

On the Kontsevich integral of Brunnian links

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The purpose of the paper is twofold. First, we give a short proof using the Kontsevich integral for the fact that the restriction of an invariant of degree $2n$ to $(n+1)$ -component Brunnian links can be expressed as a quadratic form on the Milnor $\bar{\mu}$ link-homotopy invariants of length $n+1$. Second, we describe the structure of the Brunnian part of the degree- $2n$ graded quotient of the Goussarov–Vassiliev filtration for $(n+1)$ -component links.

[57M25](#), [57M27](#)

1 Introduction

The notion of Goussarov–Vassiliev finite type link invariants provides a unifying viewpoint on the various quantum link invariants; see Bar-Natan [1], Goussarov [10; 11], Vassiliev [28]. The definition mainly relies on a descending filtration

$$(1-1) \quad \mathbb{Z}\mathcal{L}(m) = J_0(m) \supset J_1(m) \supset \dots$$

of the free abelian group $\mathbb{Z}\mathcal{L}(m)$ generated by the set $\mathcal{L}(m)$ of isotopy classes of m -component, oriented, ordered links in S^3 . Here each $J_n(m)$ is generated by alternating sums of links over n independent crossing changes. A homomorphism from $\mathbb{Z}\mathcal{L}(m)$ to an abelian group is said to be an invariant of degree $\leq n$ if it vanishes on $J_{n+1}(m)$. The Kontsevich integral [17] is a universal rational-valued finite type link invariant.

A link in S^3 is said to be *Brunnian* if every proper sublink of it is an unlink. The first author [14] proved that an $(n+1)$ -component Brunnian link cannot be distinguished from the unlink by any invariant of degree $< 2n$ with values in any abelian group. In [16] the authors study the behavior of the invariants of degree $2n$ on $(n+1)$ -component Brunnian links.

The purpose of this paper is twofold. First, we give an alternative, shorter proof using the Kontsevich integral of [Theorem 1.1](#), originally proved in [16], which relates the degree- $2n$ invariants for $(n+1)$ -component links to Milnor $\bar{\mu}$ link-homotopy

invariants. Second, we describe the structure of the *Brunnian part* of the degree- $2n$ graded quotient of the Goussarov–Vassiliev filtration for $(n+1)$ -component links.

For any $(n+1)$ -component Brunnian link L , the Milnor $\bar{\mu}$ link-homotopy invariants $\bar{\mu}_{i_1, \dots, i_{n+1}}(L) \in \mathbb{Z}$ for nonrepeating indices $i_1, \dots, i_{n+1} \in \{1, 2, \dots, n+1\}$ are well-defined and give a complete set of link-homotopy invariants [24]. It is also known that there are exactly $(n-1)!$ independent invariants

$$\bar{\mu}_\sigma(L) := \bar{\mu}_{\sigma(1), \sigma(2), \dots, \sigma(n-1), n, n+1}(L) \in \mathbb{Z} \quad \text{for } \sigma \in S_{n-1},$$

where S_{n-1} denotes the symmetric group on $\{1, 2, \dots, n-1\}$.

Theorem 1.1 ([16, Theorem 1.2], stated slightly differently) *Let $n \geq 2$ and let $f: \mathbb{Z}\mathcal{L}(n+1) \rightarrow \mathbb{Z}$ be any link invariant of degree $2n$. Then there exist integers $f_{\sigma, \sigma'}$ for $\sigma, \sigma' \in S_{n-1}$ such that for any $(n+1)$ -component Brunnian link L , we have*

$$(1-2) \quad f(L) - f(U) = \frac{1}{2} \sum_{\sigma, \sigma' \in S_{n-1}} f_{\sigma, \sigma'} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L),$$

where U is the $(n+1)$ -component unlink. Moreover, we have $f_{\sigma, \sigma'} = f_{\sigma', \sigma}$ and $f_{\sigma, \sigma} \in 2\mathbb{Z}$ for all $\sigma, \sigma' \in S_{n-1}$.

