From continua to $\mathbb{R}$–trees

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We show how to associate an $\mathbb{R}$–tree to the set of cut points of a continuum. If X is a continuum without cut points we show how to associate an $\mathbb{R}$–tree to the set of cut pairs of X.

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1 Introduction

The study of the structure of cut points of continua has a long history. Whyburn [10] in 1928 showed that the set of cut points of a Peano continuum has the structure of a "dendrite". This "dendritic" decomposition of continua has been extended and used to prove several results in continua theory.

We recall here that a continuum is a compact, connected metric space and a Peano continuum is a locally connected continuum. If X is a continuum we say that a point c is a cut point of X if $X-\{c\}$ is not connected.

Continua theory became relevant for group theory after the introduction of hyperbolic groups by Gromov [4]. The Cayley graph of a hyperbolic group G can be "compactified" and if G is one-ended its Gromov boundary ∂G is a continuum. Moreover the group G acts on ∂G as a convergence group. It turns out that algebraic properties of G are reflected in topological properties of ∂G. A fundamental contribution to the understanding of the relationship between ∂G and algebraic properties of G was made by Bowditch. In [1] Bowditch shows how to pass from the action of a hyperbolic group G on its boundary ∂G to an action on an $\mathbb{R}$–tree. The construction of the tree (under the hypothesis of the G–action) from the continuum is similar to the dendritic decomposition of Whyburn. The difficulty here comes from the fact that the continuum is not assumed to be locally connected.

The second author in [8] (see also [9]) explained how to associate to any continuum a "regular big tree" T, and conjectured that T is in fact an $\mathbb{R}$–tree. It is this conjecture that we prove in the first part of this paper.

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Let X be a continuum without cut points. If $a, b \in X$ we say that a, b is a *cut pair* if $X - \{a, b\}$ is not connected. In the second part of this paper we show how to associate an $\mathbb{R}$–tree to the set of cut pairs of X (compare [3]). We call this tree a JSJ-tree motivated by the fact that if G is a one-ended hyperbolic group then the tree associated to ∂G by this construction is the tree of the JSJ-decomposition of G (in this case one obtains in fact a simplicial tree). Continua appear in group theory also as boundaries of CAT(0) groups. In [7] we use the construction of $\mathbb{R}$–trees from cut pairs presented here to extend Bowditch's results on splittings [1] to CAT(0) groups. We show in particular that if G is a one-ended CAT(0) group such that ∂G has a cut pair then G splits over a virtually cyclic group.

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2 Preliminaries

Definition (Pretrees) Let $\mathcal{P}$ be a set with a betweenness relation. If y is between x and z we write xyz. $\mathcal{P}$ is called a pretree if the following hold:

1. There is no y such that xyx for any $x \in \mathcal{P}$.

2. $xzy \Leftrightarrow yzx$.

3. For all x, y, z, if y is between x and z then z is not between x and y.

4. If xzy and $z \neq w$ then either xzw or yzw.

Definition We say that a pretree $\mathcal{P}$ is discrete if for any $x, y \in \mathcal{P}$ there are finitely many $z \in \mathcal{P}$ such that xzy.

Definition A compact connected metric space is called a continuum.

Definition Let X be a topological space. We say that a set C *separates* the nonempty sets $A, B \subset X$ if there are disjoint open sets U, V of $X - C$, such that $A \subset U$, $B \subset V$ and $U \cup V = X - C$. We say C separates the points $a, b \in X$ if C separates $\{a\}$ and $\{b\}$. We say that C separates X if C separates two points of X. If $C = \{c\}$ then we call c a cut point. If $C = \{c, d\}$ where $c \neq d$ and *neither* c nor d is cut point, then we call $\{c, d\}$ a (unordered) cut pair.

The proof of the following Lemma is an elementary exercise in topology and will be left to the reader.

Lemma 1 *Let A be a connected subset of the space X and B closed in X. If $A \cap \mathrm{Int} B \neq \varnothing$, then either $A \subset B$ or $A \cap \partial B$ separates the subspace A.*

Lemma 2 *Let X be a continuum and $C \subset X$ be minimal with the property that $X - C$ is not connected. The set C separates $A \subset X - C$ from $B \subset X - C$ if and only if there exist continua Y, Z such that $A \subset Y$, $B \subset Z$, $Y \cup Z = X$ and $Y \cap Z = C$.*

Proof We first show that C is closed in X. There are disjoint nonempty subsets D and E open in $X - C$ with $D \cup E = X - C$. By symmetry, it suffices to show that D is open in X. Suppose that $d \in D \cap \partial D$. There is a neighborhood G of d in X such that $\bar{G} \cap \bar{E} = \varnothing$. Since $G \not\subset D$, there is $c \in C \cap G$. Notice that $D \cup \{c\}$ and E are disjoint open subsets of $X - (C - \{c\})$ with $(D \cup \{c\}) \cup E = X - (C - \{c\})$, and C is not minimal. Therefore C must be closed.

Suppose now that C separates A from B. Thus there exist disjoint nonempty U and V open in $X - C$ (this implies open in X) such that $A \subset U$, $B \subset V$, $U \cup V = X - C$. Since ∂U separates X , by the minimality of C, $\partial U = C = \partial V$. Suppose the closure $\bar{U}$ is not connected. Then $\bar{U} = P \cup Q$ where P and Q are disjoint nonempty clopen (closed and open) subsets of $\bar{U}$. Since $\bar{U}$ is closed, this implies that P and Q are closed subsets of X. Since $\bar{U} \not\subset C$, we may assume that $P \not\subset C$. The boundary of P in $\bar{U}$ is empty, so $\partial P \subset \partial \bar{U} = \partial U = C$. Again by minimality $\partial P = C$. Since P is closed in X, $C \subset P$. Thus $Q \subset U$, and Q is open in U since it is open $\bar{U}$. Thus Q is clopen in X which contradicts X being connected.

The implication in the other direction is trivial. □

This next result is just an application of the previous result.

Lemma 3 *Let X be a continuum and $A, B \subset X$.*

- *The point $c \in X$ is a cut point of X which separates A from B if and only if there exist continua $Y, Z \subset X$ such that $A \subset Y - \{c\}$, $B \subset Z - \{c\}$, $Y \cup Z = X$ and $Y \cap Z = \{c\}$.*
- *The pair of non-cut points $\{c, d\}$ is a cut pair separating A from B if and only if there exist continua $Y, Z \subset X$ such that $A \subset Y - \{c, d\}$, $B \subset Z - \{c, d\}$, $Y \cup Z = X$, $Y \cap Z = \{c, d\}$.*

3 Cutpoint trees

Let X be a metric continuum. In this section we show that the big tree constructed in [9] is always a real tree. For the reader's convenience we recall briefly the construction here.

For the remainder of this section, X will be a continuum.

Definition If $a, b \in X$ we say that $c \in (a, b)$ if the cut point c separates a from b.

We call (a, b) an interval and this relation an interval relation. We define closed and half open intervals in the obvious way, ie $[a, b) = \{a\} \cup (a, b)$, $[a, b] = \{a, b\} \cup (a, b)$ for $a \neq b$ and $[a, a) = \varnothing$, $[a, a] = \{a\}$.

Definition We define an equivalence relation on X. Each cut point is equivalent only to itself and if $a, b \in X$ are not cut points we say that a is equivalent to b if $(a, b) = \varnothing$.

Let's denote by $\mathcal{P}$ the set of equivalence classes for this relation. We can define an interval relation on $\mathcal{P}$ as follows:

Definition If $x, y \in \mathcal{P}$ and c is a cut point (so $c \in \mathcal{P}$) we say that $c \in (x, y)$ if for some (any) $a \in x$, $b \in y$, we have $c \in (a, b)$.

For $z \in \mathcal{P}$, z not a cut point we say that $z \in (x, y)$ if for some (any) $a \in x, b \in y, c \in z$ we have that

$$[a, c) \cap (c, b] = \varnothing.$$

If $x, y, z \in \mathcal{P}$ we say that z is between x, y if $z \in (x, y)$. We will show that $\mathcal{P}$ with this betweenness relation is a pretree. The first two axioms of the definition of pretree are satisfied by definition.

