Generic representations of orthogonal groups: the mixed functors

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In previous work, we defined the category of functors $\mathcal{F}_{\text{quad}}$, associated to $\mathbb{F}_2$–vector spaces equipped with a nondegenerate quadratic form. In this paper, we define a special family of objects in the category $\mathcal{F}_{\text{quad}}$, named the mixed functors. We give the complete decompositions of two elements of this family that give rise to two new infinite families of simple objects in the category $\mathcal{F}_{\text{quad}}$.

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In 1993, Henn, Lannes and Schwartz established a very strong relation between the Steenrod algebra and the category $\mathcal{F}(p)$ of functors from the category $\mathcal{E}^f$ of finite dimensional $\mathbb{F}_p$–vector spaces to the category $\mathcal{E}$ of all $\mathbb{F}_p$–vector spaces, where $\mathbb{F}_p$ is the prime field with p elements [5]. To be more precise, they study the category $\mathcal{U}$ of unstable modules over the Steenrod algebra localized away from the nilpotent unstable modules $\mathcal{N}il$; they exhibit an equivalence between the quotient category $\mathcal{U}/\mathcal{N}il$ and a full subcategory of the category of functors $\mathcal{F}(p)$. This equivalence is very useful and allows several important topological results to be derived from algebraic results in the category $\mathcal{F}(p)$. For a recent interesting application of this equivalence to the cohomology of Eilenberg MacLane spaces, we refer the reader to the results obtained by Powell [9].

An important algebraic motivation for the particular interest in the category $\mathcal{F}(p)$ follows from the link with the modular representation theory and the cohomology of finite general linear groups. Namely, the evaluation of a functor F, object in $\mathcal{F}(p)$, on a finite dimensional vector space V is a $\mathbb{F}_p[GL(V)]$–module. A fundamental result obtained by Suslin in the appendix of [4] and, independently, by Betley in [2] relates the calculation of extension groups in the category $\mathcal{F}(p)$ with certain stable cohomology groups of general linear groups.

It is natural to seek to construct other categories of functors that play a similar role for other families of algebraic groups and, in particular, for the orthogonal groups.

In [12], we constructed the functor category $\mathcal{F}_{\text{quad}}$, which has some good properties as a candidate for the orthogonal group over the field with two elements. For instance, the evaluation functors give rise to a coefficient system that allows us to define a system of

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homology groups. We obtained, in [12], two families of simple objects in $\mathcal{F}_{\mathrm{quad}}$ related, respectively, to general linear groups and to orthogonal groups. The purpose of this paper is to define a new family of objects in the category $\mathcal{F}_{\mathrm{quad}}$, named the mixed functors, which give rise to new simple objects of $\mathcal{F}_{\mathrm{quad}}$. The mixed functors are subfunctors of a tensor product between a functor coming from the category $\mathcal{F} := \mathcal{F}(2)$ and a functor coming from the subcategory $\mathcal{F}_{\mathrm{iso}}$ of $\mathcal{F}_{\mathrm{quad}}$ defined in [12]. The structure of the mixed functors is very complex, hence it is difficult to give explicit decompositions in general. However, we give the complete decompositions of two significant elements of this family: the functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$. These two mixed functors play a central role in the forthcoming paper [11] concerning the decompositions of the standard projective objects P_{H_0} and P_{H_1} of $\mathcal{F}_{\mathrm{quad}}$. We prove in [11] that these mixed functors are direct summands of P_{H_0} and P_{H_1}. The decomposition of $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$ represents a further step in our project to classify the simple objects of this category.

Recall that in [12] we constructed two families of simple objects in $\mathcal{F}_{\mathrm{quad}}$. The first one is obtained using the fully faithful, exact functor $\iota\colon \mathcal{F} \to \mathcal{F}_{\mathrm{quad}}$, which preserves simple objects. By Kuhn [7], the simple objects in $\mathcal{F}$ are in one-to-one correspondence with the irreducible representations of general linear groups. The second family is obtained using the fully-faithful, exact functor $\kappa\colon \mathcal{F}_{\mathrm{iso}} \to \mathcal{F}_{\mathrm{quad}}$ which preserves simple objects, where $\mathcal{F}_{\mathrm{iso}}$ is equivalent to the product of the categories of modules over the orthogonal groups. The results of this paper are summarized in the following theorem.

Theorem *Let α be an element in $\{0,1\}$.*

1. *The functor $\mathrm{Mix}_{\alpha,1}$ is infinite.*
2. *There exists a subfunctor $\Sigma_{\alpha,1}$ of $\mathrm{Mix}_{\alpha,1}$ such that we have the short exact sequence*
$$0 \to \Sigma_{\alpha,1} \to \mathrm{Mix}_{\alpha,1} \to \Sigma_{\alpha,1} \to 0.$$
3. *The functor $\Sigma_{\alpha,1}$ is uniserial with unique composition series given by the decreasing filtration given by the subfunctors $k_d\Sigma_{\alpha,1}$ of $\Sigma_{\alpha,1}$:*
$$\ldots \subset k_d\Sigma_{\alpha,1} \subset \ldots \subset k_1\Sigma_{\alpha,1} \subset k_0\Sigma_{\alpha,1} = \Sigma_{\alpha,1}$$
   (a) *The head of $\Sigma_{\alpha,1}$ (ie $\Sigma_{\alpha,1}/k_1\Sigma_{\alpha,1}$) is isomorphic to the functor $\kappa(\mathrm{iso}_{(x,\alpha)})$ where $\mathrm{iso}_{(x,\alpha)}$ is a simple object in $\mathcal{F}_{\mathrm{iso}}$.*

   (b) *For $d > 0$*
$$k_d\Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1} \simeq L_\alpha^{d+1}$$
   where L_α^{d+1} is a simple object of the category $\mathcal{F}_{\mathrm{quad}}$ that is neither in the image of ι nor in the image of κ.

> *The functor L_α^{d+1} is a subfunctor of $\iota(\Lambda^{d+1}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$, where Λ^{d+1} is the $(d+1)$–st exterior power functor.*

This theorem and the forthcoming paper [11] lead us to conjecture that there are only three types of simple objects in the category $\mathcal{F}_{\mathrm{quad}}$: those in the image of the functor ι, those in the image of the functor κ and those which are subfunctors of a tensor product of the form: $\iota(S) \otimes \kappa(T)$ where S is a simple object in $\mathcal{F}$ and T is a simple object in $\mathcal{F}_{\mathrm{iso}}$.

This paper is divided into seven sections. Section 1 recalls the definition of the category $\mathcal{F}_{\mathrm{quad}}$ and the results obtained in [12]. Section 2 gives a general definition of the mixed functors $\mathrm{Mix}_{V,D,\eta}$ as subfunctors of the tensor product $\iota(P_V^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_D)$ in $\mathcal{F}_{\mathrm{quad}}$, where V is an object in $\mathcal{E}^f$, D is a quadratic vector space, η is an element in the dual of $V \otimes D$, $P_V^{\mathcal{F}}$ is the standard projective object of $\mathcal{F}$ obtained by the Yoneda lemma and iso_D is an isotropic functor in $\mathcal{F}_{\mathrm{iso}}$. Section 3 studies the mixed functors $\mathrm{Mix}_{V,D,\eta}$ such that $\dim(D) = 1$ and $\dim(V) = 1$. We define, in particular, the subfunctor $\Sigma_{\alpha,1}$ of the mixed functor $\mathrm{Mix}_{\alpha,1}$ given in the second point of the previous theorem. In Section 4, we deduce a filtration of the functor $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$ from the polynomial filtration in the category $\mathcal{F}$. Section 5 gives a filtration of the functors $\Sigma_{\alpha,1}$, defined in Section 3, and we obtain the existence of a natural map from the subquotients of this filtration to the functors $\iota(\Lambda^n) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$, by relating this filtration to that introduced in the previous section. Section 6 gives the structure of the functors $\iota(\Lambda^n) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$. We define the functors L_α^n and prove their simplicity. Section 7 proves the structure of the functors $\mathrm{Mix}_{\alpha,1}$ given in the previous theorem.

The results contained in this paper extend results obtained in the author's PhD thesis [13]. The author wishes to thank her PhD supervisor, Lionel Schwartz, for his guidance, as well as Geoffrey Powell and Aurélien Djament for numerous useful discussions and Serge Bouc for suggesting that the methods used in the author's thesis should be sufficient to establish the uniseriality of the functors $\Sigma_{\alpha,1}$.

1 The category $\mathcal{F}_{\mathrm{quad}}$

We recall in this section some definitions and results about the category $\mathcal{F}_{\mathrm{quad}}$ obtained in [12].