Theorem 1.1 is proved in Section 2. The coefficients $f_{\sigma, \sigma'}$ are defined there using the weight system of f . It follows from the arguments of Section 3 that we have $f_{\sigma, \sigma'} = f_{\eta\sigma, \eta\sigma'}$ for every $\eta \in S_{n-1}$. Thus there are integers g_τ for $\tau \in S_{n-1}$ satisfying $g_\tau = g_{\tau^{-1}}$ and $g_1 \in 2\mathbb{Z}$, such that $f_{\sigma, \sigma'} = g_{\sigma^{-1}\sigma'}$. Further, f does not depend on the order of components for Brunnian links.

In Section 3 we give, in theory, a necessary and sufficient condition to extend a quadratic form of Milnor invariants to a link invariant of degree $\leq 2n$. More precisely, we study the *Brunnian part* $\text{Br}(\bar{J}_{2n}(n+1))$ of the graded quotient $\bar{J}_{2n}(n+1) = J_{2n}(n+1)/J_{2n+1}(n+1)$, which is defined as the subgroup generated by the elements $[L - U]_{J_{2n+1}}$, where L is Brunnian. We construct a homomorphism

$$h_n: \mathcal{A}_{n-1}^c(\emptyset) \rightarrow \text{Br}(\bar{J}_{2n}(n+1)),$$

where $\mathcal{A}_{n-1}^c(\emptyset)$ is a \mathbb{Z} -module of connected trivalent diagrams with $2n-2$ vertices. We show that h_n is surjective for $n \geq 3$ and is an isomorphism over \mathbb{Q} for $n \geq 2$.

A few remarks are in order.

Remark 1.2 The original proof of Theorem 1.1 involved calculus of claspers. It provided, as a byproduct, a number of results on Brunnian (string) links and the link-homotopy relation. The proof given here is shorter than the original one and may also look simpler depending on the reader's point of view.

Remark 1.3 Recall that Milnor invariants of length $n + 1$ for string links are Goussarov–Vassiliev invariants of degree $\leq n$; see Bar-Natan [2] and Lin [22]. For links, Milnor invariants are not well-defined in general. However, [Theorem 1.1](#) shows that a quadratic expression in Milnor invariants, which is well-defined at least for $(n + 1)$ -component Brunnian links, may extend to a link invariant of degree $\leq 2n$.

For example, the case $n = 2$ of [Theorem 1.1](#) states that any degree-4 invariant of 3-component links restricts for Brunnian links to a scalar multiple of the square of the triple Milnor invariant $\mu_{123}(L)$. Recall that $\mu_{123}(L)^2$ for 3-component algebraically split links extends to the degree-4 coefficient of the Alexander–Conway polynomial, which is a degree-4 invariant; see Cochran [8].

Remark 1.4 Recall that trivalent diagrams appear in the study of Ohtsuki finite type invariants of integral homology spheres; see Ohtsuki [25], Garoufalidis and Ohtsuki [9] and Le [18]. The relation between [Theorem 3.2](#) and Ohtsuki finite type invariants is discussed in [23].

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2 Proof of [Theorem 1.1](#) using the Kontsevich integral

In this section, we give a proof of [Theorem 1.1](#) using the Kontsevich integral. For an introduction to the Kontsevich integral; see for example Bar-Natan [1], Chmutov and Duzhin, [7], Lescop [21] and Ohtsuki [26].

2.1 Spaces of diagrams

We recall the definition of the spaces of diagrams in which the Kontsevich integral takes its values.

A *unitrivalent diagram* (also called *Feynman diagram* or *Jacobi diagram*) is a finite graph Γ with univalent and trivalent vertices such that each trivalent vertex is equipped with a cyclic order of the three incident edges. Throughout the paper, we assume that every component of a unitrivalent diagram has at least one univalent vertex. The *degree* of Γ is half the number of vertices in Γ .

A univalent diagram Γ is on a 1-manifold X if the univalent vertices of Γ are identified with distinct points on X up to isotopy. Let $\mathcal{A}_k(X)$ denote the \mathbb{Z} -module generated by diagrams on X of degree k , subject to the *FI relations* and the *STU relations*; see Figure 1.

