 1_{\mathsf{H}}\colon\mathsf{H}\to\mathsf{H},\quad \psi_{\mathsf{H},\mathsf{H}}^{\pm 1}\colon\mathsf{H}^{\otimes 2}\to\mathsf{H}^{\otimes 2},\quad \mu_{\mathsf{H}}\colon\mathsf{H}^{\otimes 2}\to\mathsf{H},\quad \eta_{\mathsf{H}}\colon\mathbf{1}\to\mathsf{H},$$
$$\Delta_{\mathsf{H}}\colon\mathsf{H}\to\mathsf{H}^{\otimes 2},\quad \epsilon_{\mathsf{H}}\colon\mathsf{H}\to\mathbf{1},\quad S_{\mathsf{H}}\colon\mathsf{H}\to\mathsf{H},$$

where we understand $\mathbf{1} = \mathsf{H}^{\otimes 0}$ and $\mathsf{H} = \mathsf{H}^{\otimes 1}$, and define an equivalence relation on such expressions generated by the axioms of braided category and Hopf algebra. Then the morphisms in $\langle\mathsf{H}\rangle$ are the equivalence classes of such expressions.

Note that if $F\colon \langle\mathsf{H}\rangle \to \mathcal{M}$ is a braided functor of $\langle\mathsf{H}\rangle$ into a braided category $\mathcal{M}$, then $F(\mathsf{H}) \in \mathrm{Ob}(\mathcal{M})$ is equipped with a Hopf algebra structure

$$(F(\mu_{\mathsf{H}}), F(\eta_{\mathsf{H}}), F(\Delta_{\mathsf{H}}), F(\epsilon_{\mathsf{H}}), F(S_{\mathsf{H}})).$$

Conversely, if H is a Hopf algebra in $\mathcal{M}$, then there is a unique braided functor $F\colon \langle\mathsf{H}\rangle \to \mathcal{M}$ such that F maps the Hopf algebra structure of H into that of H. Hence there is a canonical one-to-one correspondence between the Hopf algebras in a braided category $\mathcal{M}$ and the braided functors from $\langle\mathsf{H}\rangle$ to $\mathcal{M}$.

For $f \in \langle\mathsf{H}\rangle(\mathsf{H}^{\otimes m}, \mathsf{H}^{\otimes n})$ $(m,n \geq 0)$ and $i,j \geq 0$, set

$$f_{(i,j)} = \mathsf{H}^{\otimes i} \otimes f \otimes \mathsf{H}^{\otimes j} \in \langle\mathsf{H}\rangle(\mathsf{H}^{\otimes(m+i+j)}, \mathsf{H}^{\otimes(n+i+j)}).$$

Note that $\langle\mathsf{H}\rangle$ is generated as a category by the objects $\mathsf{H}^{\otimes i}$, $i \geq 0$, and the morphisms $f_{(i,j)}$ with $f \in \{\psi_{\mathsf{H},\mathsf{H}}, \psi^{-1}_{\mathsf{H},\mathsf{H}}, \mu, \eta, \Delta, \epsilon, S\}$ and $i,j \geq 0$. In the following, we write $(\psi^{\pm 1}_{\mathsf{H},\mathsf{H}})_{(i,j)} = \psi^{\pm 1}_{(i,j)}$.

Lemma 6.1 *As a category, $\langle\mathsf{H}\rangle$ has a presentation with the generators $f_{(i,j)}$ for $f \in \{\psi_{\mathsf{H},\mathsf{H}}, \psi^{-1}_{\mathsf{H},\mathsf{H}}, \mu, \eta, \Delta, \epsilon, S\}$ and $i,j \geq 0$, and the relations*

(6–1) $$f_{(i,j+q'+k)}g_{(i+p+j,k)} = g_{(i+p'+j,k)}f_{(i,j+q+k)},$$

(6–2) $$\psi_{(i,j)}\psi^{-1}_{(i,j)} = \psi^{-1}_{(i,j)}\psi_{(i,j)} = 1_{\mathsf{H}^{\otimes(i+j+2)}},$$

(6–3) $$\begin{cases} (\psi_{p',1})_{(i,j)}f_{(i,j+1)} = f_{(i+1,j)}(\psi_{p,1})_{(i,j)}, \\ (\psi_{1,p'})_{(i,j)}f_{(i+1,j)} = f_{(i,j+1)}(\psi_{1,p})_{(i,j)}, \end{cases}$$

(6–4) $$\begin{cases} \mu_{(i,j)}\eta_{(i,j+1)} = \mu_{(i,j)}\eta_{(i+1,j)} = 1_{\mathsf{H}^{\otimes(i+j+1)}}, \\ \mu_{(i,j)}\mu_{(i,j+1)} = \mu_{(i,j)}\mu_{(i+1,j)}, \end{cases}$$

(6–5) $$\begin{cases} \epsilon_{(i,j+1)}\Delta_{(i,j)} = \epsilon_{(i+1,j)}\Delta_{(i,j)} = 1_{\mathsf{H}^{\otimes(i+j+1)}}, \\ \Delta_{(i,j+1)}\Delta_{(i,j)} = \Delta_{(i+1,j)}\Delta_{(i,j)}, \end{cases}$$

(6–6) $$\begin{cases} \epsilon_{(i,j)}\eta_{(i,j)} = 1_{\mathsf{H}^{\otimes(i+j)}}, \\ \epsilon_{(i,j)}\mu_{(i,j)} = \epsilon_{(i,j)}\epsilon_{(i+1,j)}, \\ \Delta_{(i,j)}\eta_{(i,j)} = \eta_{(i+1,j)}\eta_{(i,j)}, \end{cases}$$

(6–7) $$\Delta_{(i,j)}\mu_{(i,j)} = \mu_{(i+1,j)}\mu_{(i,j+2)}\psi_{(i+1,j+1)}\Delta_{(i,j+2)}\Delta_{(i+1,j)},$$

(6–8) $$\mu_{(i,j)}S_{(i,j+1)}\Delta_{(i,j)} = \mu_{(i,j)}S_{(i+1,j)}\Delta_{(i,j)} = \eta_{(i,j)}\epsilon_{(i,j)},$$

for $i,j,k\geq 0$ *and* $f,g\in\{\psi_{\mathsf{H},\mathsf{H}},\psi^{-1}_{\mathsf{H},\mathsf{H}},\mu,\eta,\Delta,\epsilon,S\}$ *with* $f\colon \mathsf{H}^{\otimes p}\to \mathsf{H}^{\otimes p'}$, $g\colon \mathsf{H}^{\otimes q}\to \mathsf{H}^{\otimes q'}$. *Here,* $(\psi_{p,1})_{(i,j)}$ *and* $(\psi_{1,p})_{(i,j)}$ *for* $p=0,1,2$ *and* $i,j\geq 0$ *are defined by*

$$(\psi_{0,1})_{(i,j)}=(\psi_{1,0})_{(i,j)}=1_{\mathsf{H}^{\otimes(i+j+1)}},\quad (\psi_{1,1})_{(i,j)}=\psi_{(i,j)},$$
$$(\psi_{2,1})_{(i,j)}=\psi_{(i,j+1)}\psi_{(i+1,j)},\quad (\psi_{1,2})_{(i,j)}=\psi_{(i+1,j)}\psi_{(i,j+1)}.$$

Proof We only give a sketch proof, since a detailed proof is long though straightforward. The relations given in the lemma are the ones derived from the axioms for braided category and Hopf algebra, hence valid in $\langle\mathsf{H}\rangle$. We have to show, conversely, that all the relations in $\langle\mathsf{H}\rangle$ can be derived from the relations given in the lemma. It suffices to show that the category $\langle\mathsf{H}\rangle'$ with the presentation given in the lemma is a braided category with a Hopf algebra H. The relation (6–1) implies that $\langle\mathsf{H}\rangle'$ is a monoidal category, since we can define the monoidal structure for $\langle\mathsf{H}\rangle'$ by

$$f_{(i,j)}\otimes f'_{(i',j')}=f_{(i,j+i'+n'+j')}f'_{(i+m+j+i',j')}$$

for $f,f'\in\{\psi_{\mathsf{H},\mathsf{H}},\psi^{-1}_{\mathsf{H},\mathsf{H}},\mu,\eta,\Delta,\epsilon,S\}$, $f\colon \mathsf{H}^{\otimes m}\to\mathsf{H}^{\otimes n}$, $f'\colon \mathsf{H}^{\otimes m'}\to\mathsf{H}^{\otimes n'}$. The relations (6–2) and (6–3) imply that $\langle\mathsf{H}\rangle'$ is a braided category, and the other relations imply that H is a Hopf algebra in $\langle\mathsf{H}\rangle'$. □

6.2 External Hopf algebras in braided categories

Let $\langle\mathsf{A}\rangle$ denote the braided category freely generated by an algebra $\mathsf{A}=(\mathsf{A},\mu_{\mathsf{A}},\eta_{\mathsf{A}})$. For a braided category $\mathcal{M}$ and an algebra A in $\mathcal{M}$, let

$$i_{\mathcal{M},A}\colon \langle\mathsf{A}\rangle\to\mathcal{M}$$

denote the unique braided functor that maps the algebra structure of A into the algebra structure of A.

Definition 6.2 An *external Hopf algebra* (H,F) in a braided category $\mathcal{M}$ is a pair of an algebra $H=(H,\mu_H,\eta_H)$ in $\mathcal{M}$ and a functor $F\colon\langle\mathsf{H}\rangle\to\mathsf{Sets}$ into the category Sets of sets and functions such that we have a commutative square

$$\begin{array}{ccc}
\langle\mathsf{A}\rangle & \xrightarrow{i_{\mathcal{M},H}} & \mathcal{M}\\
{\scriptstyle i_{\langle\mathsf{H}\rangle,\mathsf{H}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{M}(\mathbf{1},-)}\\
\langle\mathsf{H}\rangle & \xrightarrow[F]{} & \mathsf{Sets}.
\end{array}$$

Remark 6.3 The functor $i_{\langle\mathsf{H}\rangle,\mathsf{H}}\colon\langle\mathsf{A}\rangle\to\langle\mathsf{H}\rangle$ is faithful. Hence we can identify $\langle\mathsf{A}\rangle$ with the braided subcategory of $\langle\mathsf{H}\rangle$ generated by H, μ_{H} and η_{H}. However, we do not need this fact in the rest of the paper.

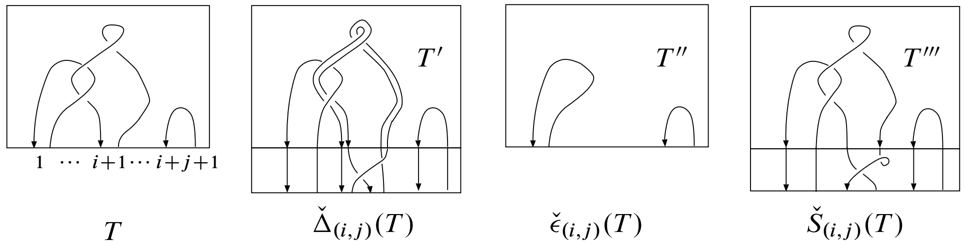


Figure 11

Remark 6.4 To each Hopf algebra H in a braided category $\mathcal{M}$, we can associate an external Hopf algebra in $\mathcal{M}$ as follows. Let $F_H\colon \langle \mathsf{H}\rangle \to \mathcal{M}$ be the braided functor which maps the Hopf algebra H into H. Then the pair $((H,\mu_H,\eta_H),\mathcal{M}(\mathbf{1},F_H(-)))$ is an external Hopf algebra in $\mathcal{M}$. Therefore, the notion of external Hopf algebra in $\mathcal{M}$ can be regarded as a generalization of the notion of Hopf algebra in $\mathcal{M}$.

6.3 The external Hopf algebra structure in B

In this subsection, we define an external Hopf algebra $(\mathsf{b}, F_{\mathsf{b}})$ in B. First note that $(\mathsf{b},\mu_{\mathsf{b}},\eta_{\mathsf{b}})$ is an algebra in B. (This algebra structure of b cannot be extended to a Hopf algebra structure in B in the usual sense, since there is no morphism $\epsilon\colon \mathsf{b}\to\mathbf{1}$ in B.)

For $i,j\geq 0$ and $T\in \mathsf{BT}_{i+j+1}$, set

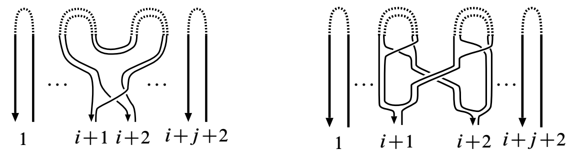
$$\begin{aligned}
\check{\Delta}_{(i,j)}(T) &= (\mathsf{b}^{\otimes i}\otimes(\downarrow\otimes\psi_{\mathsf{b},\uparrow})\otimes\mathsf{b}^{\otimes j})T',\\
\check{\epsilon}_{(i,j)}(T) &= T'',\\
\check{S}_{(i,j)}(T) &= (\mathsf{b}^{\otimes i}\otimes\psi_{\uparrow,\downarrow}(\uparrow\otimes t_{\downarrow})\otimes\mathsf{b}^{\otimes j})T''',
\end{aligned}$$

where $T'\in\mathcal{T}(\mathbf{1},\mathsf{b}^{\otimes i}\otimes\downarrow\otimes\downarrow\otimes\uparrow\otimes\uparrow\otimes\mathsf{b}^{\otimes j})$ is obtained from T by duplicating the $(i+1)$st component of T, $T''\in\mathcal{T}(\mathbf{1},\mathsf{b}^{\otimes(i+j)})$ is obtained from T by removing the $(i+1)$st component of T, and $T'''\in\mathcal{T}(\mathbf{1},\mathsf{b}^{\otimes i}\otimes\uparrow\otimes\downarrow\otimes\mathsf{b}^{\otimes j})$ is obtained from T by reversing orientation of the $(i+1)$st component of T, see Figure 11. We have

$$\check{\Delta}_{(i,j)}(T)\in\mathsf{BT}_{i+j+2},\quad \check{\epsilon}_{(i,j)}(T)\in\mathsf{BT}_{i+j},\quad \check{S}_{(i,j)}(T)\in\mathsf{BT}_{i+j+1}.$$

Hence there are functions

$$\begin{aligned}
\check{\Delta}_{(i,j)}&\colon \mathsf{BT}_{i+j+1}\to\mathsf{BT}_{i+j+2},\\
\check{\epsilon}_{(i,j)}&\colon \mathsf{BT}_{i+j+1}\to\mathsf{BT}_{i+j},\\
\check{S}_{(i,j)}&\colon \mathsf{BT}_{i+j+1}\to\mathsf{BT}_{i+j+1}.
\end{aligned}$$



(a) (b)

Figure 12

Theorem 6.5 *There is a unique external Hopf algebra* $((\mathsf{b},\mu_{\mathsf{b}},\eta_{\mathsf{b}}),F_{\mathsf{b}})$ *in* B *with* $F_{\mathsf{b}}\colon \langle \mathsf{H}\rangle \to \mathsf{Sets}$ *satisfying*

$$F_{\mathsf{b}}(\Delta_{(i,j)})=\check{\Delta}_{(i,j)},\quad F_{\mathsf{b}}(\epsilon_{(i,j)})=\check{\epsilon}_{(i,j)},\quad F_{\mathsf{b}}(S_{(i,j)})=\check{S}_{(i,j)} \tag{6–9}$$

for $i,j\geq 0$.

Proof We claim that there is a functor $F_{\mathsf{b}}\colon \langle\mathsf{H}\rangle\to\mathsf{Sets}$ satisfying (6–9) and

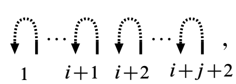
$$\begin{aligned}F_{\mathsf{b}}(\psi^{\pm1}_{(i,j)})&=(\psi^{\pm1}_{\mathsf{b},\mathsf{b}})_{(i,j)}(-),\\ F_{\mathsf{b}}(\mu_{(i,j)})&=(\mu_{\mathsf{b}})_{(i,j)}(-),\\ F_{\mathsf{b}}(\eta_{(i,j)})&=(\eta_{\mathsf{b}})_{(i,j)}(-),\end{aligned}$$

for $i,j\geq 0$. If this claim is true, then one can easily check that F_{b} is unique and $(\mathsf{b},F_{\mathsf{b}})$ is an external Hopf algebra in B.

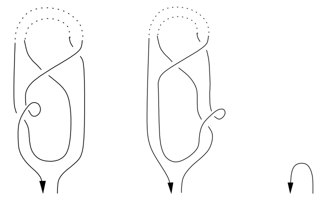
To prove the above claim, it suffices to check that each relation in Lemma 6.1 are mapped into a relation in B. For example, the relation (6–7) is mapped into

(6–10)
$$\begin{aligned}\check{\Delta}_{(i,j)}((\mu_{\mathsf{b}})_{(i,j)}g)=&(\mu_{\mathsf{b}})_{(i+1,j)}(\mu_{\mathsf{b}})_{(i,j+2)}(\psi_{\mathsf{b},\mathsf{b}})_{(i+1,j+1)}\check{\Delta}_{(i,j+2)}\check{\Delta}_{(i+1,j)}(g)\\ &\Big(=((\mu_{\mathsf{b}}\otimes\mu_{\mathsf{b}})(\mathsf{b}\otimes\psi_{\mathsf{b},\mathsf{b}}\otimes\mathsf{b}))_{(i,j)}\check{\Delta}_{(i,j+2)}\check{\Delta}_{(i+1,j)}(g)\Big)\end{aligned}$$

for $g\in\mathsf{BT}_{i+j+2}$. This can be proved in a graphical way. If

$g=$  ,

then the left and the right hand sides of (6–10) are as depicted in Figure 12 (a) and (b), respectively. As another example, (6–8) is mapped into the equivalence of Figure 13 (a), (b), and (c). It is straightforward to check the other relations, and we leave it to the reader. The detail is to a certain extent similar to the proof of the existence of the Hopf algebra in the category of cobordisms of surfaces with connected boundary



(a) (b) (c)

Figure 13: Here, only the i th component is depicted in each figure.

(see Crane and Yetter [8] and Kerler [40]) and also to the proof of the Hopf algebra relations satisfied by claspers [22]. See also Section 14.4. □

Theorem 6.5 implies that there is a Hopf algebra action on the bottom tangles as we mentioned in Section 1.6.

6.4 Adjoint actions and coactions

In this subsection, we consider the image by F_{b} of the left adjoint action and the left adjoint coaction of H, which we need later.

Let $\mathrm{ad}_{\mathsf{H}}\colon \mathsf{H}^{\otimes 2} \to \mathsf{H}$ denote the *left adjoint action* for H, which is defined by

$$\mathrm{ad}_{\mathsf{H}} = \mu_{\mathsf{H}}^{[3]}(\mathsf{H} \otimes \psi_{\mathsf{H},\mathsf{H}})(\mathsf{H} \otimes S_{\mathsf{H}} \otimes \mathsf{H})(\Delta_{\mathsf{H}} \otimes \mathsf{H}).$$

For $i, j \geq 0$ and $T \in \mathsf{BT}_{i+j+2}$, we have

$$\begin{aligned}
&F_{\mathsf{b}}((\mathrm{ad}_{\mathsf{H}})_{(i,j)})(T)\\
=&F_{\mathsf{b}}((\mu_{\mathsf{H}}^{[3]}(\mathsf{H} \otimes \psi_{\mathsf{H},\mathsf{H}})(\mathsf{H} \otimes S_{\mathsf{H}} \otimes \mathsf{H})(\Delta_{\mathsf{H}} \otimes \mathsf{H}))_{(i,j)})(T)\\
=&(\mu_{\mathsf{b}}^{[3]})_{(i,j)}(\mathsf{b} \otimes \psi_{\mathsf{b},\mathsf{b}})_{(i,j)} F_{\mathsf{b}}((\mathsf{H} \otimes S_{\mathsf{H}} \otimes \mathsf{H})_{(i,j)})(F_{\mathsf{b}}((\Delta_{\mathsf{H}} \otimes \mathsf{H})_{(i,j)})(T)).
\end{aligned}$$

Hence $F_{\mathsf{b}}((\mathrm{ad}_{\mathsf{H}})_{(i,j)})$ maps $T \in \mathsf{BT}_{i+j+1}$ to a bottom tangle as depicted in Figure 14 (a). By isotopy, we obtain a simpler tangle as in Figure 14 (b). (Note that the closures $\mathrm{cl}(T)$ and $\mathrm{cl}(F_{\mathsf{b}}((\mathrm{ad}_{\mathsf{H}})_{(i,j)})(T))$ are isotopic. This fact is used in Section 11.)

Let $\mathrm{coad}_{\mathsf{H}}\colon \mathsf{H} \to \mathsf{H}^{\otimes 2}$ denote the *left adjoint coaction* defined by

$$\mathrm{coad}_{\mathsf{H}} = (\mu_{\mathsf{H}} \otimes \mathsf{H})(\mathsf{H} \otimes \psi_{\mathsf{H},\mathsf{H}})(\mathsf{H} \otimes \mathsf{H} \otimes S_{\mathsf{H}})\Delta_{\mathsf{H}}^{[3]}.$$

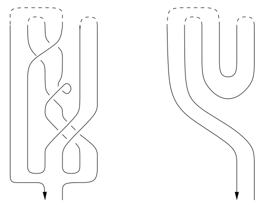


Figure 14: (a) The tangle $F_{\mathsf{b}}((\mathrm{ad}_{\mathsf{H}})_{(i,j)})(T) \in \mathsf{BT}_{i+j+1}$, with only the $(i+1)$st component depicted. The dotted lines denote (parallel copies of) the $(i+1)$st and $(i+2)$nd components of T. (b) A tangle isotopic to (a).

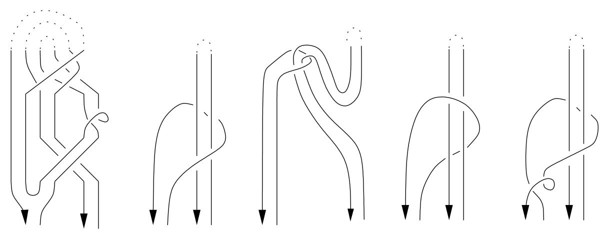


Figure 15: (a) The tangle $F_{\mathsf{b}}((\mathrm{coad}_{\mathsf{H}})_{(i,j)})(T) \in \mathsf{BT}_{i+j+2}$, where only the $(i+1)$st and $(i+2)$nd components are depicted. (b) A tangle isotopic to (a), which is $(\gamma_+)_{(i,j)}T$. (c) Another tangle isotopic to (a), which is $F_{\mathsf{b}}((\mathsf{H}\otimes \mathrm{ad}_{\mathsf{H}})_{(i,j)})((c_+\otimes \mathsf{b})_{(i,j)}T)$. (d) The tangle $(\gamma_-)_{(i,j)}T$ with only $(i+1)$st component depicted. (e) A tangle isotopic to (d).

For $i,j\geq 0$ and $T\in \mathsf{BT}_{i+j+1}$, we have

$$\begin{aligned}
&F_{\mathsf{b}}((\mathrm{coad}_{\mathsf{H}})_{(i,j)})(T)\\
=&F_{\mathsf{b}}(((\mu_{\mathsf{H}}\otimes \mathsf{H})(\mathsf{H}\otimes \psi_{\mathsf{H},\mathsf{H}})(\mathsf{H}\otimes \mathsf{H}\otimes S_{\mathsf{H}})\Delta^{[3]}_{\mathsf{H}})_{(i,j)})(T)\\
=&(\mu_{\mathsf{b}}\otimes \mathsf{b})_{(i,j)}(\mathsf{b}\otimes \psi_{\mathsf{b},\mathsf{b}})_{(i,j)}F_{\mathsf{b}}((\mathsf{H}\otimes \mathsf{H}\otimes S_{\mathsf{H}})_{(i,j)})(F_{\mathsf{b}}((\Delta^{[3]}_{\mathsf{H}})_{(i,j)})(T)).
\end{aligned}$$

Hence $F_{\mathsf{b}}((\mathrm{coad}_{\mathsf{H}})_{(i,j)})$ maps $T\in \mathsf{BT}_{i+j+1}$ to a $(i+j+2)$–component bottom tangle as depicted in Figure 15 (a). Since it is isotopic to Figure 15 (b) and (c), we have

$$F_{\mathsf{b}}((\mathrm{coad}_{\mathsf{H}})_{(i,j)})(T) = (\gamma_+)_{(i,j)}T = F_{\mathsf{b}}((\mathsf{H}\otimes \mathrm{ad}_{\mathsf{H}})_{(i,j)})((c_+\otimes \mathsf{b})_{(i,j)}T). \tag{6–11}$$

Since Figure 15 (d) and (e) are isotopic, we have

(6–12) $$(\gamma_-)_{(i,j)}T = F_{\mathsf{b}}((S_{\mathsf{H}} \otimes \mathsf{H})_{(i,j)})((\gamma_+)_{(i,j)}T).$$

7 Universal tangle invariant associated to a ribbon Hopf algebra

In this section, we give a definition of a universal tangle invariant associated to a ribbon Hopf algebra.