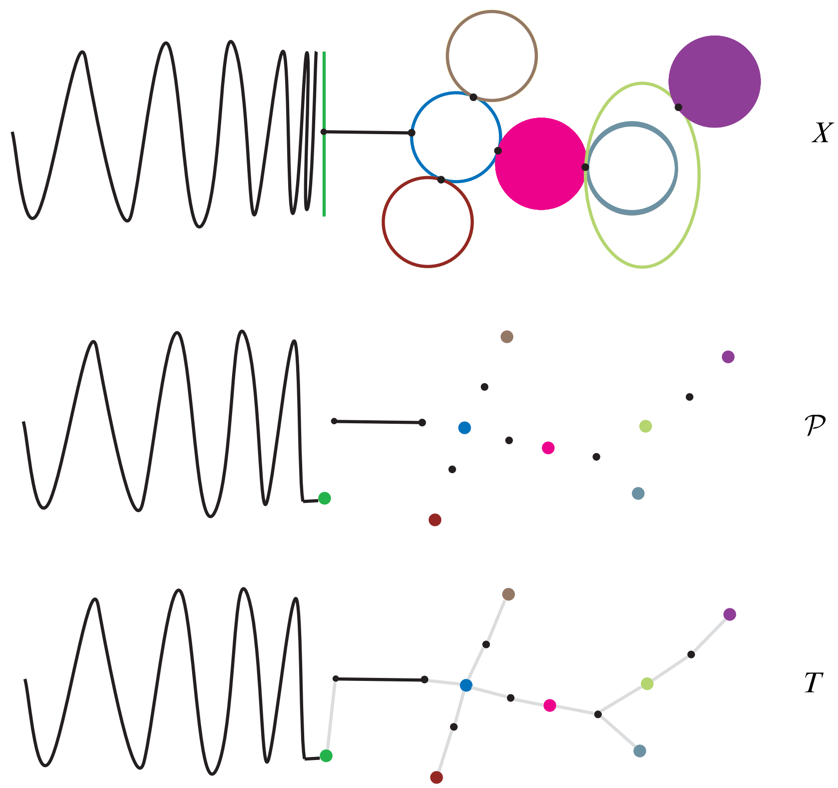
For the remaining two axioms we recall the following lemmas (for a proof see Bowditch [2] or Swenson [8]).

Lemma 4 *For any $x, y \in \mathcal{P}$, if $z \in (x, y)$ then $x \notin (y, z)$.*

Lemma 5 *For any $x, y, z \in P$, $(x, z) \subset (x, y] \cup [y, z)$.*

Axiom 3 follows from Lemma 4 and Axiom 4 from Lemma 5.

Now consider the following example where $X \subset \mathbb{R}^2$ is the union of a Topologist's sine curve, two arcs, five circles and two disks:



The tree T is obtained from $\mathcal{P}$ by "connecting the dots" according to the pretree relation on $\mathcal{P}$. We will give the rigorous definition of T later.

We have the following results about intervals in pretrees from [2]:

Lemma 6 *If $x, y, z \in \mathcal{P}$, with $y \in [x, z]$ then $[x, y] \subset [x, z]$.*

Lemma 7 *Let $[x, y]$ be an interval of $\mathcal{P}$. The interval structure induces two linear orderings on $[x, y]$, one being the opposite of the other, with the property that if $<$ is one of the orderings, then for any z and $w \in [x, y]$ satisfying $z < w$, we have $(z, w) = \{u \in [x, y] : z < u < w\}$. In other words the interval structure defined by one of the orderings is the same as our original interval structure.*

Definition If x and y are distinct points of $\mathcal{P}$ we say that x and y are *adjacent* if $(x, y) = \varnothing$. We say $x \in \mathcal{P}$ is *terminal* if there is no pair $y, z \in \mathcal{P}$ with $x \in (y, z)$.

We recall the following lemma from [8].

Lemma 8 *If $x, y \in \mathcal{P}$, are adjacent then exactly one of them is a cut point and the other is a nonsingleton equivalence class whose closure contains this cut point.*

Corollary 9 *If $p \in \mathcal{P}$ is a singleton equivalence class and p is not a cut point, then p is terminal in $\mathcal{P}$.*

Proof Let $x \in X$ with $[x] = \{x\}$, and $x \in (a,b)$ for some $a,b \in \mathcal{P}$. Suppose that x is not a cut point. By Lemma 8 there is no point of $\mathcal{P}$ adjacent to $[x]$. Thus there are infinitely many cut points in $[a,x]$. For each such cut point $c \in (a,x)$ choose a continuum $A_c \ni a$ with $\partial A_c = \{c\}$. Considered the nested union $A = \bigcup A_c$. We will show that $\partial A = \{x\}$.

First consider $y \in A$. There exists a cut point $c \in (a,x)$, $c \neq y$, with $y \in A_c$. Thus $y \in \mathrm{Int} A_c$ so $y \notin \partial A$.

Now consider $z \in X - A$ with $z \neq x$. Since $z \notin A$, by definition $x \in ([z],[y])$ for any $y \in A$. Since x and $[z]$ are not adjacent there is a cut point $d \in ([z],x)$. There exist continua Z, B such that $Z \cup B = X$, $z \in Z$, $x \in B$ and $Z \cap B = \{d\}$. Since $x \in ([z],[y])$ for any $y \in A$, by definition $A \subset B$, and $z \notin \partial A$.

The fact that $x \in \partial A$ follows since $b \not\subset A$, so $X \neq A$, and so $\partial A \neq \varnothing$. □

We have the theorem [8, Theorem 6]:

Theorem 10 *A nested union of intervals of $\mathcal{P}$ is an interval of $\mathcal{P}$.*

Corollary 11 *Any interval of $\mathcal{P}$ has the supremum property with respect to either of the linear orderings derived from the interval structure.*

Proof Let $[x,y]$ be an interval of $\mathcal{P}$ with the linear order $\leq$. Let $A \subset [x,y]$. The set $\{[x,a] : a \in A\}$ is a set of nested intervals so their union is an interval $[x,s]$ or $[x,s)$ and $s = \sup A$. □

Definition A *big arc* is the homeomorphic image of a compact connected nonsingleton linearly ordered topological space. A separable big arc is called an *arc*. A *big tree* is a uniquely big-arcwise connected topological space. If all the big arcs of a big tree are arcs, then the big tree is called a *real tree*. A metrizable real tree is called an $\mathbb{R}$–tree. An example of a real tree which is not an $\mathbb{R}$–tree is the long line [5, Section 2.5, page 56].

Definition A pretree $\mathcal{R}$ is *complete* if every closed interval is complete as a linearly ordered topological space (this is slightly weaker than the definition given in [2]). Recall that a linearly ordered topological space is complete if every bounded set has a supremum.

Let $\mathcal{R}$ be a pretree. An interval $I \subset \mathcal{R}$ is called *preseparable* if there is a countable set $Q \subset I$ such that for every nonsingleton closed interval $J \subset I$, we have $J \cap Q \neq \varnothing$. A pretree is *preseparable* if every interval in it is preseparable.

We now give a slight generalization of a construction in [8; 9]. Let $\mathcal{R}$ be a complete pretree. Set

$$T = \mathcal{R} \cup \bigsqcup_{x,\, y \text{ adjacent}} I_{x,y}$$

where $I_{x,y}$ is a copy of the real open interval $(0,1)$ glued in between x and y. We extend the interval relation of $\mathcal{R}$ to T in the obvious way (as in [8; 9]), so that in T, $(x,y) = I_{x,y}$. It is clear that T is a complete pretree with no adjacent elements. When $\mathcal{R} = \mathcal{P}$, we call the T so constructed the cut point tree of X.

Definition For A finite subset of T and $s \in T$ we define

$$U(s,A) = \{t \in T : [s,t] \cap A = \varnothing\}.$$

The following is what the proof of [8, Theorem 7] proves in this setting.

Theorem 12 *Let $\mathcal{R}$ be a complete pretree. The pretree T, defined above, with the topology defined by the basis $\{U(s,A)\}$ is a regular big tree. If in addition $\mathcal{R}$ is preseparable, then T is a real tree.*

We now prove the conjecture from [8].

Theorem 13 *The pretree $\mathcal{P}$ is preseparable, so the cut point tree T of X is a real tree.*

Proof By the proof of [8, Theorem 7], it suffices to show that there are only countably many adjacent pairs in a closed interval $[a,b]$ of $\mathcal{P}$. By Lemma 8, adjacent elements of $\mathcal{P}$ are pairs E, c where E is a nonsingleton equivalence class, c is a cut point and $c \in \bar{E} - E$. Let's assume that there are uncountably many such pairs in $[a,b]$. By symmetry we may assume that $E \in (a,c)$ for uncountably many pairs (E,c), and for each such pair we pick an $e \in E$.

Since c separates e from b choose continua A, B such that $X = A \cup B$, $\{c\} = A \cap B$, $e \in A$ and $b \in B$. Since $e \notin B$ and B is compact $d(e, B) > 0$. Let $\epsilon_e = d(e, B)$.

In this way to each pair E, c we associate $e \in E$ continua A, B and $\epsilon_e > 0$. Since there are uncountably many e, for some $n \in N$ there are uncountably many e with $\epsilon_e > 1/n$. Let's denote by S the set of all such e with $\epsilon_e > 1/n$. Consider a finite covering of X by open balls of radius $1/(2n)$. Since S is infinite there are distinct elements $e_1, e_2, e_3 \in S$ lying in the same ball. It follows that $d(e_i, e_j) < 1/n$ for all i, j.

The points e_1, e_2, e_3 correspond to adjacent elements of $\mathcal{P}$, say E_1, c_1, E_2, c_2, E_3, c_3. Since all these lie in an interval of $\mathcal{P}$ they are linearly ordered and we may assume, without loss of generality, that $E_1 \in [a, c_2)$ and $E_3 \in (c_2, b]$. Let A_1 and B_1 be the continua chosen for E_1, c_1 such that $A_1 \cap B_1 = \{c_1\}$, $A_1 \cup B_1 = X$, $e_1 \in A_1$, $b \in B_1$ and $d(e_1, B_1) = \epsilon_{e_1} > 1/n$. It follows that $e_3 \in B_1$ and so $d(e_1, e_3) \geq d(e_1, B_1) > 1/n$, which is a contradiction. □

The real tree T is not always metrizable. Take for example X to be the cone on a Cantor set C (the so-called Cantor fan). Then X has only one cut point, the cone point p, and $\mathcal{P}$ has uncountable many other elements q_c, one for each point $c \in C$. As a pretree, T consists of uncountable many arcs $\{[p, q_c] : c \in C\}$ radiating from a single central point p. However, in the topology defined from the basis $\{U(s, A)\}$, every open set containing p contains the arc $[p, q_c]$ for all but finitely many $c \in C$. There can be no metric, d, giving this topology since $d(p, q_c)$ could only be nonzero for countably many $c \in C$.