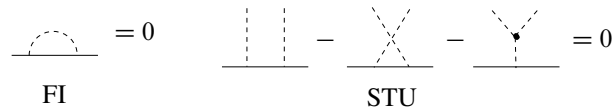


Figure 1

As is well-known, the *AS relations* and the *IHX relations* depicted in Figure 2 are valid in $\mathcal{A}_k(X)$.

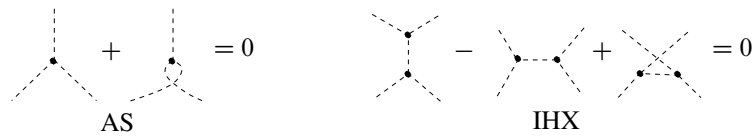


Figure 2

If X is the disjoint union of m copies of S^1 , then $\mathcal{A}_k(X)$ is denoted also by $\mathcal{A}_k(m)$.

An *open univalent diagram* for m colors $\{1, \dots, m\}$ is a univalent graph such that each univalent vertex is labeled by an element of $\{1, \dots, m\}$. Let $\mathcal{B}_k(m)$ denote the \mathbb{Q} -vector space generated by open univalent diagrams of degree k for m colors $\{1, \dots, m\}$, modulo the AS and the IHX relations.

Let $\sqcup^m I$ denote the disjoint union of m copies of the unit interval I . There is a standard isomorphism ($m \geq 1$)

$$\chi_k: \mathcal{B}_k(m) \xrightarrow{\cong} \mathcal{A}_k(\sqcup^m I) \otimes \mathbb{Q},$$

called the Poincaré–Birkhoff–Witt isomorphism; see Bar-Natan [1].

For $m = 1$, $\mathcal{A}_k(I)$ is isomorphic to $\mathcal{A}_k(1)$. For $m > 1$, let $\mathcal{B}_k^l(m)$ denote the quotient of $\mathcal{B}_k(m)$ by all *link relations* (see Section 5.2 of Bar-Natan, Garoufalidis, Rozansky and Thurston [4]). The isomorphism χ_k induces a well-defined isomorphism

$$\chi_k: \mathcal{B}_k^l(m) \xrightarrow{\cong} \mathcal{A}_k(m) \otimes \mathbb{Q}.$$

2.2 Proof of Theorem 1.1

Suppose L is an $(n + 1)$ -component Brunnian link in S^3 . By [14, Proposition 12], there is an $(n + 1)$ -component Brunnian string link T whose closure is L . (A string link is said to be *Brunnian* if every proper subtangle of it is a trivial string link.) Consider the Kontsevich integral $Z(T)$ of T , which lives in the completed \mathbb{Q} -vector space $\mathcal{A}(\sqcup^{n+1} I) = \prod_{k \geq 0} \mathcal{A}_k(\sqcup^{n+1} I) \otimes \mathbb{Q}$. Recall that $\mathcal{A}(\sqcup^{n+1} I)$ has an algebra structure with multiplication given by the ‘stacking product’ and with unit 1 given by the empty univalent diagram.

Let $p: \mathcal{A}(\sqcup^{n+1} I) \simeq \mathcal{B}(n + 1)$ denote the inverse of the Poincaré–Birkhoff–Witt isomorphism, where $\mathcal{B}(n + 1)$ is the completed \mathbb{Q} -vector space $\prod_{k \geq 0} \mathcal{B}_k(n + 1)$. Since $p(Z(T)) \in \mathcal{B}(n + 1)$ is grouplike (see Le, Murakami and Ohtsuki [20]), we have

$$(2-1) \quad p(Z(T)) = \exp_{\sqcup}(P), \quad P \in \mathcal{P}(n + 1),$$

where $\mathcal{P}(n + 1)$ denotes the primitive part of $\mathcal{B}(n + 1)$, generated by connected diagrams and where \exp_{\sqcup} denotes the exponential with respect to disjoint union of diagrams.

