Gromov's macroscopic dimension conjecture

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In this note we construct a closed 4–manifold having torsion-free fundamental group and whose universal covering is of macroscopic dimension 3. This yields a counterexample to Gromov's conjecture about the falling of macroscopic dimension.

57R19; 57R20

1 Introduction

The following definition was given by M Gromov [2]:

Definition 1.1 Let V be a metric space. We say that $\dim_\varepsilon V \leq k$ if there is a k–dimensional polyhedron P and a proper uniformly cobounded map $\phi\colon V \to P$ such that $\mathrm{Diam}(\phi^{-1}(p)) \leq \varepsilon$ for all $p \in P$. A metric space V has macroscopic $\dim_{\mathrm{mc}} V \leq k$ if $\dim_\varepsilon V \leq k$ for some possibly large $\varepsilon < \infty$. If k is minimal, we say that $\dim_{\mathrm{mc}} V = k$.

Gromov also stated the following questions which, for convenience, we state in the form of conjectures:

C1 *Let (M^n, g) be a closed Riemannian n–manifold with torsion-free fundamental group, and let $(\widetilde{M}^n, \widetilde{g})$ be the universal covering of M^n with the pullback metric. Suppose that $\dim_{\mathrm{mc}}(\widetilde{M}^n, \widetilde{g}) < n$. Then $\dim_{\mathrm{mc}}(\widetilde{M}^n, \widetilde{g}) < n-1$.*

In [1] we proved C1 for the case $n = 3$.

Evidently, the following conjecture would imply C1 (see also (C) of Section 2):

C2 *Let M^n be a closed n–manifold with torsion-free fundamental group π and let $f\colon M^n \to B\pi$ be a classifying map to the classifying space $B\pi$. Suppose that f is homotopic to a mapping into the $(n-1)$–skeleton of $B\pi$. Then f is in fact homotopic to a mapping into the $(n-2)$–skeleton of $B\pi$.*

In this note we show that both conjectures fail for $n \geq 4$.

We always assume that universal covering are equipped with the pullback metrics.

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Framed cobordism, Pontryagin manifolds and classification of mappings to the sphere

Let M be a smooth compact manifold possibly with a boundary and let (N, v) and (N', w) be closed n–submanifolds in the interior of M with trivial normal bundles and framings v and w, respectively.

Definition 1.2 Two framed submanifolds (N, v) and (N', w) are *framed cobordant* if there exists a cobordism $X \subset M \times [0,1]$ between N and N' and a framing u of X such that

$$\begin{aligned} u(x,t) &= (v(x), 0) \qquad \text{for } (x,t) \in N \times [0, \varepsilon), \\ u(x,t) &= (w(x), 1) \qquad \text{for } (x,t) \in N' \times (1-\varepsilon, 1]. \end{aligned}$$

Remark 1.3 If $(N', w) = \varnothing$ we say (N, v) is *framed cobordant to zero*.

Now let $f\colon M \to S^p$ be a smooth mapping and $y \in S^p$ be a regular value of f. Then f induces the following framing of the submanifold $f^{-1}(y) \subset M$. Choose a positively oriented basis $v = (v^1 \dots, v^p)$ for the tangent space $T(S^p)_y$. Notice that for each $x \in f^{-1}(y)$ the differential $df_x\colon TM_x \to T(S^p)_y$ vanishes on the subspace $Tf^{-1}(y)_x$ and isomorphically maps its orthogonal complement $Tf^{-1}(y)_x^{\perp}$ onto $T(S^p)_y$. Hence there exists a unique vector

$$w^i \in Tf^{-1}(y)_x^{\perp} \subset TM_x$$

which is mapped by df_x to v^i. So we have an induced framing $w = f^* v$ of $f^{-1}(y)$.

Definition 1.4 This framed manifold $(f^{-1}(y), f^* v)$ will be called the *Pontryagin manifold* associated with f.