Let $\mathcal{E}_q$ be the category having as objects finite dimensional $\mathbb{F}_2$–vector spaces equipped with a non degenerate quadratic form and with morphisms linear maps that preserve the quadratic forms. By the classification of quadratic forms over the field $\mathbb{F}_2$ (see, for instance, Pfister [8]) we know that only spaces of even dimension can be nondegenerate and, for a fixed even dimension, there are two nonequivalent nondegenerate spaces, which are distinguished by the Arf invariant. We will denote by H_0 (resp. H_1) the nondegenerate quadratic space of dimension two such that $\mathrm{Arf}(H_0)=0$ (resp. $\mathrm{Arf}(H_1)=1$). The orthogonal sum of two nondegenerate quadratic spaces (V,q_V) and (W,q_W) is, by definition, the quadratic space $(V\oplus W, q_{V\oplus W})$ where $q_{V\oplus W}(v,w)=q_V(v)+q_W(w)$. Recall that the spaces $H_0\perp H_0$ and $H_1\perp H_1$ are isomorphic. Observe that the morphisms of $\mathcal{E}_q$ are injective linear maps and this category does not admit pushouts or pullbacks. There exists a pseudo pushout in $\mathcal{E}_q$ that allows us to generalize the construction of the category of cospans of Bénabou [1] and thus to define the category $\mathcal{T}_q$ in which there exist retractions.

Definition 1.1 The category $\mathcal{T}_q$ is the category having as objects those of $\mathcal{E}_q$ and, for V and W objects in $\mathcal{T}_q$, $\mathrm{Hom}_{\mathcal{T}_q}(V,W)$ is the set of equivalence classes of diagrams in $\mathcal{E}_q$ of the form $V\xrightarrow{f}X\xleftarrow{g}W$ for the equivalence relation generated by the relation $\mathcal{R}$ defined as follows: $(V\xrightarrow{f}X_1\xleftarrow{g}W)\quad\mathcal{R}\quad(V\xrightarrow{u}X_2\xleftarrow{v}W)$ if there exists a morphism α of $\mathcal{E}_q$ such that $\alpha\circ f=u$ and $\alpha\circ g=v$. The composition is defined using the pseudo pushout. The morphism of $\mathrm{Hom}_{\mathcal{T}_q}(V,W)$ represented by the diagram $V\xrightarrow{f}X\xleftarrow{g}W$ will be denoted by $[V\xrightarrow{f}X\xleftarrow{g}W]$.

By definition, the category $\mathcal{F}_{\mathrm{quad}}$ is the category of functors from $\mathcal{T}_q$ to $\mathcal{E}$. Hence $\mathcal{F}_{\mathrm{quad}}$ is abelian and has enough projective objects. By the Yoneda lemma, for any object V of $\mathcal{T}_q$, the functor $P_V=\mathbb{F}_2[\mathrm{Hom}_{\mathcal{T}_q}(V,-)]$ is a projective object and there is a natural isomorphism: $\mathrm{Hom}_{\mathcal{F}_{\mathrm{quad}}}(P_V,F)\simeq F(V)$, for all objects F of $\mathcal{F}_{\mathrm{quad}}$. The set of functors $\{P_V|V\in\mathcal{S}\}$, named the standard projective objects in $\mathcal{F}_{\mathrm{quad}}$, is a set of projective generators of $\mathcal{F}_{\mathrm{quad}}$, where $\mathcal{S}$ is a set of representatives of isometry classes of nondegenerate quadratic spaces.

There is a forgetful functor $\epsilon\colon \mathcal{T}_q\to\mathcal{E}^f$ in $\mathcal{F}_{\mathrm{quad}}$, defined by $\epsilon(V)=\mathcal{O}(V)$ and

$$\epsilon([V\xrightarrow{f}W\perp W'\xleftarrow{g}W])=p_g\circ\mathcal{O}(f)$$

where p_g is the orthogonal projection from $W\perp W'$ to W and $\mathcal{O}\colon\mathcal{E}_q\to\mathcal{E}^f$ is the functor which forgets the quadratic form. By the fullness of the functor ϵ and an argument of essential surjectivity, we obtain the following theorem.

Theorem 1.2 [12] *There is a functor $\iota\colon\mathcal{F}\to\mathcal{F}_{\mathrm{quad}}$, which is exact, fully faithful and preserves simple objects.*

In order to define another subcategory of $\mathcal{F}_{\mathrm{quad}}$, we consider the category $\mathcal{E}_q^{\deg}$ having as objects finite dimensional $\mathbb{F}_2$–vector spaces equipped with a (possibly degenerate) quadratic form and with morphisms injective linear maps that preserve the quadratic forms. A useful relation between the categories $\mathcal{E}_q$ and $\mathcal{E}_q^{\deg}$ is given by the following theorem, which can be regarded as Witt's theorem for degenerate quadratic forms.

Theorem 1.3 *Let V be a nondegenerate quadratic space, D and D' subquadratic spaces (possibly degenerate) of V and $\underline{f}\colon D \to D'$ an isometry. Then, there exists an isometry $f\colon V \to V$ such that the following diagram is commutative:*

$$
\begin{tikzcd}
V \arrow[r, "f"] & V \\
D \arrow[u, hook] \arrow[r, "\underline{f}"'] & D' \arrow[u, hook]
\end{tikzcd}
$$

Proof For a proof of this result, refer to Bourbaki [3, Section 4, Theorem 1]. □

The category $\mathcal{E}_q^{\deg}$ admits pullbacks; consequently the category of spans $\mathrm{Sp}(\mathcal{E}_q^{\deg})$ is defined [1]. By definition, the category $\mathcal{F}_{\mathrm{iso}}$ is the category of functors from $\mathrm{Sp}(\mathcal{E}_q^{\deg})$ to $\mathcal{E}$. As in the case of the category $\mathcal{F}_{\mathrm{quad}}$, the category $\mathcal{F}_{\mathrm{iso}}$ is abelian and has enough projective objects; by the Yoneda lemma, for any object V of $\mathrm{Sp}(\mathcal{E}_q^{\deg})$, the functor $Q_V = \mathbb{F}_2[\mathrm{Hom}_{\mathrm{Sp}(\mathcal{E}_q^{\deg})}(V, -)]$ is a projective object in $\mathcal{F}_{\mathrm{iso}}$. The category $\mathcal{F}_{\mathrm{iso}}$ is related to $\mathcal{F}_{\mathrm{quad}}$ by the following theorem.

Theorem 1.4 [12] *There is a functor $\kappa\colon \mathcal{F}_{\mathrm{iso}} \to \mathcal{F}_{\mathrm{quad}}$, which is exact, fully-faithful and preserves simple objects.*

We obtain the classification of the simple objects of the category $\mathcal{F}_{\mathrm{iso}}$ from the following theorem.

Theorem 1.5 [12] *There is a natural equivalence of categories*

$$\mathcal{F}_{\mathrm{iso}} \simeq \prod_{V \in \mathcal{S}} \mathbb{F}_2[O(V)] - \mathrm{mod}$$

where $\mathcal{S}$ is a set of representatives of isometry classes of quadratic spaces (possibly degenerate) and $O(V)$ is the orthogonal group.

The object of $\mathcal{F}_{\mathrm{iso}}$ that corresponds, by this equivalence, to the module $\mathbb{F}_2[O(V)]$ is the isotropic functor iso_V, defined in [12]. The family of isotropic functors forms a set of projective generators and injective cogenerators of $\mathcal{F}_{\mathrm{iso}}$. Recall that the isotropic functor $\mathrm{iso}_V\colon \mathrm{Sp}(\mathcal{E}_q^{\deg}) \to \mathcal{E}$ of $\mathcal{F}_{\mathrm{iso}}$ is the image of Q_V by the morphism $a_V\colon Q_V \to DQ_V$

which corresponds by the Yoneda lemma to the element $(\mathrm{Id}_V)^*$ of $DQ_V(V)$, where $(\mathrm{Id}_V)^*$ is defined by

$$(\mathrm{Id}_V)^*([\mathrm{Id}_V]) = 1 \quad \text{and} \quad (\mathrm{Id}_V)^*([f]) = 0 \text{ for all } f \neq \mathrm{Id}_V$$

where we denote by $[f]$ a canonical generator of $DQ_V(V) \simeq \mathbb{F}_2[\mathrm{End}_{\mathrm{Sp}(\mathcal{E}_q^{\mathrm{deg}})}(V)]$. This definition and that of the functor $\kappa\colon \mathcal{F}_{\mathrm{iso}} \to \mathcal{F}_{\mathrm{quad}}$ give rise to the following more concrete definition of the functor iso_V which will be useful below.