For each $i = 1, \dots, n + 1$, consider the operation ϵ_i of omitting the i -th string. At the level of the string link we have $\epsilon_i(T) = \mathbf{1}_n$, since T is Brunnian. Hence we have $\epsilon_i(P) = 0$. It follows that P can be expressed as an infinite \mathbb{Q} -linear combination of connected diagrams, each having at least one univalent vertex of color i . Since we have this property for $i = 1, \dots, n + 1$, we can deduce that P is an infinite \mathbb{Q} -linear combination of connected diagrams, each involving all the colors.

By an easy counting argument, we see that P can be expressed as

$$P = P_n + P_{n+1} + P_{n+2} + \dots$$

where P_n is a linear combination of trees of degree n , and P_k for $k > n$ is a linear combination of connected diagrams of degree k . Moreover, we can use the AS and IHX relation to express P_n as a linear combination of trees t_σ as depicted in Figure 3, $\sigma \in S_{n-1}$. By Habegger and Masbaum’s result [13], P_n is thus a \mathbb{Z} -linear combination

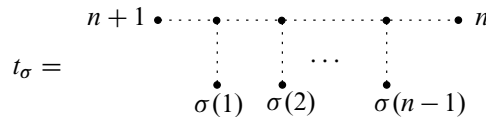


Figure 3

of trees which corresponds to the Milnor invariants of length $n + 1$ of the form

$\bar{\mu}_\sigma := \bar{\mu}_{\sigma(1),\sigma(2),\dots,\sigma(n-1),n,n+1}$, for $\sigma \in S_{n-1}$. More precisely, from [13, Theorem 6.1 and Proposition 7.1] and properties of the Milnor $\bar{\mu}$ link-homotopy invariants [24] it follows that

$$(2-2) \quad P_n = \sum_{\sigma \in S_{n-1}} \bar{\mu}_\sigma(L) t_\sigma.$$

By (2-1), we have

$$\begin{aligned} p(Z(T)) &= \exp_{\square}(P_n + P_{n+1} + \cdots + P_{2n} + \cdots) \\ &= 1 + P_n + P_{n+1} + \cdots + P_{2n-1} + (P_{2n} + \frac{1}{2}P_n^2) + O_{>2n}, \end{aligned}$$

where $O_{>2n}$ denotes a sum of terms of degree greater than $2n$. Hence we have

$$(2-3) \quad Z(T) = 1 + P'_n + P'_{n+1} + \cdots + P'_{2n-1} + P'_{2n} + p^{-1}(\frac{1}{2}P_n^2) + O_{>2n},$$

where $P'_k = p^{-1}(P_k)$ for $k = n, \dots, 2n$.

By [19, Theorem 6.1], the Kontsevich integral $Z(L)$ of L can be obtained from $Z(T)$ by first multiplying by the $(n+1)$ -parallel $\Delta^n(Z(\mathcal{O}))$ of the Kontsevich integral of the unknot $Z(\mathcal{O}) \in \mathcal{A}(1)$ and then mapping into the space $\mathcal{A}(n+1)$ by closing the strings. Thus we have

$$(2-4) \quad Z(L) - Z(U) = \pi(\Delta^n(Z(\mathcal{O}))(Z(T) - 1)),$$

where U denotes the $(n+1)$ -component unlink in the 3-sphere S^3 and where the projection $\pi: \mathcal{A}(\square^{n+1}I) \rightarrow \mathcal{A}(n+1)$ is induced by the closure operation.

As is well-known [3; 27; 5],

$$(2-5) \quad Z(\mathcal{O}) = 1 + W_2 + W_4 + W_6 + \cdots$$

where W_{2j} is a linear combination of univalent diagram of degree $2j$ consisting of finitely many *wheels*, ie, connected univalent diagrams of Euler number 1.