It is possible however to equip T with a metric that preserves the pretree structure of T. This metric is "canonical" in the sense that any homeomorphism of X induces a homeomorphism of T. The idea is to metrize T in two steps. In the first step one metrizes the subtree obtained by the span of cut points of $\mathcal{P}$. This can be written as a countable union of intervals and it is easy to equip with a metric.

T is obtained from this tree by gluing intervals to some points of T. In this step one might glue uncountably many intervals but the situation is similar to the Cantor fan described above. The new intervals are metrized in the obvious way, eg one can give all of them length one.

Theorem 14 *There is a path metric d on T, which preserves the pretree structure of T, such that (T, d) is a metric $\mathbb{R}$–tree. The topology so defined on T is canonical (and may be different from the topology with basis $\{U(s, A)\}$). Any homeomorphism ϕ of X induces a homeomorphism $\widehat{\phi}$ of T equipped with this metric.*

Proof Let $\mathcal{C}$ be the set of cut points of X and let S be a countable dense subset of $\mathcal{C}$. Choose a base point $s \in S$. Denote by T' the union of all intervals $[s, s']$ of T with $s' \in S$. Now we remark that at most countably many cut points of X are not contained in T'. Indeed if $c \in \mathcal{C}$ is a cut point not in T' then $X - c = U \cup V$ where U, V are disjoint open sets and one of the two (say U) contains no cut points. Let $\epsilon > 0$ be such that a ball $B(c)$ in X of radius ϵ is contained in U. So we associate to each c not in T' a ball $B(c)$ and we remark that to distinct c's correspond disjoint balls. Clearly there can be at most countably many such disjoint balls in X. Thus by enlarging S we may assume that T' contains all cut points, so T' is the convex hull of $\mathcal{C}$ in T, and so is canonical.

Since S is countable we can write $S = \{s_1, s_2, \dots\}$ and we metrize T' by an inductive procedure: we give $[s, s_1]$ length 1 (Choose $f\colon [0,1] \to [s, s_1]$, a homeomorphism, and define $d(f(a), f(b)) = |a - b|$). $[s, s_2]$ intersects $[s, s_1]$ along a closed interval $[s, a]$. If $[a, s_2]$ is non empty we give it length $1/2$ and we obtain a finite tree. At the n–th step of the procedure we add the interval $[s, s_{n+1}]$ to a finite tree T_n. If $[s, s_{n+1}] \cap T_n = [a, s_{n+1}]$ a nondegenerate interval we glue $[a, s_{n+1}]$ to T_n and give it length $1/2^n$. Note that if $a, b \in T'$ then $a \in [s, s_n], b \in [s, s_k]$ for some $k, n \in \mathbb{N}$. Without loss of generality, $k \leq n$ and so $a, b \in T_n$ and $d(a, b)$ is determined at some finite stage of the above procedure.

We remark that each end of T' corresponds to an element of $\mathcal{P}$ and by adding these points to T' one obtains an $\mathbb{R}$–tree that we still denote by T'. Here by end of T' we mean an ascending union $\bigcup [s, s_i]$ ($i \in \mathbb{N}$, $s_i \in S$) which is not contained in any interval of T'. If C_i is the closure of the union of all components of $X - s_i$ not containing s we have that $C_i \subset C_{i-1}$ for all i and $\bigcap C_i$ is an element of $\mathcal{P}$.

If $x \in \mathcal{P}$ does not lie in T' then x is adjacent to some cut point $c \in T'$. For each such adjacent pair (c, x), by construction (c, x) is a copy of the unit interval $(0, 1)$ and this gives the path metric on $[c, x]$. In this way we equip T with a path metric d.

Clearly a homeomorphism $\phi\colon X \to X$ induces a pretree isomorphism $\widehat{\phi}\colon \mathcal{P} \to \mathcal{P}$. By extending $\widehat{\phi}$ to the intervals corresponding to adjacent points of $\mathcal{P}$ (via the identity map on the unit interval, $(0,1) \to (0,1)$) we get a pretree isomorphism function $\widehat{\phi}\colon T \to T$ which restricts to a pretree isomorphism $\widehat{\phi}\colon T' \to T'$. For any (possibly singleton) arc $\alpha \in T'$ let $\mathcal{B}_\alpha$ be the set of components of $T' - \alpha$. By the construction of d, for any $\epsilon > 0$, the set $\{B \in \mathcal{B}_\alpha : \mathrm{diam}(B) > \epsilon\}$ is finite. It follows that $\widehat{\phi}\colon T' \to T'$ is continuous (using the metric d) and therefore a homeomorphism. We extend $\widehat{\phi}$ to T by defining it to be an isometry on the disjoint union of intervals $T - T'$. Thus we get $\widehat{\phi}\colon T \to T$ a homeomorphism. □

4 JSJ-trees

Definition Let X be a continuum without cut points. A finite set $S \subset X$ with $|S| > 2$ is called *cyclic subset* if there is an ordering $S = \{x_1, \dots x_n\}$ and continua $M_1, \dots M_n$ with the following properties:

- $M_n \cap M_1 = \{x_1\}$, and for $i > 1$, $\{x_i\} = M_{i-1} \cap M_i$
- $M_i \cap M_j = \varnothing$ for $i - j \neq \pm 1 \mod n$
- $\bigcup M_i = X$

The collection $M_1, \dots M_n$ is called the (a) *cyclic decomposition* of X by $\{x_1, \dots x_n\}$. This decomposition is unique as we show in Lemma 15 (for $n > 3$).

We also define a cut pair to be *cyclic*.

Clearly every nonempty nonsingleton subset of a cyclic set is cyclic.

If S is an infinite subset of X and every finite subset $A \subset S$ with $|A| > 1$ is cyclic, then we say S is *cyclic*.

Clearly if A is a subset of a cyclic set with $|A| > 1$, then A is cyclic.

Lemma 15 *Let X be a connected metric space without cut points. If the cut pair a, b separates the cut pair c, d then $\{a, b, c, d\}$ is cyclic, so $\{c, d\}$ separates a from b. Furthermore $X - \{c, d\}$ has exactly two components and $X - \{a, b\}$ has exactly two components.*

Proof By Lemma 3 there exist continua C, D with $C \cap D = \{a, b\}$, $X = C \cup D$, $c \in C$ and $d \in D$. Since c, d is a cut pair there exist continuum A, B such that $A \cup B = X$ and $A \cap B = \{c, d\}$. We may assume that $a \in A$. Since B is connected, with $c, d \in B$ and $a \notin B$, then b must be a cut point of B separating c and d. Similarly a is a cut point of A separating c and d. Thus by Lemma 3 there exist continua $M_{a,c}$, $M_{a,d}$ $M_{b,c}$, $M_{b,d}$ with $c \in M_{a,c}$, $c \in M_{b,c}$, $d \in M_{a,d}$, and $d \in M_{b,d}$, such that $M_{a,c} \cup M_{a,d} = A$, $M_{a,c} \cap M_{a,d} = \{a\}$, $M_{b,c} \cup M_{b,d} = B$ and $M_{b,c} \cap M_{b,d} = \{b\}$. It follows that $M_{a,c} \cap M_{b,c} = \{c\}$ and that $M_{a,d} \cap M_{b,d} = \{d\}$. Thus $\{a, b, c, d\}$ is cyclic.

Suppose that $\{c, d\}$ separated A, then there would be nonsingleton continua F, G with $A = F \cup G$ and $\{c, d\} = F \cap G$. We may assume that $a \in F$. Since a separates c from d in A, either $c \notin G$ or $d \notin G$. With no loss of generality $d \notin G$. Thus $F \cup B$ and G are continua with $X = (F \cup B) \cup G$ and $\{c\} = (F \cup B) \cap G$, making c a cut point of X. This is a contradiction, so $\{c, d\}$ doesn't separate A and similarly $\{c, d\}$ doesn't separate B. Thus $X - \{c, d\}$ has exactly two components. □

Definition Let X be a metric space without cut points. A nondegenerate nonempty set $A \subset X$ is called inseparable if no pair of points in A can be separated by a cut pair.

Every inseparable set is contained in a maximal inseparable set. A maximal inseparable subset is closed (its complement is the union of open subsets).

Example 16 A maximal inseparable set need not be connected, for example let X be the complete graph on the vertex set V with $3 < |V| < \infty$. The set V is a maximal inseparable subset of X. V also has the property that every pair in V is a cut pair of X, but V is not cyclic.