Proposition 1.6 *The following equivalent definition of the functor $\kappa(\mathrm{iso}_V)$ holds.*

- *For W an object of $\mathcal{T}_q$,*

$$\kappa(\mathrm{iso}_V)(W) = \mathbb{F}_2[\mathrm{Hom}_{\mathcal{E}_q^{\mathrm{deg}}}(V, W)].$$

- *For a morphism $m = [W \xrightarrow{f} Y \xleftarrow{g} X]$ in $\mathcal{T}_q$ and a canonical generator $[h]$ of $\kappa(\mathrm{iso}_V)(W)$, we consider the following diagram in $\mathcal{E}_q^{\mathrm{deg}}$:*

$$\begin{tikzcd} & & X \arrow[d, "g"] \\ V \arrow[r, "h"'] & W \arrow[r, "f"'] & Y \end{tikzcd}$$

  If the pullback of this diagram in $\mathcal{E}_q^{\mathrm{deg}}$ is V, this gives rise to a unique morphism $h'\colon V \to X$ in $\mathcal{E}_q^{\mathrm{deg}}$, such that $f \circ h = g \circ h'$. In this case, $\kappa(\mathrm{iso}_V)(m)[h] = [h']$. Otherwise, $\kappa(\mathrm{iso}_V)(m)[h] = 0$.

Notation In this paper, a canonical generator of $\kappa(\mathrm{iso}_D)(W)$ will be denoted by $[D \xrightarrow{h} W]$ or, more simply, by $[h]$.

We end this section by a useful corollary of Theorem 1.4 and Theorem 1.5. For $\alpha \in \{0, 1\}$, let (x, α) be the degenerate quadratic space of dimension one generated by x such that $q(x) = \alpha$.

Corollary 1.7 *The functors $\kappa(\mathrm{iso}_{(x,0)})$ and $\kappa(\mathrm{iso}_{(x,1)})$ are simple in $\mathcal{F}_{\mathrm{quad}}$.*

Proof It is a straightforward consequence of the triviality of the orthogonal groups $O(x, 0)$ and $O(x, 1)$. □

2 Definition of the mixed functors

The aim of this section is to define the mixed functors: for this, we consider the functors $\iota(P_V^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_D)$ in $\mathcal{F}_{\mathrm{quad}}$ where V is an object in $\mathcal{E}^f$, $P_V^{\mathcal{F}}$ is the standard projective object of $\mathcal{F}$ obtained by the Yoneda lemma, D is an object in $\mathcal{E}_q^{\deg}$, and $\iota\colon \mathcal{F} \to \mathcal{F}_{\mathrm{quad}}$ and $\kappa\colon \mathcal{F}_{\mathrm{iso}} \to \mathcal{F}_{\mathrm{quad}}$ are the functors defined in [12] and recalled briefly in Theorem 1.2 and Theorem 1.4 respectively. A canonical generator of $P_V^{\mathcal{F}}(W) \simeq \mathbb{F}_2[\mathrm{Hom}_{\mathcal{E}^f}(V, W)]$ will be denoted by $[f]$.

Notation In this paper, the bilinear form associated to a quadratic space V will be denoted by B_V.

Proposition 2.1 *Let D be an object in $\mathcal{E}_q^{\deg}$, V be an object in $\mathcal{E}^f$, η be an element in the dual of $V \otimes D$ and W be an object in $\mathcal{T}_q$. Then the subvector space of $(\iota(P_V^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_D))(W)$ generated by the elements*

$$[f] \otimes [D \xrightarrow{h} W]$$

such that $\qquad$ *for all $v \in V$, for all $d \in D$,* $\quad B_W(f(v), h(d)) = \eta(v \otimes d)$

defines a subfunctor of $\iota(P_V^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_D)$ which will be denoted by $\mathrm{Mix}_{V,D,\eta}$ and called the mixed functor associated to V, D and η.

Proof It is sufficient to verify that, for each morphism $M = [W \xrightarrow{k} Y \xleftarrow{l} Z]$ of $\mathcal{T}_q$ and each generator $[f] \otimes [D \xrightarrow{h} W]$ of $\mathrm{Mix}_{V,D,\eta}(W)$,

$$\mathrm{Mix}_{V,D,\eta}(M)([f] \otimes [D \xrightarrow{h} W]) \in \mathrm{Mix}_{V,D,\eta}(Z).$$

Consider the following diagram in $\mathcal{E}_q^{\deg}$:

$$\begin{tikzcd} & & Z \arrow[d, "l"] \\ D \arrow[r, "h"'] & W \arrow[r, "k"'] & Y \end{tikzcd}$$

If the pullback of this diagram in $\mathcal{E}_q^{\deg}$ is D, namely if $k \circ h(D) \subset l(Z)$, this gives rise to a unique morphism h', from D to Z in $\mathcal{E}_q^{\deg}$, such that $k \circ h = l \circ h'$ that is, the following diagram commutes:

$$\begin{tikzcd} D \arrow[rr, "h'"] \arrow[d, "\mathrm{Id}"'] & & Z \arrow[d, "l"] \\ D \arrow[r, "h"'] & W \arrow[r, "-k"'] & Y \end{tikzcd}$$

In this case, by Proposition 1.6, we have

$$\mathrm{Mix}_{V,D,\eta}(M)([f]\otimes[D\xrightarrow{h}W])=([p_l\circ k\circ f]\otimes[D\xrightarrow{h'}Z])$$

where p_l is the orthogonal projection associated to l. For an element v in V and d in D, we have

$$B_V(f(v),h(d))=B_Y(k\circ f(v),k\circ h(d)).$$

Since the pullback of the diagram considered previously is D, we have $k\circ h(D)\subset l(Z)$. Consequently,

$$B_V(f(v),h(d))=B_Z(p_l\circ k\circ f(v),p_l\circ k\circ h(d))=B_Z(p_l\circ k\circ f(v),h'(d)).$$

Thus, if $B_V(f(-),h(-))=\eta$ then $B_Z(p_l\circ k\circ f(-),h'(-))=\eta$. Therefore the element $([p_l\circ k\circ f]\otimes[D\xrightarrow{h'}Z])$ belongs to $\mathrm{Mix}_{V,D,\eta}(Z)$.

Otherwise by Proposition 1.6 we have

$$\mathrm{Mix}_{V,D,\eta}(M)([f]\otimes[D\xrightarrow{h}W])=0. \qquad \square$$

Remark The terminology "mixed functors" is chosen to reflect the fact that these functors are subfunctors of a tensor product of a functor coming from the category $\mathcal{F}$ and a functor coming from the category $\mathcal{F}_{\mathrm{iso}}$.

We obtain the following decomposition of the functors $\iota(P_V^{\mathcal{F}})\otimes\kappa(\mathrm{iso}_D)$.

Lemma 2.2 *For D an object in $\mathcal{E}_q^{\deg}$ and V an object in $\mathcal{E}^f$ we have*

$$\iota(P_V^{\mathcal{F}})\otimes\kappa(\mathrm{iso}_D)=\bigoplus_{\eta\in(V\otimes D)^*}\mathrm{Mix}_{V,D,\eta}.$$

Proof For two different elements η and η' in $(V\otimes D)^*$, we have

$$\mathrm{Mix}_{V,D,\eta}(W)\cap\mathrm{Mix}_{V,D,\eta'}(W)=\{0\}$$

for W an object in $\mathcal{T}_q$. Thus, we have the decompositions

$$\big(\iota(P_V^{\mathcal{F}})\otimes\kappa(\mathrm{iso}_D)\big)(W)=\Big(\bigoplus_{\eta\in(V\otimes D)^*}\mathrm{Mix}_{V,D,\eta}\Big)(W),$$

for all objects W in $\mathcal{T}_q$. Since $\mathrm{Mix}_{V,D,\eta}$ is a subfunctor of $\iota(P_V^{\mathcal{F}})\otimes\kappa(\mathrm{iso}_D)$ by Proposition 2.1, we deduce the result. $\square$

Remark In the definition of the mixed functors, we don't impose the condition $h(D)\cap f(V)=\{0\}$. Nevertheless, we can define similar functors with this condition, which give rise to quotient functors to the mixed functors defined in Proposition 2.1. These functors will be useful for a later general study of the mixed functors.

3 The functors $\mathrm{Mix}_{V,D,\eta}$ such that $\dim(D)=1$ and $\dim(V)=1$

The aim of this section is to give some general results about the four simplest mixed functors of $\mathcal{F}_{\mathrm{quad}}$ obtained in the case of $\dim(D) = \dim(V) = 1$. The motivation of the particular interest in this case is the study of the projective generators P_{H_0} and P_{H_1} of $\mathcal{F}_{\mathrm{quad}}$. In fact, we prove in [11], that the mixed functors that are direct summands of these two standard projective generators of $\mathcal{F}_{\mathrm{quad}}$ verify the conditions $\dim(D) = \dim(V) = 1$.

When V and D are spaces of dimension one, we will denote by $\mathrm{Mix}_{\alpha,\beta}$, where α and β are elements of $\{0,1\}$, the functor $\mathrm{Mix}_{V,D,\eta}$ such that $V \simeq \mathbb{F}_2$, $D \simeq (x,\alpha)$ and $\eta = \beta$. We have the following result.