By (2-3), (2-4) and (2-5), we have

$$(2-6) \quad Z(L) - Z(U) = \frac{1}{2}\pi p^{-1}(P_n^2) + \sum_{i=n}^{2n} \pi(P'_i) + \sum_{i,j} \pi(P'_i \Delta^n(W_{2j})) + O_{>2n},$$

where the second sum is over i, j with $n \leq i \leq 2n$, $2 \leq 2j \leq n$, $i + 2j \leq 2n$.

One can verify that the two sums in the right-hand side of (2-6) vanish, using the fact that π kills each diagram with at least one string having exactly one univalent vertex. Thus we have

$$(2-7) \quad Z(L) - Z(U) = \frac{1}{2}\pi p^{-1}(P_n^2) + O_{>2n}.$$

It follows that the first nontrivial, \mathbb{Z} -valued, finite type invariants of $(n + 1)$ -component Brunnian links are of degree $2n$. Moreover, (2-2) and (2-7) imply that

$$Z(L) - Z(U) = \frac{1}{2} \sum_{\sigma, \sigma' \in S_{n-1}} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L) \pi p^{-1}(t_\sigma t_{\sigma'}) + O_{>2n}.$$

Let f be a \mathbb{Z} -valued link invariant of degree $2n$, and denote by W_f the weight system of f . To complete the proof of Theorem 1.1, we obtain (1-2) by setting

$$f_{\sigma, \sigma'} = W_f \left(\pi p^{-1}(t_\sigma t_{\sigma'}) \right).$$

Remark 2.1 Obviously, we have $f_{\sigma, \sigma'} = f_{\sigma', \sigma}$ for all $\sigma, \sigma' \in S_{n-1}$. Observe that the identity depicted in Figure 4 can be derived from the STU relation. This identity implies that, for any $\sigma \in S_{n-1}$, we have $\pi p^{-1}(t_\sigma^2) = 2d_\sigma$ for some $d_\sigma \in \mathcal{A}_{2n}(n + 1)$. Hence $f_{\sigma, \sigma} = W_f(2d_\sigma)$ is even.

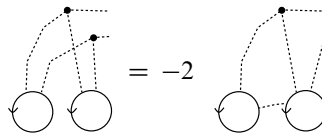


Figure 4: Hint: use the fact that, thanks to the STU relation, a circle component inserted in an edge attached to a trivalent vertex v can be moved to another edge attached to v (see Figure 5).

Remark 2.2 As recalled in the introduction, it is known that no Goussarov–Vassiliev invariant of degree $< 2n$ with values in any abelian group can distinguish L from U [14]. The above proof of Theorem 1.1 recovers, using the Kontsevich integral, a weaker version (namely, a rational version) of this result.

3 The Brunnian part of $\bar{J}_{2n}(n + 1)$

The purpose of this section is to give an almost complete description of the structure of the Brunnian part $\text{Br}(\bar{J}_{2n}(n + 1))$ of the $2n$ -th graded quotient $\bar{J}_{2n}(n + 1)$ of the Goussarov–Vassiliev filtration for $(n + 1)$ -component links, using the \mathbb{Z} -module $\mathcal{A}_{n-1}^c(\emptyset)$ of connected trivalent diagrams.

3.1 Claspers and Goussarov–Vassiliev filtration

The Goussarov–Vassiliev filtration (1–1) is usually defined in terms of singular links, where $J_k(m) \subset \mathbb{Z}\mathcal{L}(m)$ is generated by the k -fold alternating sums of links determined by singular links with k double points; see eg Bar-Natan [1]. One can also redefine the filtration using claspers; see Habiro [15] and Goussarov [12]. Here we will freely use the definitions and conventions of [15]. Recall that a *graph scheme of degree k* for a link L in S^3 is a set $S = \{G_1, \dots, G_l\}$ of l disjoint (strict) graph claspers G_1, \dots, G_l for L such that $\sum_{i=1}^l \deg G_i = k$. We set

$$(3-1) \quad [L, S] = [L; G_1, \dots, G_l] := \sum_{S' \subset S} (-1)^{|S| - |S'|} L_{\cup S'} \in \mathbb{Z}\mathcal{L}(m),$$

where the sum runs over all subsets S' of S , $|S'|$ denote the number of elements of S' , and $L_{\cup S'}$ denotes the result from L of surgery along the union of elements in S' . It is known that $J_k(m)$ is generated by the elements of the form $[L, S]$, where $L \in \mathcal{L}(m)$ and S is a graph scheme for L of degree k [15].