7.1 Ribbon Hopf algebras

In this subsection, we recall the definition of ribbon Hopf algebra (see Reshetikhin and Turaev [74]).

Let $H = (H, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf algebra over a commutative, unital ring $\mathbf{k}$.

A *universal R–matrix* for H is an invertible element $R \in H^{\otimes 2}$ satisfying the following properties:

(7–1) $$R\Delta(x)R^{-1} = P_{H,H}\Delta(x) \quad \text{for all } x \in H,$$

(7–2) $$(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}.$$

Here $P_{H,H}\colon H^{\otimes 2} \to H^{\otimes 2}$, $x \otimes y \mapsto y \otimes x$, is the $\mathbf{k}$–module homomorphism which permutes the tensor factors, and

$$R_{12} = R \otimes 1 \in H^{\otimes 3}, \quad R_{13} = (1 \otimes P_{H,H})(R_{12}) \in H^{\otimes 3}, \quad R_{23} = 1 \otimes R \in H^{\otimes 3}.$$

A Hopf algebra equipped with a universal R–matrix is called a *quasitriangular Hopf algebra*.

In what follows, we freely use the notations

$$R = \sum \alpha \otimes \beta \quad \text{and} \quad R^{-1} = \sum \bar{\alpha} \otimes \bar{\beta} \quad (= (S \otimes 1)(R)).$$

We also use the notations

$$R = \sum \alpha_i \otimes \beta_i, \quad R^{-1} = \sum \bar{\alpha}_i \otimes \bar{\beta}_i,$$

where i is any index, used to distinguish several copies of $R^{\pm 1}$.

A *ribbon element* for (H, R) is a central element $\mathbf{r} \in H$ such that

$$\mathbf{r}^2 = uS(u), \quad S(\mathbf{r}) = \mathbf{r}, \quad \epsilon(\mathbf{r}) = 1, \quad \Delta(\mathbf{r}) = (R_{21}R)^{-1}(\mathbf{r} \otimes \mathbf{r}),$$

where $R_{21} = P_{H,H}(R) = \sum \beta \otimes \alpha$ and $u = \sum S(\beta)\alpha$. Since u is invertible, so is $\mathbf{r}$. The triple $(H, R, \mathbf{r})$ is called a *ribbon Hopf algebra*.

The element $\kappa = u\mathbf{r}^{-1}$ is grouplike, ie, $\Delta(\kappa) = \kappa \otimes \kappa$, $\epsilon(\kappa) = 1$. We also have

$$\kappa x \kappa^{-1} = S^2(x) \quad \text{for all } x \in H.$$

In what follows, we often use the Sweedler notation for comultiplication. For $x \in H$, we write

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)},$$
$$\Delta^{[n]}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n)} \quad \text{for } n \geq 1.$$

7.2 Adjoint action and universal quantum trace

We regard H as a left H–module with the (left) adjoint action

$$\mathrm{ad} = \triangleright\colon H^{\otimes 2} \to H, \quad x \otimes y \mapsto x \triangleright y,$$

defined by

$$x \triangleright y = \sum x_{(1)} y S(x_{(2)}) \quad \text{for } x, y \in H.$$

Recall that ad is a left H–module homomorphism.

The function

$$H^{\otimes 2} \to H, \quad x \otimes y \mapsto x \triangleright y - \epsilon(x) y,$$

is a left H–module homomorphism. Hence the image

$$N = \mathrm{Span}_{\mathbf{k}}\{x \triangleright y - \epsilon(x) y \mid x, y \in H\} \subset H$$

is a left H–submodule of H. Note that H/N, the *module of coinvariants*, inherits from H a trivial left H–module structure.

The definition of N above is compatible with (1–1) as follows.

Lemma 7.1 *We have*

$$N = \mathrm{Span}_{\mathbf{k}}\{xy - yS^2(x) \mid x, y \in H\}.$$

Proof The assertion follows from

$$x \triangleright y - \epsilon(x) y = \sum \Big(x_{(1)} y S(x_{(2)}) - y S(x_{(2)}) S^2(x_{(1)}) \Big),$$
$$xy - yS^2(x) = \sum \Big(x_{(1)} \triangleright y S^2(x_{(2)}) - \epsilon(x_{(1)}) y S^2(x_{(2)}) \Big),$$

for $x, y \in H$. □

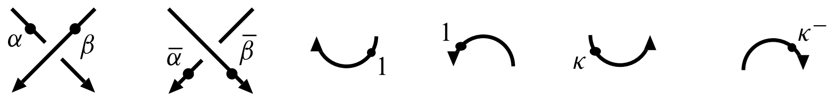


Figure 16: How to place elements in H on the strings in the tangle T

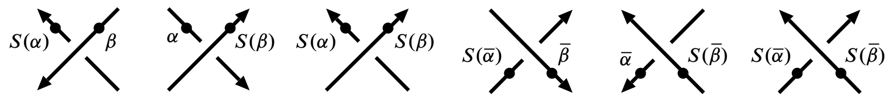


Figure 17: The cases of crossings with upward strings

As in Section 1.2.2, let $\mathrm{tr}_q\colon H \to H/N$ denote the projection, and call it the *universal quantum trace* for H. If $\mathbf{k}$ is a field and V is a finite-dimensional left H–module, then the *quantum trace* in V

$$\mathrm{tr}_q^V\colon H \to \mathbf{k}$$

factors through tr_q. Here tr_q^V is defined by

$$\mathrm{tr}_q^V(x) = \mathrm{tr}^V(\rho(\kappa x)) \quad \text{for } x \in H,$$

where $\rho\colon H \to \mathrm{End}_{\mathbf{k}}(V)$ denotes the left action of H on V, and

$$\mathrm{tr}^V\colon \mathrm{End}_{\mathbf{k}}(V) \to \mathbf{k}$$

denotes the trace in V.

7.3 Definition of the universal invariant

In this subsection, we recall the definition of the universal invariant of tangles associated to a ribbon Hopf algebra H. The definition below is close to Ohtsuki's one [65], but we use different conventions and we make some modifications. In particular, for closed components, we use the universal quantum trace instead of the universal trace.

Let $T = T_1 \cup \cdots \cup T_l \cup L_1 \cup \cdots \cup L_m$ with $l, m \geq 0$ be a (framed, oriented) tangle in a cube, consisting of l arc components $T_1, \ldots, T_l$ and m *ordered* circle components $L_1, \ldots, L_m$. First, assume that T is given by pasting copies of the tangles

$$\downarrow, \quad \uparrow, \quad \psi^{\pm 1}_{\downarrow,\downarrow}, \quad \mathrm{ev}_{\downarrow}, \quad \mathrm{ev}_{\uparrow}, \quad \mathrm{coev}_{\downarrow}, \quad \mathrm{coev}_{\uparrow}.$$

(Later we consider a more general case.) We formally put elements of H on the strings of T according to the rule depicted in Figure 16. We define

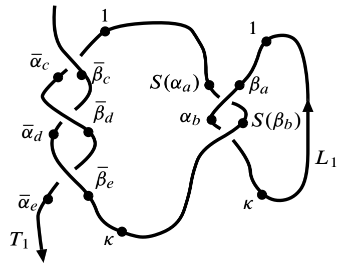


Figure 18: For the tangle $T = T_1 \cup L_1$, we have
$$J_T = \sum \bar\alpha_e \bar\beta_d \bar\alpha_c 1 S(\alpha_a) S(\beta_b)\kappa \bar\beta_e \bar\alpha_d \bar\beta_c \otimes \mathrm{tr}_q(\kappa^{-1}\kappa\alpha_b\beta_a 1),$$
where $R = \sum \alpha_a \otimes \beta_a$, and $R^{-1} = \sum \bar\alpha_c \otimes \bar\beta_c$, etc.

$$J_T = \sum J_{(T_1)} \otimes \cdots \otimes J_{(T_l)} \otimes J_{(L_1)} \otimes \cdots \otimes J_{(L_m)} \in H^{\otimes l} \otimes (H/N)^{\otimes m} \tag{7–3}$$

as follows. For each $i = 1, \dots, l$, we formally set $J_{(T_i)}$ to be the product of the elements put on the component T_i obtained by reading the elements using the order determined by the opposite orientation of T_i and writing them down from left to right. For each $j = 1, \dots, s$, we define $J_{(L_j)}$ by first obtaining a word w by reading the elements put on L_j starting from any point on L_j, and setting formally $J_{(L_j)} = \mathrm{tr}_q(\kappa^{-1}w)$. (Here, it should be noted that each of the $J_{(T_i)}$ and the $J_{(L_j)}$ has only notational meaning and does not define an element of H or H/N by itself.) For example, see Figure 18.

Now we check that J_T does not depend on where we start reading the elements on the closed components. Let L_j be a closed component of T and let $x_1, \dots, x_r$ be the elements read off from L_j. Then we have formally $J_{(L_j)} = \mathrm{tr}_q(\kappa^{-1}x_1x_2\cdots x_r)$. If we start from x_2, then the right hand side becomes
$$\mathrm{tr}_q(\kappa^{-1}x_2\cdots x_r x_1) = \mathrm{tr}_q(S^{-2}(x_1)\kappa^{-1}x_2\cdots x_r) = \mathrm{tr}_q(\kappa^{-1}x_1x_2\cdots x_r).$$
It follows that $J_{(L_i)}$ does not depend on where we start reading the elements.

Now we can follow Ohtsuki's arguments [65] to check that J_T does not depend on how we decompose T into copies of $\downarrow$, $\uparrow$, $\psi^{\pm 1}_{\downarrow,\downarrow}$, $\mathrm{ev}_\downarrow$, $\mathrm{ev}_\uparrow$, $\mathrm{coev}_\downarrow$ and $\mathrm{coev}_\uparrow$, and that J_T defines an isotopy invariant of framed, oriented, ordered tangles.

It is convenient to generalize the above definition to the case where T is given as pasting of copies of the tangles $\downarrow$, $\uparrow$, $\psi^{\pm 1}_{a,b}$ $(a, b \in \{\downarrow, \uparrow\})$, $\mathrm{ev}_\downarrow$, $\mathrm{ev}_\uparrow$, $\mathrm{coev}_\downarrow$ and $\mathrm{coev}_\uparrow$. In this case, we put elements of H on the components of T as depicted in Figures 16 and 17. Then J_T is defined in the same way as above. We can check that J_T is well defined as follows. For each tangle diagram T in Figure 17, we choose a

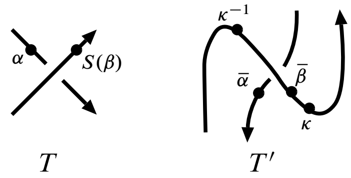


Figure 19

tangle diagram T' isotopic to T obtained by pasting copies of $\downarrow$, $\uparrow$, $\psi^{\pm 1}_{\downarrow,\downarrow}$, $\mathrm{ev}_\downarrow$, $\mathrm{ev}_\uparrow$, $\mathrm{coev}_\downarrow$ and $\mathrm{coev}_\uparrow$. Then we can verify that $J_{T'}$ in the first definition, is equal to J_T in the second definition given by Figure 17. For example, consider the second tangle in Figure 17. Then T and T' are as depicted in Figure 19. We have

$$J_{T'} = \sum \bar{\alpha} \otimes \kappa \bar{\beta} \kappa^{-1} = \sum \alpha \otimes \kappa S^{-1}(\beta)\kappa^{-1} = \sum \alpha \otimes S(\beta) = J_T,$$

where we used $\sum \bar{\alpha} \otimes \bar{\beta} = \sum \alpha \otimes S^{-1}(\beta)$. The other cases can be similarly proved.

Remark 7.2 To study invariants of tangles, it is sometimes useful to define a *functorial* invariant. One can modify Kauffman and Radford's functorial universal regular isotopy invariant [38] to define a functorial universal invariant defined on $\mathcal{T}$, ie, a braided functor $F\colon \mathcal{T} \to \mathrm{Cat}(H)$ of $\mathcal{T}$ into a category $\mathrm{Cat}(H)$ defined as in [38]. However, we do not do so here, since in the present paper we are interested in *ordered* links. Note that the categories $\mathcal{T}$ and $\mathrm{Cat}(H)$ do not care about the order of the circle components. One can still define a functorial universal invariant which distinguishes circle components by using the category of colored tangles, but it would cause unnecessary complication and we do not take this approach here.

7.4 Effect of closure operation

In Ohtsuki's definition [65] of his version of the universal invariant, the universal trace $H \to H/I$, with $I = \mathrm{Span}_{\mathbf{k}}\{xy - yx \mid x, y \in H\}$, is used. For our purposes, the universal quantum trace is more natural and more useful than the universal trace. Note that I is not a left H–submodule of H in general. The following proposition shows another reason why the universal quantum trace is more convenient.

Proposition 7.3 *If $T \in \mathsf{BT}_n$, then we have*

$$J_{\mathrm{cl}(T)} = \mathrm{tr}_q^{\otimes n}(J_T).$$

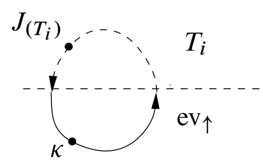


Figure 20

Proof Set $L = \mathrm{cl}(T) = L_1 \cup \cdots \cup L_n$. We write

$$J_T = \sum J_{(T_1)} \otimes \cdots \otimes J_{(T_n)} \in H^{\otimes n},$$
$$J_L = \sum J_{(L_1)} \otimes \cdots \otimes J_{(L_n)} \in (H/N)^{\otimes n},$$

For $i = 1, \dots, n$, the part $J_{(L_i)}$ is computed as follows. The diagram of L_i is divided into the diagram of T_i and the diagram of $\mathrm{ev}_\uparrow$. See Figure 20. Since we have $J_{\mathrm{ev}_\uparrow} = \kappa$, it follows that

$$J_{(L_i)} = \mathrm{tr}_q(\kappa^{-1} J_{\mathrm{ev}_\uparrow} J_{(T_i)}) = \mathrm{tr}_q(\kappa^{-1}\kappa J_{(T_i)}) = \mathrm{tr}_q(J_{(T_i)}).$$

This implies the assertion. □

In Section 11, we give a definition of a more refined version of the universal invariants of links.

7.5 Duplication, removal, and orientation-reversal

Let T be a tangle and let T_i be an arc component of T. Define a **k**–module homomorphism $\widetilde{S}_{T_i}\colon H \to H$ by

$$\widetilde{S}_{T_i}(x) = \kappa^{-r(T_i)} S(x) \kappa^{s(T_i)} \quad \text{for } x \in H,$$

where $r(T_i) = 0$ if T_i starts at the top and $r(T_i) = 1$ otherwise, and $s(T_i) = 0$ if T_i ends at the bottom, and $s(T_i) = 1$ otherwise. For example, we have

$$\widetilde{S}_\downarrow(x) = S(x), \qquad \widetilde{S}_\uparrow(x) = \kappa^{-1} S(x)\kappa = S^{-1}(x),$$
$$\widetilde{S}_\cup(x) = \widetilde{S}_{\cup}(x) = S(x)\kappa, \qquad \widetilde{S}_\cap(x) = \widetilde{S}_{\cap}(x) = \kappa^{-1} S(x)$$

for $x \in H$. We need the following result, which is almost standard.

Lemma 7.4 *Let $T = T_1 \cup \cdots \cup T_n$ be a tangle with n arcs with $n \geq 1$. For $i = 1, \dots, n$, let $\Delta_i(T)$ (resp. $\epsilon_i(T)$, $S_i(T)$) denote the tangle obtained from T by duplicating* *(resp. removing, orientation-reversing) the i th component T_i. Then we have*

(7–4) $$J_{\Delta_i(T)} = (1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(n-i)})(J_T),$$

(7–5) $$J_{\epsilon_i(T)} = (1^{\otimes(i-1)} \otimes \epsilon \otimes 1^{\otimes(n-i)})(J_T),$$

(7–6) $$J_{S_i(T)} = (1^{\otimes(i-1)} \otimes \widetilde{S}_{T_i} \otimes 1^{\otimes(n-i)})(J_T).$$

Proof The cases of $\Delta_i(T)$ and $\epsilon_i(T)$ are standard. We prove the case of $S_i(T)$, which may probably be well known to the experts but does not seem to have appeared in a way as general as here.

We can easily check (7–6) for $T = \psi_{\mathsf{b},\mathsf{b}}, \psi^{-1}_{\mathsf{b},\mathsf{b}}, \overset{\uparrow}{\cup}, \cup^{\uparrow}, \overset{\downarrow}{\cap}, \cap^{\downarrow}$. For the general case, we express T as an iterated composition and tensor product of finitely many copies of the morphisms $\downarrow, \uparrow, \psi^{\pm 1}_{\downarrow,\downarrow}, \overset{\uparrow}{\cup}, \cup^{\uparrow}, \overset{\downarrow}{\cap}, \cap^{\downarrow}$. We may assume that T_i involves at least one crossing or critical point, since otherwise the assertion is obvious. We decompose the component T_i into finitely many intervals $T_{i,1}, \ldots, T_{i,p}$ with $p \geq 1$, where

- if one goes along T_i in the opposite direction to the orientation, then one encounter the intervals in the order $T_{i,1}, \ldots, T_{i,p}$, and
- for each $j = 1, \ldots, p$, there is just one crossing or critical points in $T_{i,j}$.

For each $j = 1, \ldots, p$, let $x_j = J_{(T_{i,j})}$ denote the formal element put on the interval $T_{i,j}$ in the definition of J_T. Then we have $J_{(T_i)} = x_1 x_2 \cdots x_p$. Let $-T_{i,j}$ denote the orientation reversal of $T_{i,j}$. Then it follows from the cases of $T = \psi_{\mathsf{b},\mathsf{b}}, \psi^{-1}_{\mathsf{b},\mathsf{b}}, \overset{\uparrow}{\cup}, \cup^{\uparrow}, \overset{\downarrow}{\cap}, \cap^{\downarrow}$ that the formal element put on $-T_{i,j}$ in the definition of $J_{S_i(T)}$ is $\widetilde{S}_{T_{i,j}}(x_j)$. Hence we have $J_{(-T_i)} = x'_p \cdots x'_1$, where

$$x'_j = J_{(-T_{i,j})} = \widetilde{S}_{T_{i,j}}(x_j) = \kappa^{-r(T_{i,j})} S(x_j) \kappa^{s(T_{i,j})}$$

for $j = 1, \ldots, p$. We have $s(T_{i,j}) = r(T_{i,j-1})$ for $j = 2, \ldots, p$. We also have $s(T_{i,1}) = s(T_i)$ and $r(T_{i,p}) = r(T_i)$. Hence it follows that

$$\begin{aligned}
J_{(-T_i)} &= x'_p x'_{p-1} \ldots x'_1 \\
&= \big(\kappa^{-r(T_{i,p})} S(x_p) \kappa^{s(T_{i,p})}\big)\big(\kappa^{-r(T_{i,p-1})} S(x_{p-1}) \kappa^{s(T_{i,p-1})}\big) \\
&\qquad \ldots \big(\kappa^{-r(T_{i,1})} S(x_1) \kappa^{s(T_{i,1})}\big) \\
&= \kappa^{-r(T_i)} S(x_p) S(x_{p-1}) \cdots S(x_1) \kappa^{s(T_i)} \\
&= \kappa^{-r(T_i)} S(x_1 \ldots x_{p-1} x_p) \kappa^{s(T_i)} \\
&= \kappa^{-r(T_i)} S(J_{(T_i)}) \kappa^{s(T_i)} \\
&= \widetilde{S}_{T_i}(J_{(T_i)}).
\end{aligned}$$

Now the assertion immediately follows. □

8 The braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$

In this section, we fix a ribbon Hopf algebra H over a commutative, unital ring $\mathbf{k}$.

8.1 The category Mod_H of left H–modules

In this subsection, we recall some algebraic facts about the category Mod_H of left H–modules. For details, see Majid [55; 56].

Let Mod_H denote the category of left H–modules and left H–module homomorphisms. The category Mod_H is equipped with a (non-strict) monoidal category structure with the tensor functor $\otimes\colon \mathsf{Mod}_H \times \mathsf{Mod}_H \to \mathsf{Mod}_H$ given by tensor product over $\mathbf{k}$ with the usual left H–module structure defined using comultiplication. The unit object is $\mathbf{k}$ with the trivial left H–module structure. The braiding $\psi_{V,W}$ and its inverse of two objects V and W are given by

$$\psi_{V,W}(v \otimes w) = \sum \beta w \otimes \alpha v, \qquad \psi^{-1}_{V,W}(v \otimes w) = \sum \bar{\alpha} w \otimes \bar{\beta} v, \tag{8–1}$$

for $v \in V$, $w \in W$.

We regard H as a left H–module using the adjoint action $\mathrm{ad} = \triangleright\colon H^{\otimes 2} \to H$. By (8–1), the braiding $\psi_{H,H}\colon H^{\otimes 2} \to H^{\otimes 2}$ and its inverse are given by

$$\psi_{H,H}(x \otimes y) = \sum (\beta \triangleright y) \otimes (\alpha \triangleright x), \qquad \psi^{-1}_{H,H}(x \otimes y) = \sum (\bar{\alpha} \triangleright y) \otimes (\bar{\beta} \triangleright x)$$

for $x, y \in H$.

The *transmutation* [55; 56] of a quasitriangular Hopf algebra H is a Hopf algebra $\underline{H} = (H, \mu, \eta, \underline{\Delta}, \epsilon, \underline{S})$ in the braided category Mod_H, which is obtained by modifying the Hopf algebra structure of H as follows. The algebra structure morphisms μ and η, and the counit ϵ of $\underline{H}$ are the same as those of H. The comultiplication $\underline{\Delta}\colon H \to H^{\otimes 2}$ and the antipode $\underline{S}\colon H \to H$ are defined by

$$\underline{\Delta}(x) = \sum x_{(1)} S(\beta) \otimes (\alpha \triangleright x_{(2)}), \tag{8–2}$$

$$\underline{S}(x) = \sum \beta S(\alpha \triangleright x), \tag{8–3}$$

for $x \in H$. The morphisms $\mu, \eta, \underline{\Delta}, \epsilon, \underline{S}$ are all left H–module homomorphisms, and $\underline{H}$ is a Hopf algebra in the braided category Mod_H.

Define $c^H_{\pm} \in H^{\otimes 2}$ by

$$c^H_{\pm} = (S \otimes 1)((R_{21} R)^{\pm 1}). \tag{8–4}$$

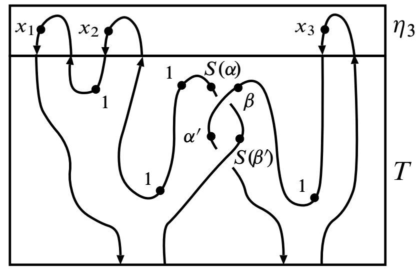


Figure 21: For the tangle $T=(\mu_{\mathsf{b}}\otimes\mu_{\mathsf{b}})(\mu_{\mathsf{b}}\otimes c_+\otimes \mathsf{b})\colon \mathsf{b}^{\otimes 3}\to \mathsf{b}^{\otimes 2}$ depicted, and $\sum x_1\otimes x_2\otimes x_3\in H^{\otimes 3}$, we have $\mathsf{J}(T)(\sum x_1\otimes x_2\otimes x_3)=\sum x_1x_2S(\alpha)S(\beta')\otimes\alpha'\beta x_3$, where $R=\sum\alpha\otimes\beta=\sum\alpha'\otimes\beta'$.

By abuse of notation, we denote by $c_{\pm}^{H}$ the $\mathbf{k}$–module homomorphism $\mathbf{k}\to H^{\otimes 2}$ which maps 1 to $c_{\pm}^{H}$.

Using $\Delta(x)R_{21}R=R_{21}R\Delta(x)$, $x\in H$, one can verify

$$c_{\pm}^{H}\in \mathsf{Mod}_H(\mathbf{k},H^{\otimes 2}). \tag{8–5}$$

8.2 Definition of $\mathsf{J}\colon \mathsf{B}\to\mathsf{Mod}_H$

In this subsection, we define a braided functor

$$\mathsf{J}\colon \mathsf{B}\to\mathsf{Mod}_H, \tag{8–6}$$

which maps $\mathsf{b}\in\mathrm{Ob}(\mathsf{B})$ to $H\in\mathrm{Ob}(\mathsf{Mod}_H)$.