Lemma 3.1 *Let W be an object in $\mathcal{T}_q$, if $[f] \otimes [(x,\alpha) \xrightarrow{h} W]$ is a canonical generator of $\mathrm{Mix}_{\alpha,\beta}(W)$, then $[f + h(x)] \otimes [(x,\alpha) \xrightarrow{h} W]$ is a canonical generator of $\mathrm{Mix}_{\alpha,\beta}(W)$.*

Proof This is a straightforward consequence of the fact that the bilinear form associated to a quadratic form is alternating. □

In order to make this symmetry clearer in the set of canonical generators of $\mathrm{Mix}_{\alpha,\beta}(W)$ and to introduce an action of the symmetric group $\mathfrak{S}_2$ on this set, we use a slightly different description of the canonical generators of $\mathrm{Mix}_{\alpha,\beta}(W)$ corresponding to a reindexing of these canonical generators.

Definition 3.2 For α and β elements of $\{0,1\}$, we consider the following set:

$$N^W_{\alpha,\beta} = \{(w_1,w_2) \mid w_1 \in W, w_2 \in W,\ q(w_1+w_2) = \alpha,\ B(w_1,w_2) = \beta\}.$$

We have the following result.

Lemma 3.3 *For $D \simeq (x,\alpha)$ and $\eta = \beta$, we have*

$$\mathrm{Mix}_{\alpha,\beta}(W) \simeq \mathbb{F}_2[N^W_{\alpha,\beta}]$$

where W is an object in $\mathcal{T}_q$.

Furthermore, for a morphism $m = [W \xrightarrow{f} Y \xleftarrow{g} X]$ in $\mathcal{T}_q$ and a canonical generator $[(w_1,w_2)]$ of $\mathbb{F}_2[N^W_{\alpha,\beta}]$, we consider the diagram in $\mathcal{E}_q^{\deg}$

$$\begin{tikzcd} & & X \arrow[d, "g"] \\ (x,\alpha) \arrow[r, "l"'] & W \arrow[r, "f"'] & Y \end{tikzcd}$$

where l is the morphism of $\mathcal{E}_q^{\deg}$ given by $l(x) = w_1 + w_2$. If the pullback of this diagram in $\mathcal{E}_q^{\deg}$ is (x,α) then $\mathrm{Mix}_{\alpha,\beta}(m)[(w_1,w_2)] = [(p_g \circ f(w_1), p_g \circ f(w_2)]$ where p_g is the orthogonal projection associated to g. Otherwise, $\mathrm{Mix}_{\alpha,\beta}(m)[(w_1,w_2)] = 0$.

Proof The generator of the vector space V of dimension one will be denoted by a. There is an isomorphism

$$\begin{aligned} f_W\colon \mathrm{Mix}_{\alpha,\beta}(W) &\to \mathbb{F}_2[N^W_{\alpha,\beta}] \\ [f]\otimes[(x,\alpha)\xrightarrow{h} W] &\mapsto [(f(a)+h(x), f(a))], \end{aligned}$$

of which the inverse is given by

$$\begin{aligned} f_W^{-1}\colon \mathbb{F}_2[N^W_{\alpha,\beta}] &\to \mathrm{Mix}_{\alpha,\beta}(W) \\ [(w_1,w_2)] &\mapsto [k]\otimes[(x,\alpha)\xrightarrow{l} W] \end{aligned}$$

where $k\colon V\to W$ is defined by $k(a) = w_2$ and $l\colon (x,\alpha)\to W$ is defined by $l(x) = w_1 + w_2$.

The second statement of the lemma is only a translation of the definition of the mixed functors on the sets of morphisms in terms of the sets $N^W_{\alpha,\beta}$. □

Notation Henceforth, we will use the basis given by the set $N^W_{\alpha,\beta}$ to represent the elements of $\mathrm{Mix}_{\alpha,\beta}(W)$.

Thus, the canonical generator $[f]\otimes[(x,\alpha)\xrightarrow{h} W]$ of $\mathrm{Mix}_{\alpha,\beta}(W)$ is represented by $[(f(a)+h(x), f(a))]$ and $[f+h(x)]\otimes[(x,\alpha)\xrightarrow{h} W]$, which is also a canonical generator of $\mathrm{Mix}_{\alpha,\beta}(W)$ by Lemma 3.1, is represented by $[(f(a), f(a)+h(x))]$.

We have the following lemma.

Lemma 3.4 *The symmetric group $\mathfrak{S}_2$ acts on the functor $\mathrm{Mix}_{\alpha,\beta}$.*

Proof Let W be an object of $\mathcal{T}_q$. Define an action of $\mathfrak{S}_2 = \{\mathrm{Id}, \tau\}$ on $\mathrm{Mix}_{\alpha,\beta}(W)$ by

$$\tau\cdot[(w_1,w_2)] = [(w_2,w_1)].$$

We leave the reader to verify that the linear maps

$$\begin{aligned} \tau_W\colon \mathrm{Mix}_{\alpha,\beta}(W) &\to \mathrm{Mix}_{\alpha,\beta}(W) \\ [(w_1,w_2)] &\mapsto [(w_2,w_1)] \end{aligned}$$

define a natural transformation. □

This lemma allows us to define an object in $\mathcal{F}_{\text{quad}}$ by considering the invariants by this action.

Definition 3.5 Let $\Sigma_{\alpha,\beta}$ be the subfunctor of $\text{Mix}_{\alpha,\beta}$ defined by considering the invariants of $\text{Mix}_{\alpha,\beta}(W)$ by the action of the symmetric group $\mathfrak{S}_2$.

In the following, we will focus on study the functors $\text{Mix}_{0,1}$ and $\text{Mix}_{1,1}$. These two functors are particularly interesting since they are direct summands of P_{H_0} and P_{H_1} (see [11]).

We have the following lemma.

Lemma 3.6 *Let W be an object in $\mathcal{T}_q$ and $[(w_1, w_2)]$ be a generator of $\text{Mix}_{\alpha,1}(W)$, then the vectors w_1 and w_2 are linearly independent.*

Proof This is a straightforward consequence of the fact that the bilinear form B is alternating. □

We deduce the following lemma.

Lemma 3.7 *Let W be an object in $\mathcal{T}_q$, the action of $\mathfrak{S}_2$ on the set of canonical generators of $\text{Mix}_{\alpha,1}(W)$ is free.*

Proof For a canonical generator $[(w_1, w_2)]$ of $\text{Mix}_{\alpha,1}(W)$, since the vectors w_1 and w_2 are linearly independent by Lemma 3.6, we have $w_1 \neq w_2$. Hence, the action of $\mathfrak{S}_2$ is free. □

Remark We deduce from Lemma 3.6 that the two functors $\text{Mix}_{\alpha,1}$, coincide with the functors mentioned in the last remark of the Section 2.

We give the following general result about the free actions of the group $\mathfrak{S}_2$.

Lemma 3.8 *If A is a finite set equipped with a free action of the group $\mathfrak{S}_2$ then there exists a short exact sequence of $\mathfrak{S}_2$–modules:*

$$0 \to \mathbb{F}_2[A]^{\mathfrak{S}_2} \to \mathbb{F}_2[A] \to \mathbb{F}_2[A]^{\mathfrak{S}_2} \to 0.$$

Proof We deduce from the action of $\mathfrak{S}_2$ on A, the existence of the canonical inclusion $\mathbb{F}_2[A]^{\mathfrak{S}_2} \overset{f}{\hookrightarrow} \mathbb{F}_2[A]$ of the invariants in $\mathbb{F}_2[A]$. The norm $\mathbb{F}_2[A] \xrightarrow{1+\tau} \mathbb{F}_2[A]$ induces a linear map $\mathbb{F}_2[A] \xrightarrow{g} F[A]^{\mathfrak{S}_2}$ such that the composition

$$\mathbb{F}_2[A]^{\mathfrak{S}_2} \overset{f}{\hookrightarrow} \mathbb{F}_2[A] \xrightarrow{g} \mathbb{F}_2[A]^{\mathfrak{S}_2}$$

is trivial. We verify that this defines a short exact sequence. □

We deduce the following proposition.

Proposition 3.9 *There exists a short exact sequence*

$$0 \to \Sigma_{\alpha,1} \to \mathrm{Mix}_{\alpha,1} \to \Sigma_{\alpha,1} \to 0. \tag{3–1}$$

Proof This is a straightforward consequence of Lemma 3.7 and Lemma 3.8. □

Notation We will denote by $[\{w_1, w_2\}]$ the image of the element $[(w_1, w_2)]$ of $\mathrm{Mix}_{\alpha,1}(W)$ in $\Sigma_{\alpha,1}(W)$ by the surjection $\mathrm{Mix}_{\alpha,1}(W) \twoheadrightarrow \Sigma_{\alpha,1}(W)$.

