Algebraic & Geometric Topology 7 (2007) 301–308 **301**

Super-exponential distortion of subgroups of CAT(−1) groups

JOSH BARNARD
NOEL BRADY
PALLAVI DANI

We construct 2–dimensional CAT(−1) groups which contain free subgroups with arbitrary iterated exponential distortion, and with distortion higher than any iterated exponential.

20F65; 20F67, 57M20

1 Introduction

The purpose of this note is to produce explicit examples of CAT(−1) groups containing free subgroups with arbitrary iterated exponential distortion, and distortion higher than any iterated exponential. The construction parallels that of Mahan Mitra in [2] but our groups are the fundamental groups of locally CAT(−1) 2–complexes. The building blocks used in [2] are hyperbolic $F_3 \rtimes F_3$ groups, which are not known to be CAT(0). Our building blocks are graphs of groups where the vertex and edge groups are all free groups of equal rank and the underlying graph is a bouquet of a finite number of circles. We use the combinatorial and geometric techniques from Dani Wise's version of the Rips construction [3] to ensure that our building blocks glue together in a locally CAT(−1) fashion.

One of our motivations for producing these examples was that it was not immediate from the description that the examples in [2] had the appropriate iterated exponential distortions. Mitra's examples are graphs of groups with underlying graph a segment of length n, where the vertex groups are hyperbolic $F_3 \rtimes F_3$ groups, and each edge identifies the kernel F_3 in one vertex group with the second F_3 factor in the adjacent vertex group.

While it is easy to see that the nth power of a hyperbolic automorphism of a free group will send a generator of the free group into a word which grows exponentially in n, it appears to be harder to see (without using bounded cancellation properties of carefully chosen automorphisms) that a word of length n in three hyperbolic automorphisms

Published: 29 March 2007 DOI: 10.2140/agt.2007.7.301

(and their inverses) will send a generator of the free group to a word which grows exponentially in n. In contrast, the monomorphisms in the multiple HNN extensions in our construction are all defined using positive words. This makes it easy to see that the exponential distortions compose as required. Also, the example in [2] with distortion higher than any iterated exponential is of the form $(F_3 \rtimes F_3) \rtimes \mathbb{Z}$, with the generator of $\mathbb{Z}$ conjugating the generators of the first F_3 to "sufficiently random" words in the generators of the second F_3. In contrast, our group can be described explicitly, without recourse to random words, allowing for an explicit check that our group is $\mathrm{CAT}(-1)$.

Recall that if $H \subset G$ is a pair of finitely generated groups with word metrics d_H and d_G respectively, the *distortion* of H in G is given by

$$\delta_H^G(n) = \max\{d_H(1,h) \mid h \in H \text{ with } d_G(1,h) \leq n\}.$$

Up to Lipschitz equivalence, this function is independent of the choice of word metrics. Background on $\mathrm{CAT}(-1)$ spaces and the large link condition may be found in Bridson and Haefliger [1].

Acknowledgements

Noel Brady acknowledges support from NSF grant no. DMS-0505707.

2 The building blocks

For each positive integer n we define a building block group

$$G_n = \langle a_1, \dots, a_m, t_1, \dots, t_n \mid t_i a_j t_i^{-1} = W_{ij} \ ; \ 1 \leq i \leq n, 1 \leq j \leq m \rangle,$$

where $m = 14n$ and $\{W_{ij}\}$ is a collection of positive words of length 14 in $a_1, \dots, a_n$, such that each two-letter word $a_k a_l$ appears at most once among the W_{ij}'s. One way of ensuring this is to choose these words to be consecutive subwords of the following word, defined by Dani Wise in [3].