For $T\in\mathsf{B}(m,n)$ with $m,n\geq 0$, we define a $\mathbf{k}$–module homomorphism

$$\mathsf{J}(T)\colon H^{\otimes m}\to H^{\otimes n}$$

as follows. Consider a tangle diagram of $T\eta_m$, see Figure 21. Given an element $\sum x_1\otimes\cdots\otimes x_m\in H^{\otimes m}$, we put x_i on the ith component in η_m for each $i=1,\dots,m$. Moreover, we put elements in H to the components in T as in the definition of J_T. Then we obtain a tangle diagram consisting of n arcs, decorated with elements of H. For $i=1,\dots,n$, let y_i denote the word obtained by reading the elements on the ith component of $T\eta_m$. Then we set

$$\mathsf{J}(T)\Big(\sum x_1\otimes\cdots\otimes x_m\Big)=\sum y_1\otimes\cdots\otimes y_n.$$

Clearly, $\mathsf{J}(T)$ is a $\mathbf{k}$–module homomorphism, and does not depend on the choice of the diagram of T. Note that if $m=0$, then we have

$$\mathsf{J}(T)(1)=J_T.$$

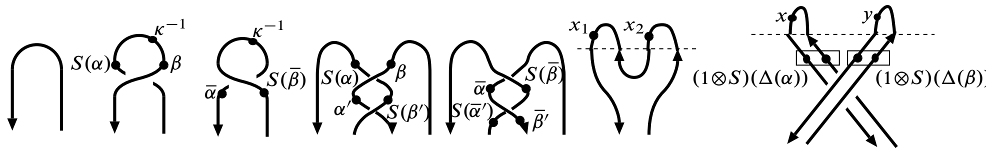


Figure 22

It is also clear that $\mathsf{J}(TT') = \mathsf{J}(T)\mathsf{J}(T')$ for any two composable pair of morphisms T and T' in B, and that $\mathsf{J}(1_{\mathsf{b}}^{\otimes n}) = 1_{H^{\otimes n}}$. This means that the correspondence $T \mapsto \mathsf{J}(T)$ defines a functor

$$\mathsf{J}_{\mathbf{k}}\colon \mathsf{B} \to \mathsf{Mod}_{\mathbf{k}},$$

where $\mathsf{Mod}_{\mathbf{k}}$ denotes the category of $\mathbf{k}$–modules and $\mathbf{k}$–module homomorphisms. We give the category $\mathsf{Mod}_{\mathbf{k}}$ the standard symmetric monoidal category structure. Then we can easily check that $\mathsf{J}_{\mathbf{k}}$ is a monoidal functor.

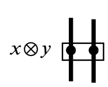
To prove that $\mathsf{J}_{\mathbf{k}}$ lifts along the forgetful functor $\mathsf{Mod}_H \to \mathsf{Mod}_{\mathbf{k}}$ to a monoidal functor (8–6), it suffices to show that if T is a morphism in B, then $\mathsf{J}(T)$ is a left H–module homomorphism. By Theorem 5.16, we have only to check this property for $T \in \{\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}, \mu_{\mathsf{b}}, \eta_{\mathsf{b}}, v_{\pm}, c_{\pm}\}$. This follows from Proposition 8.1 below, since $\eta, v^{\pm 1}, c_{\pm}^H, \mu, \psi_{H,H}^{\pm 1}$ are left H–module homomorphisms. Proposition 8.1 also shows that J is a braided functor.

Proposition 8.1 *We have*

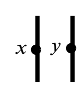
$$\mathsf{J}(\eta_{\mathsf{b}}) = \eta, \quad \mathsf{J}(v_{\pm}) = \mathbf{r}^{\pm 1}, \quad \mathsf{J}(c_{\pm}) = c_{\pm}^H, \quad \mathsf{J}(\mu_{\mathsf{b}}) = \mu, \quad \mathsf{J}(\psi_{\mathsf{b},\mathsf{b}}^{\pm 1}) = \psi_{H,H}^{\pm 1},$$

for $x, y \in H$. Here, by abuse of notation, we denote by $\mathbf{r}^{\pm 1}$ the corresponding morphism in $\mathsf{Mod}_H(\mathbf{k}, H)$.

Proof For $T = \eta_{\mathsf{b}}, v_{\pm}, c_{\pm}, \mu_{\mathsf{b}}$, the homomorphism $\mathsf{J}(T)$ are easily computed using Figure 22 (a)–(f). The case $T = \psi_{H,H}$ is computed using Figure 22 (g), where



means



.

We have

$$\begin{aligned}\mathsf{J}(\psi_{\mathsf{b},\mathsf{b}})(x \otimes y) &= \sum \beta_{(1)} y S(\beta_{(2)}) \otimes \alpha_{(1)} x S(\alpha_{(2)}) \\ &= \sum (\beta \triangleright y) \otimes (\alpha \triangleright x) = \psi_{H,H}(x \otimes y),\end{aligned}$$

where we write $(\Delta\otimes\Delta)(R)=\sum\alpha_{(1)}\otimes\alpha_{(2)}\otimes\beta_{(1)}\otimes\beta_{(2)}$. We can similarly check the case $T=\psi^{-1}_{\mathsf{b},\mathsf{b}}$. □

An easy consequence of the braided functor J is the following, which is essentially well known.

Proposition 8.2 (See Kerler [39, Corollary 12]) *If $T\in\mathsf{BT}_n$, then we have $J_T\in\mathsf{Mod}_H(\mathbf{k},H^{\otimes n})$. In particular, if $T\in\mathsf{BT}_1$, then $J_T\in H$ is central.*

8.3 The functor J as a morphism of external Hopf algebras

Note that, by Remark 6.4, the Hopf algebra structure of the transmutation $\underline{H}=(H,\mu,\eta,\underline{\Delta},\epsilon,\underline{S})$ of H determines an external Hopf algebra $((H,\mu,\eta),F_{\underline{H}})$ in the canonical way.

Theorem 8.3 *The braided functor $\mathsf{J}\colon\mathsf{B}\to\mathsf{Mod}_H$ maps the external Hopf algebra $(\mathsf{b},F_{\mathsf{b}})$ in B into the external Hopf algebra $(H,F_{\underline{H}})$ in Mod_H in the following sense.*

(1) *J maps the algebra $(\mathsf{b},\mu_{\mathsf{b}},\eta_{\mathsf{b}})$ into the algebra (H,μ,η).*

(2) *By defining $\mathsf{J}'_{\mathsf{H}^{\otimes m}}=\mathsf{J}\colon\mathsf{BT}_m\to\mathsf{Mod}_H(\mathbf{k},H^{\otimes m})$ for $m\geq 0$, we obtain a natural transformation $\mathsf{J}'\colon F_{\mathsf{b}}\Rightarrow F_{\underline{H}}$.*

Proof The condition (1) follows immediately from Proposition 8.1.

The condition (2) is equivalent to that, for any morphism $f\colon\mathsf{H}^{\otimes m}\to\mathsf{H}^{\otimes n}$ in $\langle\mathsf{H}\rangle$, the diagram

$$\begin{array}{ccc}\mathsf{BT}_m & \xrightarrow{\mathsf{J}} & \mathsf{Mod}_H(\mathbf{k},H^{\otimes m})\\ {\scriptstyle F_{\mathsf{b}}(f)}\big\downarrow & & \big\downarrow{\scriptstyle F_{\underline{H}}(f)}\\ \mathsf{BT}_n & \xrightarrow{\mathsf{J}} & \mathsf{Mod}_H(\mathbf{k},H^{\otimes n})\end{array}\tag{8–7}$$

commutes. It suffices to prove (8–7) for f in a set of generators of $\langle\mathsf{H}\rangle$ as a category. Hence we can assume $f=g_{(i,j)}$ with $g\in\{\psi_{\mathsf{H},\mathsf{H}},\psi^{-1}_{\mathsf{H},\mathsf{H}},\mu_{\mathsf{H}},\eta_{\mathsf{H}},\Delta_{\mathsf{H}},\epsilon_{\mathsf{H}},S_{\mathsf{H}}\}$ and $i,j\geq 0$. The condition (1) implies that we have (8–7) if $g=\psi^{\pm1}_{\mathsf{H},\mathsf{H}}$, μ_{H} or η_{H}. To prove the cases $g=\Delta_{\mathsf{H}},\epsilon_{\mathsf{H}},S_{\mathsf{H}}$, it suffices to prove

$$J_{\check{\Delta}_{(i,j)}(T)}=(1^{\otimes i}\otimes\underline{\Delta}\otimes 1^{\otimes j})(J_T),\tag{8–8}$$

$$J_{\check{\epsilon}_{(i,j)}(T)}=(1^{\otimes i}\otimes\epsilon\otimes 1^{\otimes j})(J_T),\tag{8–9}$$

$$J_{\check{S}_{(i,j)}(T)}=(1^{\otimes i}\otimes\underline{S}\otimes 1^{\otimes j})(J_T)\tag{8–10}$$

for $T \in \mathsf{BT}_{i+j+1}$, $i, j \geq 0$. Note that (8–9) follows from Lemma 7.4.

We write
$$J_T = \sum J_{(T_1)} \otimes \cdots \otimes J_{(T_{i+j+1})} = \sum y \otimes x \otimes y',$$
where
$$\begin{aligned} x &= J_{(T_{i+1})} \in H, \\ y &= J_{(T_1)} \otimes \cdots \otimes J_{(T_i)} \in H^{\otimes i}, \\ y' &= J_{(T_{i+2})} \otimes \cdots \otimes J_{(T_{i+j+1})} \in H^{\otimes j}. \end{aligned}$$
By Figure 23, we have
$$\begin{aligned} J_{\check{\Delta}_{(i,j)}(T)} &= \sum y \otimes x_{(1)} S(\beta) \otimes \alpha_{(1)} x_{(2)} S(\alpha_{(2)}) \otimes y' \\ &= \sum y \otimes x_{(1)} S(\beta) \otimes (\alpha \triangleright x_{(2)}) \otimes y' \\ &= \sum y \otimes \underline{\Delta}(x) \otimes y' \\ &= (1^{\otimes i} \otimes \underline{\Delta} \otimes 1^{\otimes j})(J_T), \end{aligned}$$
and
$$\begin{aligned} J_{\check{S}_{(i,j)}(T)} &= \sum y \otimes \beta \kappa \alpha_{(2)} \kappa^{-1} S(x) S(\alpha_{(1)}) \otimes y' \\ &= \sum y \otimes \beta S^2(\alpha_{(2)}) S(x) S(\alpha_{(1)}) \otimes y' \\ &= \sum y \otimes \beta S(\alpha_{(1)} x S(\alpha_{(2)})) \otimes y' \\ &= \sum y \otimes \beta S(\alpha \triangleright x) \otimes y' \\ &= \sum y \otimes \underline{S}(x) \otimes y' \\ &= (1^{\otimes i} \otimes \underline{S} \otimes 1^{\otimes j})(J_T). \end{aligned}$$
Hence we have (8–8) and (8–10). □

Theorem 8.3 implies the relationship between the Hopf algebra action on the bottom tangles and the functor J that we mentioned in the latter half of Section 1.6.

8.4 Topological proofs of algebraic identities

Theorem 8.3 means that, to a certain extent, the braided Hopf algebra structure of $\underline{H}$ is explained in terms of the external Hopf algebra structure in B, which is defined topologically. Thus, Theorem 8.3 can be regarded as a *topological interpretation* of

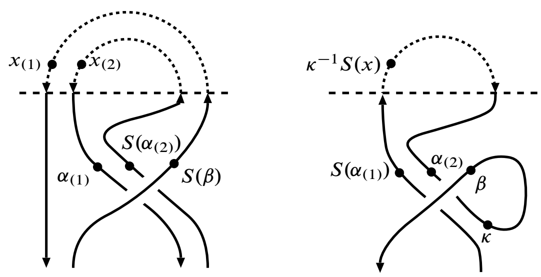


Figure 23

transmutation of a ribbon Hopf algebra. We explain below that Theorem 8.3 can be used in proving various identities for transmutation using isotopy of tangles.

For a **k**–module homomorphism $f\colon H^{\otimes m} \to H^{\otimes n}$ and $i,j \geq 0$, we set

$$f_{(i,j)} = 1_H^{\otimes i} \otimes f \otimes 1_H^{\otimes j}\colon H^{\otimes(m+i+j)} \to H^{\otimes(n+i+j)}.$$

For $f,g \in \mathsf{Mod}_H(H^{\otimes m}, H^{\otimes n})$, we write $f \equiv g$ if we have

$$f_{(i,j)}(J_T) = g_{(i,j)}(J_T)$$

for all $i,j \geq 0$ and $T \in \mathsf{BT}_{i+j+m}$. Note that if $m=0$, then $f \equiv g$ and $f=g$ are equivalent.

Remark 8.4 *All the formulas of the form "$f \equiv g$" that appear in what follows can be replaced with "$f = g$".* One can prove this fact either by direct computation or using the functor $\mathsf{J}^{\mathcal{B}}$ mentioned in Section 14.4 below. In the present paper, we content ourselves with the weaker form "$f \equiv g$".

Let $\underline{\mathrm{ad}} \in \mathsf{Mod}_H(H^{\otimes 2}, H)$ denote the left adjoint action for $\underline{H}$ defined by

$$\underline{\mathrm{ad}} = \mu^{[3]}(1 \otimes \psi_{H,H})((1 \otimes \underline{S})\underline{\Delta} \otimes 1).$$

It is well known that $\mathrm{ad} = \underline{\mathrm{ad}}$. As a first example of topological proofs, we show that $\underline{\mathrm{ad}} \equiv \mathrm{ad}$. Since Figure 14 (a) and (b) are isotopic, we see that, for $i,j \geq 0$ and $T \in \mathsf{BT}_{i+j+2}$,

$$J_{F_{\mathrm{b}}((\mathrm{ad}_{\mathrm{H}})_{(i,j)})(T)} = \underline{\mathrm{ad}}_{(i,j)}(J_T)$$

is calculated using Figure 24. Hence we have

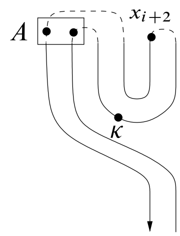


Figure 24: The tangle $F_{\mathsf{b}}((\mathrm{ad}_{\mathsf{H}})_{(i,j)})(T)$ with only the $(i+1)$st component depicted. Here we write $J_T = \sum x_1 \otimes \cdots \otimes x_{i+j+2}$. The box labeled "A" contains the tensor $\sum (x_{i+1})_{(1)} \otimes \kappa^{-1} S((x_{i+1})_{(2)})$.

$$
\begin{aligned}
&\underline{\mathrm{ad}}_{(i,j)}(J_T) \\
&\quad = \sum x_1 \otimes \cdots \otimes x_i \otimes (x_{i+1})_{(1)} x_{i+2} S((x_{i+1})_{(2)}) \otimes x_{i+3} \otimes \cdots \otimes x_{i+j+2} \\
&\quad = \sum x_1 \otimes \cdots \otimes x_i \otimes (x_{i+1} \triangleright x_{i+2}) \otimes x_{i+3} \otimes \cdots \otimes x_{i+j+2} \\
&\quad = \mathrm{ad}_{(i,j)}(J_T).
\end{aligned}
$$

Hence we have $\underline{\mathrm{ad}} \equiv \mathrm{ad}$.

Let $\underline{\mathrm{coad}} \in \mathrm{Mod}_H(H, H^{\otimes 2})$ denote the left adjoint coaction for $\underline{H}$, defined by

$$\underline{\mathrm{coad}} = (\mu \otimes 1)(1 \otimes \psi_{H,H}(1 \otimes \underline{S}))\underline{\Delta}^{[3]}.$$

By taking J of (6–11), we have

$$\underline{\mathrm{coad}}_{(i,j)}(J_T) = (\mathsf{J}(\gamma_+))_{(i,j)}(J_T) = (1_H \otimes \mathrm{ad})_{(i,j)}(c_+^H \otimes 1_H)_{(i,j)}(J_T),$$

where we used $\mathrm{ad} \equiv \underline{\mathrm{ad}}$. Hence we have

(8–11) $$\mathsf{J}(\gamma_+) \equiv \underline{\mathrm{coad}} \equiv (1_H \otimes \mathrm{ad})(c_+^H \otimes 1_H).$$

By taking J of (6–12), we have

$$\mathsf{J}(\gamma_-)_{(i,j)}(J_T) = (\underline{S} \otimes 1_H)_{(i,j)} \mathsf{J}(\gamma_+)_{(i,j)}(J_T).$$

Hence we have

(8–12) $$\mathsf{J}(\gamma_-) \equiv (\underline{S} \otimes 1_H)\underline{\mathrm{coad}}.$$

We also note that the proof of external Hopf algebra axioms in B yields a topological proof of the weak "$\equiv$–version" of the identities in the axiom of Hopf algebra for $\underline{H}$. For example, one can derive the formula

$$\underline{\Delta}\mu \equiv (\mu \otimes \mu)(1_H \otimes \psi_{H,H} \otimes 1_H)(\underline{\Delta} \otimes \underline{\Delta})$$

from Figure 12, and

$$\mu(1_H \otimes \underline{S})\underline{\Delta} \equiv \mu(\underline{S} \otimes 1_H)\underline{\Delta} \equiv \eta\epsilon$$

from Figure 13.

9 Values of universal invariants of bottom tangles

In this subsection, we study the set of values of universal invariants of bottom tangles. In Section 9.1, we give several general results, and in later subsections we give applications to some specific cases.

We fix a ribbon Hopf algebra H over a commutative, unital ring $\mathbf{k}$.

9.1 Values of J_T

We use the following notation. Let $K \subset H^{\otimes m}$ and $L \subset H^{\otimes n}$ be subsets. Set

$$K \otimes L = \{x \otimes y \mid x \in K, y \in L\} \subset H^{\otimes(m+n)}.$$

If $x \in H$, then set

$$K \otimes x = K \otimes \{x\}, \quad x \otimes K = \{x\} \otimes K.$$

The category B acts on the left H–modules $H^{\otimes n}$, $n \geq 0$, by the functions

$$j_{m,n}\colon \mathsf{B}(m,n) \times H^{\otimes m} \to H^{\otimes n}, \quad (T,x) \mapsto Tx = \mathsf{J}(T)(x).$$

If C is a subcategory of B, and if $K \subset \bigcup_{i\geq 0} H^{\otimes i}$, set

$$C \cdot K = \bigcup_{m,n\geq 0} j_{m,n}(C(m,n) \times (K \cap H^{\otimes m})).$$

Recall that $\mathcal{A}$ denotes the braided subcategory of B generated by the object b and the morphisms μ_{b} and η_{b}.

We have the following characterization of the possible values of the universal invariants of bottom tangles.

Theorem 9.1 *The set $\{J_U \mid U \in \mathsf{BT}\}$ of the values of J_U for all the bottom tangles $U \in \mathsf{BT}$ is given by*

$$\{J_U \mid U \in \mathsf{BT}\} = \mathcal{A} \cdot \{J_T \mid T \in \{v_\pm, c_\pm\}^*\}. \tag{9–1}$$

Proof The result follows immediately from Corollary 5.21 and functoriality of J. □

Using Theorem 9.1, we obtain the following, which will be useful in studying the universal invariants of bottom tangles.

Corollary 9.2 *Let $K_i \subset H^{\otimes i}$ for $i \geq 0$, be subsets satisfying the following.*

(1) $1 \in K_0$, $1, v^{\pm 1} \in K_1$, *and* $c^H_{\pm} \in K_2$.

(2) *For* $m, n \geq 0$, *we have* $K_m \otimes K_n \subset K_{m+n}$.

(3) *For* $p, q \geq 0$ *we have*

$$\begin{aligned}(\psi^{\pm 1}_{H,H})_{(p,q)}(K_{p+q+2}) &\subset K_{p+q+2},\\ \mu_{(p,q)}(K_{p+q+2}) &\subset K_{p+q+1}.\end{aligned}$$

Then, for any $U \in \mathsf{BT}_n$, $n \geq 0$, *we have* $J_U \in K_n$.

Proof By (1) and (2), the K_i contain J_T for $T \in \{v_{\pm}, c_{\pm}\}^*$. By (1), (2) and (3), the K_i are invariant under the action of $\mathcal{A}$. Hence we have the assertion. □

Using Theorem 5.14, we see that, for $T \in \mathsf{ABT}$, the set of the values of the universal invariant of bottom tangles obtained from η_n $(n \geq 0)$ by one T–move is equal to $\check{\mathsf{B}}_0 \cdot \{J_T\}$. Similarly, by Corollary 5.15, we see that, for $M \subset \mathsf{ABT}$ inversion-closed, the set of the values of the universal invariant of the M–trivial bottom tangles is equal to $\check{\mathsf{B}}_0 \cdot \{J_U \mid U \in M^*\}$. From these observations we have the following results, which will be useful in applications.

Corollary 9.3 *Let* $T \in \mathsf{ABT}_m$, *and let* $K_i \subset H^{\otimes i}$ *for* $i \geq 0$ *be subsets satisfying the following conditions.*

(1) $J_T \in K_m$.

(2) *For* $p, q \geq 0$ *and* $f \in \{\psi^{\pm 1}_{H,H}, \eta, \mu, \underline{\mathrm{coad}}, (\underline{S} \otimes 1)\underline{\mathrm{coad}}\}$ *with* $f \colon \mathsf{b}^{\otimes i} \to \mathsf{b}^{\otimes j}$, *we have*

$$f_{(p,q)}(K_{p+q+i}) \subset K_{p+q+j}.$$

Then, for any $U \in \mathsf{BT}_n$ *obtained from* η_n *by one* T*–move, we have* $J_U \in K_n$.

Proof We have to show that $\check{\mathsf{B}}_0 \cdot \{J_T\} \subset \bigcup_n K_n$. Since $J_T \in K_m$, it suffices to show that $\bigcup_n K_n$ is stable under the action of generators of $\check{\mathsf{B}}_0$. Since $\check{\mathsf{B}}_0$ is as a category generated by $f_{(p,q)}$ with $p, q \geq 0$ and $f \in \{\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-\}$, the condition (2) implies that $\bigcup_n K_n$ is stable under the action of $\check{\mathsf{B}}_0$. Here we use (8–11) and (8–12) with $\equiv$ replaced by $=$. (See Remark 8.4.) □

Corollary 9.4 *Let $M \subset \mathsf{ABT}$ be inversion-closed, and let $K_i \subset H^{\otimes i}$ for $i \geq 0$ be subsets satisfying the following conditions.*

(1) $1 \in K_0$ *and* $1 \in K_1$.

(2) *For $V \in M$, we have $J_V \in K_{|V|}$.*

(3) *If $k, l \geq 0$, then we have $K_k \otimes K_l \subset K_{k+l}$.*

(4) *For $p, q \geq 0$ and $f \in \{\psi^{\pm 1}_{H,H}, \mu, \underline{\mathrm{coad}}, (\underline{S} \otimes 1)\underline{\mathrm{coad}}\}$ with $f\colon H^{\otimes i} \to H^{\otimes j}$, we have*

$$f_{(p,q)}(K_{p+q+i}) \subset K_{p+q+j}.$$

Then, for any M–trivial $U \in \mathsf{BT}_n$, we have $J_U \in K_n$.

Proof It suffices to check the conditions in Corollary 9.3, where T is an element of M^*. The condition (1) in Corollary 9.3, ie, $J_T \in K_{|T|}$, follows from (1), (2) and (3). The condition (2) in Corollary 9.3 with $f \neq \eta$ follows from (4). The condition (2) with $f = \eta$ in Corollary 9.3 follows from (1), (3), and (4), since we have

$$\eta_{(p,q)}(K_{p+q}) = (1^{\otimes p} \otimes \psi_{H^{\otimes q},H})(K_{p+q} \otimes 1). \qquad \square$$

The following will be useful in studying the set of M–equivalence classes of bottom tangles for $M \subset \mathsf{ABT}$.

Corollary 9.5 *Let $M \subset \mathsf{ABT}$, not necessarily inversion-closed. Let $K_i \subset H^{\otimes i}$ for $i \geq 0$ be $\mathbb{Z}$–submodules satisfying the following conditions.*

(1) *For each $T \in M$ we have $J_T - 1^{\otimes |T|} \in K_{|T|}$.*

(2) *For $i \geq 0$, we have*

$$K_i \otimes 1 \subset K_{i+1}, \quad K_i \otimes v^{\pm 1} \subset K_{i+1}, \quad K_i \otimes c^H_{\pm} \subset K_{i+2}.$$

(3) *For $p, q \geq 0$, we have*

$$(\psi^{\pm 1}_{H,H})_{(p,q)}(K_{p+q+2}) \subset K_{p+q+2}, \tag{9–2}$$

$$\mu_{(p,q)}(K_{p+q+2}) \subset K_{p+q+1}. \tag{9–3}$$

Then, for any pair $U, U' \in \mathsf{BT}_n$ of M–equivalent bottom tangles, we have $J_{U'} - J_U \in K_n$. Hence there is a well-defined function

$$\mathsf{BT}_n/(M\text{–equivalence}) \to H^{\otimes n}/K_n, \quad [U] \mapsto [J_U]$$

for each $n \geq 0$.