Remark Lemma 3.7 has no analogue for the functors $\mathrm{Mix}_{0,0}$ and $\mathrm{Mix}_{1,0}$ since, in these two cases, the action of the group $\mathfrak{S}_2$ is not free. Nevertheless, we can apply similar arguments to the functors $\mathrm{Mix}'_{\alpha,0}$ such that $\mathrm{Mix}_{\alpha,0} \twoheadrightarrow \mathrm{Mix}'_{\alpha,0}$ mentioned in the last remark of Section 2. In fact the condition $h(D) \cap f(V) = \{0\}$ implies the freedom of the action of the group $\mathfrak{S}_2$ on these functors.

Remark It is shown in [11] that the functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$ are indecomposable. Consequently, the short exact sequence (3–1) is not split for the functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$.

4 Study of the functor $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$

By Proposition 2.1, the functor $\mathrm{Mix}_{\alpha,1}$ is a subfunctor of $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$. Consequently, in order to obtain the decomposition of $\mathrm{Mix}_{\alpha,1}$, we study, in this section, the functor $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$.

4.1 Filtration of the functors $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$

We define, below, the filtration of the functors $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$ induced by the polynomial filtration of the functor $P_{\mathbb{F}_2}^{\mathcal{F}}$ in the category $\mathcal{F}$. First we recall the essential results concerning the polynomial functors in the category $\mathcal{F}$. We refer the interested reader to Henn, Lannes and Schwartz [5], Kuhn [6] and Schwartz [10] for details on the subject.

Notation Henceforth, in order to simplify the notation, we will denote the functor $\iota(P_{\mathbb{F}_2}^{\mathcal{F}}) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$ by $P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$ and, if $F \neq P_{\mathbb{F}_2}^{\mathcal{F}}$, we will denote the functor $\iota(F) \otimes \kappa(\mathrm{iso}_{(x,\alpha)})$ by $F \otimes \mathrm{iso}_\alpha$.

Definition 4.1 Let F be an object in $\mathcal{F}$ and d an integer, the functor $q_d F$ is the largest polynomial quotient of degree d of the functor F.

Notation We denote by $k_d F$ the kernel of $F \twoheadrightarrow q_d F$.

We have the following result.

Proposition 4.2 *The functors $k_d F$ define a decreasing filtration of the functor F, indexed by natural numbers.*

Thus, for the standard projective functor $P_{\mathbb{F}}$, we have the short exact sequence

$$0 \to k_d P_{\mathbb{F}} \to P_{\mathbb{F}} \xrightarrow{f_d} q_d P_{\mathbb{F}} \to 0. \tag{4–1}$$

Furthermore, the decreasing filtration of $P_{\mathbb{F}}$ given by the functors $k_d P_{\mathbb{F}}$ is separated (that is $\bigcap k_d P_{\mathbb{F}} = 0$).

We recall below the description of the vector space $k_d P_{\mathbb{F}}(V)$ for V an object in $\mathcal{E}^f$.

Proposition 4.3 *The vector space $k_d P_{\mathbb{F}}(V)$ is generated by the elements*

$$\sum_{z \in \mathcal{L}} [z]$$

where $\mathcal{L}$ is a subvector space of V of dimension $d+1$.

Notation The subvector space of V or subquadratic space of (V, q_V) generated by $v_1, \dots v_n$ will be denoted by $\mathrm{Vect}(v_1, \dots v_n)$.

The subquotients of the filtration of the functor $P_{\mathbb{F}}$ are given in the following proposition.

Proposition 4.4 [7, Theorem 7.8] *For d a nonnegative integer, there exists a short exact sequence*

$$0 \to k_{d+1} P_{\mathbb{F}} \to k_d P_{\mathbb{F}} \xrightarrow{g_d} \Lambda^{d+1} \to 0 \tag{4–2}$$

where Λ^{d+1} is the $(d+1)$–th exterior power and the map g_d is defined in the following way: for V an object in $\mathcal{E}^f$ and $\mathcal{L}$ the subvector space of V of dimension $d+1$ generated by the elements $l_1, \dots, l_{d+1}$ of V, we have

$$(g_d)_V\Big(\sum_{z \in \mathcal{L}} [z]\Big) = l_1 \wedge \dots \wedge l_{d+1}.$$

By taking tensor product with the isotropic functors iso_α, Proposition 4.2 and the short exact sequences (4–1) and (4–2) give rise to the following result.

Corollary 4.5 (1) *For d an integer, the functors $(k_d P_{\mathbb{F}}) \otimes \mathrm{iso}_\alpha$ define a decreasing separated filtration of the functor $P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$.*

(2) *There exist the following short exact sequences in $\mathcal{F}_{\mathrm{quad}}$:*

$$0 \to (k_d P_{\mathbb{F}}) \otimes \mathrm{iso}_\alpha \to P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha \xrightarrow{f_d \otimes \mathrm{iso}_\alpha} (q_d P_{\mathbb{F}}) \otimes \mathrm{iso}_\alpha \to 0 \tag{4–3}$$

$$0 \to (k_{d+1} P_{\mathbb{F}}) \otimes \mathrm{iso}_\alpha \to (k_d P_{\mathbb{F}}) \otimes \mathrm{iso}_\alpha \xrightarrow{g_d \otimes \mathrm{iso}_\alpha} (\Lambda^{d+1}) \otimes \mathrm{iso}_\alpha \to 0. \tag{4–4}$$

Remark We obtain similar results by taking the tensor product between the short exact sequence (4–2) and the isotropic functors iso_D. This will be useful for a general study of the mixed functors.

5 Filtration of the functors $\Sigma_{\alpha,1}$

In this section, we define a filtration of the functors $\Sigma_{\alpha,1}$ that we will relate below to the filtration of $P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$ obtained in the previous section.

Definition 5.1 For V an object of $\mathcal{T}_q$ and d an integer, let $k_d \Sigma_{\alpha,1}(V)$ be the subvector space of $\Sigma_{\alpha,1}(V)$ generated by the elements

$$\sum_{z \in \mathcal{L}} [\{x+z, y+z\}],$$

where $[\{x,y\}] \in \Sigma_{\alpha,1}(V)$ and $\mathcal{L}$ is a subvector space of $\mathrm{Vect}(x+y)^{\perp}$ of dimension d.

Proposition 5.2 *The spaces $k_d \Sigma_{\alpha,1}(V)$, for V an object in $\mathcal{T}_q$, define a subfunctor of $\Sigma_{\alpha,1}$.*

Proof For a morphism T in $\mathrm{Hom}_{\mathcal{T}_q}(V,W)$, it is straightforward to check, by definition of $\mathrm{Mix}_{\alpha,1}(T)$, that the image of $k_d \Sigma_{\alpha,1}(V)$ by

$$\Sigma_{\alpha,1}(T)\colon k_d \Sigma_{\alpha,1}(V) \to \Sigma_{\alpha,1}(W)$$

is a subvector space of $k_d \Sigma_{\alpha,1}(W)$. □

Proposition 5.3 *The functors $k_d \Sigma_{\alpha,1}$ define a separated decreasing filtration of the functor $\Sigma_{\alpha,1}$:*

$$\ldots \subset k_d \Sigma_{\alpha,1} \subset \ldots \subset k_1 \Sigma_{\alpha,1} \subset k_0 \Sigma_{\alpha,1} = \Sigma_{\alpha,1}.$$

Proof To show that the filtration is decreasing, it is sufficient to prove, for an object V of $\mathcal{T}_q$, that there is an inclusion of vector spaces

$$k_{d+1}\Sigma_{\alpha,1}(V) \subset k_d\Sigma_{\alpha,1}(V).$$

We consider a generator of $k_{d+1}\Sigma_{\alpha,1}(V)$: $v = \sum_{z\in\mathcal{L}}[\{x+z, y+z\}]$ where $[\{x,y\}]$ is in $\Sigma_{\alpha,1}(V)$ and $\mathcal{L}$ is the subvector space of $\mathrm{Vect}(x+y)^{\perp}$ of dimension $d+1$ generated by the elements $l_1,\dots,l_{d+1}$. We also consider the following decomposition into direct summands: $\mathcal{L} = \mathrm{Vect}(l_1,\dots,l_d) \oplus \mathrm{Vect}(l_{d+1}) = \mathcal{L}' \oplus \mathrm{Vect}(l_{d+1})$. By considering separately the elements z in $\mathcal{L}$ with an nonzero component on l_{d+1} and those with a zero component on l_{d+1}, we obtain

$$v = \sum_{z\in\mathcal{L}'}[\{x+z, y+z\}] + \sum_{z\in\mathcal{L}'}[\{x+l_{d+1}+z, y+l_{d+1}+z\}].$$

By definition of $k_d\Sigma_{\alpha,1}$, we have

$$\textstyle\sum_{z\in\mathcal{L}'}[\{x+z, y+z\}] \in k_d\Sigma_{\alpha,1}(V)$$

and
$$\textstyle\sum_{z\in\mathcal{L}'}[\{x+l_{d+1}+z, y+l_{d+1}+z\}] \in k_d\Sigma_{\alpha,1}(V)$$

since $[\{x+l_{d+1}, y+l_{d+1}\}] \in \Sigma_{\alpha,1}(V)$. Consequently $v \in k_d\Sigma_{\alpha,1}(V)$.