Definition 2.1 (Wise's long word with no two-letter repetitions) Given the set of letters $\{a_1, \dots, a_m\}$, define

$$\Sigma(a_1, \dots, a_m) = (a_1 a_1 a_2 a_1 a_3 \dots a_1 a_m)(a_2 a_2 a_3 a_2 a_4 \dots a_2 a_m) \dots (a_{m-1} a_{m-1} a_m) a_m.$$

It is easy to see that $\Sigma(a_1, \dots, a_m)$ is a positive word of length m^2, such that each two-letter word $a_k a_l$ appears as a subword in at most one place. Following Dani Wise, we simply chop $\Sigma(a_1, \dots, a_m)$ into subwords of length 14. In order to obtain all mn relator words of G_n from $\Sigma(a_1, \dots, a_m)$ we must have $m^2 \geq 14mn$, which explains our choice of m above.

Proposition 2.2 *The presentation 2–complex X_n for G_n can be given a locally* CAT(-1) *structure. Furthermore,*

(1) *The a_j 's generate a free subgroup $F(a_j)$ whose distortion in G_n is exponential.*

(2) *The t_i 's generate a free subgroup $F(t_i)$ of G_n that is highly convex in the following sense:*
Let v be the vertex of X_n. Then

$$\mathrm{d}_{\mathrm{Lk}(v,X_n)}(t_i^{\epsilon_1}, t_j^{\epsilon_2}) \geq 2\pi, \text{ where } \epsilon_1, \epsilon_2 \in \{+,-\} \text{ and if } i = j \text{ then } \epsilon_1 \neq \epsilon_2.$$

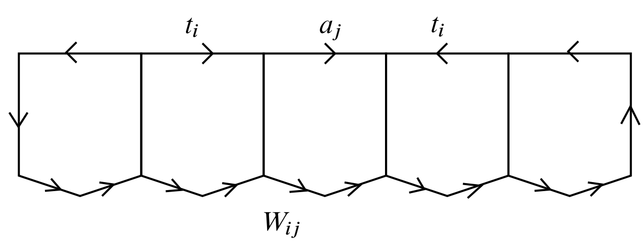

Figure 1: A 2–cell of X_n decomposed into right-angled pentagons.

Proof Each disk in X_n is given a piecewise hyperbolic structure by expressing it as a concatenation of right-angled hyperbolic pentagons, as shown in Figure 1. The fact that X_n satisfies the large link condition is a consequence of the condition that the W_{ij} 's are positive words with no two-letter repetitions. The details are in [3]. Thus X_n is locally CAT(-1) and G_n is hyperbolic.

To prove (1), first note that the group G_n can be viewed as the fundamental group of a graph of groups. The underlying graph is a bouquet of n circles, and the edge and vertex groups are all equal to $F(a_j)$, the free group on $a_1, \dots, a_m$. The two maps associated with the i th edge are $\mathrm{id}_{F(a_j)}$ and $\phi_i \colon F(a_j) \to F(a_j)$, defined by $\phi_i(a_j) = W_{ij}$ for $1 \leq j \leq m$. To see that ϕ_i is injective, note that ϕ_i induces a map on a subdivided bouquet of m circles. By Stallings' algorithm this factors through a sequence of folds followed by an immersion. The no two-letter repetitions condition restricts the amount of folding that can occur and ensures that no non-trivial loops are killed by the sequence of folds. Thus the folding maps induce isomorphisms at the level of π_1 and the final immersion induces an injection. The theory of graphs of groups now implies that the subgroup of G_n generated by $a_1,\dots,a_m$ is $F(a_j)$.

The distortion of $F(a_j)$ in G_n is at least exponential because, for example, the element $t_1^k a_1 t_1^{-k}$ of $F(a_j)$, when expressed in terms of the a_j's, is a positive (hence reduced) word of length 14^k. To see that the distortion is at most exponential consider a word $w(a_j,t_i)$ which represents an element of $F(a_j)$ and has length k in G_n. It can be reduced to a word in the a_j's by successively cancelling at most $k/2$ innermost $t_i\cdots t_i^{-1}$ pairs. Each such cancellation multiplies the word length by at most a factor of 14. So the length of $w(a_j,t_i)$ in $F(a_j)$ is at most $14^{k/2}k$.