Proof Since $K_n \subset H^{\otimes n}$ are $\mathbb{Z}$–submodules for $n \geq 0$, we may assume without loss of generality that U and U' are related by one T–move for $T \in M$. Set $r = |T|$. By Theorem 5.8, there is $W \in \mathsf{B}(r,n)$ such that $U = W\eta_r$ and $U' = WT$. Hence we have

$$J_{U'} - J_U = \mathsf{J}(W)(J_T - 1^{\otimes r}) \in \mathsf{J}(W)(K_r),$$

Therefore we have only to prove that $\mathsf{J}(W)(K_r) \subset K_n$. By the assumptions, this holds for each generator W of B as a subcategory of $\mathcal{T}$, described in Corollary 5.18. Hence we have the assertion. □

9.2 Unknotting number

A *positive crossing change* is a local move on a tangle which replaces a negative crossing with a positive crossing. A *negative crossing change* is the inverse operation. In our terminology, a positive (resp. negative) crossing change is equivalent to a c_-–move (resp. c_+–move).

A *bottom knot* is a 1–component bottom tangle.

Corollary 9.6 *Let $n_+, n_- \geq 0$, and let $K_i \subset H^{\otimes i}$, $i \geq 1$, be subsets satisfying the following conditions.*

(1) $(c_+^H)^{\otimes n_-} \otimes (c_-^H)^{\otimes n_+} \in K_{2(n_+ + n_-)}$.

(2) *For $p, q \geq 0$ and $f \in \{\psi^{\pm 1}_{H,H}, \eta, \mu, \underline{\mathrm{coad}}, (\underline{S} \otimes 1)\underline{\mathrm{coad}}\}$ with $f\colon \mathsf{b}^{\otimes i} \to \mathsf{b}^{\otimes j}$, we have*

$$f_{(p,q)}(K_{p+q+i}) \subset K_{p+q+j}.$$

Then, if a bottom knot $T \in \mathsf{BT}_1$ of framing 0 is obtained from η_{b} by n_+ positive crossing changes and n_- negative crossing changes up to framing change, we have $J_T \in \mathbf{r}^{2(n_+ - n_-)} K_1$.

Proof The result follows from Corollary 9.3, since T is obtained from η_{b} by a $(c_+^{\otimes n_-} \otimes c_-^{\otimes n_+})$–move and framing change by $-2(n_+ - n_-)$. □

Corollary 9.6 can be used to obtain an obstruction for a bottom knot T to be of unknotting number at most n, since a bottom knot is of unknotting number n if and only if, for some $n_+, n_- \geq 0$ with $n_+ + n_- = n$, we have the situation in the statement of Corollary 9.6. Also, we can use Corollary 9.6 to obtain an obstruction for a bottom tangle to be positively unknottable, ie, obtained from η_{b} by finitely many positive crossing change up to framing change.

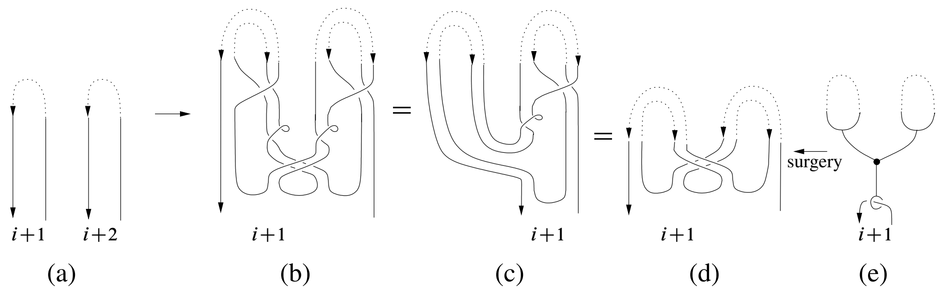


Figure 25: (a) An $(i+j+2)$–component bottom tangle T. (b) The $(i+j+1)$–component bottom tangle $T' = F((Y_{\mathsf{H}})_{(i,j)})(T)$, calculated using (9–4). (c) T' calculated using (9–5). (d) Another picture of T', in which the $(i+1)$st component bounds a Seifert surface of genus 1. (d) A presentation of T' using a clasper.

In the literature, versions of unknotting numbers with respect to various kinds of admissible local moves are studied, see, for example, Murakami [58] or Murakami–Nakanishi [59]. For $M \subset \mathsf{ABT}$, a bottom knot $T \in \mathsf{BT}_1$ is said to be of "M–unknotting number n" if T can be obtained from η_1 by n applications of M–moves. One can easily modify Corollary 9.6 to give obstructions for a bottom knot to be of M–unknotting number $\leq n$.

9.3 Commutators and Seifert surfaces

For any Hopf algebra A in a braided category $\mathcal{M}$, we define the *commutator morphism* [22, Section 8.1] $Y_A \in \mathcal{M}(A^{\otimes 2}, A)$ by

$$Y_A = \mu_A^{[4]}(A \otimes \psi_{A,A} \otimes A)(A \otimes S_A \otimes S_A \otimes A)(\Delta_A \otimes \Delta_A). \tag{9–4}$$

Using adjoint action, we obtain a simpler formula, which is sometimes more useful:

$$Y_A = \mu_A(\mathrm{ad}_A \otimes A)(A \otimes S_A \otimes A)(A \otimes \Delta_A). \tag{9–5}$$

The function

$$F_{\mathsf{b}}((Y_{\mathsf{H}})_{(i,j)}) \colon \mathsf{BT}_{i+j+2} \to \mathsf{BT}_{i+j+1}$$

transforms a bottom tangle into another as illustrated in Figure 25.

For a ribbon Hopf algebra H, a (relatively) simple formula for the left H–module homomorphism $Y_{\underline{H}}$ is as follows.

Proposition 9.7 *For* $\sum x \otimes y \in H^{\otimes 2}$, *we have*

$$Y_{\underline{H}}\Big(\sum x \otimes y\Big) = \sum\Big(x \triangleright \beta S((\alpha \triangleright y)_{(1)})\Big)(\alpha \triangleright y)_{(2)},$$

where $\Delta(\alpha \triangleright y) = \sum (\alpha \triangleright y)_{(1)} \otimes (\alpha \triangleright y)_{(2)}$.

Proof By computation, we have

$$(\underline{S} \otimes 1)\underline{\Delta}(y) = \sum \beta S((\alpha \triangleright y)_{(1)}) \otimes (\alpha \triangleright y)_{(2)} \tag{9–6}$$

for $y \in H$. Hence we have by (9–5)

$$\begin{aligned} Y_{\underline{H}}\Big(\sum x \otimes y\Big) &= \mu(\mathrm{ad} \otimes H)(1 \otimes (\underline{S} \otimes 1)\underline{\Delta})\Big(\sum x \otimes y\Big) \\ &= \sum (x \triangleright \beta S((\alpha \triangleright y)_{(1)}))(\alpha \triangleright y)_{(2)}. \end{aligned}$$

This completes the proof. □

Remark 9.8 Proposition 9.7 holds also for the transmutation of a quasitriangular Hopf algebra H which are not ribbon.

A *Seifert surface* of a bottom knot T in a cube $[0,1]^3$ is a compact, connected, oriented surface F in $[0,1]^3$ such that $\partial F = T \cup \gamma$ and $F \cap ([0,1]^2 \times \{0\}) = \gamma$, where $\gamma \subset [0,1]^2 \times \{0\}$ is the line segment with $\partial\gamma = \partial T$. Note that a Seifert surface of a bottom knot T determines in the canonical way a Seifert surface of the closure of T.

Recall that a link L in S^3 is *boundary* if the components of L bounds mutually disjoint Seifert surfaces. Similarly, a bottom tangle $T \in \mathsf{BT}_n$ is said to be *boundary* if the components of T are of framing 0 and bound mutually disjoint Seifert surfaces in $[0,1]^3$.

Theorem 9.9 *Let* $K_i \subset H^{\otimes i}$, $i \geq 0$, *be as in Corollary 9.2. Then, for any boundary bottom tangle* $T = T_1 \cup \cdots \cup T_n$ *bounding mutually disjoint Seifert surfaces* $F_1, \dots, F_n$ *of genus* $g_1, \dots, g_n$, *we have*

$$J_T \in (\mu^{[g_1]} \otimes \cdots \otimes \mu^{[g_n]}) Y_{\underline{H}}^{\otimes (g_1 + \cdots + g_n)}(K_{2(g_1 + \cdots + g_n)}).$$

In particular, if a bottom knot bounds a Seifert surface of genus g, *then we have*

$$J_T \in \mu^{[g]} Y_{\underline{H}}^{\otimes g}(K_{2g}).$$

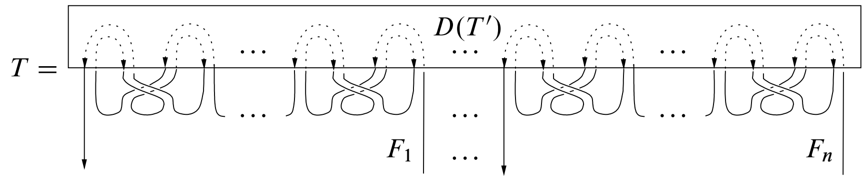


Figure 26

Proof Set $g = g_1 + \cdots + g_n$. By isotopy, we can arrange $F_1, \ldots, F_n$ as depicted in Figure 26, where $D(T') \in \mathcal{T}(\mathbf{1}, \mathsf{b}^{\otimes 4g})$ is obtained from a bottom tangle $T' \in \mathsf{BT}_{2g}$ by doubling the components. (The surfaces bounded by the components of T should be obvious from the figure.) We have

$$T = F_{\mathsf{b}}((\mu_{\mathsf{H}}^{[g_1]} \otimes \cdots \otimes \mu_{\mathsf{H}}^{[g_n]}) Y_{\mathsf{H}}^{\otimes g})(T').$$

Hence

$$\begin{aligned} J_T &= F_{\underline{H}}((\mu_{\mathsf{H}}^{[g_1]} \otimes \cdots \otimes \mu_{\mathsf{H}}^{[g_n]}) Y_{\mathsf{H}}^{\otimes g})(J_{T'}) \\ &= (\mu_{H}^{[g_1]} \otimes \cdots \otimes \mu_{H}^{[g_n]}) Y_{\underline{H}}^{\otimes g}(J_{T'}). \end{aligned}$$

By Corollary 9.2, we have $J_{T'} \in K_{2g}$. Hence we have the assertion. □

It is easy to verify that a link L is boundary if and only if there is a boundary bottom tangle T such that the closure of T is equivalent to L. (However, there are many non-boundary bottom tangles whose closures are boundary.) Hence we can use Theorem 9.9 to obtain J_L for boundary links L. Also, the latter part of Theorem 9.9 can be used to obtain an obstruction for a knot from being of genus $\leq g$.

9.4 Unoriented spanning surfaces

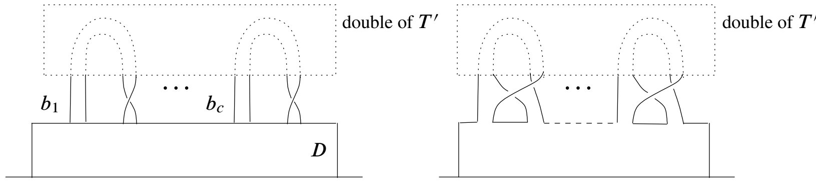
Here we consider the "unorientable version" of the previous subsection.

The *crosscap number* (see Clark [7] and Murakami–Yasuhara [60]) of an unframed nontrivial knot K is the minimum number of the first Betti numbers of unorientable surfaces bounded by K. The crosscap number of an unknot is defined to be 0.

Proposition 9.10 *Let T be a 0–framed bottom knot of crosscap number $c \geq 0$ (ie, the closure of T is of crosscap number c). Then there is $T' \in \mathsf{BT}_c$ such that*

(9–7) $$J_T = \mathbf{r}^{4w(T')}(\mu \underline{\Delta})^{\otimes c}(J_{T'}),$$

where $w(T') \in \mathbb{Z}$ is the writhe of the tangle T'.



(a) (b)

Figure 27

Proof By assumption, the union of the bottom knot T and the line segment bounded by the endpoints of T bounds a connected, compact, unorientable surface N of genus c in the cube. Here the framing of T which is determined by N may differ from the 0–framing. (We ignore the framing until the end of this proof.) As is well known, N can be obtained from a disc D by attaching c bands $b_1, \dots, b_c$ such that, for each $i = 1, \dots, c$, the union $D \cup b_i$ is a Möbius band, and between the two components of $D \cap b_i$ there are no attaching region of the other band, see Figure 27 (a). Here the dotted part is obtained from a c–component bottom tangle T' by replacing the components with bands, using the framings. T can be isotoped as in Figure 27 (b). Since the framing of the tangle depicted in Figure 27 (b) is $4w(T')$, we have

$$T = (t_{\downarrow} \otimes \uparrow)^{-4w(T')} F_{\mathsf{b}}((\mu_{\mathsf{H}} \Delta_{\mathsf{H}})^{\otimes c})(J_{T'}).$$

Hence we have the assertion. □

The unorientable version of boundary link is $\mathbb{Z}_2$*–boundary link* (see Hillman [27]). A link L in S^3 is called $\mathbb{Z}_2$–boundary if the components of L bounds mutually disjoint possibly unorientable surfaces. Similarly, a bottom tangle $T \in \mathsf{BT}_n$ is said to be $\mathbb{Z}_2$*–boundary* if the components of T are of framing 0 and bound mutually disjoint possibly unorientable surfaces in $[0,1]^3$. In the above definitions, “possibly unorientable” can be replaced with “unorientable”. One can easily modify Theorem 9.9 into the $\mathbb{Z}_2$–boundary case.

9.5 Borromean tangle and delta moves

We consider *delta moves* (see Murakami and Nakanishi [59]) or *Borromean transformation* (see Matveev [57]) on bottom tangles, which we mentioned in Section 1.7. In our setting, a delta move can be defined as a B–move, where $B \in \mathsf{BT}_3$ is the Borromean tangle defined in Section 1.7.

The following is an easily verified variant of a theorem of Murakami and Nakanishi [59], which makes delta moves especially useful.

Proposition 9.11 (Murakami–Nakanishi [59]) *Two n–component bottom tangles T and T' have the same linking matrix if and only if there is a sequence of finitely many delta moves (and isotopies) from T to T'. (Here the linking matrix of an n–component bottom tangle T is defined to be the linking matrix of the closure of T.)*

Using Proposition 9.11, we obtain the following results.

Corollary 9.12 *For two n–component bottom tangles T and T', the following conditions are equivalent.*

(1) *T and T' have the same linking matrix.*

(2) *T and T' are delta move equivalent, ie, B–equivalent.*

(3) *For some $k \geq 0$ and $W \in \mathsf{B}(3k, n)$, we have*

$$T = W\eta_{3k}, \quad T' = WB^{\otimes k}.$$

Proof The equivalence of (1) and (2) is just Proposition 9.11. The equivalence of (2) and (3) follows from Proposition 5.12, since the set $\{B\}$ is inversion-closed (see [59]). $\square$

Corollary 9.13 *The linking matrix of an n–component bottom tangle T is zero if and only if there are $k \geq 0$ and $W \in \check{\mathsf{B}}_0(3k, n)$ such that we have $T = WB^{\otimes k}$.*

In particular, a bottom tangle with zero linking matrix is obtained by pasting finitely many copies of $1_{\mathsf{b}}, \psi_{\mathsf{b},\mathsf{b}}, \psi_{\mathsf{b},\mathsf{b}}^{-1}, \mu_{\mathsf{b}}, \eta_{\mathsf{b}}, \gamma_+, \gamma_-, B$.

Proof The result follows from Corollary 5.15, since T is of linking matrix 0 if and only if T is B–trivial. $\square$

In some applications, the following form may be more useful.

Corollary 9.14 *The linking matrix of an n–component bottom tangle T is zero if and only if there are $k \geq 0$ and $f \in \langle \mathsf{H} \rangle(3k, n)$ such that we have $T = F_{\mathsf{b}}(f)(B^{\otimes k})$.*

Proof The “only if” part follows easily from Corollary 9.13, using

$$(\gamma_+)_{(i,j)}U = F_{\mathsf{b}}((\mathrm{coad}_{\mathsf{H}})_{(i,j)})(U),$$
$$(\gamma_-)_{(i,j)}U = F_{\mathsf{b}}(((S_{\mathsf{H}} \otimes \mathsf{H})\mathrm{coad}_{\mathsf{H}})_{(i,j)})(U),$$

for $i, j \geq 0$, $U \in \mathsf{BT}_{i+j+1}$.

The “if” part follows from the easily verified fact that the set of bottom tangles with zero linking matrices is closed under the Hopf algebra action. $\square$

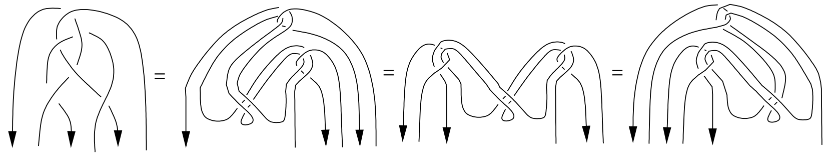


Figure 28

Now we apply the above results to the universal invariant. First we give a few formulas for $J_B \in H^{\otimes 3}$. Using Figure 3, we can easily see that

$$J_B = \sum S^2(\alpha_5)\beta_2\alpha_6 S(\beta_1) \otimes \alpha_1\beta_4\alpha_2 S(\beta_3) \otimes \alpha_3 S^{-2}(\beta_6)\alpha_4 S(\beta_5), \tag{9–8}$$

where $R = \sum \alpha_i \otimes \beta_i$ for $i = 1, \dots, 6$. By Figure 28, we have

(9–9)

$$B = F_{\mathsf{b}}(Y_{\mathsf{H}} \otimes \mathsf{H} \otimes \mathsf{H})(c_{+,2}) = F_{\mathsf{b}}(\mathsf{H} \otimes Y_{\mathsf{H}} \otimes \mathsf{H})((c_+)^{\otimes 2}) = F_{\mathsf{b}}(\mathsf{H} \otimes \mathsf{H} \otimes Y_{\mathsf{H}})(c_{+,2}),$$

where $c_{+,2} = (\mathsf{b} \otimes c_+ \otimes \mathsf{b})c_+ \in \mathsf{BT}_4$. By (9–9), it follows that

$$J_B = (Y_{\underline{H}} \otimes 1_H^{\otimes 2})(c_{+,2}^H) = (1_H \otimes Y_{\underline{H}} \otimes 1_H)((c_+^H)^{\otimes 2}) = (1_H^{\otimes 2} \otimes Y_{\underline{H}})(c_{+,2}^H),$$

where $c_{+,2}^H = \sum (c_+^H)_{[1]} \otimes c_+^H \otimes (c_+^H)_{[2]} \in H^{\otimes 4}$ with $c_+^H = \sum (c_+^H)_{[1]} \otimes (c_+^H)_{[2]}$. One can apply Corollary 9.5 to the case $M = \{B\}$ to obtain a result about the difference of the universal invariants of bottom tangles which have the same linking matrix. We do not give the explicit statement here.

For the bottom tangles with zero linking matrices, we can easily derive the following result from Corollaries 9.13 and 9.14.

Corollary 9.15 *Set either*

$$X = \{\psi_{H,H}^{\pm 1},\ \mu,\ \underline{\mathrm{coad}},\ (\underline{S} \otimes 1)\underline{\mathrm{coad}}\} \quad \text{or} \quad X = \{\psi_{H,H}^{\pm 1},\ \mu,\ \underline{\Delta},\ \underline{S}\}.$$

Let $K_i \subset H^{\otimes i}$, $i \geq 0$, be subsets satisfying the following conditions.

(1) $1 \in K_0$, $1 \in K_1$, *and* $J_B \in K_3$.

(2) *If* $k, l \geq 0$, *then we have* $K_k \otimes K_l \subset K_{k+l}$.

(3) *For* $p, q \geq 0$ and $f \in X$ with $f\colon H^{\otimes i} \to H^{\otimes j}$, *we have*

$$f_{(p,q)}(K_{p+q+i}) \subset K_{p+q+j}.$$

Then, for any $U \in \mathsf{BT}_n$ with zero linking matrix, we have $J_U \in K_n$.

As mentioned in Section 1.7, in future publications we will apply Corollary 9.15 to the case where H is a quantized enveloping algebra.

9.6 Clasper moves

In this subsection, we apply the settings in this paper to the *clasper moves* or *C_n–moves* (see Goussarov [18] and Habiro [22]) which are closely related to the Goussarov–Vassiliev finite type link invariants (see Vassiliev [82], Goussarov [16; 17], Birman [3], Birman and Lin [4] and Bar-Natan [1]).

Recall that a simple C_n–moves in the sense of [22] is a local move on a tangle T defined as surgery on a strict tree clasper C of degree n (ie, with $n+1$ disc-leaves) such that each disc-leaf of T intersects transversely with T by one point. A simple C_n–move is a generalization of a crossing change ($n=1$) and a delta move ($n=2$). In this subsection, for simplicity, we slightly modify the definition of a simple C_n–move so that the sign of the intersection of C (which is defined as a surface homeomorphic to a disc) and the strings of T are all positive or all negative. It is known (see, for example, the author's master's thesis [21]) that this does not make any essential difference if $n \geq 2$. Ie, the relations on tangles defined by the moves are the same.

We can use the results in the previous sections in the study of simple C_n–moves, by redefining a simple C_n–move as an M_n–move, where M_n is an inversion-closed subset of ABT defined as follows. Define $\mathcal{Y}_n \subset \langle \mathsf{H}\rangle(\mathsf{H}^{\otimes n}, \mathsf{H})$ for $n \geq 1$ inductively by $\mathcal{Y}_1 = \{1_{\mathsf{H}}\}$ and

$$\mathcal{Y}_n = \{Y_{\mathsf{H}}(f \otimes g) \mid f \in \mathcal{Y}_i, g \in \mathcal{Y}_j, i+j=n\} \quad \text{for } n \geq 2.$$

Thus $\mathcal{Y}_n$ is the set of iterated commutators of class n. For example, we have $\mathcal{Y}_2 = \{Y_{\mathsf{H}}\}$ and $\mathcal{Y}_3 = \{Y_{\mathsf{H}}(Y_{\mathsf{H}} \otimes \mathsf{H}), Y_{\mathsf{H}}(\mathsf{H} \otimes Y_{\mathsf{H}})\}$. For $n \geq 1$, define $M_n \subset \mathsf{ABT}_{n+1}$ by

$$M_n = \{F_{\mathsf{b}}(f \otimes \mathsf{H}^{\otimes n})(c_{+,n}) \mid f \in \mathcal{Y}_n\},$$

where we set

$$c_{+,n} = (\mathsf{b}^{\otimes(n-1)} \otimes c_+ \otimes \mathsf{b}^{\otimes(n-1)}) \cdots (\mathsf{b} \otimes c_+ \otimes \mathsf{b})c_+ \in \mathsf{BT}_{2n}$$

for $n \geq 1$. (Here, the fact that each element of M_n is admissible follows from [22, Lemma 3.20].) In particular, we have $M_1 = \{c_+\}$ and $M_2 = \{B\}$. For example, Figure 29 shows a clasper C for η_5 such that surgery along C yields the tangle

$$F_{\mathsf{b}}((Y_{\mathsf{H}}(Y_{\mathsf{H}} \otimes \mathsf{H})(Y_{\mathsf{H}} \otimes \mathsf{H}^{\otimes 2})) \otimes \mathsf{H}^{\otimes 4})(c_{+,4}) \in M_4.$$

We can also define the M_n using the *cocommutator morphism* [22]

$$Y_{\mathsf{H}}^*\colon \mathsf{H} \to \mathsf{H} \otimes \mathsf{H}$$

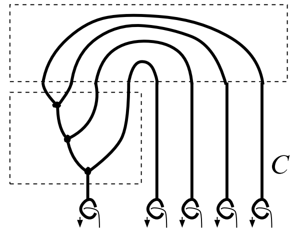


Figure 29: The upper rectangle corresponds to $c_{+,4} \in \mathsf{BT}_8$. The lower rectangle corresponds to $Y_{\mathsf{H}}(Y_{\mathsf{H}} \otimes \mathsf{H})(Y_{\mathsf{H}} \otimes \mathsf{H}^{\otimes 2}) \in \mathcal{Y}_4$.