Lemma 17 *Let S be a subset of X with $|S| > 1$. If every pair of points in S is a cut pair and $\{a, b\}$ is a cut pair separating points of S, then $S \cup \{a, b\}$ is cyclic.*

Proof It suffices to prove this when S is finite. Let $c, d \in S$ be separated by $\{a, b\}$. By Lemma 15 , $X - \{c, d\}$ has exactly two components, $X - \{a, b\}$ has exactly two components, and there are continua $M_{a,c}$, $M_{a,d}$ $M_{b,c}$, $M_{b,d}$ whose union is X such that $M_{a,c} \cap M_{a,d} = \{a\}$, $M_{b,c} \cap M_{b,d} = \{b\}$, $M_{a,c} \cap M_{b,c} = \{c\}$, $M_{a,d} \cap M_{b,d} = \{d\}$.

Now let $e \in S - \{a, b, c, d\}$. We may assume $e \in M_{a,c}$. Now $\{a, b\}$ separates the cut pair $\{d, e\}$ and so by Lemma 15, $\{d, e\}$ separates a from b. It follows that e is a cut point of the continuum $M_{a,c}$. Thus there exist continua $M_{a,e} \ni a$ and $M_{e,c} \ni c$ such that $M_{a,e} \cup M_{e,c} = M_{a,c}$ and $M_{a,e} \cap M_{e,c} = \{e\}$. The set $\{a, b, c, d, e\}$ is now known to be cyclic. Continuing this process, we see that $S \cup \{a, b\}$ is cyclic. □

Corollary 18 *If $S \subset X$ with $|S| > 1$ and S has the property that every pair of points in S is a cut pair, then either S is inseparable or S is cyclic.*

Definition By Zorn's Lemma, every cyclic subset of X is contained in a maximal cyclic subset. A maximal cyclic subset with more than two elements is called a *necklace*. In particular, every separable cut pair is contained in a necklace.

Lemma 19 *Let S be a cyclic subset of X, a continuum without cut points. If S separates the point x from y in X, then there is a cut pair in S separating x from y.*

Proof Suppose not, then for any finite subset $\{x_1, \dots x_n\} \subset S$ with $M_1, \dots M_n$ the cyclic decomposition of X by $\{x_1, \dots x_n\}$, x and y are contained in the same element M_i of this cyclic decomposition. There are two cases.

In the first case, we can find a strictly nested intersection of (cyclic decomposition) continua $C \ni x, y$, with the property that $|C \cap S| \leq 1$. Nested intersections of continua are continua, so C connected, and similarly using Lemma 15, $C-(C\cap S)$ is connected, so S doesn't separate x from y.

In the second case there is a cut pair $\{a,b\}\subset S$ and continua M,N such that $x,y\in M$, $N\cup M=X$, $N\cap M=\{a,b\}$ and $S\subset N$. It follows that $\{a,b\}$ separates x from y in M, so there exist continua Y,Z with $M=Y\cup Z$ where $Y\cap Z=\{a,b\}$, $y\in Y$ and $x\in Z$. Since $(N\cup Y)\cap Z=\{a,b\}$ and $(N\cup Y)\cup Z=X$, it follows that $\{a,b\}$ separates x from y in X. □

Definition Let S be a necklace of X. We say $y,z\in X-S$ are S equivalent, denoted $y\sim_S z$, if for every cyclic decomposition $M_1,\dots M_n$ of X by $\{x_1,\dots x_n\}\subset S$, both $y,z\in M_i$ for some $1\le i\le n$. The relation $\sim_S$ is clearly an equivalence relation on $X-S$. By Lemma 19, if y,z are separated by S then $y\not\sim_S z$, but the converse is false.

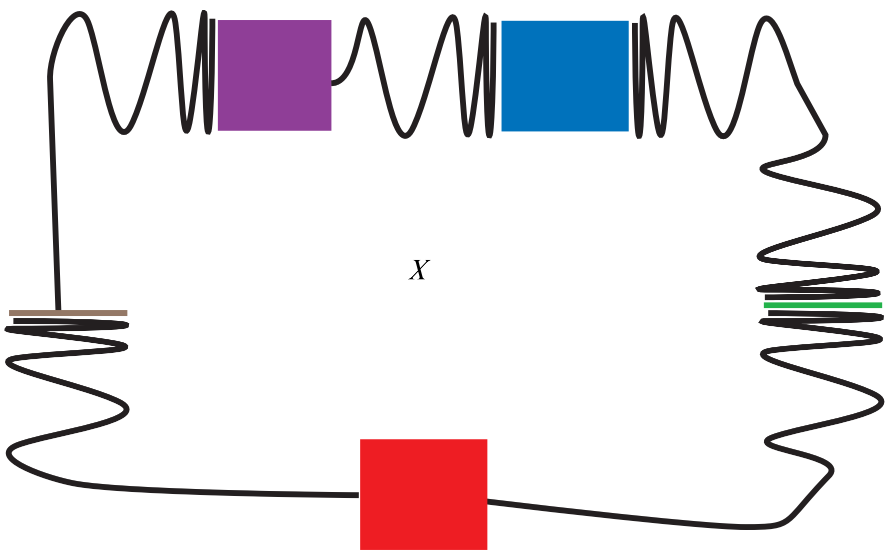
The closure (in X) of a ($\sim_S$)–equivalence class of $X-S$ is called a *gap* of S. Notice that every gap is a nested intersection of continua, and so is a continuum. Every inseparable cut pair in S defines a unique gap. The converse is true if X is locally connected, but false in the nonlocally connected case.

Let $s\in S$. Choose distinct $x,y\in S-\{s\}$, and take the cyclic decomposition M_1,M_2,M_3 of X by $\{s,x,y\}$ with $M_1\cap M_3=\{s\}$. For each i, take a copy $\widehat{M}_i$ of M_i. Let $\widehat{M}$ be the disjoint union of the $\widehat{M}_i$. For $i=1,2,3$ let s_i,y_i,x_i be the points of $\widehat{M}_i$ which correspond to s,x,y respectively whenever they exist (for instance there is no s_2 since $s\notin M_2$). Let $\widehat{X}$ be the quotient space of $\widehat{M}$ under the identification $y_i=y_j$ and $x_i=x_j$ for all i,j. The metrizable continuum $\widehat{X}$ is clearly independent of the choice of x and y. The obvious map $q\colon \widehat{X}\to X$ is one to one except that $\{s_1,s_3\}=q^{-1}(s)$. We will abuse notation and refer to points of $X-\{s\}$ as points of $\widehat{X}-\{s_1,s_3\}$ and vice versa.

The cut points of the continuum $\widehat{X}$ are exactly $S-\{s\}$. Consider the cut point pretree $\mathcal{P}$ for $\widehat{X}$. By Corollary 9, the cut points of $\widehat{X}$ will be exactly the singleton equivalence classes in $\mathcal{P}$ other than $\{s_1\}$ and $\{s_3\}$. The closures of nonsingleton equivalence classes in $\mathcal{P}$ are exactly the gaps of S. *Thus every gap of S has more than one point*. The cut point real tree T is in this case an arc (see Lemma 15), so there is a linear order on $\mathcal{P}$ corresponding to the pretree structure. Let $A\in\mathcal{P}$ be a nonsingleton equivalence class (so $\overline{A}\subset X$ is a gap of S) with $s\notin\overline{A}$. Let $U=\{x\in\widehat{X}:[x]<A\}$ and let $B=q(\overline{U}\cap\overline{A})$. Similarly let $O=\{x\in\widehat{X}:A<[x]\}$ and let $C=q(\overline{O}\cap\overline{A})$. The two closed sets B and C are called the *sides* of the gap $\overline{A}$. Notice that $\partial A=C\cup B$. Since X has no cut points B and C are nonempty.

Definition Let D be a gap of S with sides B and C. If $B \cap C = \varnothing$, then we say D is a *fat* gap of S. Each fat gap is a continuum whose boundary is the disjoint union of its sides. It follows that every fat gap has nonempty interior. Distinct fat gaps of S will have disjoint interiors. Since the compact metric space X is Lindelöf (every collection of nonempty disjoint open sets is countable), S has only countably many fat gaps. If X is locally connected then there are only fat gaps because the sides of a gap form (with local connectivity) an inseparable cut pair.

Consider the following example where X is a continuum in $\mathbb{R}^2$ containing a single necklace S and five gaps of S. The three solid rectangles are fat gaps, and the two thin gaps are limit arcs of Topologist's sine curves.



Lemma 20 *The union of the sides of a gap of* S *is a nonsingleton inseparable set.*

Proof Take A, U, O, B, C, s and $q\colon \hat{X} \to X$ as above. We show that $B \cup C$ is a nonsingleton inseparable set. Suppose that $B \cup C = \{b\}$. Then $\partial A = \{b\}$ and since gaps are not singletons, b is a cut point of X. Thus $B \cup C$ is not a singleton.

Now suppose that $d, e \in B \cup C$ and $\{r, t\}$ is a cut pair separating d and e. Let $E = q(O) \cup q(U)$. Since O and U are nested unions of connected sets they are connected and since $s \in q(O) \cap q(U)$, E is connected. Thus for any $P \subset X$, with $E \subseteq P \subseteq \overline{E}$, P is connected. Since $d, e \in \overline{E}$ it follows that $\{r, t\}$ must separate E. Since the gap $\overline{A} \ni d, e$ is connected it follows that $\{r, t\}$ must separate $\overline{A}$. Notice that

$E \cap \overline{A}$ consists of sides of A which are points of S, so $|E \cap \overline{A}| \leq 2$ and if $|E \cap \overline{A}| = 2$ then $E \cap \overline{A} = \{e, d\}$. It follows that $\{r, t\} \not\subset E\overline{A}$.