3.2 The Brunnian part of the Goussarov–Vassiliev filtration

If L is an $(n+1)$ -component Brunnian link in S^3 , then, by [14, Theorem 3], we have $L - U \in J_{2n}(n+1)$. Define $\text{Br}(\bar{J}_{2n}(n+1))$ to be the \mathbb{Z} -submodule of $\bar{J}_{2n}(n+1)$ generated by the elements $[L - U]_{J_{2n+1}}$, where L is an $(n+1)$ -component Brunnian link. We call $\text{Br}(\bar{J}_{2n}(n+1))$ the *Brunnian part* of $\bar{J}_{2n}(n+1)$.

3.3 Trivalent diagrams

A *trivalent diagram* is a finite trivalent graph Γ such that each vertex is equipped with a cyclic order on the three incident edges. The *degree* of Γ is half the number of its vertices.

For $k \geq 0$, let $\mathcal{A}_k(\emptyset)$ denote the \mathbb{Z} -module generated by trivalent diagrams of degree k , subject to the AS and IHX relations (see Figure 2).

Let $\mathcal{A}_n^c(\emptyset)$ denote the \mathbb{Z} -submodule of $\mathcal{A}_n(\emptyset)$ generated by connected trivalent diagrams.

3.4 The circle-insertion map g_n

For $n \geq 2$, denote the homomorphism which maps each trivalent diagram Γ to the result of inserting $n+1$ ordered copies of S^1 in its edges by

$$g_n: \mathcal{A}_{n-1}^c(\emptyset) \rightarrow \mathcal{A}_{2n}(n+1).$$

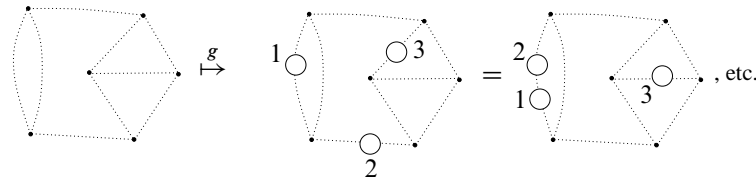


Figure 5

See Figure 5. This map is well-defined thanks to the STU relation, since Γ is connected.

We need the following result.

Proposition 3.1 For $n \geq 2$, the map

$$g_n \otimes \mathbb{Q}: \mathcal{A}_{n-1}^c(\emptyset) \otimes \mathbb{Q} \rightarrow \mathcal{A}_{2n}(n+1) \otimes \mathbb{Q}$$

is injective.

Proof Define a homomorphism

$$P: \mathcal{B}_{2n}^l(n+1) \longrightarrow \mathcal{A}_{n-1}(\emptyset) \otimes \mathbb{Q}$$

as follows. Let Γ be a diagram in $\mathcal{B}_{2n}^l(n+1)$. If there are exactly two univalent vertices colored by i for each $i = 1, \dots, n+1$, then we set $P(\Gamma)$ to be the trivalent diagram obtained from Γ by joining each pair of univalent vertices of the same color. Otherwise, set $P(\Gamma) = 0$. Let

$$\pi: \mathcal{A}_{n-1}(\emptyset) \otimes \mathbb{Q} \longrightarrow \mathcal{A}_{n-1}^c(\emptyset) \otimes \mathbb{Q}$$

denote the projection which maps each connected diagram into itself and maps non-connected diagrams to 0. One can easily check that the composition

$$\mathcal{A}_{n-1}^c(\emptyset) \otimes \mathbb{Q} \xrightarrow{g_n \otimes \mathbb{Q}} \mathcal{A}_{2n}(n+1) \otimes \mathbb{Q} \xrightarrow{\chi_{2n}^{-1}} \mathcal{B}_{2n}^l(n+1) \xrightarrow{P} \mathcal{A}_{n-1}(\emptyset) \otimes \mathbb{Q} \xrightarrow{\pi} \mathcal{A}_{n-1}^c(\emptyset) \otimes \mathbb{Q}$$