Theorem 1.5 (Milnor [3]) *If y' is another regular value of f and v' is a positively oriented basis for $T(S^p)_{y'}$, then the framed manifold $(f^{-1}(y'), f^* v')$ is framed cobordant to $(f^{-1}(y), f^* v)$.*

Theorem 1.6 (Milnor [3]) *Two mappings from $(M, \partial M)$ to (S^p, s_0) are smoothly homotopic if and only if the associated Pontryagin manifolds are framed cobordant.*

2 The construction of an example

Consider a circle bundle $S^3 \times S^1 \to S^2 \times S^1$ obtained by multiplying the Hopf circle bundle $S^3 \to S^2$ by S^1. Take also the trivial circle bundle $T^4 = S^1 \times T^3 \to T^3$ and produce a connected sum

$$M^4 = S^3 \times S^1 \#_{S^1} T^4$$

of these circle bundles along small tubes consisting of the circle fibers equipped with natural trivialization. Clearly

(A) M^4 is the total space of the circle bundle

$$p\colon M^4 \to M^3 = S^2 \times S^1 \# T^3;$$

(B) $\pi_1(M^4) = \pi_1(M^3)$. Denote this group by π;

(C) $B\pi = S^1 \vee T^3$ and $\dim_{\mathrm{mc}} M^4 \le 3$. Indeed, the classifying map $f\colon M^4 \to B\pi$ can be lifted to the proper cobounded (by $\mathrm{Diam}(M^4)$) map $\widetilde{f}\colon \widetilde{M}^4 \to \widetilde{B\pi}$ of the universal coverings;

(D) the classifying map $f\colon M^4 \to B\pi$ can be defined as the composition

$$M^4 \xrightarrow{p} S^2 \times S^1 \# T^3 \xrightarrow{f_1} S^2 \times S^1 \vee T^3 \xrightarrow{f_2} S^1 \vee T^3,$$

where f_1 is a quotient map which maps a separating sphere S^2 to a point, and f_2 is the mapping which coincides with the projection onto the generating circle of $S^2 \times S^1$ and is the identity on T^3–component.

Let $g\colon S^1 \vee T^3 \to S^3$ be a degree one map which maps S^1 to a point. Then the following composition $J = g \circ f_2 \circ f_1\colon M^3 \to S^3$ also has degree one.

Theorem 2.1 *The mapping $f\colon M^4 \to B\pi$ is not homotopic into the 2–skeleton of $B\pi$.*

Proof Let $\pi\colon E \to M^3$ be a two-dimensional vector bundle associated with the circle bundle $p\colon M^4 \to M^3$. Let E_0 denote E without zero section $s\colon M^3 \hookrightarrow E$ and $j\colon M^4 \hookrightarrow E_0$ be a unit circle subbundle of E.

The following diagram is homotopically commutative:

$$\begin{array}{ccc} M^4 & \overset{j}{\hookrightarrow} & E_0 \\ \downarrow p & & \downarrow \text{embedding} \\ M^3 & \overset{s}{\hookrightarrow} & E \end{array}$$

Obviously, j and s are homotopy equivalences.

Recall that we have the Thom isomorphism (see Milnor and Stasheff [4])

$$\Phi\colon H^k(M^3;\Lambda) \to H^{k+2}(E, E_0;\Lambda)$$

defined by

$$\Phi(x) = (\pi^* x) \cup u,$$

where Λ is a ring with unity, and u denotes the Thom class.

The Thom class u has the following properties [4]:

(a) If e is the Euler class of E then we have the Thom–Wu formula

$$\Phi(e) = u \cup u.$$

(b) $s^*(u) = e$.

Let

$$M_p = M^4 \times I/(x \times 1 \sim p(x))$$

be the cylinder of the map $p\colon M^4 \to M^3$. Then we have natural embeddings

$$i_1\colon M^4 \to M^4 \times 0 \subset M_p \qquad \text{and} \qquad i_2\colon M^3 \to M^3 \times 1 \subset M_p$$

and a natural retraction $r\colon M_p \to M_3$. It is easy to see that M_p is just a D^2–bundle associated to the circle bundle $p\colon M^4 \to M^3$ and $r|_{M^4} = p$.

Recall that the Thom space $(T(E), \infty)$ is the one point compactification of E. Denote $T(E)$ by T. Clearly, T is homeomorphic to the quotient space M_p/M^4 and

$$H^*(T, \infty; \Lambda) \cong H^*(E, E_0; \Lambda) \tag{1}$$

is a ring isomorphism (see Milnor and Stasheff [4] for more details).