To prove (2), note that any path in the link of v connecting $t_i^{\pm}$ and $a_j^{\pm}$ within a disk as in Figure 1 has combinatorial length 2. So any path in the link connecting $t_i^{\epsilon_1}$ and $t_j^{\epsilon_2}$ has combinatorial length at least 4. Since we are using right-angled pentagons, combinatorial length 4 corresponds to a spherical metric length of $4(\pi/2)$ or 2π. Thus $\langle t_1,\dots,t_n\rangle$ is highly convex. As a consequence, we have that the map from the bouquet of circles with edges labelled $t_1,\dots,t_n$ into X_n is a local isometric embedding. This implies that it is π_1–injective, i.e. $t_1,\dots,t_n$ generate the free group $F(t_i)$. To see this algebraically, note that the homomorphism $\psi\colon G_n\to F(t_i)$ defined by $\psi(t_i)=t_i$ and $\psi(a_j)=1$ is a retraction of G_n onto $F(t_i)$. $\square$

3 Iterated exponential distortion

In this section we see how to get arbitrary iterated exponential distortions. The idea is to amalgamate a chain of building block groups together, identifying the distorted free group in one with the highly convex free subgroup in the next. This identification of distorted with highly convex can be made in a non-positively curved way, and the distortion functions compose as expected. Here are the details.

Theorem 3.1 *For any integer $l>0$, there exists a 2–dimensional* CAT(-1) *group H_l with a free subgroup F such that $\delta_F^{H_l}(x)\simeq \exp^l(x)$.*

We will actually show that $\delta_F^{H_l}(x)\simeq f^l(x)$, where $f(x)=14^x$.

Proof The group H_l is defined using the building blocks from Proposition 2.2 as

$$H_l = G_1 *_{F_1} G_{14} *_{F_2} G_{14^2} * \cdots\cdots *_{F_{l-1}} G_{14^{l-1}},$$

where F_k, for $1\le k\le l-1$, is a free group of rank 14^k which is identified with the exponentially distorted free subgroup of $G_{14^{k-1}}$ and the highly convex free subgroup of G_{14^k}. Let $a_j^{(k)}$, with $1 \le j \le 14^k$, denote the generators of F_k, and let t denote the stable letter of G_1, so that $G_1 = \langle a_j^{(1)}, t \rangle$ and $G_{14^k} = \langle a_j^{(k+1)}, a_j^{(k)} \rangle$. We shall use this notation in the upper and lower bound arguments below.

Let Y_l denote the presentation complex of H_l. Then $Y_1 = X_1$, which is locally CAT(-1) by Proposition 2.2. Further, there are inclusions $Y_1 \subset Y_2 \subset Y_3 \cdots$, so that the large link condition can be checked inductively. The space Y_{k+1} is obtained by gluing X_{14^k} to Y_k along a rose R_k with 14^k petals, and R_k is highly convex in X_{14^k}. It follows that the link of the base vertex of Y_{k+1} is obtained by gluing together the links of base vertices in Y_k and X_{14^k}, along a set of $2(14^k)$ points which is 2π–separated in the latter link. By induction, the link of the base vertex in Y_k is large, and hence the union is large.

The group with the desired distortion in H_l is F_l, the free group generated by $a_j^{(l)}$, with $1 \le j \le 14^l$, which is exponentially distorted in $G_{14^{l-1}}$. We prove the lower bound as follows. Given a positive integer n consider the sequence given by $w_1 = t^n a_1^{(1)} t^{-n} \in F_1$ and $w_k = w_{(k-1)} a_1^{(k)} w_{(k-1)}^{-1} \in F_k$, and set $w = w_l$. Given $g \in H_l$ let $\ell_{H_l}(g)$ denote the distance from 1 to g in H_l. Observe that $\ell_{H_l}(w_1) \le 2n+1$, and inductively that $\ell_{H_l}(w) \le 2^l n + 2^l - 1$ (a linear function of n). On the other hand, each w_k can be expressed as a positive word in the generators of F_k by using the W_{ij}'s from the definition of the building blocks. Since there is no cancellation among positive words, we obtain that $|w_1|_{F_1} = 14^n$ and inductively that $|w|_{F_l} = f^l(n)$. This gives the lower bound $f^l(x) \preceq \delta_F^{H_l}(x)$.