One verifies easily that the filtration is separated. □

In the following result, we relate the filtration of the functors $\Sigma_{\alpha,1}$ and the filtration of $P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$ obtained from the polynomial filtration in Corollary 4.5.

Lemma 5.4 *The composition*

$$k_d\Sigma_{\alpha,1} \hookrightarrow \Sigma_{\alpha,1} \hookrightarrow \mathrm{Mix}_{\alpha,1} \hookrightarrow P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha \xrightarrow{f_d\otimes \mathrm{iso}_\alpha} q_d P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$$

is zero.

Proof Let V be an object in $\mathcal{T}_q$ and v a generator of $k_d\Sigma_{\alpha,1}(V)$. Then $v = \sum_{z\in\mathcal{L}}[\{y+z, y'+z\}]$ where $[\{y,y'\}] \in \Sigma_{\alpha,1}(V)$ and $\mathcal{L}$ is the subvector space of $\mathrm{Vect}(y+y')^{\perp}$ of dimension d.

Let h be the element of $\mathrm{Hom}_{\mathcal{E}_q^{\deg}}((x,\alpha),V)$ such that $h(x) = y+y'$, we have:

$$\begin{aligned}\sum_{z\in\mathcal{L}}([y+z]+[y'+z])\otimes[h] &= \sum_{z\in\mathcal{L}}([y+z]+[z]+[y'+z]+[z])\otimes[h]\\ &= \sum_{z\in\mathcal{L}}([y+z]+[z])\otimes[h] + \sum_{z\in\mathcal{L}}([y'+z]+[z])\otimes[h]\\ &= \sum_{z'\in\mathcal{L}\oplus\mathrm{Vect}(y)}[z']\otimes[h] + \sum_{z''\in\mathcal{L}\oplus\mathrm{Vect}(y')}[z'']\otimes[h]\end{aligned}$$

By Proposition 4.3 we have:

$$\sum_{z' \in \mathcal{L} \oplus \mathrm{Vect}(y)} [z'] \in k_d P_{\mathbb{F}}(V) \quad \text{and} \quad \sum_{z'' \in \mathcal{L} \oplus \mathrm{Vect}(y')} [z''] \in k_d P_{\mathbb{F}}(V)$$

Hence $f_d \otimes \mathrm{iso}_\alpha(v) = 0$. □

We have the following result.

Proposition 5.5 (1) *There exists a monomorphism $i_d \colon k_d \Sigma_{\alpha,1} \to k_d P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$.*

(2) *There exists a natural map*

$$k_d \Sigma_{\alpha,1} / k_{d+1} \Sigma_{\alpha,1} \to \Lambda^{d+1} \otimes \mathrm{iso}_\alpha$$

induced by $i_d \colon k_d \Sigma_{\alpha,1} \to k_d P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$.

Proof The first point is a direct consequence of Lemma 5.4 and the short exact sequence (4–3).

We deduce the second point from the following commutative diagram given by the first point and from the short exact sequence (4–4).

$$\begin{tikzcd}
0 \arrow[r] & k_{d+1}\Sigma_{\alpha,1} \arrow[r] \arrow[d, hook, "i_{d+1}"'] & k_d \Sigma_{\alpha,1} \arrow[r] \arrow[d, hook, "i_d"'] & (k_d \Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k_{d+1} P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha \arrow[r] & k_d P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha \arrow[r, "g_d \otimes \mathrm{iso}_\alpha"'] & \Lambda^{d+1} \otimes \mathrm{iso}_\alpha \arrow[r] & 0
\end{tikzcd} \tag{5–1}$$
□

Hence, to obtain the composition factors of the functors $\mathrm{Mix}_{\alpha,1}$ we study the functors $\Lambda^n \otimes \mathrm{iso}_\alpha$ in the following section.

Remark We obtain that the natural map $k_d \Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1} \to \Lambda^{d+1} \otimes \mathrm{iso}_\alpha$ is a monomorphism as a consequence of Theorem 7.1.

To conclude this section, we prove that the functors $\mathrm{Mix}_{\alpha,1}$ are infinite. For this, we need the following lemma.

Lemma 5.6 *Let V be an object in $\mathcal{T}_q$ of dimension greater than $d+1$ such that $\Sigma_{\alpha,1}(V) \neq \{0\}$, let $[\{y, y'\}]$ be a canonical generator of $\Sigma_{\alpha,1}(V)$ and let $v_1, \dots v_d$ be d linearly independent elements in $\mathrm{Vect}(y+y')^\perp$. Then*

$$(g_d \otimes \mathrm{iso}_\alpha) \circ i_d \Big(\sum_{z \in \mathrm{Vect}(v_1, \dots, v_d)} [\{y+z, y'+z\}] \Big) = v_1 \wedge \dots \wedge v_d \wedge (y+y') \otimes [h]$$

where i_d is the monomorphism from $k_d \Sigma_{\alpha,1}$ to $k_d P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$ defined in the first point of Proposition 5.5 and h the element of $\mathrm{Hom}_{\mathcal{E}_q}((x,\alpha), V)$ such that $h(x) = y+y'$.

Proof We have, by definition of i_d and $g_d \otimes \mathrm{iso}_\alpha$,

$$\begin{aligned}(g_d \otimes \mathrm{iso}_\alpha) \circ i_d\Big(\sum_{z \in \mathrm{Vect}(v_1,\dots,v_d)} [\{y+z, y'+z\}]\Big) &= (g_d \otimes \mathrm{iso}_\alpha)\Big(\sum_{z \in \mathrm{Vect}(v_1,\dots,v_d)} ([y+z]+[y'+z]) \otimes [h]\Big) \\ &= (v_1 \wedge \dots \wedge v_d \wedge y + v_1 \wedge \dots \wedge v_d \wedge y') \otimes [h] \\ &= v_1 \wedge \dots \wedge v_d \wedge (y+y') \otimes [h] \end{aligned}$$ □

Proposition 5.7 *The functors* $\mathrm{Mix}_{\alpha,1}$ *are infinite.*

Proof It is sufficient to prove that the quotients of the filtration of $\Sigma_{\alpha,1}$ are nonzero. For an object V in $\mathcal{T}_q$ of dimension greater than d, the space $\Sigma_{1,1}(H_0 \perp V)$ contains the nonzero element $[\{a_0, b_0\}]$. Hence, the element of $k_d \Sigma_{1,1}(H_0 \perp V)$

$$x = \sum_{z \in \mathrm{Vect}(v_1,\dots,v_d)} [\{a_0+z, b_0+z\}]$$

verifies $$(g_d \otimes \mathrm{iso}_\alpha) \circ i_d(x) = v_1 \wedge \dots \wedge v_d \wedge (a_0+b_0) \otimes [h] \neq 0$$

by Lemma 5.6. Consequently, $(g_d \otimes \mathrm{iso}_\alpha) \circ i_d(k_d \Sigma_{1,1}) \neq \{0\}$ and, by the commutativity of the diagram (5–1) given in the proof of the second point of Proposition 5.5, we have $k_d \Sigma_{1,1}/k_{d+1}\Sigma_{1,1} \neq \{0\}$.

In the same way, by considering the element

$$\sum_{z \in \mathrm{Vect}(v_1,\dots,v_d)} [\{a_0+z, a_0+b_0+z\}] \in k_d \Sigma_{0,1}(H_0 \perp V),$$

we show that $k_d \Sigma_{0,1}/k_{d+1}\Sigma_{0,1} \neq \{0\}$. □

6 Structure of the functors $\Lambda^n \otimes \mathrm{iso}_\alpha$

By the second point of Proposition 5.5, there exists a natural map from the subquotients $k_d \Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1}$ of the filtration of the functor $\Sigma_{\alpha,1}$ by the functors $k_d \Sigma_{\alpha,1}$ to the functors $\Lambda^{d+1} \otimes \mathrm{iso}_\alpha$. The aim of this section is to study the functors $\Lambda^n \otimes \mathrm{Iso}_\alpha$ in order to obtain the composition factors of the functors $\Sigma_{\alpha,1}$. It is divided into two subsections: the first concerns the decompositions of the functors $\Lambda^n \otimes \mathrm{Iso}_\alpha$ by the functors denoted by L^n_α, and the second concerns the simplicity of the functors L^n_α.