If $g \circ f\colon M^4 \to S^3$ is nullhomotopic then we can extend the map $J\colon M_3 \times 1 \to S^3$ to a mapping $G\colon T \to S^3$. This means that the composition

$$M^3 \xrightarrow{i_2} M_p \xrightarrow{\text{quotient}} T \xrightarrow{G} S^3$$

has degree 1 and $G^*\colon H^3(S^3, s_0; \Lambda) \to H^3(T, \infty; \Lambda)$ is nontrivial.

Let $a \in H^*(E, E_0; \Lambda)$ denote a class corresponding to the class $G^*(\overline{s})$ by isomorphism (1), where $\overline{s}$ is a generator of $H^3(S^3, \Lambda)$.

Let us consider the following exact sequence of pair :

$$H^3(E, E_0; \Lambda) \overset{\xi}{\to} H^3(E; \Lambda) \overset{\psi}{\to} H^3(E_0; \Lambda)$$

Since E is homotopy equivalent to M^3, we have $H^i(E; \Lambda) = H^i(M^3; \Lambda)$. Clearly $s^*\xi(a) = J^*(\overline{s})$. (Note that $J^*(\overline{s})$ is a generator of $H^3(M^3; \Lambda)$).

Let us note that e mod 2 is equal to the Stiefel–Whitney class w_2 which is nonzero. Indeed, the restriction of E onto the embedded sphere $i\colon S^2 \subset M^3$ is the vector bundle associated with the Hopf circle bundle, and so $i^* w_2 \neq 0$. By the Thom construction above there exists a class $z \in H^1(M^3;\mathbb{Z}_2)$ such that $\Phi(z) = a$. Thus

$$s^*\xi(a) = z \cup w_2 = \{\text{generator of } H^3(M^3;\mathbb{Z}_2)\}.$$

Recall the basic properties of Steenrod squares [6; 4]:

(1) For each n, i and $Y \subset X$ there exists an additive homomorphism

$$\mathrm{Sq}^i\colon H^n(X,Y;\mathbb{Z}_2) \to H^{n+i}(X,Y;\mathbb{Z}_2).$$

(2) If $f\colon (X,Y) \to (X',Y')$ is a continuous map of pairs, then

$$\mathrm{Sq}^i \circ f^* = f^* \circ \mathrm{Sq}^i .$$

(3) If $a \in H^n(X,Y;\mathbb{Z}_2)$, then $\mathrm{Sq}^0(a) = a$, $\mathrm{Sq}^n(a) = a \cup a$ and $\mathrm{Sq}^i(a) = 0$ for $i > n$.

(4) We have Cartan's formula:

$$\mathrm{Sq}^k(a \cup b) = \sum_{i+j=k} \mathrm{Sq}^i(a) \cup \mathrm{Sq}^j(b).$$

(5) $\mathrm{Sq}^1 = w_1 \cup\colon H^{m-1}(M;\mathbb{Z}_2) \to H^m(M;\mathbb{Z}_2)$, where M is a closed smooth manifold and w_1 is the first Stiefel–Whitney class of the tangent bundle TM. This follows from the coincidence of the class w_1 with the first Wu class v_1 [4]. It is well known that $w_1 = 0$ if M is an orientable manifold.

Let us show that $\mathrm{Sq}^2(\Phi(z)) \neq 0$. Using the properties above, it is easy to see that $\mathrm{Sq}^1(z) = \mathrm{Sq}^2(z) = 0$. Using the Thom–Wu formula (a), we have

$$\begin{aligned}\mathrm{Sq}^2(\Phi(z)) &= \pi_* z \cup \mathrm{Sq}^2(u)\\ &= \pi_* z \cup u \cup u\\ &= \pi_* z \cup \Phi(w_2) = \Phi(z \cup w_2) \neq 0.\end{aligned}$$

Whence $0 = G^*(\mathrm{Sq}^2(\overline{s})) = \mathrm{Sq}^2(G^*(\overline{s})) \neq 0$. This contradiction implies that the composition $g \circ f\colon M^4 \to S^3$ is not homotopic to zero and $f\colon M^4 \to B\pi$ can not be deformed into the 2–skeleton of $B\pi$. □