To prove the upper bound it is more convenient to do the induction in the opposite direction. Proposition 2.2 provides the base case. Let w be an element whose length in H_l is at most n. Then by successively cancelling at most $n/2$ innermost $t \cdots t^{-1}$ pairs, w can be represented by a word in $G_{14} *_{F_2} G_{14^2} * \cdots G_{14^{l-1}}$ of length at most $14^{n/2} n$. At each stage of the previous cancellation procedure, we may assume (by replacement if necessary) that the subword enclosed by an innermost $t \dots t^{-1}$ involves only the $a_i^{(1)}$. This is because the free group on the $a_i^{(1)}$ is a retract of $G_{14} *_{F_2} G_{14^2} * \cdots G_{14^{l-1}}$. The upper bound follows by induction. □

4 Distortion higher than any iterated exponential

In this section we produce a 2–dimensional CAT(-1) group containing a free subgroup with distortion more than any iterated exponential. The idea is to take a suitable building block group from section 2 with base group free on a_i and stable letters t_j, and to form a new HNN extension with stable letter s which sends the free group on the a_i into a subgroup of the free group on the t_j.

Theorem 4.1 *There exists a* 2*–dimensional* $\mathrm{CAT}(-1)$ *group* G *with a subgroup* H*, such that* $\delta_H^G(x)$ *is a function that is bigger than any iterated exponential.*

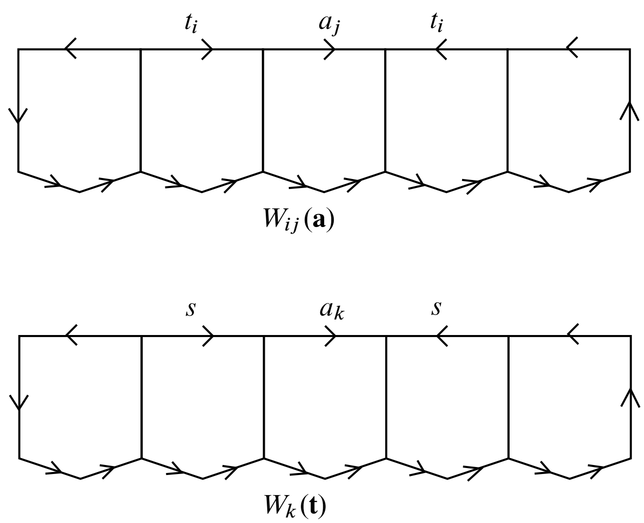

Figure 2: Relator 2–cells of the group G decomposed into right-angled pentagons.

Proof Define

$$G = \langle a_1, \dots, a_m, t_1, \dots, t_n, s \mid t_i a_j t_i^{-1} = W_{ij} \; ; \; s a_k s^{-1} = W_k \rangle,$$

where $\{W_{ij}\}$ (resp. $\{W_k\}$) consists of mn (resp. m) positive words of length 14 in the letters a_i (resp. t_j), with no 2–letter repetitions. Thus we may choose the W_{ij}'s and W_k's to be disjoint subwords of $\Sigma(a_1, \dots, a_m)$ and $\Sigma(t_1, \dots, t_n)$ (see Definition 2.1) respectively. This gives the two conditions

$$14mn \le m^2 \quad \text{and} \quad 14m \le n^2.$$

So, for example, $n = 14^2$ and $m = 14^3$ is a possible choice.