First consider the case where one of $\{r, t\}$, say r is in $E \cap \overline{A}$. It follows that $r \in S$ is one of the sides of $\overline{A}$, say $\{r\} = B$. Thus $d, e \in C$, the other side of $\overline{A}$. Since $q(O)$ is connected and its closure contains C, it follows that $t \in q(O) \cap S$. Since B is not a point of S, there are infinitely many elements $u \in S$ such that $\{r, u\}$ separates t from C. Replacing t with such an u, we may assume that r are t are not inseparable and so $X - \{r, t\}$ has exactly two components by Lemma 15. One of these components will contain s and the other will contain $\overline{A} - \{r\}$. Thus $\{r, t\}$ doesn't separate d from e. Contradiction.

Now we have the case where $\{r, t\} \cap (E \cap \overline{A}) = \varnothing$. It follows that one of them (say r) is a cut point of E and the other t is a cut point of $\overline{A}$. Since r is a cut point of E, it follows that $r \in S$, and since r is not a side of $\overline{A}$, with no loss of generality $r = s$. Thus t is a cut point in $\widehat{X}$ and so $t \in S$. But $S \subset E$ and $t \notin E$. Contradiction. □

Corollary 21 *Let X be a continuum without cut points. Suppose that for every pair of points $c, d \in X$ there is a pair of points a, b that separates c, d. Then X is homeomorphic to the circle.*

Proof Let S be a necklace of X. Using Lemma 15, we can show that S is infinite and in fact any two points of S are separated by a cut pair in S. If $X - S \neq \varnothing$ then there is a gap A of S. The union of sides of the gap A is a nonsingleton inseparable subset of X. There are no nonsingleton inseparable subsets of X, so $S = X$. Thus X is homeomorphic to the circle. This follows from [5, Theorem 2-28, page 55] (our Theorem 22 also proves this). □

Theorem 22 *Let S be a necklace of X. There exists a continuous surjective function $f\colon X \to S^1$, with the following properties:*

(1) *The function f is one to one on S.*

(2) *The image of a fat gap of S is a nondegenerate arc of S^1.*

(3) *For $x, y \in X$ and $a, b \in S$:*

  (a) *If $\{f(a), f(b)\}$ separates $f(x)$ from $f(y)$ then $\{a, b\}$ separates x from y.*

  (b) *If $x \in S$ and $\{a, b\} \subset S$ separates x from y, then $\{f(a), f(b)\}$ separates $f(x)$ from $f(y)$.*

The function f is unique up to homotopy and reflection in S^1. In addition, if G is a group acting by homeomorphisms on X which stabilizes S, then the action of G on S extends to an action of G on S^1.

Proof We use the strong Urysohn Lemma [6, 4.4 Exercise 5] If A and B are disjoint closed G_δ subsets of a normal space Y, then there is a continuous $f\colon Y \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. In a metric space, all closed sets are G_δ. Since X has a countable basis, the subspace S has a countable dense subset $\widehat{S}$. Since the fat gaps of S are countable, the collection R of all sides of fat gaps of S is countable. Let $\{s_n : n \in \mathbb{N}\} = \widehat{S} \cup R$. Notice that now some of the elements of $\{s_n : n \in \mathbb{N}\}$ are points (singleton sets) of S and some of them are sides of gaps (and therefore closed sets of X). In particular all inseparable cut pairs of S are in $\{s_n : n \in \mathbb{N}\}$.

For the remainder of this proof, we will maintain the useful fiction that each element of $\{s_n : n \in \mathbb{N}\}$ is a point (which would be true if X were locally connected), and leave it to the reader (with some hints) to check the details for the nonsingleton sides of gaps.

Notice that the elements of $\{s_n : n \in \mathbb{N}\}$ are pairwise disjoint.

We inductively construct the map f. We take as S^1, quotient space of the interval $[0,1]/(0=1)$ with 0 identified with 1. Since $\{s_1, s_2, s_3\}$ is cyclic, there exist cyclic decomposition M_1, M_2, M_3 of X with respect to $\{s_1, s_2, s_3\}$. We define the map $f_3\colon M_1 \to [0, 1/3]$ by $f_3(s_1) = 0$, $f_3(s_2) = 1/3$ and then extend to M_1 using the strong Urysohn Lemma so that $f_3^{-1}(0) = \{s_1\}$ and $f_3^{-1}(1/3) = \{s_2\}$. Similarly we define the continuous map $f_3\colon M_2 \to [1/3, 2/3]$ such that $f_3^{-1}(1/3) = \{s_2\}$ and $f_3^{-1}(2/3) = \{s_3\}$. Lastly we define $f_3\colon M_3 \to [2/3, 1]$ such that $f_3^{-1}(2/3) = \{s_3\}$ and $f_3^{-1}(1) = \{s_1\}$. Since $0 = 1$ we paste to get the function $f_3\colon X \to S^1$.

Now inductively suppose that we have $N_1, \dots N_k$ a cyclic decomposition of X with respect to $\{s_i : i \le k\}$ (when the s_i are sides of gaps the definition of cyclic decomposition will be similar), and a map $f_k\colon X \to S^1$ such that $f_k(N_j) = [f_k(s_p), f_k(s_q)]$ for each $1 \le j \le k$, where $\partial N_j = \{s_p, s_q\}$ and $q, p \le k$, satisfying $f_k^{-1}(f(s_j)) = \{s_j\}$ for all $j \le k$. If $s_{k+1} \in N_j$ with $\partial N_j = \{s_p, s_q\}$ then there exists continua A, B such that $A \cup B = N_j$, $A \cap B = \{s_{k+1}\}$, $s_p \in A$ and $s_q \in B$ (in the case where s_{k+1} is the side of a gap, then one of A, B will be a nested union of continua, and the other will be a nested intersection). Using the strong Urysohn Lemma, we define $f_{k+1}\colon N_j \to [f_k(s_p), f_k(s_q)]$ such that $f_{k+1}^{-1}(f_k(s_p)) = \{s_p\}$, $f_{k+1}^{-1}(f_k(s_q)) = \{s_q\}$, $f_{k+1}^{-1}((s_q + s_p)/2) = \{s_{k+1}\}$, $f_{k+1}(A) = [f_{k+1}(s_p), f_{k+1}(s_{k+1})]$ and lastly $f_{k+1}(B) = [f_{k+1}(s_{k+1}), f_{k+1}(s_q)]$. We define f_{k+1} to be equal to f_k on $X - N_j$ and by pasting we obtain $f_{k+1}\colon X \to S^1$. By construction, the sequence of functions f_k converges uniformly to a continuous function $f\colon X \to S^1$. Property (2) follows from the construction of f.

For uniqueness, consider $h\colon X \to S^1$ satisfying these properties. Since the cyclic ordering on S implies that $f\colon S \to S^1$ is unique up to isotopy and reflection [1], we may assume that h and f agree on S. Thus for any fat gap O of S, we have

$h(O) = f(O) = J$, an interval. Since f and h agree on the sides of O, which are sent to the endpoints of J, we simply straight line homotope h to f on each fat gap. Clearly after the homotopy they are the same.

The action of G on S gives an action on $\overline{f(S)} \subset S^1$, which preserves the cyclic order. Thus by extending linearly on the complementary intervals, we get an action of G on S^1. This action has the property that for any $g \in G$, $f \circ g \simeq g \circ f$. □

Notation Let X be a continuum without cut points. We define $\mathcal{R} \subset 2^X$ to be the collection of all necklaces of X, all maximal inseparable subsets of X, and all inseparable cut pairs of X. For the remainder of this section, X is fixed.

Lemma 23 *Let E be a nonsingleton subcontinuum of X. There exists $Q \in \mathcal{R}$ with $Q \cap E \neq \varnothing$.*

Proof Let $c, d \in E$ distinct. If $\{c, d\}$ is an inseparable set, then there is a maximal inseparable set $D \in \mathcal{R}$ with $c, d \in D$.

If not then there is a cut pair $\{a, b\}$ separating c from d. It follows that $E \cap \{a, b\} \neq \varnothing$. Either $\{a, b\}$ is inseparable or there is a necklace $N \in \mathcal{R}$ with $\{a, b\} \subset N$, and so $E \cap N \neq \varnothing$. □

Theorem 24 *Let X be a continuum without cut points. If $S, T \in \mathcal{R}$ are distinct then $|S \cap T| < 3$ and if $|S \cap T| = 2$, then $S \cap T$ is an inseparable cut pair.*

Proof If S or T is an inseparable cut pair, then the result is trivial. We are left with three cases.