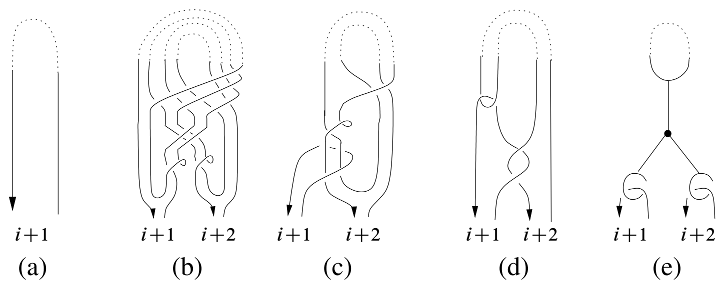


Figure 30: (a) An $(i+j+1)$–component bottom tangle T. (b) The $(i+j+2)$–component bottom tangle $T' = F((Y^*_{\mathsf{H}})_{(i,j)})(T)$, calculated using (9–10) (upper). (c) T' calculated using (9–10) (lower). (d) Another picture of T'. (d) A presentation of T' using a clasper.

defined by

$$Y^*_{\mathsf{H}} = (\mu_{\mathsf{H}} \otimes \mu_{\mathsf{H}})(\mathsf{H} \otimes S_{\mathsf{H}} \otimes S_{\mathsf{H}} \otimes \mathsf{H})(\mathsf{H} \otimes \psi_{\mathsf{H},\mathsf{H}} \otimes \mathsf{H})\Delta^{[4]}_{\mathsf{H}} \tag{9–10}$$
$$= (\mathsf{H} \otimes \mu_{\mathsf{H}})(\mathrm{coad}_{\mathsf{H}} \otimes \mathsf{H})(S_{\mathsf{H}} \otimes \mathsf{H})\Delta_{\mathsf{H}}.$$

Note that the notion of cocommutator is dual to the notion of commutator. For $i, j \geq 0$, the function

$$F_{\mathsf{b}}((Y^*_{\mathsf{H}})_{(i,j)})\colon \mathsf{BT}_{i+j+1} \to \mathsf{BT}_{i+j+2}$$

transforms a bottom tangle into another as illustrated in Figure 30. For $n \geq 1$, define $\mathcal{Y}^*_n \subset \langle \mathsf{H} \rangle(\mathsf{H}, \mathsf{H}^{\otimes n})$ inductively by $\mathcal{Y}^*_1 = \{1_{\mathsf{H}}\}$ and

$$\mathcal{Y}^*_n = \{(f \otimes g)Y^*_{\mathsf{H}} \mid f \in \mathcal{Y}^*_i, g \in \mathcal{Y}^*_j, i + j = n\} \quad \text{for } n \geq 2.$$

Then we have for $n \geq 1$,

$$M_n = \{F_{\mathsf{b}}(f \otimes g)(c_+) \mid f \in \mathcal{Y}^*_i, g \in \mathcal{Y}^*_j, i + j = n + 1\},$$

which follows by induction using (9–9) and

$$B = F_{\mathsf{b}}(Y_{\mathsf{H}}^* \otimes \mathsf{H})(c_+) = F_{\mathsf{b}}(\mathsf{H} \otimes Y_{\mathsf{H}}^*)(c_+). \tag{9–11}$$

(The above definition of M_n using Y_{H}^* is similar to the definition of local moves in [21], where we defined a family of local moves without using claspers. See also Taniyama and Yasuhara [79] for a similar definition.)

One can show that the notion of simple C_n–move and that of M_n–move are the same.

Remark 9.16 A general C_n–move, which may not be simple, is obtained by allowing removal, orientation reversal and parallelization of strings in the tangles which define the move. Hence it can be redefined as an M'_n–move, where the set $M'_n \subset \mathsf{ABT}$ is defined by

$$M'_n = \Big\{ F_{\mathsf{b}}\Big(\bigotimes_{i=1}^{n+1} \Delta_{\mathsf{H}}^{[c_i;d_i]}\Big)(f) \;\Big|\; c_1, \dots, c_{n+1}, d_1, \dots, d_{n+1} \geq 0,\, f \in M_n \Big\},$$

where we set

$$\Delta_{\mathsf{H}}^{[c;d]} = (\mathsf{H}^{\otimes c} \otimes S_{\mathsf{H}}^{\otimes d})\Delta^{[c+d]} \in \langle \mathsf{H} \rangle(\mathsf{H}, \mathsf{H}^{\otimes(c+d)})$$

for $c, d \geq 0$. As special cases, the following local moves in the literature can be redefined algebraically:

(1) A *pass-move* (see Kauffman [33] and Figure 31 (a)), which characterizes the Arf invariant of knots, is the same as a $F_{\mathsf{b}}((\Delta_{\mathsf{H}}^{[1;1]})^{\otimes 2})(c_+)$–move.

(2) A $\sharp$*–move* (see Murakami [58] and Figure 31 (b)) is the same as a $F_{\mathsf{b}}(\Delta_{\mathsf{H}}^{\otimes 2})(c_+)$–move. (This is a framed version. In applications to unframed or 0–framed knots, one should take framings into account.)

(3) A $D(\Delta)$*–move* (see Nakanishi [63] and Figure 31 (c)), which preserves the stable equivalence class of the Goeritz matrix (see Goeritz [12] and Gordon–Litherland [13]) of (possibly unorientable) spanning surfaces of knots, can be redefined by setting

$$D(\Delta) = \{F_{\mathsf{b}}((\Delta_{\mathsf{H}})^{[i;2-i]} \otimes (\Delta_{\mathsf{H}})^{[j;2-j]} \otimes (\Delta_{\mathsf{H}})^{[k;2-k]})(B) \mid 0 \leq i, j, k \leq 2\}.$$

It is known that the $D(\Delta)$–equivalence is the same as an oriented version of it, which can be defined as the $F_{\mathsf{b}}(\Delta_{\mathsf{H}}^{\otimes 3})(B)$–equivalence, ie, the case $i = j = k = 2$.

(4) A *doubled-delta move* (see Naik–Stanford [61] and Figure 31 (d)), which characterizes the S–equivalence class of Seifert matrices of knots, can be defined as a $F_{\mathsf{b}}((\Delta_{\mathsf{H}}^{[1;1]})^{\otimes 3})(B)$–move, which is the case $i = j = k = 1$ of $D(\Delta)$–move.

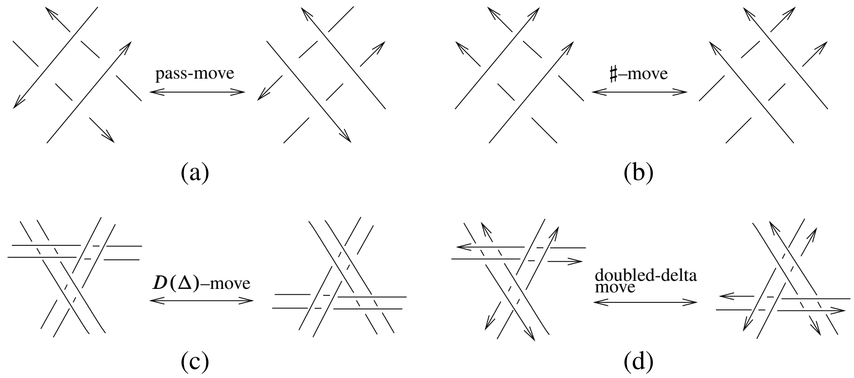


Figure 31: (a) A pass-move. (b) A ♯–move. (c) A $D(\Delta)$–move. (Here the orientations of strings are arbitrary.) (d) A doubled-delta move.

We postpone to future publications a more systematic study of the clasper moves in a category-theoretical setting, which was announced in [22]. For this purpose the category $\mathcal{B}$ (see Section 14.4 below) is more useful than B.

9.7 Goussarov–Vassiliev filtrations on tangles

In this subsection, we give an algebraic formulation of Goussarov–Vassiliev invariants using the setting of the category B.

9.7.1 Four-sided ideals in a monoidal Ab**–category** Here we recall the notion of *four-sided ideal* in a monoidal Ab–category, which can be regarded as the linearized version of the notion of four-sided congruence in a monoidal category.

Let C be a (strict) *monoidal* Ab*–category*, ie, a monoidal category C such that for each pair $X, Y \in \mathrm{Ob}(C)$ the set $C(X, Y)$ is equipped with a structure of a $\mathbb{Z}$–module, and the composition and the tensor product are bilinear.

A *four-sided ideal* $I = (I(X, Y))_{X,Y \in \mathrm{Ob}(C)}$ in a monoidal Ab–category C is a family of $\mathbb{Z}$–submodules $I(X, Y)$ of $C(X, Y)$ for $X, Y \in \mathrm{Ob}(C)$ such that

(1) if $f \in I(X, Y)$ and $g \in C(Y, Z)$ (resp. $g \in C(Z, X)$), then we have $gf \in I(X, Z)$ (resp. $fg \in I(Z, Y)$),

(2) if $f \in I(X, Y)$ and $g \in C(X', Y')$, then we have $f \otimes g \in I(X \otimes X', Y \otimes Y')$ and $g \otimes f \in I(X' \otimes X, Y' \otimes Y)$.

By abuse of notation, we denote by I, the union $\bigcup_{X,Y \in \mathrm{Ob}(C)} I(X, Y)$.

Let $S \subset \mathrm{Mor}(C)$ be a set of morphisms in C. Then there is the smallest four-sided ideal I_S in C such that $S \subset I_S$. The four-sided ideal I_S is said to be *generated* by S. For $X, X' \in \mathrm{Ob}(C)$, $I_S(X, X')$ is $\mathbb{Z}$–spanned by the elements

$$f'(g \otimes s \otimes g')f, \tag{9–12}$$

where $s \in S$, and $f, f', g, g' \in \mathrm{Mor}(C)$ are such that the expression (9–12) gives a well-defined morphisms in $C(X, X')$.

For two four-sided ideals I and I' in a monoidal Ab–category C, the *product* $I'I$ of I' and I is defined to be the smallest four-sided ideal in C such that if $(g, f) \in I' \times I$ is a composable pair, then $gf \in II'$. It follows that $f \in I$ and $g \in I'$ implies $f \otimes g, g \otimes f \in I'I$. For $X, Y \in \mathrm{Ob}(C)$, then we have

$$I'I(X,Y) = \sum_{Z \in \mathrm{Ob}(C)} I'(Z,Y)I(X,Z).$$

For $n \geq 0$, let I^n denote the nth power of I, which is defined by $I^0 = \mathrm{Mor}(C)$, $I^1 = I$, and $I^n = I^{n-1}I$ for $n \geq 2$.

Lemma 9.17 *Let C be a braided* Ab*–category and let I be a four-sided ideal in C generated by $S \subset \prod_{X \in \mathrm{Ob}(C)} C(\mathbf{1}, X)$. Then $I^n(X,Y)$ $(X, Y \in \mathrm{Ob}(C))$ is $\mathbb{Z}$–spanned by the elements of the form $f(X \otimes s_1 \otimes \cdots \otimes s_n)$, where $s_1, \dots, s_n \in S$ and $f \in C(X \otimes \mathrm{target}(s_1 \otimes \cdots \otimes s_n), Y)$.*

Proof The proof is sketched as follows. Each element of I^n is a $\mathbb{Z}$–linear combination of morphisms, each obtained as an iterated composition and tensor product of finitely many morphisms of C involving n copies of elements $s_1, \dots, s_n$ of S. By the assumption, one can arrange (using braidings) the copies $s_1, \dots, s_n$ involved in each term of an element of I^n to be placed side by side as in $s_1 \otimes \cdots \otimes s_n$ in the upper right corner, ie, we obtain a term of the form $f(X \otimes s_1 \otimes \cdots \otimes s_n)$, as desired. □

9.7.2 Goussarov–Vassiliev filtration for $\mathbb{Z}\mathcal{T}$ Here we recall a formulation of Goussarov–Vassiliev filtration using the category $\mathcal{T}$ of framed, oriented tangles, which is given by Kassel and Turaev [32].

Let $\mathbb{Z}\mathcal{T}$ denote the category of $\mathbb{Z}$*–linear tangles*. Ie, we have $\mathrm{Ob}(\mathbb{Z}\mathcal{T}) = \mathrm{Ob}(\mathcal{T})$, and for $w, w' \in \mathrm{Ob}(\mathcal{T})$, the set $\mathbb{Z}\mathcal{T}(w, w')$ is the free $\mathbb{Z}$–module generated by the set $\mathcal{T}(w, w')$. $\mathbb{Z}\mathcal{T}$ is a braided Ab–category.

Let $\mathcal{I}$ denote the four-sided ideal in $\mathbb{Z}\mathcal{T}$ generated by the morphism

$$\psi^{\times} = \psi_{\downarrow,\downarrow} - \psi_{\downarrow,\downarrow}^{-1} \in \mathbb{Z}\mathcal{T}(\downarrow^{\otimes 2}, \downarrow^{\otimes 2}).$$

For $n \geq 0$, let $\mathcal{I}^n$ denote the nth power of $\mathcal{I}$. $\mathcal{I}^n$ is equal to the four-sided ideal in $\mathbb{Z}\mathcal{T}$ generated by the morphism $(\psi^\times)^{\otimes n}$. For $w, w' \in \mathrm{Ob}(\mathcal{T})$, the filtration

$$\mathbb{Z}\mathcal{T}(w,w') = \mathcal{I}^0(w,w') \supset \mathcal{I}^1(w,w') \supset \mathcal{I}^2(w,w') \supset \cdots \tag{9–13}$$

is known [32] to be the same as the Goussarov–Vassiliev filtration for $\mathbb{Z}\mathcal{T}(w,w')$.

Remark 9.18 There is an alternative, perhaps more natural, definition of the Goussarov–Vassiliev filtration for framed tangles, which involves the difference $t_\downarrow - 1_\downarrow$ of framing change as well as $\psi^\times$. In the present paper, we do not consider this version for simplicity.

9.7.3 Goussarov–Vassiliev filtration for $\mathbb{Z}\mathsf{B}$ Now we consider the case of tangles in B. The definition of the category $\mathbb{Z}\mathsf{B}$ of $\mathbb{Z}$–linear tangles in B is obvious. For $i, j \geq 0$, the Goussarov–Vassiliev filtration for the tangles in $\mathsf{B}(i,j)$ is given by the $\mathbb{Z}$–submodules

$$(\mathbb{Z}\mathsf{B} \cap \mathcal{I}^n)(i,j) := \mathbb{Z}\mathsf{B}(i,j) \cap \mathcal{I}^n(\mathsf{b}^{\otimes i}, \mathsf{b}^{\otimes j})$$

for $n \geq 0$. Clearly, this defines a four-sided ideal $\mathbb{Z}\mathsf{B} \cap \mathcal{I}^n$ in $\mathbb{Z}\mathsf{B}$.

Set

$$c^\times = \eta_2 - c_+ \in \mathbb{Z}\mathsf{B}(0,2),$$

and let $\mathcal{I}_\mathsf{B}$ denote the four-sided ideal in $\mathbb{Z}\mathsf{B}$ generated by $c^\times$.

The following result gives a definition of the Goussarov–Vassiliev filtration for tangles in B, and in particular for bottom tangles, *defined algebraically in* $\mathbb{Z}\mathsf{B}$. Thus the setting in the present paper is expected to be useful in the study of Goussarov–Vassiliev finite type invariants.

Theorem 9.19 *For each $n \geq 0$, we have*

$$\mathbb{Z}\mathsf{B} \cap \mathcal{I}^n = \mathcal{I}_\mathsf{B}^n. \tag{9–14}$$

For bottom tangles, we also have

$$\mathcal{I}_\mathsf{B}^n(0,m) = \mathbb{Z}\mathsf{B}(2n,m)(c^\times)^{\otimes n}\Big(= \{f(c^\times)^{\otimes n} \mid f \in \mathbb{Z}\mathsf{B}(2n,m)\}\Big). \tag{9–15}$$

Proof We have

$$c^\times = (\downarrow \otimes \psi_{\downarrow,\uparrow} \otimes \uparrow)(\psi^\times \otimes \uparrow \otimes \uparrow)\mathrm{coev}_{\downarrow\otimes\downarrow} \in \mathcal{I}(\mathbf{1}, \mathsf{b}^{\otimes 2}), \tag{9–16}$$

$$\psi^\times = (\downarrow \otimes \downarrow \otimes \mathrm{ev}_{\downarrow\otimes\downarrow})(\downarrow \otimes \psi_{\downarrow,\uparrow}^{-1} \otimes \uparrow \otimes \downarrow \otimes \downarrow)(c^\times \otimes \downarrow \otimes \downarrow). \tag{9–17}$$

By (9–16), we have $\mathcal{I}_\mathsf{B} \subset \mathcal{I}$, and hence $\mathcal{I}_\mathsf{B}^n \subset \mathcal{I}^n$ for $n \geq 0$. Since $\mathcal{I}_\mathsf{B}^n \subset \mathbb{Z}\mathsf{B}$, we have $\mathcal{I}_\mathsf{B}^n \subset \mathbb{Z}\mathsf{B} \cap \mathcal{I}^n$.

We show the other inclusion. Suppose that $f \in (\mathbb{Z}\mathsf{B} \cap \mathcal{I}^n)(l,m)$. By (9–16) and (9–17), $\mathcal{I}$ is generated by $c^\times$ as a four-sided ideal in $\mathbb{Z}\mathcal{T}$. By Lemma 9.17, we have

$$f = g'(\mathsf{b}^{\otimes l} \otimes (c^\times)^{\otimes n}),$$

where $g' \in \mathbb{Z}\mathcal{T}(\mathsf{b}^{\otimes(l+2n)}, \mathsf{b}^{\otimes m})$. We can write

$$g' = \sum_{h \in \mathcal{T}(\mathsf{b}^{\otimes(l+2n)}, \mathsf{b}^{\otimes m})} p_h h, \quad p_h \in \mathbb{Z}.$$

Set

$$g = \sum_{h \in \mathsf{B}(l+2n,m)} p_h h \in \mathbb{Z}\mathsf{B}(l+2n,m)$$

We have $(g-g')(\mathsf{b}^{\otimes l} \otimes (c^\times)^{\otimes n}) = 0$, since if $h \in \mathcal{T}(\mathsf{b}^{\otimes(l+2n)}, \mathsf{b}^{\otimes m}) \setminus \mathsf{B}(l+2n,m)$, then $h\eta_{l+2n}$ is not homotopic to η_m. Hence

$$f = g(\mathsf{b}^{\otimes l} \otimes (c^\times)^{\otimes n}) \in \mathcal{I}^n_{\mathsf{B}}.$$

Hence we have $\mathbb{Z}\mathsf{B} \cap \mathcal{I}^n \subset \mathcal{I}^n_{\mathsf{B}}$.

The identity (9–15) follows from the above argument with $l = 0$. This completes the proof. $\square$

Remark 9.20 It is easy to generalize this subsection to the case of *skein modules* (see Przytycki [69]) involving bottom tangles. Let $\mathbf{k}$ be a commutative, unital ring, and consider the $\mathbf{k}$–linear braided categories $\mathbf{k}\mathcal{T}$ and $\mathbf{k}\mathsf{B}$. A *skein element* is just a morphism $f \in \mathbf{k}\mathcal{T}(w,w')$, $w,w' \in \mathrm{Ob}(\mathbf{k}\mathcal{T}) = \mathrm{Ob}(\mathcal{T})$. For a set $S \subset \mathrm{Mor}(\mathbf{k}\mathcal{T})$ of skein elements, let I_S denote the four-sided ideal in $\mathbf{k}\mathcal{T}$ generated by S. Then the quotient ($\mathbf{k}$–linear, braided) category $\mathbf{k}\mathcal{T}/I_S$ is known as the *skein category* defined by S as the set of skein relations.

Suppose $S \subset \bigcup_{n \geq 0} \mathbf{k}\,\mathsf{BT}_n \subset \mathrm{Mor}(\mathbf{k}\mathsf{B})$. Thus S is a set of skein elements involving only bottom tangles. Let I^{B}_S denote the four-sided ideal in $\mathbf{k}\mathsf{B}$ generated by S. Then we have the following generalization of Theorem 9.19:

$$\mathbf{k}\mathsf{B} \cap I_S = I^{\mathsf{B}}_S,$$
$$I^{\mathsf{B}}_S(0,n) = \sum_{l \geq 0} \mathbf{k}\mathsf{B}(l,n)(S \cap \mathbf{k}\,\mathsf{BT}_l).$$

Thus, analogously to the case of local moves, it follows that skein theory defined by skein elements of compatible tangles consisting of arcs can be formulated within the setting of $\mathbf{k}\mathsf{B}$ using skein elements of bottom tangles.

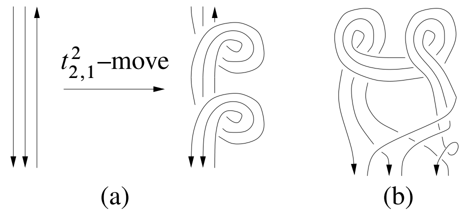


Figure 32: (a) A $t^2_{2,1}$–move. (b) The tangle $t^2_{2,1} \in \mathsf{BT}_3$.

9.8 Twist moves

A *twist move* is a local move on a tangle which performs a power of full twist on a parallel family of strings. A type of a twist move is determined by a triple of integers (n, i, j) with $i, j \geq 0$, where the move performs n full twists on a parallel family of i downward strings and j upward strings. Let us call it a $t^n_{i,j}$–move. In our notation, a $t^n_{i,j}$–move is the same as a $(\downarrow^{\otimes i} \otimes \uparrow^{\otimes j}, t^n_{\downarrow^{\otimes i} \otimes \uparrow^{\otimes j}})$–move. For example, see Figure 32 (a).

Note that a $t^n_{1,0}$–move is just an n–full twist of a string, and is the same as v_n–move, where $v_n \in \mathsf{BT}_1$ is the nth convolution power of v_- defined by

$$v_n = \begin{cases} \mu_{\mathsf{b}}^{[n]} v_-^{\otimes n} & \text{if } n \geq 0, \\ \mu_{\mathsf{b}}^{[-n]} v_+^{\otimes(-n)} & \text{if } n \leq 0. \end{cases}$$

Using an idea similar to the one in Remark 9.16, we see that an $t^n_{i,j}$–move is the same as $F_{\mathsf{b}}(\Delta_{\mathsf{H}}^{[i;j]})(v_n)$–move. ($\Delta_{\mathsf{H}}^{[i;j]}$ is defined in Remark 9.16.) By abuse of notation, set

$$t^n_{i,j} = F_{\mathsf{b}}(\Delta_{\mathsf{H}}^{[i;j]})(v_n) \in \mathsf{BT}_{i+j},$$

which should not cause confusion. Note that $t^n_{i,j}$ is admissible.

Note that a $t^n_{i,j}$–move changes the writhe of a tangle by $n(i-j)^2$. (Here the *writhe* of a tangle is the number of positive crossing minus the number of negative crossings.) In the literature, twist moves are often considered in the unframed context. The modification to the unframed case is easy. For example, two 0–framed knots are related by unframed $t^n_{i,j}$–move if they are related by a sequence of a framed $t^n_{i,j}$–move and a framed $t^{-n(i-j)^2}_{1,0}$–move. The latter move multiplies the universal invariant associated to a ribbon Hopf algebra by the factor of the power $\mathbf{r}^{n(i-j)^2}$ of the ribbon element $\mathbf{r}$.

Twist moves have long been studied in knot theory. For a recent survey, see Przytycki [70]. Here we give a few examples from the literature with translations into our setting. For simplicity, we only give suitable framed versions of the notions in the literature.

For integers $n,k\geq 0$, a framed version of Fox's notion of *congruence modulo* (n,k) (see Fox [9], Nakanishi–Suzuki [64] and Nakanishi [62]) can be defined as the $FC_{n,k}$–equivalence, where we set

$$FC_{n,k}=\{t^n_{i,j}\mid i-j\equiv 0 \pmod{k}\}\subset \mathsf{ABT}\,.$$

For integer n, a framed version of t_{2n}*–move* (see Przytycki [68]) can be defined as $t^n_{2,0}$–move, and a framed version of $\bar{t}_{2n}$*–move* can be defined as $t^n_{1,1}$–move. Nakanishi's 4–move conjecture [64; 62], which is still open, can be restated that any knot is $\{t^2_{2,0},t^2_{1,1}\}$–equivalent to an unknot.