is the identity. Hence $g_n \otimes \mathbb{Q}$ is injective. □

3.5 Structure of $\text{Br}(\bar{\mathcal{J}}_{2n}(n+1))$

We need the well-known surjective surgery map

$$\xi_k: \mathcal{A}_k(m) \rightarrow \bar{\mathcal{J}}_k(m)$$

which maps each univalent diagram $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_l$ to the coset of alternating sum $[U; G_1, \dots, G_l]$, where G_1, \dots, G_l are “realizations” of $\Gamma_1, \dots, \Gamma_l$; see Figure 6.

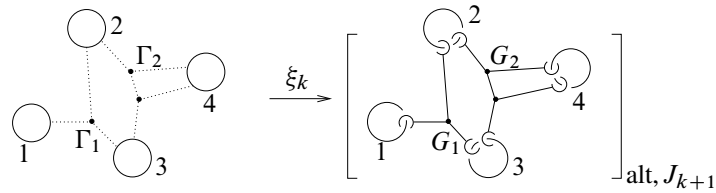


Figure 6: Realization of univalent diagrams and the map ξ_k . Here $[-]_{\text{alt}, J_{k+1}}$ denotes the J_{k+1} -coset of the alternating sum of the form (3-1) defined by the link and graph clasps.

This map coincides with the standard map which “replaces chords with double points” [15, Section 8.2]. $\xi_k \otimes \mathbb{Q}$ is an isomorphism, with inverse given by the Kontsevich integral.

For $n \geq 2$, let h_n denote the composition

$$h_n: \mathcal{A}_{n-1}^c(\emptyset) \xrightarrow{g_n} \mathcal{A}_{2n}(n+1) \xrightarrow{\xi_{2n}} \bar{J}_{2n}(n+1).$$

Theorem 3.2 (1) For $n \geq 3$ we have $h_n(\mathcal{A}_{n-1}^c(\emptyset)) = \text{Br}(\bar{J}_{2n}(n+1))$. For $n = 2$, $h_2(\mathcal{A}_1^c(\emptyset))$ is an index 2 subgroup of $\text{Br}(\bar{J}_4(3))$.

(2) For $n \geq 2$, the \mathbb{Q} -linear map

$$(3-2) \quad h_n \otimes \mathbb{Q}: \mathcal{A}_{n-1}^c(\emptyset) \otimes \mathbb{Q} \rightarrow \text{Br}(\bar{J}_{2n}(n+1)) \otimes \mathbb{Q}$$

is an isomorphism.

Proof (1) Suppose $n \geq 3$. First we show that

$$(3-3) \quad \text{Br}(\bar{J}_{2n}(n+1)) \subset h_n(\mathcal{A}_{n-1}^c(\emptyset)).$$

As outlined in [16, Section 7], $\text{Br}(\bar{J}_{2n}(n+1))$ is \mathbb{Z} -spanned by

$$\begin{aligned} \frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma] & \text{ for } \sigma \in S_{n-1}, \\ [U; T_\sigma, \tilde{T}_{\sigma'}] & \text{ for } \sigma, \sigma' \in S_{n-1}, \end{aligned}$$

where for all σ, σ' in the symmetric group S_{n-1} , T_σ is the simple degree- n tree clasper for the $(n+1)$ -component unlink U depicted in Figure 7, and $\tilde{T}_{\sigma'}$ is obtained from $T_{\sigma'}$ by a small isotopy so that it is disjoint from T_σ . ($\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]$ means an element $x \in \text{Br}(\bar{J}_{2n}(n+1))$ such that $2x = [U; T_\sigma, \tilde{T}_\sigma]$. Existence of such an element is shown in [16].)