Corollary 2.2 *The Pontryagin manifold $(p^{-1}(m), p^*(w))$ is not cobordant to zero, where (m,w) is any framed point of M^3.*

Proof Indeed, from Theorem 2.1 and Theorem 1.6 it follows that if $s \in S^3$ is a regular point of $g \circ f\colon M^4 \to S^3$, then the Pontryagin manifold $(f^{-1}(g^{-1}(s)), f^*(g^*(v))$ is not cobordant to zero, where v is a framing at s. Thus the Pontryagin manifold

$(p^{-1}(m), p^*(w))$ for $(m, w) = (J^{-1}(s), (J^*(v))$ is also not cobordant to zero. Now the statement follows from Theorem 1.5 and regularity of the map $p\colon M^4 \to M^3$. $\square$

3 The main theorem

Definition 3.1 A metric space is called uniformly contractible (UC) if there exists an increasing function $Q\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that each ball of radius r contracts to a point inside a ball of radius $Q(r)$.

It is well known that the universal covering of a compact $K(\tau, 1)$ space is UC (see Gromov [2] for more details).

Denote by ρ the distance function on $\widetilde{B\pi}$.

Lemma 3.2 *Let $\widetilde{f}\colon \widetilde{M}^4 \to \widetilde{B\pi}$ be a lifting of a classifying map to the universal coverings. If $\dim_{\mathrm{mc}} \widetilde{M}^4 \le 2$, then there exists a short homotopy $\widetilde{F}\colon \widetilde{M}^4 \times I \to \widetilde{B\pi}$ of $\widetilde{f}$ such that $\widetilde{F}(x,0) = \widetilde{f}(x)$ and $\widetilde{F}(x,1)$ is a through mapping*

$$\widetilde{F}(x,1)\colon \widetilde{M}^4 \to P^2 \to \widetilde{B\pi},$$

where P^2 is a 2–dimensional polyhedron and "short homotopy" means that we have $\rho(\widetilde{f}(x), \widetilde{F}(x,t)) \le \mathrm{const}$ for each $x \in \widetilde{M}^4$, $t \in I$.

Proof Let $h\colon \widetilde{M}^4 \to P$ be a proper cobounded continuous map to some 2–dimensional polyhedron P. Using a simplicial approximation of h, we can suppose that h is a simplicial map between such triangulations of $\widetilde{M}^4$ and P, that the preimage of the star of each vertex is uniformly bounded (recall that h is proper). Since $\widetilde{f}$ is a quasi-isometry, the $\widetilde{f}$–image $\widetilde{f}(h^{-1}(St(v)))$ of the preimage of the star of each vertex $v \in P$ is bounded by some constant d. Let M_h be the cylinder of h with natural triangulation consisting of the triangulations of $\widetilde{M}^4$ and P and the triangulations of the simplices $\{v_0, \dots, v_k, h(v_k), \dots, h(v_p)\}$, where $\{v_0, \dots, v_p\}$ is a simplex in $\widetilde{M}^4$ with $v_0 < v_1 < \dots, < v_p$ [5].

Consider the map $\widetilde{f}_0\colon (M_h)^0 \to \widetilde{B\pi}$ from 0–skeleton $(M_h)^0$ of M_h which coincides with $\widetilde{f}$ on the lower base of $(M_h)^0$ and with the composition $\widetilde{f} \circ t_0$ on the upper base of $(M_h)^0$, where $t_0\colon (P)^0 \to \widetilde{M}^4$ is a section of h defined on the 0–skeleton $(P)^0$ of P. Since $\widetilde{B\pi}$ is uniformly contractible, we can extend $\widetilde{f}_0$ to M_h using the function Q of the definition of UC-spaces as follows:

By the construction above, $\widetilde{f}_0$–image of every two neighbouring vertexes of M_h lies into a ball of radius d. Therefore we can extend the map $\widetilde{f}_0$ to a mapping