First consider the case where S and T are necklaces of X Suppose there are distinct $a, b, c \in S \cap T$. Since S, T are distinct necklaces, there exists $d \in S - T$. Since $\{a, b, c, d\} \subset S$ is cyclic, renaming a, b, c if needed, we have $X = A \cup B \cup C \cup D$ where A, B, C, D are continua and $A \cap B = \{b\}$, $B \cap C = \{c\}$, $C \cap D = \{d\}$, and $D \cap A = \{a\}$, and all other pairwise intersections are empty. Thus $\{b, d\}$ separates a and c, points of T. It follows by Lemma 17 that $d \in T$. This contradicts the choice of d so $|S \cap T| < 3$. Now suppose we have distinct $a, b \in S \cap T$. If $\{y, z\}$ is a cut pair separating a from b in X then, by Lemma 17, $\{y, z\} \subset T$ and $\{y, z\} \subset S$, so $|S \cap T| > 3$. This is a contradiction, so $\{a, b\}$ is an inseparable cut pair.

Now consider the case where S and T are maximal inseparable subsets of X. Since S and T are distinct maximal inseparable sets, there exist $y \in S$, $z \in T$ and a cut pair $\{a, b\}$ separating y from z. It follows that $y \notin S$ and $z \notin T$. Thus $X = C \cup D$

where C and D are continua, $y \in C$, $z \in D$ and $C \cap D = \{a, b\}$. By inseparability, $S \subset C$ and $T \subset D$. Clearly $S \cap T \subset C \cap D = \{a, b\}$. If $S \cap T = \{a, b\}$ then $\{a, b\}$ is inseparable.

Finally consider the case where S is a necklace of X, and T is a maximal inseparable set of X. By definition, every cyclic subset with more than three elements is not inseparable. It follows that $|S \cap T| < 4$. The only way that $|S \cap T| = 3$ is if $S = T$ which is not allowed. If $|S \cap T| = 2$ then $S \cap T$ is inseparable (since T is) and cyclic (since S is) and therefore $S \cap T$ is an inseparable cut pair. □

Lemma 25 *If $S, T \in \mathcal{R}$, then S doesn't separate points of T.*

Proof Suppose that $r, t \in T - S$ with S separating r and t. First suppose that S is cyclic (so S is a necklace or an inseparable cut pair). In this case by Lemma 19, there exists a cut pair $\{a, b\} \subset S$ such that $\{a, b\}$ separates r from t.

If $\{r, t\}$ is a cut pair, then by Lemma 15, a and b are separated by $\{r, t\}$, so S is not an inseparable pair. Thus S is a necklace and it follows by Lemma 17 that $r, t \in S$ (contradiction). Thus $\{r, t\}$ is not a cut pair, and so T is a maximal inseparable set, but $\{a, b\}$ separates points of T which is a contradiction.

We are left with the case where S is a maximal inseparable set.

If T is also a maximal inseparable set, then there is a cut pair A separating a point of S from a point of T. Thus there exist continua N and M such that $N \cup M = X$, $N \cap M = A$ and, since S and T are inseparable, we may assume $T \subset N$ and $S \subset M$. Since A doesn't separate points of T, and $S \cap N \subset A$, it follows that S doesn't separate points of T.

Lastly we have the case where S is maximal inseparable, and T is cyclic. Thus $\{r, t\}$ is a cut pair. So there exist continua N, M such that $N \cup M = X$ and $N \cap M = \{r, t\}$. Since S is maximal inseparable, S is contained in one of N or M (say $S \subset M$). However $r, t \subset N$ and since $r, t \notin S$, we have $S \cap N = \varnothing$. Thus S doesn't separate r from t. □

Definition We now define a symmetric betweenness relation on $\mathcal{R}$ under which $\mathcal{R}$ is a pretree. Let R, S, T be distinct elements of $\mathcal{R}$. We say S is between R and T, denoted RST or TSR, provided:

(1) S is an inseparable cut pair and S separates a point of R from a point of T.

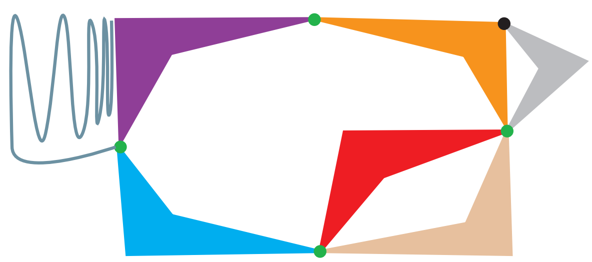
(2) S is not an inseparable pair and:

(a) $R \subset S$, so R is an inseparable cut pair, and R isn't between S and T (see case (1)).

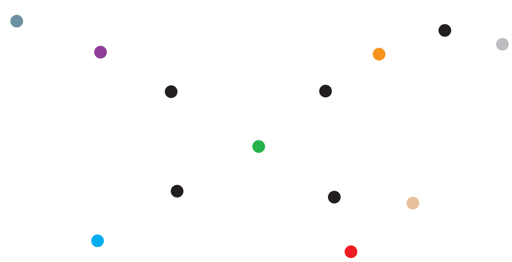
(b) S separates a point of R from a point of T, and there is no cut pair $Q \in \mathcal{R}$ with RQS and TQS (see case (1)).

For $R, T \in \mathcal{R}$ we define the open interval $(R, T) = \{S \in \mathcal{R} : RST\}$. We now defined the closed interval $[R, S] = (R, S) \cup \{R, T\}$ and we define the half-open intervals analogously. We will show that $\mathcal{R}$ with this betweenness relation forms a pretree [2]. Clearly for any $R, S \in \mathcal{R}$, by definition $[R, S] = [S, R]$ and $R \notin (R, S)$.

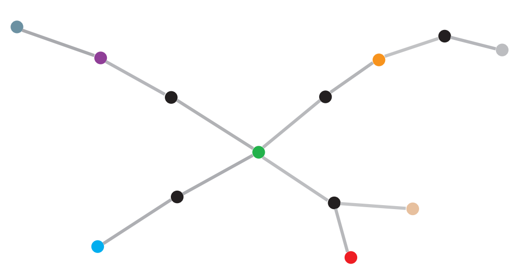
Consider the following example in where $X \subset \mathbb{R}^2$ is the union of 6 nonconvex quadrilaterals (meeting only at vertices) and a Topologist's sine curve limiting up to one of them. There are two necklaces, one being the Topologist's sine curve and the other consisting of the four green points. The cut pair tree T is obtained from $\mathcal{R}$ by connecting the dots (definition to be given later).



X



The pretree $\mathcal{R}$



The cut pair tree T

Lemma 26 *For any* $R, S, T \in \mathcal{R}$*, we have that* $[R, T] \subset [R, S] \cup [S, T]$*.*

Proof We may assume R, S, T are distinct. Let $Q \in (R, T)$ with $Q \neq S$.

If Q is an inseparable pair then Q separates a point $r \in R$ from a point $t \in T$. Thus there exist continua N, M such that $N \cup M = X$, $N \cap M = Q$, $r \in N$ and $t \in M$. Since $S \not\subset Q$, either $(S - Q) \cap N \neq \varnothing$ implying $Q \in (S, T)$, or $(S - Q) \cap M \neq \varnothing$ implying $Q \in (R, S)$.

Now consider the case where Q is not an inseparable pair.

Suppose that one of R, T (say R) is contained in Q, so $R \subset Q$ is an inseparable cut pair and $R \notin (Q, T)$. If $R \notin (Q, S)$ then by definition $Q \in (R, S)$ as required. If on the other hand $R \in (Q, S)$ then there exist continua N, M and $q \in Q - R$ and $s \in S - R$ such that $q \in N$, $s \in M$, $N \cup M = X$ and $N \cap M = R$. Since $R \notin (Q, T)$, it follows that $(T - R) \subset N$. If $T \subset Q$, then since $T \neq R$, it follows that $T \notin (Q, S)$, and so $Q \in (T, S)$. If $T \not\subset Q$, then there is $t \in (T - Q) \subset (T - R) \subset N$ and it follows that Q separates s from t since R separated them, thus $Q \in (S, T)$ as required.

We are now left with the case where Q is not an inseparable pair, $R \not\subset Q$, and $T \not\subset Q$ (see Property (2)(b) of the betweenness relation). Thus by definition there exists $r \in R - Q$, $t \in T - Q$ and disjoint continua M, N with $r \in M$ and $t \in N$, $N \cup M = X$ and $N \cap M \subset Q$. If $S \not\subset Q$ then there exists $s \in S - Q$ and either $s \in M$ in which case $Q \in (S, T)$ or $s \in N$ in which case $Q \in (S, R)$. If on the other hand $S \subset Q$, then by Property (2)(b), either $S \notin (Q, R)$ implying $Q \in (S, R)$ or $S \notin (Q, T)$ implying $Q \in (S, T)$. □

Lemma 27 *For any* $R, T \in \mathcal{R}$*, if* $S \in (R, T)$ *then* $R \notin (S, T)$*.*

Proof First consider the case where S is an inseparable cut pair. We have $r \in R - S$, $t \in T - S$ and continua $N \ni r$ and $M \ni t$ such that $N \cup M = X$, $N \cap M = S$. In fact by Lemma 25 $R \subset N$ and $T \subset M$.

If $S \not\subset R$, then $|R \cap S| < 2$. Since X has no cut points, no point in S is a cut point of M, so $M - R$ is connected. Thus R doesn't separate S from T, so $R \notin (S, T)$.

If $S \subset R$, then by definition since $S \in (R, T)$ then $R \notin (S, T)$.