We expect that the above "algebraic redefinitions" of twist moves and equivalence relations are useful in the study of these notions in terms of quantum invariants, by applying the results in Section 9.1.

10 The functor $\tilde{\mathsf{J}}\colon \mathsf{B}\to\mathsf{Mod}_H$ and universal invariants of bottom knots

The following idea may be useful in studying the universal invariants of bottom knots.

10.1 The functor $\tilde{\mathsf{J}}\colon \mathsf{B}\to\mathsf{Mod}_H$

Let H be a ribbon Hopf algebra over a commutative, unital ring $\mathbf{k}$, and let $Z(H)$ denote the center of H. Let $\tilde{H}$ denote H regarded as a $Z(H)$–algebra. For $n\geq 0$, let $\tilde{H}^{\otimes n}$ denote the n–fold iterated tensor product of $\tilde{H}$, ie, n–fold tensor product of H over $Z(H)$, regarded as a $Z(H)$–algebra. In particular, we have $\tilde{H}^{\otimes 0}=Z(H)$. Let

$$\iota_n\colon H^{\otimes n}\to\tilde{H}^{\otimes n}$$

denote the natural map, which is surjective if $n\geq 1$.

The functor $\mathsf{J}\colon \mathsf{B}\to\mathsf{Mod}_H$ induces another functor $\tilde{\mathsf{J}}\colon \mathsf{B}\to\mathsf{Mod}_H$ as follows. For $n\geq 0$, set $\tilde{\mathsf{J}}(\mathsf{b}^{\otimes n})=\tilde{H}^{\otimes n}$, which is given the left H–module structure induced by that of $H^{\otimes n}$. (This left H–module structure of $\tilde{H}^{\otimes n}$ does *not* restrict to the $Z(H)$–module structure of $\tilde{H}^{\otimes n}$.)

For each $f\in\mathsf{B}(m,n)$, the left H–module homomorphism $\tilde{\mathsf{J}}(f)\colon \tilde{H}^{\otimes m}\to\tilde{H}^{\otimes n}$ is induced by $\mathsf{J}(f)\colon H^{\otimes m}\to H^{\otimes n}$ as follows. If $m>0$, then $\tilde{\mathsf{J}}(f)$ is defined to be the unique map such that the following diagram commutes

$$\begin{array}{ccc} H^{\otimes m} & \xrightarrow{\mathsf{J}(f)} & H^{\otimes n} \\ {\scriptstyle \iota_m}\downarrow & & \downarrow{\scriptstyle \iota_n} \\ \widetilde{H}^{\otimes m} & \xrightarrow[\widetilde{\mathsf{J}}(f)]{} & \widetilde{H}^{\otimes n}. \end{array} \tag{10–1}$$

If $m=0$, then set

$$\widetilde{\mathsf{J}}(f)(z) = z\iota_n(\mathsf{J}(f)(1)) \quad \text{for } z \in Z(H). \tag{10–2}$$

Note that commutativity of the diagram (10–1) holds also for $m=0$. It is straightforward to check that the above defines a well-defined functor $\widetilde{\mathsf{J}}$.

For $m,n \geq 0$, let

$$\xi_{m,n}\colon \widetilde{H}^{\otimes m} \otimes_{\mathbf{k}} \widetilde{H}^{\otimes n} \to \widetilde{H}^{\otimes (m+n)}$$

denote the natural map. The $\xi_{m,n}$ form a natural transformation

$$\xi\colon \widetilde{\mathsf{J}}(-) \otimes \widetilde{\mathsf{J}}(-) \to \widetilde{\mathsf{J}}(- \otimes -)$$

of functors from $\mathsf{B} \times \mathsf{B}$ to Mod_H.

It is straightforward to see that the triple $(\widetilde{\mathsf{J}}, \xi, \iota_0)$ is an *ordinary* braided functor, ie a ordinary monoidal functor which preserves braiding. (By "ordinary monoidal functor", we mean a "monoidal functor" in the ordinary sense, see Mac Lane [54, Chapter XI, Section 2].)

The ι_n form a monoidal natural transformation $\iota\colon \mathsf{J} \Rightarrow \widetilde{\mathsf{J}}$ (in the ordinary sense) of ordinary monoidal functors from B to Mod_H.

10.2 Universal invariant of bottom knots

Since $\iota_1\colon H = H^{\otimes 1} \to \widetilde{H}^{\otimes 1} = H$ is the identity, the functor $\widetilde{\mathsf{J}}$ can be used in computing $J_T = \mathsf{J}(T)(1)$ for a bottom knot $T \in \mathsf{BT}_1$. For example, we have the following version of Corollary 9.2 for $n=1$.

Proposition 10.1 *Let $K_i \subset \widetilde{H}^{\otimes i}$ for $i \geq 0$, be $Z(H)$–submodules satisfying the following.*

1. $1 \in K_0$, $1, v^{\pm 1} \in K_1$, *and* $\iota_2(c_{\pm}^H) \in K_2$.
2. *For $m,n \geq 0$, we have* $K_m \otimes_{Z(H)} K_n \subset K_{m+n}$.

(3) *For* $p, q \geq 0$ *we have*

$$(\widetilde{\psi}^{\pm 1}_{H,H})_{(p,q)}(K_{p+q+2}) \subset K_{p+q+2},$$
$$\widetilde{\mu}_{(p,q)}(K_{p+q+2}) \subset K_{p+q+1},$$

where

$$(\widetilde{\psi}_{H,H})_{(p,q)} = \widetilde{\mathsf{J}}((\psi_{\mathsf{b},\mathsf{b}})_{(p,q)})\colon \widetilde{H}^{\otimes(p+q+2)} \to \widetilde{H}^{\otimes(p+q+2)}$$

is induced by $(\psi_{H,H})_{(p,q)}\colon H^{\otimes(p+q+2)} \to H^{\otimes(p+q+2)}$, *and*

$$\widetilde{\mu}_{(p,q)} = \widetilde{\mathsf{J}}((\mu_{\mathsf{b}})_{(p,q)})\colon \widetilde{H}^{\otimes(p+q+2)} \to \widetilde{H}^{\otimes(p+q+1)}$$

is induced by $\mu_{(p,q)}\colon H^{\otimes(p+q+2)} \to H^{\otimes(p+q+1)}$.

Then, for any bottom knot $U \in \mathsf{BT}_1$, *we have* $J_U \in K_1$.

Proof This is easily verified using Corollary 9.2. □

Corollaries 9.3, 9.4, 9.5, 9.6 and 9.15 have similar versions for bottom knots.

Remark 10.2 One can replace $Z(H)$ in this section with any $\mathbf{k}$–subalgebra of $Z(H)$.

Remark 10.3 For $n \geq 0$, $\widetilde{H}^{\otimes n}$ has a natural $Z(H)$–module structure induced by multiplication of elements of $Z(H)$ on one of the tensor factors in $H^{\otimes n}$. For each $f \in \mathsf{B}(m,n)$, the map $\widetilde{\mathsf{J}}(f)$ is a $Z(H)$–module map. One can show that there is a monoidal functor $\widetilde{\mathsf{J}}'\colon \mathsf{B} \to \mathsf{Mod}_{Z(H)}$ of B into the category $\mathsf{Mod}_{Z(H)}$ of $Z(H)$–modules which maps each object $\mathsf{b}^{\otimes n}$ into $\widetilde{H}^{\otimes n}$ and each morphism f into $\widetilde{\mathsf{J}}'(f) = \widetilde{\mathsf{J}}(f)$.

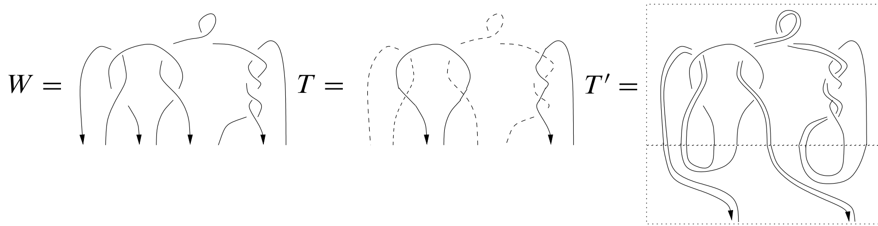
11 Band-reembedding of bottom tangles

11.1 Refined universal invariants of links

As in Section 1.1, for $n \geq 0$, let L_n denote the set of isotopy classes of n–component, framed, oriented, ordered links for $n \geq 0$. There is a surjective function

$$\mathrm{cl}\colon \mathsf{BT}_n \to \mathsf{L}_n, \quad T \mapsto \mathrm{cl}(T).$$

We study an algebraic condition for two bottom tangles to yield the same closure.



(a) (b) (c)

Figure 33: (a) A bottom tangle $W \in \mathsf{BT}_4$. (b) The tangle $T = F_\mathsf{b}((\epsilon_\mathsf{H} \otimes \mathsf{H})^{\otimes 2})(W) \in \mathsf{BT}_2$. (c) The tangle $T' = F_\mathsf{b}(\mathrm{ad}_\mathsf{H}^{\otimes 2})(W) \in \mathsf{BT}_2$.

Definition 11.1 Two bottom tangles $T, T' \in \mathsf{BT}_n$ are said to be related by a *band-reembedding* if there is $W \in \mathsf{BT}_{2n}$ such that

$$T = F_\mathsf{b}((\epsilon_\mathsf{H} \otimes \mathsf{H})^{\otimes n})(W), \quad T' = F_\mathsf{b}(\mathrm{ad}_\mathsf{H}^{\otimes n})(W). \tag{11–1}$$

See Figure 33 for an example.

If we regard a bottom tangle as a based link in a natural way, then band-reembedding corresponds to changing the basing.

Proposition 11.2 *Two bottom tangles $T, T' \in \mathsf{BT}_n$ are related by a band-reembedding if and only if $\mathrm{cl}(T) = \mathrm{cl}(T')$.*

Proof Suppose that T and T' are related by a band-reembedding with $W \in \mathsf{BT}_{2n}$ as in Definition 11.1. Note that the tangle T is obtained from W by removing the components of W of odd indices, and the tangle T' is obtained from the composition tangle $1_\mathsf{b}^{\otimes n} T$ by reembedding the n bands in $1_{\mathsf{b}^{\otimes n}}$ along the components of W of odd indices. Hence we easily see that $\mathrm{cl}(T) = \mathrm{cl}(T')$.

Conversely, if $\mathrm{cl}(T) = \mathrm{cl}(T')$, then we can express T' as a result from $1_{\mathsf{b}^{\otimes n}} T$ by reembedding the n bands in $1_{\mathsf{b}^{\otimes n}}$, and we can arrange by isotopy that there is $W \in \mathsf{BT}_{2n}$ satisfying (11–1). □

Let H be a ribbon Hopf algebra. Proposition 11.2 implies that if two bottom tangles $T, T' \in \mathsf{BT}_n$ satisfies $\mathrm{cl}(T) = \mathrm{cl}(T')$, then there is $W \in \mathsf{BT}_{2n}$ such that

$$J_T = (\epsilon \otimes 1_H)^{\otimes n}(J_W), \quad J_{T'} = \mathrm{ad}^{\otimes n}(J_W).$$

If $K_i \subset H^{\otimes i}$ for $i = 0, 1, 2, \ldots$ are as in Corollary 9.2, then we have

$$J_{T'} - J_T \in (\mathrm{ad}^{\otimes n} - (\epsilon \otimes 1_H)^{\otimes n})(K_{2n}). \tag{11–2}$$

Hence we have the following.

Theorem 11.3 *Let $K_n \subset H^{\otimes n}$ for $n = 0, 1, 2, \dots$ be $\mathbb{Z}$–submodules satisfying the conditions of Corollary 9.2. Set*

$$K'_n = (\mathrm{ad}^{\otimes n} - (\epsilon \otimes 1_H)^{\otimes n})(K_{2n}). \tag{11–3}$$

Then, for each $n \geq 0$, the function

$$J\colon \mathsf{BT}_n \to K_n, \quad T \mapsto J_T,$$

induces a link invariant

$$\bar{J}\colon \mathsf{L}_n \to K_n/K'_n$$

Note that if we set $K_n = H^{\otimes n}$ in Theorem 11.3, then we get the usual definition of universal invariant of links.

Remark 11.4 The idea of Theorem 11.3 can also be used to obtain "more refined" universal invariants for more special classes of links. For example, let us consider links and bottom tangles of zero linking matrices. Suppose in Proposition 11.2 that T and T' are of zero linking matrices. Note that the tangle $W \in \mathsf{BT}_{2n}$ satisfying (11–1) is not necessarily of zero linking matrix, but the n–component bottom tangle $F_{\mathsf{b}}((\epsilon_{\mathsf{H}} \otimes \mathsf{H})^{\otimes n})(W)$, which is equivalent to T, is of zero linking matrix. Hence we can replace the conditions for the K_n in Corollary 9.2 with weaker ones. We hope to give details of this idea in future publications.

11.2 Ribbon discs

We close this section with a result which is closely related to Proposition 11.2.

A *ribbon disc* for a bottom knot T is a ribbon disc for the knot $T \cup \gamma$, where $\gamma \subset [0,1]^2 \times \{0\}$ is the line segment such that $\partial\gamma = \partial T$. Clearly, a bottom knot admits a ribbon disc if and only if the closure $\mathrm{cl}(T)$ of T is a ribbon knot.

Theorem 11.5 *For any bottom knot $T \in \mathsf{BT}_1$, the following conditions are equivalent.*

(1) *T admits a ribbon disc.*

(2) *There is an integer $n \geq 0$ and a bottom tangle $W \in \mathsf{BT}_{2n}$ such that*

$$\eta_{\mathsf{b}}^{\otimes n} = F_{\mathsf{b}}((\epsilon_{\mathsf{H}} \otimes \mathsf{H})^{\otimes n})(W), \tag{11–4}$$

$$T = \mu_{\mathsf{b}}^{[n]} F_{\mathsf{b}}(\mathrm{ad}_{\mathsf{H}}^{\otimes n})(W). \tag{11–5}$$

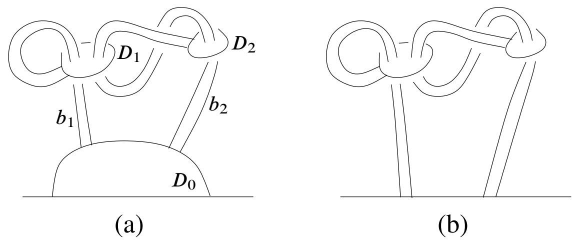


Figure 34: (a) A ribbon bottom knot T and a ribbon disc decomposed as $D_0 \cup D_1 \cup D_2 \cup b_1 \cup b_2$. (b) The bottom tangle T'.

Proof A ribbon disc bounded by T can be decomposed into $n+1$ mutually disjoint discs $D_0, D_1, \ldots, D_n$ and n mutually disjoint bands $b_1, \ldots, b_n$ for some $n \geq 0$ satisfying the following conditions.

(1) For each $i = 1, \ldots, n$ the band b_i joins D_0 and D_i,

(2) D_0 is a disc attached to the bottom square of the cube along a line segment,

(3) The only singularities of the ribbon disc are ribbon singularities in $D_i \cap b_j$ for $1 \leq i, j \leq n$. (We do not allow ribbon singularity in D_0.)

For example, see Figure 34 (a).

Let $T' \in \mathsf{BT}_n$ be the bottom tangle obtained from T by removing D_0 and regarding the rest as an n–component bottom tangle, see Figure 34 (b). Then $\mathrm{cl}(T') = \mathrm{cl}(\eta_n)$ is an unlink. It follows from Proposition 11.2 that T' and η_n are related by band-reembedding. Since $T = \mu_{\mathsf{b}}^{[n]} T'$, we have the assertion. □

Remark 11.6 In Theorem 11.5, one can replace (11–4) with

$$T = \mu_{\mathsf{b}}^{[n]} F_{\mathsf{b}}(Y_{\mathsf{H}}^{\otimes n})(W). \tag{11–6}$$

We sketch how to prove this claim. For a ribbon bottom knot T, there is a Seifert surface F of genus $n \geq 0$ for T and simple closed curves $c_1, \ldots, c_n$ in F satisfying the following conditions.

(1) $c_1, \ldots, c_n$ generates a Lagrangian subgroup of $H_1(F; \mathbb{Z}) \simeq \mathbb{Z}^{2n}$.

(2) As a framed link in the cube, $c_1 \cup \cdots \cup c_n$ is a 0–framed unlink. Here the framings of c_i are induced by the surface F.

(The Seifert surface obtained from a ribbon disc by smoothing the singularities in a canonical way has the above property.) By isotopy one can arrange the curves $c_1, \ldots, c_n$ as in Figure 35. The part bounded by a rectangle is a double of a bottom tangle $W \in \mathsf{BT}_{2n}$ satisfying (11–4) and (11–6).

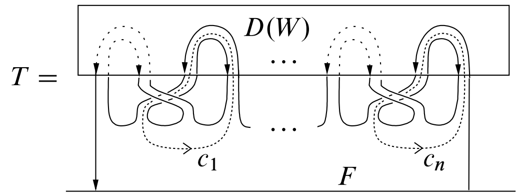


Figure 35

12 Algebraic versions of Kirby moves and Hennings 3–manifold invariants

An important application of the universal invariants is to the Hennings invariant of 3–manifolds and its generalizations. Hennings [26] introduced a class of invariants of 3–manifolds associated to quantum groups, which use right integrals and no finite-dimensional representations. The Hennings invariants are studied further by Kauffman and Radford [37], Ohtsuki [66], Lyubashenko [53], Kerler [39; 41], Sawin [77], Virelizier [83], etc. As mentioned in, or at least obvious from, these papers, the Hennings invariants can be formulated using universal link invariants.

In this section, we reformulate the Hennings 3–manifold invariants using universal invariants of bottom tangles. For closely related constructions, see Kerler [39] and Virelizier [83].

For a Hopf algebra A in a braided category, we set

$$h_A = (\mu_A \otimes A)(A \otimes \Delta_A)\colon A^{\otimes 2} \to A^{\otimes 2}.$$

The morphism h_A is invertible with the inverse

$$h_A^{-1} = (\mu_A \otimes A)(A \otimes S_A \otimes A)(A \otimes \Delta_A).$$

For the transmutation $\underline{H}$ of a ribbon Hopf algebra H, we have

$$h_{\underline{H}} = (\mu_H \otimes 1_H)(1_H \otimes \underline{\Delta}),$$

which should not be confused with

$$h_H = (\mu_H \otimes 1_H)(1_H \otimes \Delta_H)$$

defined for the Hopf algebra H in the symmetric monoidal category $\mathsf{Mod}_{\mathbf{k}}$.

The following is a version of Kirby's theorem [44]. Note that each move is formulated in an algebraic way. Therefore we may regard the following as an *algebraic version of Kirby's theorem*.

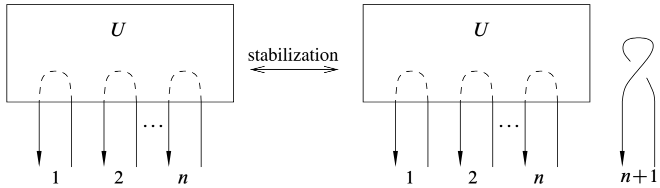


Figure 36: A stabilization move $U \leftrightarrow U \otimes v_+$

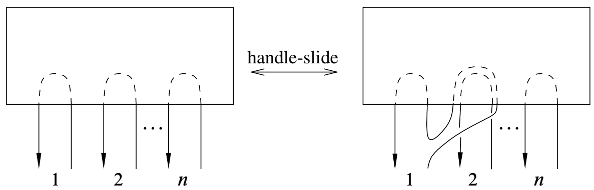


Figure 37: A handle slide move

Theorem 12.1 *For two bottom tangles T and T', the two 3–manifolds $M_T = (S^3)_{\mathrm{cl}(T)}$ and $M_{T'} = (S^3)_{\mathrm{cl}(T')}$ obtained from S^3 by surgery along $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$, respectively, are orientation-preserving homeomorphic if and only if T and T' are related by a sequence of the following moves.*

(1) *Band-reembedding.*

(2) *Stabilization: Replacing $U \in \mathsf{BT}_n$ with $U \otimes v_\pm \in \mathsf{BT}_{n+1}$, or its inverse operation.*

(3) *Handle slide: Replacing $U \in \mathsf{BT}_n$ $(n \geq 2)$ with $F_{\mathsf{b}}(h_{\mathsf{H}} \otimes \mathsf{H}^{\otimes(n-2)})(U)$, or its inverse operation.*

(4) *Braiding: Replacing $U \in \mathsf{BT}_n$ with βU, where $\beta \in \mathsf{B}(n,n)$ is a doubled braid.*

Proof First we see the effects of the moves listed above.

(1) A band-reembedding does not change the closure, hence the result of surgery.

(2) The effect of stabilization move $U \leftrightarrow U \otimes v_+$ is depicted in Figure 36. The case of v_- is similar. The closures $\mathrm{cl}(U)$ and $\mathrm{cl}(U \otimes v_\pm) = \mathrm{cl}(U) \sqcup \mathrm{cl}(v_\pm)$ are related by Kirby's stabilization move. Hence they have the same result of surgery.

(3) The effect of handle slide move $U \leftrightarrow U' := F_{\mathsf{b}}(h_{\mathsf{H}} \otimes \mathsf{H}^{\otimes(n-2)})(U)$ is depicted in Figure 37. It is easy to see that the closures $\mathrm{cl}(U)$ and $\mathrm{cl}(U')$ are related by a Kirby handle slide move of the first component over the second. Hence they have the same result of surgery.

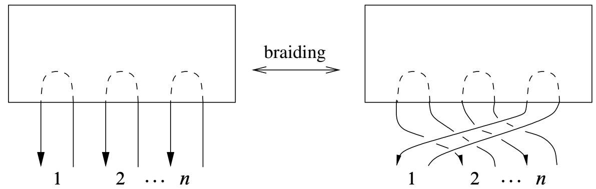


Figure 38: An example of braiding move

(4) A braiding move (see Figure 38) just changes the order of the components at the closure level, and hence does not change the result of surgery.

The "if" part of the theorem follows from the above observations. To prove the "only if" part, we assume that M_T and $M_{T'}$ are orientation-preserving homeomorphic to each other. By Kirby's theorem, there is a sequence from $\mathrm{cl}(T)$ to $\mathrm{cl}(T')$ of stabilizations, handle slides, orientation changes of components, and changes of ordering. It suffices to prove that if $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by one of these moves, then T and T' are related by the moves listed in the theorem.

If $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by change of ordering, then it is easy to see that T and T' are related by a braiding move and a band-reembedding.

If $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by stabilization, ie, $\mathrm{cl}(T') = \mathrm{cl}(T) \sqcup O_\pm$, where $O_\pm$ is an unknot of framing ± 1, then T' and $T \otimes v_\mp$ are related by a band-reembedding.

Suppose $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by handle slide of a component of $\mathrm{cl}(T)$ over another component of $\mathrm{cl}(T)$. By conjugating with change of ordering, we may assume that the first component of $\mathrm{cl}(T)$ is slid over the second component of $\mathrm{cl}(T)$. Then there is $T'' \in \mathsf{BT}_n$ such that T'' is obtained from T by a band-reembedding, and T' is obtained from T'' by a handle slide move or its inverse. For example, see Figure 39.