Each disk in the presentation complex is given a piecewise hyperbolic structure by concatenating a sequence of right-angled pentagons as shown in Figure 2. Observe that loops in the link of the unique vertex which involve $s^{\pm}$ have combinatorial length at least 4. Similarly, any loop involving an $a_j^{\pm}$ and a $t_i^{\pm}$, for some i and j, has

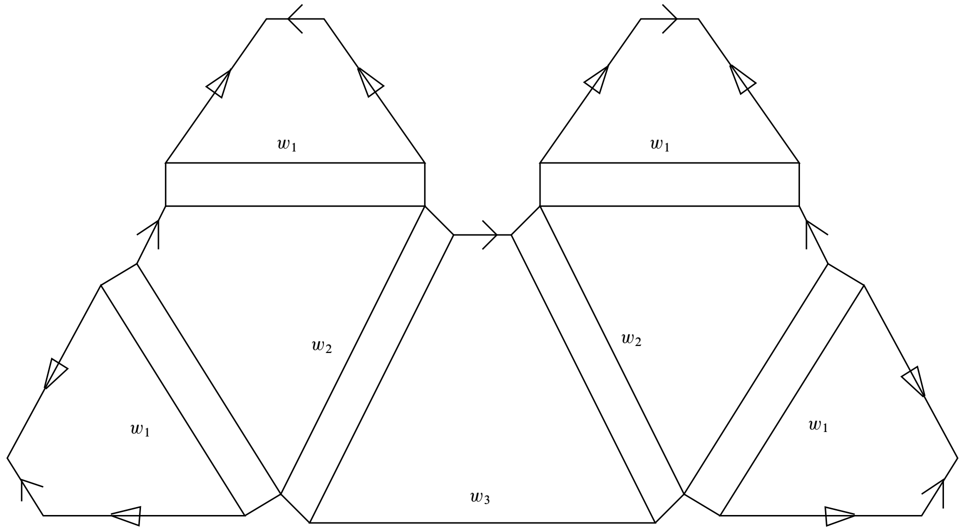

Figure 3: The word w_k (in this case $k = 3$).

length at least 4. Thus to show that the complex is locally CAT(-1), it is enough to consider loops involving only $a_i^{\pm}$'s or only $t_j^{\pm}$'s. The fact that such loops are large is a consequence of the condition that the W_{ij}'s and W_k's are positive words with no two-letter repetitions. The details are exactly as in [3].

Let H be the subgroup of G generated by $a_1, \dots, a_m$. Then $\delta_H^G(x)$ is higher than any iterated exponential. To see this, consider the sequence of words $w_k \in H$ given by $w_1 = t_1 a_1 t_1^{-1}$ and $w_k = (s w_{k-1} s^{-1}) a_1 (s w_{k-1}^{-1} s^{-1})$ for $k > 1$. The word w_3 is shown in Figure 3. The label of each single arrow edge is a_1, the label of each solid arrow edge is t_1, and the edges along the strips are all labeled by s and are oriented from w_j toward w_{j-1}. Let ℓ_G and ℓ_H denote geodesic lengths in G and H respectively. Note that $\ell_G(w_1) = 3$ and $\ell_G(w_k) \leq 2\ell_G(w_{k-1}) + 5$ for $k > 1$. So, for example, $\ell_G(w_k) \leq 4^k$ is true. On the other hand, $\ell_H(w_1) = 14$ and $\ell_H(w_k) = 14^{14\ell_H(w_{k-1})} > 14^{\ell_H(w_{k-1})}$. So by induction $\ell_H(w_k) \geq f^k(1)$. (Recall that $f(x) = 14^x$.) This shows that $\delta_H^G(x) \geq f^{\lfloor \log_4 x \rfloor}(1)$, which is a function that grows faster than any iterated exponential.
□

References

[1] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer, Berlin (1999) MR1744486

[2] **M Mitra**, *Cannon–Thurston maps for trees of hyperbolic metric spaces*, J. Differential Geom. 48 (1998) 135–164 MR1622603

[3] **D T Wise**, *Incoherent negatively curved groups*, Proc. Amer. Math. Soc. 126 (1998) 957–964 MR1423338

Department of Mathematics, University of Oklahoma
Norman OK 73019, USA

jbarnard@math.ou.edu, nbrady@math.ou.edu, pdani@math.ou.edu

http://www.math.ou.edu/~jbarnard/, http://www.math.ou.edu/~nbrady/,
http://www.math.ou.edu/~pdani/

Received: 15 December 2006