Now consider the case where S is not an inseparable pair. If $R \subset S$, then R is an inseparable pair and $R \notin (S, T)$ as required. We may now assume that $R \not\subset S$. If $T \subset S$, then T is an inseparable pair and $T \notin (R, S)$. By Lemma 25 R cannot separate a point of T from a point of S, since $R \not\subset S$, it follows that $R \notin (S, T)$.

We are left with case (2,b) , so S separates a point $r \in R - S$ from a point $t \in T - S$. Thus there exists continua M, N with $r \in M$, $t \in N$, $N \cup M = X$ and $N \cap M \subset S$. In fact by Lemma 25 $R \subset M$ and $T \subset N$. Since X has no cut points and $|R \cap S| < 2$, then $N - R$ is connected, and so $R \notin (S, T)$. □

Definition We say distinct $R, S \in \mathcal{R}$ are adjacent if $(R, S) = \varnothing$.

Lemma 28 *If $R, S \in \mathcal{R}$ are adjacent then $R \subset S$, $S \subset R$, or (interchanging if need be) R is a necklace and S is maximal inseparable with $[\overline{R} - R] \cap S \neq \varnothing$.*

Proof We need only consider the case where R, S are adjacent and neither is a subset of the other.

First consider the case where one of R, S (say S) is an inseparable set. There is no maximal inseparable set containing both R and S, so there exists $r \in R - S$ and cut pair A separating r from a point of S. Notice that A is contained in some necklace T. Since $A, T \notin (R, S)$, it follows that $T = R$ and that S is maximal inseparable.

Let G be the gap of R with $S \subset G$. Let Q be a side of G and $p \in Q$. If $p \notin S$, then there exists a cut pair B separating p from S. Since $(R, S) = \varnothing$, B doesn't separate R from S. It follows by definition of side, that B separates points of R which implies that $B \subset R$. This contradicts the fact that Q is a side of the gap $G \supset S$. If both sides of G are points, then they form an inseparable cut pair in (R, S). Thus they are not both points so $[\overline{R} - R] \cap S \neq \varnothing$.

We are left with the case where R and S are each necklaces with more than 2 elements. Again let G be the gap of R with $S \subset G$, and let Q, P be sides of G. Since $Q \cup P$ is inseparable, there is a maximal inseparable set $A \supset Q \cup P$. It follows that $A \in (R, S)$ which is a contradiction. □

Using the pretree structure on $\mathcal{R}$, we can put a linear order (two actually) on any interval of $\mathcal{R}$. We recall that the order topology on a linearly ordered set I is the topology with basis $I_y = \{x : x > y\}$, $J_y = \{x : x < y\}$ and $K_{y,z} = \{x : z < x < y\}$ where y, z range over elements of I. The suspension of a Cantor set is a continuum with uncountably many maximal inseparable sets, but this doesn't happen for inseparable cut pair and necklaces.

Lemma 29 *Only countably many elements of $\mathcal{R}$ are inseparable pairs or necklaces.*

Proof We first show that any interval I of $\mathcal{R}$ contains only countably many necklaces and inseparable pairs.

Let Q be the set of all cut pairs in I which have more than two complementary components, union the set of necklaces in I. Let $A \in Q$.

- If A is a cut pair, then since $X - A$ has more than two components, and $\bigcup Q$ will intersect two of the components, A will separate $(\bigcup Q) - A$ from some other point of X. Using Lemma 3, we find subcontinua Y, Z of X such that $Y \cup Z = X$, $Y \cap Z = A$ where $(\bigcup Q) - A \subset Y$. We define the open set $U_A = Z - A$
- If A is a necklace, then $|A| > 2$ and there is a cut pair $\{a, b\} \subset A$ which doesn't separate $(\bigcup Q) - A$. Using Lemma 3, we find subcontinua Y, Z of X such that $Y \cup Z = X$, $Y \cap Z = A$ where $(\bigcup Q) - A \subset Y$. We define the open set $U_A = Z - \{a, b\}$

Notice that for any $A, B \in Q$, $U_A \cap U_B = \varnothing$. Since X is Lindelöf, the collection $\{U_A : A \in Q\}$ is countable and therefore Q is countable.

It is more involved to show that inseparable cut pairs $\{a, b\}$ of I such that $X - \{a, b\}$ has 2 components are countable. Let S be the set of inseparable cut pairs $\{a, b\}$ in I such that $X - \{a, b\}$ has 2 components. We argue by contradiction, so we assume that S is uncountable.

Let $\{a, b\}$ be a cut pair of S and let C_L, C_R be the components of $X - \{a, b\}$.

We say that $\{a, b\}$ is a limit pair if there are inseparable cut pairs $\{a_i, b_i\}$ and $\{a_i', b_i'\}$ in S such that $\{a_i, b_i\} \subset C_L$, $\{a_i', b_i'\} \subset C_R$, and for each limit pair $\{a', b'\} \neq \{a, b\}$ of S one of the two components of $X - \{a', b'\}$ contains at most finitely many elements of the sequences $\{a_i, b_i\}$ and $\{a_i', b_i'\}$.

We claim that there are at most countable pairs in S which are not limit pairs. Indeed if $\{a, b\}$ is not a limit pair and $I = [x, y]$ let C_L, C_R be the components of $X - \{a, b\}$ containing, respectively, x, y (L, R stand for left, right). Since $\{a, b\}$ is not a limit pair for some $\epsilon > 0$ one of the four sets

$$C_L \cap B_\epsilon(a),\ C_R \cap B_\epsilon(a),\ C_L \cap B_\epsilon(b,) \text{ or } C_R \cap B_\epsilon(b)$$

intersects the union of all cut pairs of S at either a or b.

We remark now that for fixed $\epsilon > 0$ there are at most finitely many pairs $\{a, b\}$ in S such that (say) $C_L \cap B_\epsilon(a)$ intersects the union of all cut pairs of S in a subset of $\{a, b\}$. Indeed if we take all pairs $\{a, b\}$ with this property the balls $B_{(\epsilon/2)}(a)$ are mutually disjoint so there are finitely many such pairs. The same argument applies to each one of the three other sets $C_R \cap B_\epsilon(a)$, $C_L \cap B_\epsilon(b,)$ or $C_R \cap B_\epsilon(b)$. This implies that non limit cut pairs are countable.

So we may assume S has uncountably many limit pairs. Let $\{c, d\}$ be a limit pair in S, let C_L, C_R be the components of $X - \{c, d\}$ and let $\{c_i, d_i\} \in \bar{C}_L$, $\{c_i', d_i'\} \in \bar{C}_R$ be sequences of distinct pairs in S provided by the definition of limit pair.

Let C_R^i be the component of $X - \{c_i, d_i\}$ containing c, d. We claim that there is an ϵ such that for all i there is some $x_i \in C_L \cap C_R^i$ with $d(x_i, \{c, d\}) > \epsilon$. Indeed this is clear if the accumulation points of the sequences c_i and d_i are not contained in the set $\{c, d\}$. Otherwise by passing to a subsequence and relabelling, if necessary, we may assume that either $c_i \to c, d_i \to d$ or both c_i, d_i converge to, say, c.

In the first case we remark that there is a component C_i of $X - \{c, d, c_i, d_i\}$, such that its closure contains both c, d_i or both d, c_i. Indeed otherwise $\{c, d, c_i, d_i\}$ is a cyclic subset which is impossible since we assume that $\{c_j, d_j\}$ $(j > i)$ are all inseparable cut pairs.

Since $d_i \to d$ and $c_i \to c$ there exists $\epsilon > 0$ and $x_i \in C_i$ such that $d(x_i, c) > \epsilon$ and $d(x_i, d) > \epsilon$ for all i.

In the second case we remark that since c is not a cut point there is some $e > 0$ such that for each i there is a component C_i of $C_L \cap C_R^i$ with diameter bigger than e. It follows that there is an $\epsilon > 0$ and $x_i \in C_i$ so that $d(x_i, c) > \epsilon, d(x_i, d) > \epsilon$ for all i.

By passing to subsequence we may assume that x_i converges to some $x_L \in C_L$. Clearly $d(x_L, c) \geq \epsilon, d(x_L, d) \geq \epsilon$. It follows that $d(x_L, C_R) > 0$.

We associate in this way to a limit pair $\{c, d\}$ in S a point x_L and a $\delta > 0$ such that:

(1) $x_L \in C_L$

(2) $d(x_L, C_R) > \delta$

Since there are uncountably many limit pairs in S there are infinitely many such pairs for which $1, 2$ above hold for some fixed $\delta > 0$. But then the corresponding x_L's are at distance greater than δ (by property 2 above). This is impossible since X is compact. Thus S is countable.

Thus for any interval I of $\mathcal{R}$, the set of necklaces and inseparable cut pairs of I is countable.

Let E be a countable dense subset of X. For any A, a necklace with more than one gap or an inseparable pair, there exist $a, b \in E$ separated by A. Thus the interval $I = [[a], [b]]$ contains A. There are countably many such intervals, so the set of necklaces with more than one gap is countable, and the set of inseparable cut pairs of X is countable.