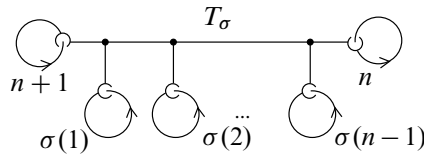
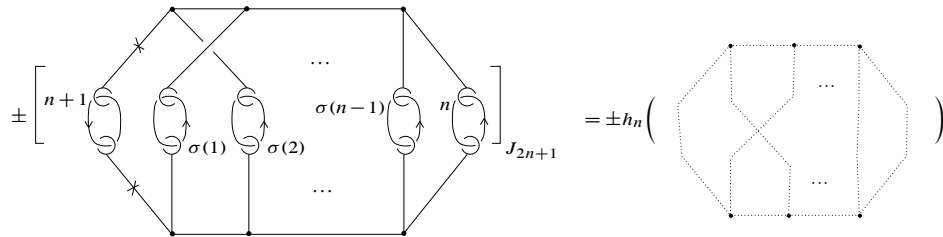
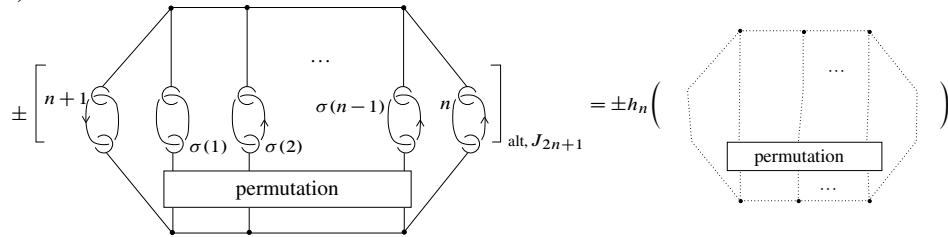


Figure 7: The tree clasper T_σ

For $n \geq 3$, one can check using the AS and IHX relations that $\frac{1}{2}[U; T_\sigma, \tilde{T}_\sigma]_{J_{2n+1}}$ is equal to:



For $n \geq 2$, $[U; T_\sigma, \tilde{T}'_\sigma]_{J_{2n+1}}$ is equal to:
(3-4)



Hence we have (3-3) for $n \geq 3$.

Now we prove

$$h_n(\mathcal{A}_{n-1}^c(\emptyset)) \subset \text{Br}(\bar{J}_{2n}(n+1)).$$

It suffices to prove that every element of $\mathcal{A}_{n-1}^c(\emptyset)$ is a \mathbb{Z} -linear combination of elements of the form depicted in the right hand side of (3-4). This follows from the fact that any chord diagram on a circle is, up to $4T$ relations [1], a linear combination of chord diagrams of this form (where the dotted circle is replaced by a solid one). Chmutov and Duzhin proved a quite similar statement [6, Theorem 2.2], where they also assume the FI relation (see Figure 1). For trivalent diagrams, a $4T$ relation can be easily derived from the IHX relation, but we do not have the FI relation. It is easy to avoid the latter in the proof of Chmutov–Duzhin (it is only used in one place [6, Section 3.4.2.2]) to prove the above-mentioned result on chord diagrams.

The case $n = 2$ follows, since $\text{Br}(\bar{J}_4(3))$ is \mathbb{Z} -spanned by $[U; T_1]_{J_5} = \frac{1}{2}[U; T_1, \tilde{T}_1]_{J_5}$, and $h_2(\mathcal{A}_1^c(\emptyset))$ is \mathbb{Z} -spanned by $[U; T_1, \tilde{T}_1]_{J_5}$. (Here ‘1’ in ‘ T_1 ’ denotes the unit in the (trivial) symmetric group S_1 of order 1.)

(2) Since ξ_k is an isomorphism over \mathbb{Q} , the assertion follows immediately from (1) above and [Proposition 3.1](#). \square

Corollary 3.3 *Let f be a \mathbb{Z} -valued invariant of degree $2n$ for $(n+1)$ -component links in S^3 . Then $f(L)$ restricted to Brunnian links is invariant under permutation of components and under orientation reversal of any component.*

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