6.1 Decomposition

In order to identify the composition factors of the functor $\Lambda^n \otimes \mathrm{Iso}_\alpha$, we define the following morphisms of $\mathcal{F}_{\text{quad}}$.

Lemma 6.1 (1) *For an object V in $\mathcal{T}_q$, the linear maps*

$$\mu_V\colon \mathrm{iso}_\alpha(V) \to (\Lambda^1 \otimes \mathrm{iso}_\alpha)(V)$$

defined by $$\mu_V([(x,\alpha)\xrightarrow{h} V]) = h(x) \otimes [(x,\alpha)\xrightarrow{h} V]$$

for a canonical generator $[(x,\alpha)\xrightarrow{h} V]$ in $\mathrm{iso}_\alpha(V)$ give rise to a monomorphism μ from iso_α to $\Lambda^1 \otimes \mathrm{iso}_\alpha$ of $\mathcal{F}_{\text{quad}}$.

(2) *For an object V in $\mathcal{T}_q$, the linear maps*

$$\nu_V\colon (\Lambda^1 \otimes \mathrm{iso}_\alpha)(V) \to \mathrm{iso}_\alpha(V)$$

defined by $$\nu_V(w \otimes [(x,\alpha)\xrightarrow{h} V]) = B(w, h(x))[(x,\alpha)\xrightarrow{h} V]$$

for a canonical generator $[(x,\alpha)\xrightarrow{h} V]$ in $\mathrm{iso}_\alpha(V)$ give rise to an epimorphism ν from $\Lambda^1 \otimes \mathrm{iso}_\alpha$ to iso_α of $\mathcal{F}_{\text{quad}}$.

Proof (1) We check the commutativity of the following diagram, for a morphism $T = [V \xrightarrow{f} X \xleftarrow{g} W]$ of $\mathrm{Hom}_{\mathcal{T}_q}(V, W)$:

$$\begin{tikzcd}
\mathrm{iso}_\alpha(V) \arrow[r, "\mu_V"] \arrow[d, "\mathrm{iso}_\alpha(T)"'] & (\Lambda^1 \otimes \mathrm{iso}_\alpha)(V) \arrow[d, "(\Lambda\otimes\mathrm{iso}_\alpha)(T)"] \\
\mathrm{iso}_\alpha(W) \arrow[r, "\mu_W"'] & (\Lambda^1 \otimes \mathrm{iso}_\alpha)(W)
\end{tikzcd}$$

For a canonical generator $[(x,\alpha)\xrightarrow{h} V]$ of $\mathrm{iso}_\alpha(V)$, we denote by P the pullback of the diagram $(x,\alpha)\xrightarrow{f\circ h} X \xleftarrow{g} W$. If $P = (x,\alpha)$, we denote by h' the morphism making the following diagram commutative:

$$\begin{tikzcd}
(x,\alpha) \arrow[rr, "h'"] \arrow[d, "\mathrm{Id}"'] & & W \arrow[d, "g"] \\
(x,\alpha) \arrow[r, "h"'] & V \arrow[r, "f"'] & X
\end{tikzcd}$$

We have

$$(\Lambda^1 \otimes \mathrm{iso}_\alpha)(T) \circ \mu_V([(x,\alpha) \xrightarrow{h} V]) = (\Lambda^1 \otimes \mathrm{iso}_\alpha)(T)(h(x) \otimes [(x,\alpha) \xrightarrow{h} V])$$
$$= \begin{cases} p_g \circ f \circ h(x) \otimes [(x,\alpha) \xrightarrow{h'} W] & \text{if } P = (x,\alpha) \\ 0 & \text{otherwise} \end{cases}$$

and $$\mu_W \circ \mathrm{iso}_\alpha(T)([(x,\alpha) \xrightarrow{h} V]) = \mu_W \begin{cases} [(x,\alpha) \xrightarrow{h'} W] & \text{if } P = (x,\alpha) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} h'(x) \otimes [(x,\alpha) \xrightarrow{h'} W] & \text{if } P = (x,\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

By commutativity of the previous cartesian diagram, we have $g \circ h' = f \circ h$ hence, by composition with p_g, we obtain: $h' = p_g \circ f \circ h$. Consequently, the linear maps μ_W give rise to a nonzero natural map $\mu\colon \mathrm{iso}_\alpha \to \Lambda^1 \otimes \mathrm{iso}_\alpha$. We deduce from the simplicity of the functors iso_α given in Corollary 1.7 that the natural map μ is a monomorphism in $\mathcal{F}_{\mathrm{quad}}$.

(2) We check the commutativity of the following diagram, for a morphism $T = [V \xrightarrow{f} X \xleftarrow{g} W]$ of $\mathrm{Hom}_{\mathcal{T}_q}(V,W)$:

$$\begin{tikzcd} (\Lambda^1 \otimes \mathrm{iso}_\alpha)(V) \arrow[r, "\nu_V"] \arrow[d, "(\Lambda\otimes\mathrm{iso}_\alpha)(T)"'] & \mathrm{iso}_\alpha(V) \arrow[d, "\mathrm{iso}_\alpha(T)"] \\ (\Lambda^1 \otimes \mathrm{iso}_\alpha)(W) \arrow[r, "\nu_W"'] & \mathrm{iso}_\alpha(W) \end{tikzcd}$$

With the same notations as above, we have:

$$\mathrm{iso}_\alpha(T) \circ \nu_V(v \otimes [h]) = \mathrm{iso}_\alpha(T)(B(h(x),v)[h])$$
$$= \begin{cases} B(h(x),v)[h'] & \text{if } P \simeq \mathrm{Vect}(x) \\ 0 & \text{otherwise} \end{cases}$$
$$\nu_W \circ (\Lambda^1 \otimes \mathrm{iso}_\alpha)(T)(v \otimes [h]) = \nu_W \begin{cases} p_g \circ f(v) \otimes [h'] & \text{if } P \simeq \mathrm{Vect}(x) \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} B(h'(x), p_g \circ f(v))[h'] & \text{if } P \simeq \mathrm{Vect}(x) \\ 0 & \text{otherwise} \end{cases}$$

We also have:

$$\begin{aligned} B(h(x),v) &= B(f \circ h(x), f(v)) && \text{since } f \text{ preserves quadratic forms} \\ &= B(g \circ h'(x), f(v)) && \text{by commutativity of the cartesian diagram} \\ &= B(g \circ h'(x), g \circ p_g \circ f(v)) && \text{by orthogonality of } W \text{ and } L \\ &= B(h'(x), p_g \circ f(v)) && \text{since } g \text{ preserves quadratic forms.} \end{aligned}$$

Hence, the linear maps ν_W define a natural map $\nu\colon \Lambda^1 \otimes \mathrm{iso}_\alpha \to \mathrm{iso}_\alpha$, which is clearly nonzero. We deduce from the simplicity of the functors iso_α given in Corollary 1.7, that ν is an epimorphism of $\mathcal{F}_{\mathrm{quad}}$. □

The natural maps μ and ν defined in the previous lemma allow us to define the following morphisms in $\mathcal{F}_{\mathrm{quad}}$.

Definition 6.2 Let n be a nonnegative integer.

(1) The natural map $\mu_n\colon \Lambda^n \otimes \mathrm{iso}_\alpha \to \Lambda^{n+1} \otimes \mathrm{iso}_\alpha$ is obtained by the composition

$$\Lambda^n \otimes \mathrm{iso}_\alpha \xrightarrow{1\otimes\mu} \Lambda^n \otimes \Lambda^1 \otimes \mathrm{iso}_\alpha \xrightarrow{m\otimes 1} \Lambda^{n+1} \otimes \mathrm{iso}_\alpha$$

where $m\colon \Lambda^n \otimes \Lambda^1 \to \Lambda^{n+1}$ is the product in the exterior algebra.

(2) The natural map $\nu_n\colon \Lambda^{n+1} \otimes \mathrm{iso}_\alpha \to \Lambda^n \otimes \mathrm{iso}_\alpha$ is obtained by the composition

$$\Lambda^{n+1} \otimes \mathrm{iso}_\alpha \xrightarrow{\Delta\otimes 1} \Lambda^n \otimes \Lambda^1 \otimes \mathrm{iso}_\alpha \xrightarrow{1\otimes\nu} \Lambda^n \otimes \mathrm{iso}_\alpha$$

where $\Delta\colon \Lambda^{n+1} \to \Lambda^n \otimes \Lambda^1$ is the coproduct.

We have the following proposition.

Proposition 6.3 *The following sequence is an exact complex:*

$$\ldots \to \Lambda^n \otimes \mathrm{iso}_\alpha \xrightarrow{\mu_n} \Lambda^{n+1} \otimes \mathrm{iso}_\alpha \xrightarrow{\mu_{n+1}} \Lambda^{n+2} \otimes \mathrm{iso}_\alpha \to \ldots$$

Proof We prove, according to the definition, that the kernel of $(\mu_{n+1})_V$ is the vector space generated by the set

$$\{v_1 \wedge \ldots \wedge v_n \wedge h(x) \otimes [h] \text{ for } [h] \text{ a generator of } \mathrm{iso}_\alpha(V);\ v_1, \ldots, v_n \text{ elements in } V\}$$

and this space coincide with the image of $(\mu_n)_V$. □

Proposition 6.3 justifies the introduction of the following functor.