If a necklace N has less than two gaps, there is an open set $U \subset N$. By Lindelöf, there are at most countably many such necklaces, and thus there are at most countably many inseparable cut pairs and necklaces in X. □

Lemma 30 *The pretree $\mathcal{R}$ is preseparable and complete.*

Proof Let $[R, W]$ be a closed interval of $\mathcal{R}$.

We first show that any bounded strictly increasing sequence in $[R, W]$ converges. Let $(S_n) \subset [R, W]$ be strictly increasing. Let C_n be the component of $X - S_n$ which contains R. Let $C = \overline{\bigcup C_n}$. Clearly C is contained in the closure Q of the component of $X - W$ which contains R, and so $\partial C \subset Q$. Clearly ∂C is not a point (since it would by definition be a cut point separating R from W). The set ∂C is inseparable, and so $\partial C \subset A$ is a maximal inseparable set. It follows that $A \in [R, W]$. If $S_n \not\to A$, then there is $B \in [R, A)$ with $S_n < B$ for all n. As before, we have C contained in the closure D of the component of $X - B$ containing R. This would imply that $A \in [R, B]$, a contradiction. Thus every strictly increasing sequence in $[R, W]$ converges.

We now show that there are only countably many adjacent pairs in $[R, W]$. We remark that if A, B is an adjacent pair in $\mathcal{R}$ at most one of the sets A, B is a maximal inseparable set. By Lemma 29 there are only countably many inseparable cut pairs and necklaces in X. It follows that there are only countably many inseparable pairs in $[R, W]$. □

We have shown thus that $\mathcal{R}$ is a complete preseparable pretree. By gluing in intervals to adjacent pairs of $\mathcal{R}$ we obtain a real tree T as in Theorem 13.

Corollary 31 *There is a metric on T, which preserves the pretree structure of T, such that T is an ℝ–tree. The topology so defined on T is canonical.*

Proof We metrize T as in Theorem 14. We metrize first the subtree spanned by the set of inseparable cut pairs and necklaces (which is countable) and then we glue intervals for the inseparable subsets of $\mathcal{R}$ which are not contained in this subtree. □

We call this ℝ–tree the *JSJ- tree* of the continuum X since in the case $X = \partial G$ with G one-ended hyperbolic our construction produces a simplicial tree corresponding to the JSJ decomposition of G.

5 Combining the two trees

When X is locally connected, one can combine the constructions of the previous 2 sections to obtain a tree for both the cut points and the cut pairs of a continuum X. The obvious application would be to relatively hyperbolic groups, and we should note that in that setting, the action of the tree may be nesting. We explain briefly how to construct this tree.

Let X be a Peano continuum and let $\mathcal{P}$ be the cut point pretree.

Lemma 32 *Let $A \in \mathcal{P}$ be a nonsingleton equivalence class of $\mathcal{P}$. Then the closure $\bar{A}$ is a Peano continuum without cut points.*

Proof We first show that $\bar{A}$ is a Peano continuum. Since $\bar{A}$ is compact, and X is (locally) arcwise connected, it suffices to show that A is convex in the sense that every arc joining points of $\bar{A}$ is contained in $\bar{A}$.

Let a, b be distinct points of $\bar{A}$ and let I be an arc from a to b. Suppose $d \in I - A$. Thus either c is a cut point adjacent to A, or there is a cut point $c \in A$ separating d from A, but then I cannot be an arc since it must run through d twice.

Let $a, b, e \in \bar{A}$. Since e doesn't separate a from b in X, there is an arc in X from a to b missing e. By convexity, this arc is contained in $\bar{A}$, and so e doesn't separate a from b in $\bar{A}$. It follows that the continuum $\bar{A}$ has no cut points. □

Let A be a nonsingleton equivalence class of the cut point pretree $\mathcal{P}$, and let T_A be the ends compactification (well, it will not be compact, but we glue the ends to the tree anyway) of the cut pair tree for $\bar{A}$. Since X is locally connected, for any interval $(B, D) \ni A$ there are cut points $a_1, a_2 \in \bar{A}$ with $a_1, a_2 \in (B, D)$. Not every point of $\bar{A}$ is contained in one of the defining sets of the cut pair pretree for $\bar{A}$. Some of the points of $\bar{A}$ are not contained in a cut pair, or in a maximal inseparable set with more than two elements, and these appear as ends of the cut pair tree $\mathcal{R}_{\bar{A}}$ for $\bar{A}$.

For each nonsingleton class A of a the cut point tree T we replace A by $T - A$. The end of the component of $T - A$ corresponding to a cut point $a_1 \in \bar{A}$ is glued to the minimal point or end of T_A containing a_1.

To see that this construction yields a tree, we use the following Lemma.

Lemma 33 *The set of classes of $\mathcal{P}$ with nontrivial relative JSJ-tree in any interval of $\mathcal{P}$ is countable.*

Proof Let $[u, v]$ be an interval of $\mathcal{P}$ and let A be a class of $[u, v]$ with nontrivial JSJ-tree. Since X is locally connected, $\bar{A}$ contains some cut point of $[u, v]$. If c is a cut point of $[u, v]$ in $\bar{A}$ we have that c, A are adjacent elements of $[u, v]$. But we have shown in Theorem 13 that there are at most countable such pairs. □

Clearly any group of homeomorphisms of X acts on this combined tree.

6 Group actions

The ℝ–trees we construct in the previous sections usually come from group boundaries and the group action on them is induced from the action on the boundary, so it's an action by homeomorphisms. In this section we examine such actions and generalize some results from the more familiar setting of isometric actions.

We recall that the action of a group G on an ℝ–tree T is called non-nesting if there is no interval $[a,b]$ in T and $g \in G$ such that $g([a,b])$ is properly contained in $[a,b]$. An element $g \in G$ is called elliptic if $gx = x$ for some $x \in T$. If g is elliptic we denote by $\mathrm{fix}(g)$ the fixed set of g. An element which is not elliptic is called hyperbolic.

Lemma 34 *Let G be a group acting on an ℝ–tree T by homeomorphisms. Suppose that the action is non-nesting. Then if g is elliptic* $\mathrm{fix}(g)$ *is connected. If g is hyperbolic then g has an "axis", ie there is a subtree L invariant by g which is homeomorphic to* $\mathbb{R}$.

Proof Let g be elliptic. We argue by contradiction. If A, B are distinct connected components of $\mathrm{fix}(g)$ let $[a,b]$ be an interval joining them ($a \in A, b \in B$). Then $g([a,b]) = [a,b]$. Since $[a,b]$ is not fixed pointwise there is a $c \in [a,b]$ such that $g(c) \neq c$. So $g(c) \in [a,c)$ or $g(c) \in (c,b]$. In the first case $g([a,c]) \subset [a,c)$ and in the second $g([c,b]) \subset (c,b]$. This is a contradiction since the action is non-nesting.

Let g be hyperbolic. If $a \in T$ consider the interval $[a, g(a)]$. The set of all $x \in [a, g(a)]$ such that $g(x) \in [a, g(a)]$ is a closed set. If c is the supremum of this set then there is no $x \in [c, g(c)]$ such that $g(x) \in [c, g(c)]$. We take L to be the union of all $g^n([c, g(c)])$ ($n \in \mathbb{Z}$). Clearly L is homeomorphic to $\mathbb{R}$ and is invariant by g. □

Proposition 35 *Let G be a finitely generated group acting on an ℝ–tree T by homeomorphisms. Suppose that the action is non-nesting. Then if every element of G is elliptic there is an $x \in T$ fixed by G.*

Proof We argue by contradiction. Let $G = \langle a_1, a_2, ..., a_n \rangle$. If the intersection $\mathrm{fix}(a_1) \cap \mathrm{fix}(a_2) \cap ... \cap \mathrm{fix}(a_n) = \varnothing$ then $\mathrm{fix}(a_i) \cap \mathrm{fix}(a_j) = \varnothing$ for some a_i, a_j. We claim that $a_i^{-1} a_j^{-1} a_i a_j$ is hyperbolic. Indeed if $a_i^{-1} a_j^{-1} a_i a_j(x) = x$ then $a_i a_j(x) = a_j a_i(x)$. Let $A = \mathrm{fix}(a_i)$, $B = \mathrm{fix}(a_j)$. We remark that the smallest interval joining $a_i a_j(x)$ to $A \cup B$ has one endpoint in A while the smallest interval joining $a_j a_i(x)$ to $A \cup B$ has one endpoint in B so these two points can not be equal. This is a contradiction. □

Proposition 36 *Let G be a group acting on an ℝ–tree T by homeomorphisms. Suppose that the action is non-nesting. Then if every element of G is elliptic G fixes either an $x \in T$ or an end of T.*

Proof Suppose that G does not fix any $x \in T$. Then there is a sequence $g_n \in G$ and $x_n \in T$ such that $x_n \in \mathrm{fix}(g_n)$, $x_n \notin \mathrm{fix}(g_{n-1})$ and x_n goes to infinity. The sequence x_n defines an end e of T. If r is a ray from $x_0 \in T$ to e then any $g \in G$ fixes a ray r_g contained in r. Indeed if this is not the case, for some n, $\mathrm{fix}(g)$ and $\mathrm{fix}(g_n)$ are disjoint. It follows as in the previous proposition that $g^{-1}g_n^{-1}gg_n$ is hyperbolic, which is a contradiction. □

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