Suppose $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by orientation change of i th components with $1 \le i \le n$. It suffices to show that $\mathrm{cl}(T)$ and $\mathrm{cl}(T')$ are related by a sequence of handle slides, changes of orientation, changes of ordering and stabilizations. We may assume $i = 1$ by change of ordering. Using stabilization, we may safely assume $n \ge 2$. Now we see that change of orientation of the first component can be achieved by a sequence of handle slides and change of ordering. Suppose that $L = L_1 \cup L_2$ is an 2–component link. There is a sequence of moves $L = L^0 \to L^1 \to \cdots \to L^4 = -L_1 \cup L_2$, with $L^i = L^i_1 \cup L^i_2$ as depicted in Figure 40. (Note here that, in the move $L^1 \to L^2$, we can slide the second component over the first by conjugating the handle slide move by changing of the ordering.) Hence we have the assertion. □

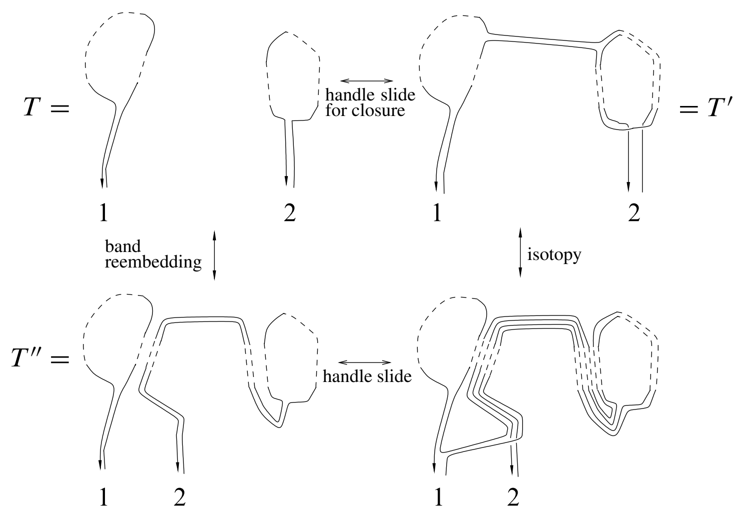


Figure 39: Here only the first and the second components are depicted in each figure

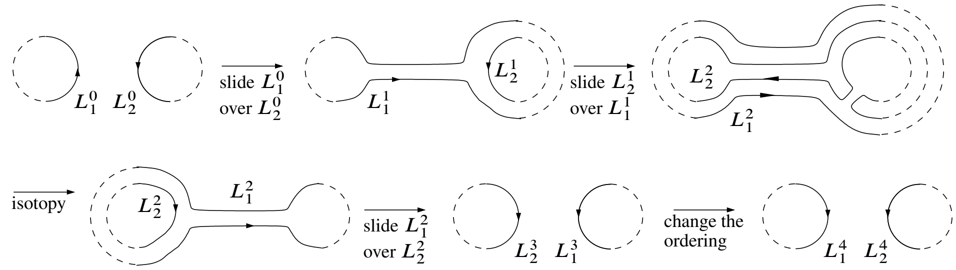


Figure 40

Lemma 12.2 *If $f\colon H \to \mathbf{k}$ is a left H–module homomorphism, then we have*

$$(1_H \otimes f)\Delta = (1_H \otimes f)\underline{\Delta} = (f \otimes 1_H)\underline{\Delta}.$$

Proof The lemma is probably well known. We prove it for completeness.

The first identity is proved using (8–2) as follows.

$$\begin{aligned}(1 \otimes f)\underline{\Delta}(x) &= (1 \otimes f)(\sum x_{(1)} S(\beta) \otimes \alpha \triangleright x_{(2)})\\ &= \sum x_{(1)} S(\beta) \otimes f(\alpha \triangleright x_{(2)}) = \sum x_{(1)} S(\beta) \otimes \epsilon(\alpha) f(x_{(2)})\\ &= \sum x_{(1)} \otimes f(x_{(2)}) = (1 \otimes f)\Delta(x).\end{aligned}$$

The identity $(f\otimes 1)\underline{\Delta}(x)=(1\otimes f)\Delta(x)$ follows similarly by using the identity

$$\underline{\Delta}(x)=\sum(\beta\triangleright x_{(2)})\otimes\alpha x_{(1)}\quad\text{for } x\in H.\qquad\square$$

Now we formulate the Hennings invariant in our setting. If $\chi\colon H\to\mathbf{k}$ is a left H–module homomorphism and a left integral on H, then we can define a 3–manifold invariant. This invariant is essentially the same as the Hennings invariant defined using the right integral, since left and right integrals interchange under application of the antipode.

Proposition 12.3 *Let $\chi\colon H\to\mathbf{k}$ be a left H–module homomorphism. Then the following conditions are equivalent.*

(1) *χ is a left integral on H, ie,*

$$(1\otimes\chi)\Delta=\eta\chi\colon H\to H. \tag{12–1}$$

(2) *χ is a two-sided integral on $\underline{H}$ in Mod_H, ie,*

$$(1\otimes\chi)\underline{\Delta}=(\chi\otimes 1)\underline{\Delta}=\eta\chi\colon H\to H. \tag{12–2}$$

Suppose either (hence both) of the above holds, and also suppose that $\chi(\mathbf{r}^{\pm1})\in\mathbf{k}$ is invertible. Then there is a unique invariant $\tau_{H,\chi}(M)\in\mathbf{k}$ of connected, oriented, closed 3–manifolds M such that for each bottom tangle $T\in\mathsf{BT}_n$ we have

$$\tau_{H,\chi}(M_T)=\frac{\chi^{\otimes n}(J_T)}{\chi(\mathbf{r}^{-1})^{\sigma_+(T)}\chi(\mathbf{r})^{\sigma_-(T)}}, \tag{12–3}$$

where $\sigma_+(T)$ (resp. $\sigma_-(T)$) is the number (with multiplicity) of the positive (resp. negative) eigenvalues of the linking matrix of T, and $M_T=(S^3)_{\mathrm{cl}(T)}$ denote the result from S^3 of surgery along $\mathrm{cl}(T)$.

Proof The first assertion follows from Lemma 12.2.

In the following, we show that the right hand side of (12–3) is invariant under the moves described in Theorem 12.1.

First we consider the stabilization move. Suppose $T\in\mathsf{BT}_n$ and $T'=T\otimes v_\pm\in\mathsf{BT}_{n+1}$. Then one can easily verify $\tau_{H,\chi}(M_T)=\tau_{H,\chi}(M_{T'})$ using

$$\chi^{\otimes(n+1)}(J_{T\otimes v_\pm})=\chi^{\otimes n}(J_T)\cdot\chi(\mathbf{r}^{\pm1}).$$

Since the other moves does not change the number of components and the number of positive (resp. negative) eigenvalues of the linking matrix, it suffices to verify that $\chi^{\otimes n}(J_T)=\chi^{\otimes n}(J_{T'})$ for $T,T'\in\mathsf{BT}_n$ related by each of the other moves.

Suppose T and T' are related by a band-reembedding. Then there is $W \in \mathsf{BT}_{2n}$ satisfying (11–1). Since χ is a left H–module homomorphism, we have

$$\chi^{\otimes n}(J_T) = \chi^{\otimes n}(\epsilon \otimes 1_H)^{\otimes n}(J_W) = \chi^{\otimes n}\mathrm{ad}^{\otimes n}(J_W) = \chi^{\otimes n}(J_{T'}).$$

Suppose that T and T' are related by a handle slide, ie, $T' = F_{\mathsf{b}}(h_{\mathsf{H}} \otimes \mathsf{H}^{\otimes(n-2)})(T)$. Since χ is a two-sided integral on $\underline{H}$, we have

$$\begin{aligned}\chi^{\otimes n}(J_{T'}) &= \chi^{\otimes n}(h_{\underline{H}} \otimes 1_H^{\otimes(n-2)})(J_T)\\ &= \chi^{\otimes n}((\mu_H \otimes 1_H)(1_H \otimes \underline{\Delta}) \otimes 1_H^{\otimes(n-2)})(J_T)\\ &= (\chi\mu_H \otimes \chi^{\otimes(n-2)})(1_H \otimes (1_H \otimes \chi)\underline{\Delta} \otimes 1_H^{\otimes(n-2)})(J_T)\\ &= (\chi\mu_H \otimes \chi^{\otimes(n-2)})(1_H \otimes \eta_H\chi \otimes 1_H^{\otimes(n-2)})(J_T)\\ &= \chi^{\otimes n}(J_T).\end{aligned}$$

Suppose that T and T' are related by a braiding move. We may assume that $T' = (\psi_{\mathsf{b},\mathsf{b}}^{\pm1})_{(i-1,n-i-1)}T$ with $1 \le i \le n-1$. Since $(\chi \otimes \chi)\psi_{H,H} = \chi \otimes \chi$, it follows that

$$\chi^{\otimes n}(J_{T'}) = \chi^{\otimes n}(\psi_{H,H}^{\pm1})_{(i-1,n-i-1)}(J_T) = \chi^{\otimes n}(J_T).$$

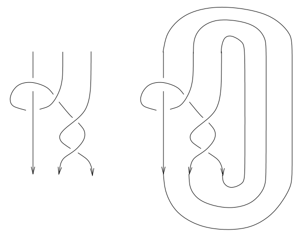
This completes the proof. □

Remark 12.4 One can verify that the invariant $\tau_{H,\chi}(M)$ is equal (up to a factor determined only by the first Betti number of M) to the Hennings invariant of M associated to the right integral $\chi S\colon H \to \mathbf{k}$.

Remark 12.5 Some results in Section 9 can be used together with Proposition 12.3 to obtain results on the range of values of the Hennings invariants for various class of 3–manifolds. Recall from Hennings [26] and Ohtsuki [65] (see also Virelizier [83]) that the sl_2 Reshetikhin–Turaev invariants can be defined using a universal link invariant associated to a finite-dimensional quantum group $U_q(sl_2)'$ at a root of unity, and a certain trace function on $U_q(sl_2)'$. Hence results in Section 9 can also be used to study the range of values of the Reshetikhin–Turaev invariants.

13 String links and bottom tangles

An n–component string link $T = T_1 \cup \cdots \cup T_n$ is a tangle consisting n arcs $T_1, \dots, T_n$, such that for $i = 1, \dots, n$ the ith component T_i runs from the ith upper endpoint to the ith lower endpoint. In other words, T is a morphism $T \in \mathcal{T}(\downarrow^{\otimes n}, \downarrow^{\otimes n})$ homotopic to $\downarrow^{\otimes n}$. (Here and in what follows, the endpoints are counted from the left.) The



(a) (b)

Figure 41: (a) A string link. (b) Its closure.

closure operation is as depicted in Figure 41 (a), (b). One can use string links to study links via the closure operation.

As in Section 4.2, we denote by SL_n the submonoid of $\mathcal{T}(\downarrow^{\otimes n}, \downarrow^{\otimes n})$ consisting of the isotopy classes of the n–component framed string links.

Of course, there are many orientation-preserving self-homeomorphisms of a cube $[0,1]^3$, which transform n–component string links into n–component bottom tangles and induces a bijection $\mathsf{SL}_n \cong \mathsf{BT}_n$. In this sense, one can think of the notion of string links and the notion of bottom tangles are equivalent. However, SL_n and BT_n are *not* equally convenient. For example, the monoid structure in the SL_n can not be defined in each BT_n as conveniently as in SL_n, and also that the external Hopf algebra structure in the BT_n can not be defined in the SL_n as conveniently as in the BT_n. It depends on the contexts which is more useful.

In the following, we define a *preferred bijection*

$$\tau_n\colon \mathsf{BT}_n \to \mathsf{SL}_n,$$

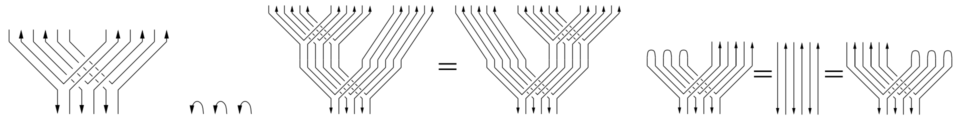
which enables one to translate results about the bottom tangles into results about the string links and vice versa. We define a monoid structure of each BT_n such that τ_n is a monoid homomorphism. We also study several other structures on BT_n and SL_n and consider the algebraic counterparts for a ribbon Hopf algebra. The proofs are straightforward and left to the reader.

For $n \geq 0$, we give $\mathsf{b}^{\otimes n} \in \mathrm{Ob}(\mathcal{T})$ the standard tensor product algebra structure (see Majid [55, Section 2])

$$\mu_{\mathsf{b}^{\otimes n}}\colon \mathsf{b}^{\otimes n} \otimes \mathsf{b}^{\otimes n} \to \mathsf{b}^{\otimes n}, \quad \eta_{\mathsf{b}^{\otimes n}}\colon \mathbf{1} \to \mathsf{b}^{\otimes n},$$

induced by the algebra structure $(\mathsf{b}, \mu_{\mathsf{b}}, \eta_{\mathsf{b}})$, ie, $\mu_{\mathsf{b}^{\otimes 0}} = 1_{\mathbf{1}}$, $\mu_{\mathsf{b}^{\otimes 1}} = \mu$,

$$\mu_{\mathsf{b}^{\otimes n}} = (\mu_{\mathsf{b}} \otimes \mu_{\mathsf{b}^{\otimes (n-1)}})(\mathsf{b} \otimes \psi_{\mathsf{b}^{\otimes (n-1)},\mathsf{b}} \otimes \mathsf{b}^{\otimes (n-1)}) \quad \text{for } n \geq 2,$$



(a) (b) (c) (d)

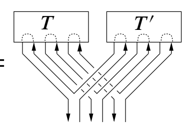
Figure 42: (a) Multiplication $\mu_{\mathsf{b}^{\otimes 3}}$. (b) Unit $\eta_{\mathsf{b}^{\otimes 3}}$. (c) Associativity. (d) Unitality.

and $\eta_{\mathsf{b}^{\otimes n}} = \eta_n$ for $n \geq 0$. See Figure 42 for example.

We define a monoid structure for BT_n with multiplication

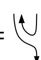
$$\tilde{\mu}_n = *\colon \mathsf{BT}_n \times \mathsf{BT}_n \to \mathsf{BT}_n, \quad (T, T') \mapsto T * T'$$

defined by

(13–1) $$T * T' = \mu_{\mathsf{b}^{\otimes n}}(T \otimes T') =$$ 

for $T, T' \in \mathsf{BT}_n$, where the figure in the right hand side is for $n = 3$. Then the set BT_n has a monoid structure with multiplication $\tilde{\mu}_n$ and with unit η_n.

We give $\downarrow \in \mathrm{Ob}(\mathcal{T})$ a left b–module structure defined by the left action

$$\alpha_{\downarrow} = \downarrow \otimes \mathrm{ev}_{\downarrow} =$$  $$\colon \mathsf{b} \otimes \downarrow \to \downarrow .$$

For $n \geq 0$, this left b–module structure induces in the canonical way a left $\mathsf{b}^{\otimes n}$–module structure for $\downarrow^{\otimes n}$

$$\alpha_{\downarrow^{\otimes n}}\colon \mathsf{b}^{\otimes n} \otimes \downarrow^{\otimes n} \to \downarrow^{\otimes n},$$

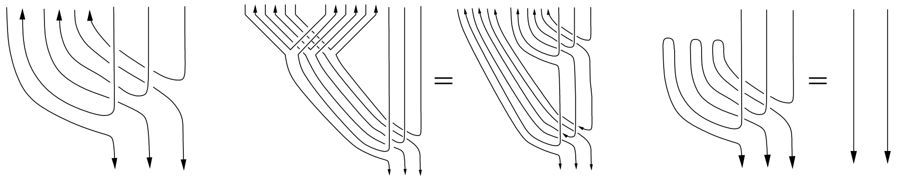
ie, $\alpha_{\downarrow^{\otimes n}}$ is defined inductively by

$$\alpha_{\downarrow^{\otimes 0}} = 1_{\mathbf{1}}, \quad \alpha_{\downarrow^{\otimes 1}} = \alpha,$$
$$\alpha_{\downarrow^{\otimes n}} = (\alpha_{\downarrow} \otimes \alpha_{\downarrow^{\otimes (n-1)}})(\mathsf{b} \otimes \psi_{\mathsf{b}^{\otimes (n-1)}, \downarrow} \otimes \downarrow^{\otimes (n-1)}) \quad \text{for } n \geq 2.$$

For example, see Figure 43.

Now we define a function $\tau_n\colon \mathsf{BT}_n \to \mathsf{SL}_n$ for $n \geq 0$ by

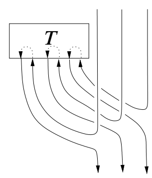
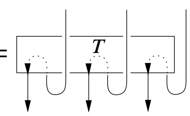
$$\tau_n(T) = \alpha_{\downarrow^{\otimes n}}(T \otimes \downarrow^{\otimes n})$$



(a) (b) (c)

Figure 43: (a) Left action $\alpha_{\downarrow^{\otimes 3}}$. (b) Associativity. (c) Unitality.

for $T \in \mathsf{BT}_n$. In a certain sense, $\tau_n(T)$ is the result of "letting T act on $\downarrow^{\otimes n}$". For example, if $T \in \mathsf{BT}_3$, then

$\tau_3(T) =$  $=$  .

The function τ_n is invertible with the inverse $\tau_n^{-1}\colon \mathsf{SL}_n \to \mathsf{BT}_n$ given by

$$\tau_n^{-1}(L) = \theta_n(T \otimes \uparrow^{\otimes n})\mathrm{coev}_{\downarrow^{\otimes n}},$$

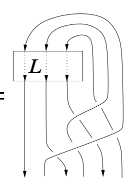
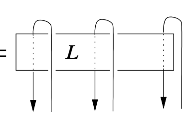
where

$$\theta_n\colon \downarrow^{\otimes n} \otimes \uparrow^{\otimes n} \to \mathsf{b}^{\otimes n}$$

is defined inductively by

$$\theta_0 = 1_{\mathbf{1}}, \quad \theta_{n+1} = (\downarrow \otimes \psi_{\mathsf{b}^{\otimes n},\uparrow})(\downarrow \otimes \theta_n \otimes \uparrow)$$

for $n \geq 1$. For example, if $L \in \mathsf{SL}_3$, then

$\tau_3^{-1}(L) =$  $=$  .

The function τ_n is a monoid isomorphism, ie,

$$\tau_n(T * T') = \tau_n(T)\tau_n(T'), \quad \tau_n(\eta_n) = \downarrow^{\otimes n}$$

for $T, T' \in \mathsf{BT}_n$.

As is well known, there is a "coalgebra-like" structure on the SL_n. For $T \in \mathsf{SL}_n$ and $i = 1, \ldots, n$, let $\Delta_i(T) \in \mathsf{SL}_{n+1}$ (resp. $\epsilon_i(T) \in \mathsf{SL}_{n-1}$) be obtained from T by duplicating (resp. removing) the i th component. These operations define monoid homomorphisms

$$\Delta_i\colon \mathsf{SL}_n \to \mathsf{SL}_{n+1}, \quad \epsilon_i\colon \mathsf{SL}_n \to \mathsf{SL}_{n-1}.$$

The following diagrams commutes.

(13–2)
$$\begin{tikzcd} \mathsf{BT}_n \arrow[r, "\check{\Delta}_{(i-1,n-i)}"] \arrow[d, "\tau_n"'] & \mathsf{BT}_{n+1} \arrow[d, "\tau_{n+1}"] & \mathsf{BT}_n \arrow[r, "\check{\epsilon}_{(i-1,n-i)}"] \arrow[d, "\tau_n"'] & \mathsf{BT}_{n-1} \arrow[d, "\tau_{n-1}"] \\ \mathsf{SL}_n \arrow[r, "\Delta_i"'] & \mathsf{SL}_{n+1} & \mathsf{SL}_n \arrow[r, "\epsilon_i"'] & \mathsf{SL}_{n-1}. \end{tikzcd}$$

Thus the "coalgebra-like" structure of the SL_n corresponds via the τ_n to the "coalgebra-like" structure in the BT_n.

Now we translate the above observations into the universal invariant level. Let H be a ribbon Hopf algebra over a commutative, unital ring $\mathbf{k}$.

Define a $\mathbf{k}$–module homomorphism $\tau'_n\colon H^{\otimes n} \to H^{\otimes n}$, $n \geq 0$, by $\tau'_0 = 1_{\mathbf{k}}$, $\tau'_1 = 1_H$, and for $n \geq 2$

$$\tau'_n = \big(1_H^{\otimes(n-2)} \otimes \lambda_2\big)\big(1_H^{\otimes(n-3)} \otimes \lambda_3\big) \cdots (1_H \otimes \lambda_{n-1})\lambda_n,$$

where $\lambda_n\colon H^{\otimes n} \to H^{\otimes n}$, $n \geq 2$, is defined by

$$\lambda_n\Big(\sum x_1 \otimes \cdots \otimes x_n\Big) = \sum x_1\beta \otimes (\alpha_{(1)} \triangleright x_2) \otimes \cdots \otimes (\alpha_{(n-1)} \triangleright x_n).$$

Then the effects of τ_n on the universal invariants is given by

(13–3) $$J_{\tau_n(T)} = \tau'_n(J_T) \quad \text{for } T \in \mathsf{BT}_n,$$

which can be used in translating results about the universal invariant of bottom tangles into results about the universal invariant of string links.

We denote by $\underline{H}^{\otimes n}$ the n–fold tensor product of $\underline{H}$ in the braided category Mod_H. Thus $\underline{H}^{\otimes n}$ is equipped with the standard algebra structure in Mod_H, with the multiplication

$$\underline{\mu}_n\colon \underline{H}^{\otimes n} \otimes \underline{H}^{\otimes n} \to \underline{H}^{\otimes n}$$

given by $\underline{\mu}_n = \mathsf{J}(\mu_{\mathsf{b}^{\otimes n}})$ and with the unit in $\underline{H}^{\otimes n}$ by $1_H^{\otimes n}$.

The map τ'_n defines a $\mathbf{k}$–algebra isomorphism

$$\tau'_n\colon \underline{H}^{\otimes n} \to H^{\otimes n},$$

where $H^{\otimes n}$ is equipped with the standard algebra structure. In other words, we have

$$\tau'_n(\underline{\mu}_n(x \otimes y)) = \tau'_n(x)\tau'_n(y)$$

for $x, y \in H^{\otimes n}$, and $\tau'_n(1_H^{\otimes n}) = 1_H^{\otimes n}$.

We have algebraic analogues of the diagrams in (13–2)

$$\begin{tikzcd}
\underline{H}^{\otimes n} \arrow[r, "\underline{\Delta}_i"] \arrow[d, "\tau'_n"'] & \underline{H}^{\otimes (n+1)} \arrow[d, "\tau'_{n+1}"] \\
H^{\otimes n} \arrow[r, "\Delta_i"'] & H^{\otimes (n+1)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{H}^{\otimes n} \arrow[r, "\epsilon_i"] \arrow[d, "\tau'_n"'] & \underline{H}^{\otimes (n-1)} \arrow[d, "\tau'_{n-1}"] \\
H^{\otimes n} \arrow[r, "\epsilon_i"'] & H^{\otimes (n-1)}
\end{tikzcd}$$

where $\underline{\Delta}_i = 1^{\otimes(i-1)} \otimes \underline{\Delta} \otimes 1^{\otimes(n-i)}$, $\Delta_i = 1^{\otimes(i-1)} \otimes \Delta \otimes 1^{\otimes(n-i)}$ and $\epsilon_i = 1^{\otimes(i-1)} \otimes \epsilon \otimes 1^{\otimes(n-i)}$. If $n = i = 1$, then the commutativity of the diagram on the left, $\underline{\Delta} = \tau_2^{-1}\Delta$, above coincides (8–2), the definition of the transmuted comultiplication $\underline{\Delta}$.

14 Remarks

14.1 Direct applications of the category B for representation-colored link invariants

We have argued that the setting of universal invariants is useful in the study of representation-colored link invariants. However, it would be worth describing how to apply the setting of the category B to the study of representation-colored link invariants without using universal invariants.

Let H be a ribbon Hopf algebra over a field $\mathbf{k}$, and let V be a finite-dimensional left H–module. Let $F_V^{\mathcal{T}}\colon \mathcal{T} \to \mathsf{Mod}_H$ denote the canonical braided functor from the category $\mathcal{T}$ of framed, oriented tangles to the category Mod_H of left H–modules, which maps the object $\downarrow$ to V. Let us denote the restriction of $F_V^{\mathcal{T}}$ to B by $F_V\colon \mathsf{B} \to \mathsf{Mod}_H$. Then F_V maps the object b into $V \otimes V^*$, which we identify with the $\mathbf{k}$–algebra $E = \mathrm{End}_{\mathbf{k}}(V)$ of $\mathbf{k}$–vector space endomorphisms of V. As one can easily verify, the algebra structure of b is mapped into that of E, ie, $F_V(\mu_{\mathsf{b}}) = \mu_E$, and $F_V(\eta_{\mathsf{b}}) = \eta_E$, where $\mu_E\colon E \otimes E \to E$ and $\eta_E\colon \mathbf{k} \to E$ are the structure morphisms for the algebra E. Also, the images by F_V of the other generating morphisms of B are determined by

$$F_V(v_\pm)(1_{\mathbf{k}}) = \rho_V(\mathbf{r}^{\pm 1}),$$
$$F_V(c_\pm)(1_{\mathbf{k}}) = (\rho_V \otimes \rho_V)(c_\pm^H),$$

where $\rho_V\colon H \to E$ denotes the left action of H on V, ie, $\rho_V(x)(v) = x \cdot v$ for $x \in H$, $v \in V$. In fact, we have for each $T \in \mathsf{BT}_n$

$$F_V(T)(1_{\mathbf{k}}) = \rho_V^{\otimes n}(J_T).$$

Many results for universal invariants in the previous sections can be modified into versions for F_V. For example, the following is a version of Corollary 9.2.