$\widetilde{f}_1\colon (M_h)^1 \to \widetilde{B\pi}$ such that $\rho(\widetilde{f}(x), \widetilde{f}_1(x,t)) \le d$, $x \in (\widetilde{M}^4)^0$. The $\widetilde{f}_1$–image of the boundary of arbitrary 2–simplex of M_h lies into a ball of radius $3d$. So we can extend $\widetilde{f}_1$ to a mapping $\widetilde{f}_2\colon (M_h)^2 \to \widetilde{B\pi}$ so that $\rho(\widetilde{f}(x), \widetilde{f}_2(x,t)) \le 4Q(3d)$, $x \in (\widetilde{M}^4)^1$. Similarly, continue $\widetilde{f}_2$ to mappings $\widetilde{f}_3, \dots, \widetilde{f}_5$ defined on skeletons $(M_h)^3, \dots, (M_h)^5 = M_h$ respectively, so that $\rho(\widetilde{f}(x), \widetilde{f}_5(x,t)) \le c$, where c is a constant. □

Main Theorem $\dim_{\mathrm{mc}} \widetilde{M}^4 = 3$.

Proof Let $q\colon \widetilde{B\pi} \to \widetilde{B\pi}/(\widetilde{B\pi} \setminus D^3) \cong S^3$ be a quotient map, where D^3 is an embedded open 3–dimensional ball.

Suppose that $\dim_{\mathrm{mc}} \widetilde{M}^4 \le 2$ and let $h\colon \widetilde{M}^4 \to P$ be a proper cobounded continuous map to some 2–dimensional polyhedron P as in Lemma 3.2. It is not difficult to find a compact smooth submanifold with boundary $W \subset \widetilde{M}^4$ such that W contains a ball of arbitrary fixed radius r. Since $\widetilde{f}$ is a quasi-isometry, using Lemma 3.2 we can choose r big enough such that $\bar{D}^3 \subset \widetilde{f}(W)$ and $\widetilde{F}(\partial W \times I) \cap \bar{D}^3 = \varnothing$, where $\widetilde{F}$ denotes the short homotopy from Lemma 3.2. Thus we have a homotopy

$$q \circ \widetilde{F}\colon (W, \partial W) \times I \to (S^3, s_0)$$

which maps $\partial W \times I$ into the base point s_0. Since $\dim P = 2$, from Lemma 3.2 it follows that $q \circ \widetilde{F}(x,1)$ is homotopic to zero. Therefore $q \circ \widetilde{F}(x,0) = q \circ \widetilde{f}$ is homotopic to zero (and $q \circ \widetilde{f}$ is smoothly homotopic to zero [3]). Let (s, v) be a framed regular point in S^3 for the map $q \circ \widetilde{f}$. Then the Pontryagin manifold

$$(\widetilde{f}^{-1} \circ q^{-1}(s), \widetilde{f}^* q^*(v))$$

must be cobordant to zero (see Theorem 1.6). Let $(\widetilde{\Omega}, w)$ be a framed nullcobordism which is embedded in $W \times I$ with the boundary $(\widetilde{f}^{-1} \circ q^{-1}(s), \widetilde{f}^* q^*(v))$.

Consider the covering map $\tau\colon \widetilde{M}^4 \times I \to M^4 \times I$. Then $\tau(\widetilde{f}^{-1} \circ q^{-1}(s), \widetilde{f}^* q^*(v))$ is an embedded framed submanifold of M^4 which coincides with the Pontryagin manifold $(p^{-1}(m), p^*(\nu))$ of some framed point $(m, \nu) \in M^3$. And $\tau(\widetilde{\Omega}, w)$ is an immersed framed submanifold of $M^4 \times I$. Using the Whitney Embedding Theorem [7], we can make a small perturbation of $\tau(\widetilde{\Omega}, w)$ identically on the small collar of the boundary to obtain a framed nullcobordism with the boundary $\tau(\widetilde{f}^{-1} \circ q^{-1}(s), \widetilde{f}^* q^*(v))$. But this is impossible by Corollary 2.2. □

Remark 3.3 By similar arguments one can prove that

$$\dim_{\mathrm{mc}}(\widetilde{M^4 \times T^p}) = p + 3.$$

Question Does $M^4 \times T^p$ admit a PSC–metric for some p?

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