Proposition 14.1 *Let $K_i \subset E^{\otimes i}$ for $i \geq 0$, be subsets satisfying the following.*

1. $1_{\mathbf{k}} \in K_0$, $1_E, \rho_V(\mathbf{r}^{\pm 1}) \in K_1$, *and* $(\rho_V \otimes \rho_V)(c_{\pm}^H) \in K_2$.
2. *For $m, n \geq 0$, we have $K_m \otimes K_n \subset K_{m+n}$.*
3. *For $p, q \geq 0$ we have*

$$(1^{\otimes p} \otimes \psi_{E,E}^{\pm 1} \otimes 1^{\otimes q})(K_{p+q+2}) \subset K_{p+q+2},$$
$$(1^{\otimes p} \otimes \mu_E \otimes 1^{\otimes q})(K_{p+q+2}) \subset K_{p+q+1},$$

where $\psi_{E,E} = F_V(\psi_{\mathsf{b},\mathsf{b}})\colon E \otimes E \to E \otimes E$ is the braiding of two copies of E in Mod_H.

Then, for any $T \in \mathsf{BT}_n$, $n \geq 0$, we have $F_V(T) \in K_n$.

Note that ρ_V can be regarded as a morphism $\rho_V\colon \underline{H} \to E$ in Mod_H. In fact, ρ_V is an algebra-morphism, ie, $\rho_V \mu_H = \mu_E(\rho_V \otimes \rho_V)$, $\rho_V \eta_H = \eta_E$. One can check that the morphisms $\rho_V^{\otimes i}\colon \underline{H}^{\otimes i} \to E^{\otimes i}$, $i \geq 0$, form a natural transformation

$$\rho\colon \mathsf{J} \Rightarrow F_V\colon \mathsf{B} \to \mathsf{Mod}_H,$$

ie, the following diagram is commutative for $i, j \geq 0$, $T \in \mathsf{B}(i, j)$

$$\begin{array}{ccc}
\underline{H}^{\otimes i} & \xrightarrow{\mathsf{J}(T)} & \underline{H}^{\otimes j} \\
{\scriptstyle \rho_V^{\otimes i}} \downarrow & & \downarrow {\scriptstyle \rho_V^{\otimes j}} \\
E^{\otimes i} & \xrightarrow[F_V(T)]{} & E^{\otimes j}.
\end{array}$$

14.2 Generalizations

14.2.1 Non-strict version B^q of B and the Kontsevich invariant Recall that there is a non-strict version $\mathcal{T}^q$ of $\mathcal{T}$, ie, the category of *q–tangles* (see Le and Murakami [47]), whose objects are parenthesized tensor words in $\downarrow$ and $\uparrow$ such as

$$(\downarrow \otimes \downarrow) \otimes (\uparrow \otimes (\downarrow \otimes \uparrow))$$

and whose morphisms are isotopy classes of tangles. In a natural way, one can define non-strict braided subcategories B^q (resp. B^q_0) of B whose objects are parenthesized tensor words of $\mathsf{b} = \downarrow \otimes \uparrow$, such as $(\mathsf{b} \otimes \mathsf{b}) \otimes \mathsf{b}$, and the morphisms are isotopy classes of tangles in B (resp. B_0). The results for B and B_0 in Sections 4, 5 and 6 can be easily generalized to results for the non-strict braided categories B^q and B^q_0. Recall [47] that the Kontsevich invariant can be formulated as a (non-strict) monoidal functor $Z\colon \mathcal{T}^q \to \mathcal{A}$ of $\mathcal{T}^q$ into a certain "category of diagrams". It is natural to expect that the non-strict versions of the results for B and B_0 in the present paper can be applied to the Kontsevich invariant and can give some integrality results for the Kontsevich invariant.

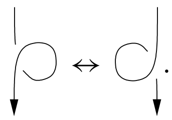
14.2.2 Ribbon Hopf algebras in symmetric monoidal category Universal invariant of links and tangles can be defined for any ribbon Hopf algebra H in any symmetric monoidal category $\mathcal{M}$. If T is a tangle consisting of n arcs and no circles, then the universal invariant J_T takes values in $\mathcal{M}(\mathbf{1}_{\mathcal{M}}, H^{\otimes n})$, where $\mathbf{1}_{\mathcal{M}}$ is the unit object in $\mathcal{M}$. Most of the results in Section 8 can be generalized to this setting. In particular, there is a braided functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$ such that $\mathsf{J}(\mathsf{b}) = H$, where Mod_H is the category of left H–modules in $\mathcal{M}$, and H is given the left H–module structure via the adjoint action.

We comment on two interesting special cases below.

14.2.3 Complete ribbon Hopf algebras and quantized enveloping algebras The universal invariant of tangles can also be defined for any ribbon *complete* Hopf algebra H over a linearly topologized, commutative, unital ring $\mathbf{k}$. The construction of universal invariant can be generalized to ribbon complete Hopf algebras in an obvious way. This case may be considered as the special case of Section 14.2.2, since H is a ribbon Hopf algebra in the category of complete $\mathbf{k}$–modules.

An important example of a complete ribbon Hopf algebra is the h–adic quantized enveloping algebra $U_h(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$. In future papers [20; 25], we will consider this case and prove some integrality results of the universal invariants.

14.2.4 Universal invariants and virtual tangles Kauffman [36] introduced *virtual knot theory* (see also Goussarov, Polyak and Viro [15], Kamada and Kamada [30], and Sakai [75; 76]). A virtual link is a diagram in a plane similar to a link diagram but allowing "virtual crossings". There is a preferred equivalence relation among virtual links called "virtual isotopy", and two virtually isotopic virtual links are usually regarded as the same. There is also a weaker notion of equivalence called "virtual regular isotopy", and it is observed by Kauffman [36] that many quantum link invariants can be extended to invariants of virtual regular isotopy classes of virtual links. The notion of virtual links are naturally generalized to that of virtual tangles. *Virtual framed isotopy* is generated by virtual regular isotopy and the move



An extreme case of Section 14.2.2 is the case of the symmetric monoidal category $\langle \mathsf{H}_r \rangle$ freely generated by a ribbon Hopf algebra H_r. The universal tangle invariant J_T in this case is very closely related to virtual tangles. We can construct a canonical bijection between the set $\langle \mathsf{H}_r \rangle(\mathbf{1}, \mathsf{H}_r^{\otimes n})$ and the set of the virtual framed isotopy classes of n–component "virtual bottom tangles". For $T \in \mathsf{BT}_n$, the universal invariant J_T takes values in $\mathsf{Mod}_{\mathsf{H}_r}(\mathbf{1}, \mathsf{H}_r^{\otimes n}) \subset \langle \mathsf{H}_r \rangle(\mathbf{1}, \mathsf{H}_r^{\otimes n})$. Thus, the universal invariant associated to H_r takes values in the virtual bottom tangles. For $n \geq 0$, the function

$$\mathsf{J}_{0,n} = \mathsf{J}\colon \mathsf{BT}_n\big(= \mathsf{B}(0,n)\big) \to \mathsf{Mod}_{\mathsf{H}_r}(\mathbf{1}, \mathsf{H}_r^{\otimes n}), \quad T \mapsto \mathsf{J}(T),$$

is injective. We conjecture that $\mathsf{J}_{0,n}$ is surjective. If this is true, then we can regard it as an algebraic characterization of bottom tangles among virtual bottom tangles. We can formulate similar conjectures for general tangles and links. This may be regarded as a new way to view virtual knot theory in an algebraically natural way. (We also remark here that there is another (perhaps more natural) way to formulate virtual knot theory in a category-theoretic setting, which uses the tangle invariant associated to the symmetric monoidal category $\langle \mathsf{H}_r, \mathsf{V} \rangle$ freely generated by a ribbon Hopf algebra H_r and a left H_r–module V with left dual.) We plan to give the details of the above in future publications.

14.2.5 Quasitriangular Hopf algebras and even bottom tangles We can generalize our setting to a quasitriangular Hopf algebra, which may not be ribbon. For a similar idea of defining quantum invariants associated to quasitriangular Hopf algebras, see Sawin [77].

It is convenient to restrict our attention to *even-framed* bottom tangles up to regular isotopy. Here a tangle T is even-framed if the closure of each component of T is of even framing, or, in other words, each component of T has even number of self crossings. Let B^{ev} denote the subcategory of B such that $\mathrm{Ob}(\mathsf{B}^{\mathrm{ev}}) = \mathrm{Ob}(\mathsf{B})$ and $\mathsf{B}^{\mathrm{ev}}(m,n)$ consists of $T \in \mathsf{B}(m,n)$ even-framed. Then we have the following.

(1) B^{ev} is a braided subcategory of B.

(2) B^{ev} is generated as a braided subcategory of B by the object b and the morphism $\mu_{\mathsf{b}}, \eta_{\mathsf{b}}, c_+, c_-$. (This follows from Remark 5.20.)

(3) B^{ev} inherits from B an external Hopf algebra structure.

(4) There is a braided functor $\mathsf{J}^{\mathrm{ev}}\colon \mathsf{B}^{\mathrm{ev}} \to \mathsf{Mod}_H$, defined similarly to the ribbon case, and we have an analogue of Theorem 8.3. Hence we have a topological interpretation of transmutation of a quasitriangular Hopf algebra. If H is ribbon, then J^{ev} is the restriction of J to B^{ev}.

14.3 The functor $\check{\mathsf{J}}$

The functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$ is not faithful for any ribbon Hopf algebra H. For example, we have $\mathsf{J}(t_\downarrow \otimes \uparrow) = \mathsf{J}(\downarrow \otimes t_\uparrow)$ for any H but we have $t_\downarrow \otimes \uparrow \neq \downarrow \otimes t_\uparrow$. Using the construction by Kauffman [35], one can construct a functor $\check{\mathsf{J}}\colon \mathsf{B} \to \mathrm{Cat}(H)$, which distinguishes more tangles than J. Here $\mathrm{Cat}(H)$ is the category defined in [35], and $\check{\mathsf{J}}$ is just the restriction to B of the functor $F\colon \mathcal{T} \to \mathrm{Cat}(H)$ defined in [35]. Since each $T \in \mathsf{B}(m,n)$ consists of $m+n$ arc components, $\check{\mathsf{J}}(T)$ can be defined as an element of $H^{\otimes(m+n)}$. If we take H as the Hopf algebra in the braided category $\langle \mathsf{H}_r \rangle$ freely generated by a ribbon Hopf algebra as in Section 14.2.4, then $\check{\mathsf{J}}$ is faithful.

We have not studied the functor $\check{\mathsf{J}}$ in the present paper because our aim of introducing the category B is to provide a useful tool to study bottom tangles. The functor $\mathsf{J}\colon \mathsf{B} \to \mathsf{Mod}_H$ is more suitable than $\check{\mathsf{J}}$ for this purpose.

14.4 The category $\mathcal{B}$ of bottom tangles in handlebodies

In future papers, we will give details of the following.

Let $\mathcal{B}$ denote the category of *bottom tangles in handlebodies*, which is roughly defined as follows. For $n \geq 0$, let V_n denote a "standard handlebody of genus n", which is obtained from the cube $[0,1]^3$ by adding n handles in a canonical way, see Figure 44 (a). An n–component *bottom tangle in* V_n is a framed, oriented tangle T in V_n consisting of n arc components $T_1, \dots, T_n$, such that, for $i = 1, \dots, n$, T_i starts at the $2i$ th endpoint on the bottom and end at the $(2i-1)$ st endpoint on the bottom. See Figure 44 (b) for example, which we usually draw as the projected diagram as in (c).

The category $\mathcal{B}$ is defined as follows. Set $\mathrm{Ob}(\mathcal{B}) = \{0, 1, 2, \dots\}$. For $m, n \geq 0$, the set $\mathcal{B}(m,n)$ is the set of isotopy classes of n–component bottom tangles in V_m. For $T \in \mathcal{B}(l,m)$ and $T' \in \mathcal{B}(m,n)$, the composition $T'T \in \mathcal{B}(l,n)$ is represented by the l–component tangle in V_n obtained as follows. First let E_T denote the "exterior of T in V_l", ie, the closure of $V_l \setminus N_T$, with N_T a tubular neighborhood of T in V_l. Note that E_T may be regarded as a cobordism from the connected oriented surfaces $F_{l,1}$ of genus l with one boundary component to $F_{m,1}$. Thus there is a natural way to identify the "bottom surface" of N_T with the "top surface" of V_m. By gluing E_T and

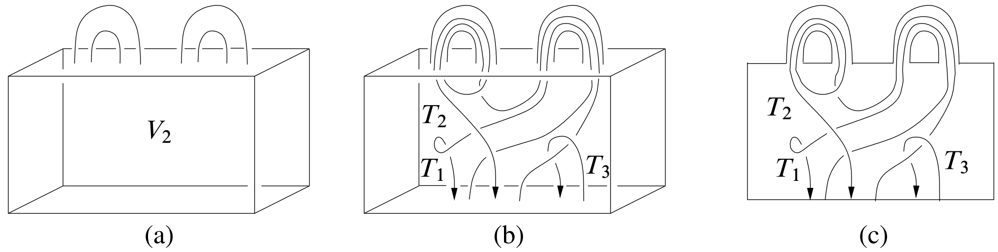


Figure 44: (a) A standard handlebody V_2 of genus 2. (b) A 3–component bottom tangle $T = T_1 \cup T_2 \cup T_3$ in V_2. (c) A diagram for T.

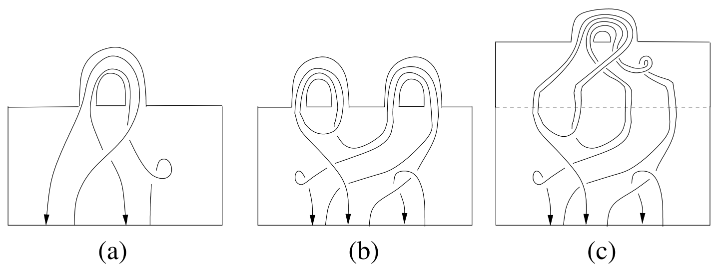


Figure 45: (a) A bottom tangle $T \in \mathcal{B}(1,2)$. (b) A bottom tangle $T' \in \mathcal{B}(2,3)$. (c) The composition $T'T \in \mathcal{B}(1,3)$.

V_m along these surfaces, we obtain a 3–manifold $E_T \cup V_m$, naturally identified with V_l. The tangle T', viewed as a tangle in $E_T \cup V_m \cong V_l$, represents the composition $T'T$. Figure 45 shows an example. For $n \geq 0$, the identity morphism $1_n\colon n \to n$ is represented by the bottom tangle depicted in Figure 46. We can prove that the category $\mathcal{B}$ is well defined. The category $\mathcal{B}$ has the monoidal structure given by horizontal pasting.

There is a functor $\xi\colon \mathsf{B} \to \mathcal{B}$ such that $\xi(\mathsf{b}^{\otimes n}) = n$, and, for $T \in \mathsf{B}(m,n)$, the tangle $\xi(T) \in \mathcal{B}(m,n)$ is obtained from T by pasting a copy of the identity bottom tangle $1_{\mathsf{b}^{\otimes m}}$ on the top of T. This functor is monoidal, and the braiding structure for B induces that for $\mathcal{B}$ via ξ, see Figure 46.

Also, there is a Hopf algebra structure $H_{\mathcal{B}} = (1, \mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \Delta_{\mathcal{B}}, \epsilon_{\mathcal{B}}, S_{\mathcal{B}})$ in the usual sense for the object $1 \in \mathrm{Ob}(\mathcal{B})$. Graphically, the structure morphisms $\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \Delta_{\mathcal{B}}, \epsilon_{\mathcal{B}}, S_{\mathcal{B}}$ for $\mathcal{B}$ is as depicted in Figure 46.

The external Hopf algebra structure in B is mapped by ξ into the external Hopf algebra structure in $\mathcal{B}$ associated to $H_{\mathcal{B}}$.

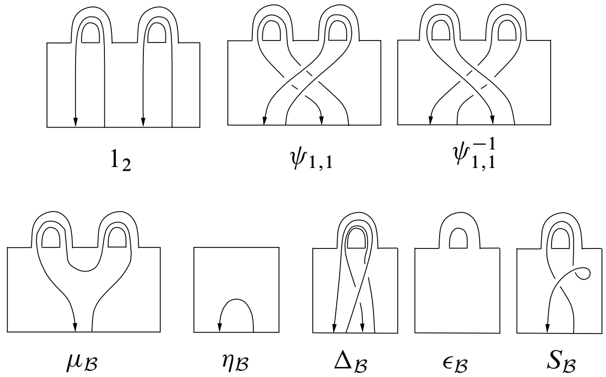


Figure 46: The morphisms 1_2, $\psi_{1,1}$, $\psi_{1,1}^{-1}$, $\mu_{\mathcal{B}}$, $\eta_{\mathcal{B}}$, $\Delta_{\mathcal{B}}$, $\epsilon_{\mathcal{B}}$ and $S_{\mathcal{B}}$ in $\mathcal{B}$

For $n \geq 0$, the function

$$\xi\colon \mathsf{BT}_n \to \mathcal{B}(0,n)$$

is bijective. Hence we can identify the set $\mathcal{B}(0,n)$ with the set of n–component bottom tangles. (However, $\xi\colon \mathsf{B}(m,n) \to \mathcal{B}(m,n)$ is neither injective nor surjective in general. Hence the functor ξ is neither full nor faithful.) Using Theorem 5.16, we can prove that $\mathcal{B}$ is generated as a braided category by the morphisms

$$\mu_{\mathcal{B}}, \eta_{\mathcal{B}}, \Delta_{\mathcal{B}}, \epsilon_{\mathcal{B}}, S_{\mathcal{B}}, S_{\mathcal{B}}^{-1}, v_{\mathcal{B},+}, v_{\mathcal{B},-}, \tag{14–1}$$

where $v_{\mathcal{B},\pm} = \xi(v_\pm)$.

There is a natural, faithful, braided functor $i\colon \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is the category of cobordisms of surfaces with connected boundary as introduced by Crane and Yetter [8] and by Kerler [40], independently. The objects of $\mathcal{C}$ are the nonnegative integers $0, 1, 2, \ldots$, the morphisms from m to n are (certain equivalence classes of) cobordisms from F_m to F_n, where F_m is a compact, connected, oriented surface of genus n with $\partial F_m \cong S^1$. (See also Habiro [22], Kerler [39; 42], Kerler–Lyubashenko [43] and Yetter [85] for descriptions of $\mathcal{C}$.) This functor i maps $T \in \mathcal{B}(m,n)$ into the cobordism E_T defined above. In the following, we regard $\mathcal{B}$ as a braided subcategory of $\mathcal{C}$ via i. Recall from [8] and [40] that $\mathcal{C}$ is a braided category, and there is a Hopf algebra $\mathsf{h} = (1, \mu_{\mathsf{h}}, \eta_{\mathsf{h}}, \Delta_{\mathsf{h}}, \epsilon_{\mathsf{h}}, S_{\mathsf{h}})$ in $\mathcal{C}$ with the underlying object 1, which corresponds to the punctured torus F_1.

The category $\mathcal{B}$ can be identified with the subcategory of $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{B}) = \mathrm{Ob}(\mathcal{C})$ and

$$\mathcal{B}(m,n) = \left\{ f \in \mathcal{C}(m,n) \mid \epsilon_{\mathsf{h}}^{\otimes n} f = \epsilon_{\mathsf{h}}^{\otimes m} \right\}$$

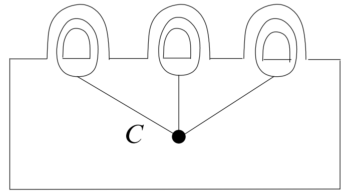


Figure 47: The cobordism B^* is obtained from the trivial cobordism $\epsilon_{\mathcal{B}}^{\otimes 3}$ by surgery along the Y–graph C.

for $m,n \geq 0$. Recall from [42] that $\mathcal{C}$ is generated as a braided category by the generators of $\mathcal{B}$ listed in (14–1) and an integral χ_{h} of the Hopf algebra h. This integral χ_{h} for h is not contained in $\mathcal{B}$.

For each ribbon Hopf algebra H, we can define a braided functor

$$\mathsf{J}^{\mathcal{B}}\colon \mathcal{B} \to \mathsf{Mod}_H$$

such that $\mathsf{J} = \mathsf{J}^{\mathcal{B}}\xi$. The functor $\mathsf{J}^{\mathcal{B}}$ maps the Hopf algebra $H_{\mathcal{B}}$ in $\mathcal{B}$ into the transmutation $\underline{H}$ of H. If H is finite-dimensional ribbon Hopf algebra over a field $\mathbf{k}$, and is *factorizable* (see Reshetikhin and Semenov-Tian-Shansky [73]) – that is, the function $\mathrm{Hom}_{\mathbf{k}}(H,\mathbf{k}) \to H$, $f \mapsto (1\otimes f)(c_+^H)$, is an isomorphism – then $\mathsf{J}^{\mathcal{B}}$ extends to Kerler's functorial version of the Hennings invariant [39; 41]

$$\mathsf{J}^{\widetilde{\mathcal{C}}}\colon \widetilde{\mathcal{C}} \to \mathsf{Mod}_H. \tag{14–2}$$

An interesting extension of $\mathcal{B}$ is the braided subcategory $\overline{\mathcal{B}}$ of $\mathcal{C}$ generated by the objects and morphisms of $\mathcal{B}$ and the morphism $B^* \in \mathcal{C}(3,0)$ described in Figure 47. One can show that the category $\overline{\mathcal{B}}$ is the same as the category of *bottom tangles in homology handlebodies*, which is defined in the same way as $\mathcal{B}$ but the bottom tangles are contained in a homology handlebody. (Recall that a homology handlebody can be characterized as a 3–manifold which is obtainable as the result from a standard handlebody of surgery along finitely many Y–graphs, see Habegger [19].) For each $n \geq 0$, the monoid $\overline{\mathcal{B}}(n,n)$ contains the Lagrangian submonoid $\mathcal{L}_n$ (see Levine [52]) of the monoid of homology cobordisms (see Goussarov [14] and Habiro [22]) of a compact, connected, oriented surface $\Sigma_{n,1}$ of genus 1 and with one boundary component, and hence contains the monoid of homology cylinders (or homologically trivial cobordisms) $\mathcal{HC}_{n,1}$ over $\Sigma_{n,1}$ and the Torelli group $\mathcal{I}_{n,1}$ of $\Sigma_{n,1}$. Here the Lagrangian subgroup of $H_1\Sigma_{n,1} \simeq \mathbb{Z}^{2n}$, which is necessary to specify $\mathcal{L}_n$, is generated by the meridians of the handles in V_n. The monoid $\overline{\mathcal{B}}(n,n)$ does not contain the whole mapping class group $\mathcal{M}_{n,1}$. In fact, $\overline{\mathcal{B}}(n,n) \cap \mathcal{M}_{n,1}$ is precisely the Lagrangian subgroup of $\mathcal{M}_{n,1}$. The subgroup $\mathcal{B}(n,n)\cap\mathcal{M}_{n,1}$ of $\bar{\mathcal{B}}(n,n)\cap\mathcal{M}_{n,1}$ corresponds to the handlebody group $H_{n,1}$, which is the group of isotopy classes of self-homeomorphisms of handlebody of genus n fixing a disc in the boundary pointwise.

14.5 Surgery on 3–manifolds as monoidal relation

The idea of identifying the relations on tangles defined by local moves with monoidal relations in the monoidal category $\mathcal{T}$ of tangles can be generalized to 3–manifolds as follows. Matveev [57] defined a class of surgery operations on 3–manifolds called *$\mathcal{V}$–surgeries*. A special case of $\mathcal{V}$–surgery removes a handlebody from a 3–manifold and reglues it back in a different way. Here $\mathcal{V}=(V_1,V_2)$ is a pair of two handlebodies V_1, V_2 of the same genus with boundaries identified, and determines a type of surgery. We call such surgery *admissible*. For each such $\mathcal{V}$, there is a (not unique) pair (f_1, f_2) of morphisms f_1 and f_2 of the same source and target in the monoidal category $\mathcal{C}$ of cobordisms of surfaces with connected boundary (see Crane–Yetter [8], Kerler [40] and Section 14.4). For 3–manifolds representing morphisms in $\mathcal{C}$, the equivalence relations of 3–manifolds generated by $\mathcal{V}$–surgeries is the same as the monoidal relation in $\mathcal{C}$ generated by the pair (f_1, f_2). Thus, one can formulate the theory of admissible surgeries in an algebraic way.

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