Definition 6.4 The functor K^n_α is the kernel of the map μ_n from $\Lambda^n \otimes \mathrm{iso}_\alpha$ to $\Lambda^{n+1} \otimes \mathrm{iso}_\alpha$.

As observed in the proof of Proposition 6.3, we have the following characterization of the spaces $K^n_\alpha(V)$.

Lemma 6.5 *For an object V in $\mathcal{T}_q$, the space $K^n_\alpha(V)$ is generated by the elements*

$$z \wedge h(x) \otimes [h]$$

where $[h]$ is a canonical generator of the space $\mathrm{iso}_\alpha(V)$ and z is an element in $\Lambda^{n-1}(V)$.

The result below is a straightforward consequence of Proposition 6.3.

Corollary 6.6 *For n a nonzero integer, we have the short exact sequence*

$$0 \to K^n_\alpha \to \Lambda^n \otimes \mathrm{iso}_\alpha \to K^{n+1}_\alpha \to 0.$$

Remark We will prove that this short exact sequence is not split in a subsequent paper concerning the calculation of $\mathrm{Hom}_{\mathcal{F}_{\mathrm{quad}}}(\Lambda^n \otimes \mathrm{iso}_\alpha, \Lambda^m \otimes \mathrm{iso}_\alpha)$.

We next explain how to decompose the functors K^n_α. We begin by investigating the case of the functor K^1_α.

Lemma 6.7 *The functor K^1_α is equivalent to the functor iso_α.*

Proof Let V be an object in $\mathcal{T}_q$. A basis of the vector space $K^1_\alpha(V)$ is given by the set of elements of the following form: $h(x) \otimes [h]$ for $[h]$ a canonical generator of $\mathrm{iso}_\alpha(V)$. Then we define the linear map

$$\begin{aligned} K^1_\alpha(V) &\xrightarrow{\sigma_V} \mathrm{iso}_\alpha(V) \\ h(x) \otimes [h] &\longmapsto [h] \end{aligned}$$

and we leave the reader to check that σ_V is an isomorphism and that these linear maps are natural. □

In order to identify the composition factors of the functors K^n_α for $n > 1$, we need the following lemma.

Lemma 6.8 *For n a nonzero integer, the morphism $\nu_n \colon \Lambda^{n+1} \otimes \mathrm{iso}_\alpha \to \Lambda^n \otimes \mathrm{iso}_\alpha$ induces a morphism $\nu^K_n \colon K^{n+1}_\alpha \to K^n_\alpha$ making the following diagram commute:*

$$\begin{tikzcd} K^{n+1}_\alpha \arrow[r, "\nu^K_n"] \arrow[d, hook] & K^n_\alpha \arrow[d, hook] \\ \Lambda^{n+1} \otimes \mathrm{iso}_\alpha \arrow[r, "\nu_n"'] & \Lambda^n \otimes \mathrm{iso}_\alpha \end{tikzcd}$$

Proof For an object V in $\mathcal{T}_q$ and $v_1 \wedge \ldots \wedge v_n \wedge h(x) \otimes [h]$ a generator of $K_\alpha^{n+1}(V)$, we have

$$(\nu_{nV})(v_1 \wedge \ldots \wedge v_n \wedge h(x) \otimes [h])$$
$$= \Big(\sum_{i=1}^{n} v_1 \wedge \ldots \wedge \hat{v_i} \wedge \ldots \wedge v_n \wedge h(x) \otimes B(v_i, h(x))[h] \Big) + v_1 \wedge \ldots \wedge v_n \otimes B(h(x), h(x))[h].$$

Since B is alternating, we have $B(h(x), h(x)) = 0$. Hence,

$$(\nu_{nV})(v_1 \wedge \ldots \wedge v_n \wedge h(x) \otimes [h])$$
$$= \sum_{i=1}^{n} v_1 \wedge \ldots \wedge \hat{v_i} \wedge \ldots \wedge v_n \wedge h(x) \otimes B(v_i, h(x))[h] \in K_\alpha^n(V).$$

We deduce the existence of the induced morphism ν_n^K. □

This lemma justifies the introduction of the following definition.

Definition 6.9 For $n \geq 2$ an integer, let L_α^n be the kernel of the morphism $\nu_{n-1}^K \colon K_\alpha^n \to K_\alpha^{n-1}$.

We have the following characterization of the spaces $L_\alpha^n(V)$ which is useful below.

Lemma 6.10 *For an object V in $\mathcal{T}_q$, the space $L_\alpha^n(V)$ is generated by the elements of the form*
$$z \wedge h(x) \otimes [h],$$
where $[h]$ is a canonical generator of the space $\mathrm{iso}_\alpha(V)$ and z is an element of $\Lambda^{n-1}(\mathrm{Vect}(h(x))^\perp)$.

Proof Let V be an object in $\mathcal{T}_q$. Since the space $L_\alpha^n(V)$ is a subvector space of $K_\alpha^n(V)$, we deduce from Lemma 6.5 that the vector space $L_\alpha^n(V)$ is generated by the elements of the following form: $z \wedge h(x) \otimes [h]$.

For a given canonical generator $[h]$ of $\mathrm{iso}_\alpha(V)$, since the quadratic space V is nondegenerate, there is an element w in V such that
$$B(h(x), w) = 1.$$

Let W be the space $\mathrm{Vect}(h(x), w)$ and $V \simeq W \perp W^\perp$ be an orthogonal decomposition of the space V. The canonical generator $z \wedge h(x) \otimes [h]$ of $K_\alpha^n(V)$ can be written in the form
$$z' \wedge h(x) \otimes [h] + z'' \wedge w \wedge h(x) \otimes [h]$$

where $z' \in \Lambda^{n-1}(W^{\perp})$ and $z'' \in \Lambda^{n-2}(W^{\perp})$.

Let x be an element of $L^n_\alpha(V)$. Then

$$x = \sum_{[h]\in \mathcal{G}(\mathrm{iso}_\alpha(V))} z_h \wedge h(x) \otimes [h] = \sum_{[h]\in \mathcal{G}(\mathrm{iso}_\alpha(V))} (z'_h \wedge h(x) \otimes [h] + z''_h \wedge w \wedge h(x) \otimes [h])$$

where $\mathcal{G}(\mathrm{iso}_\alpha(V))$ is the set of the canonical generators of $\mathrm{iso}_\alpha(V)$. Consequently,

$$\nu^K_{n-1}(z'_h \wedge h(x) \otimes [h]) = 0$$

since $z'_h \in \Lambda^{n-1}(W^{\perp}) \subset \Lambda^{n-1}(\mathrm{Vect}(h(x))^{\perp})$ and

$$\nu^K_{n-1}(z''_h \wedge w \wedge h(x) \otimes [h]) = z''_h \wedge h(x) \otimes [h].$$

Since the element x is in the kernel of ν^K_{n-1}, we deduce that $z''_h = 0$. Hence

$$x = \sum_{[h]\in \mathcal{G}(\mathrm{iso}_\alpha(V))} z'_h \wedge h(x) \otimes [h]$$

for $z'_h \in \Lambda^{n-1}(W^{\perp}) \subset \Lambda^{n-1}(\mathrm{Vect}(h(x))^{\perp})$. □

We have the following lemma.

Lemma 6.11 *The composition* $\Lambda^{n+2} \otimes \mathrm{iso}_\alpha \xrightarrow{\nu_{n+1}} \Lambda^{n+1} \otimes \mathrm{iso}_\alpha \xrightarrow{\nu_n} \Lambda^n \otimes \mathrm{iso}_\alpha$ *is zero.*

Proof Let V be an object in $\mathcal{T}_q$ and let $v_1 \wedge \ldots \wedge v_{n+2} \otimes [h]$ be an element of $\Lambda^{n+2} \otimes \mathrm{iso}_\alpha(V)$. Then

$$\begin{aligned}
&\nu_n \circ \nu_{n+1}\big(v_1 \wedge \ldots \wedge v_{n+2} \otimes [h]\big) \\
&\quad = \nu_n\Big(\sum_{i=1}^{n+2} \big(v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_{n+2} \otimes B(v_i, h(x))[h]\big)\Big) \\
&\quad = \sum_{j\neq i} \sum_{i=1}^{n+2} \Big(v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_{n+2} \otimes B(v_i, h(x)) B(v_j, h(x))[h]\Big) \\
&\quad = 0
\end{aligned}$$

since the characteristic is equal to 2. □

We deduce the following result.

Lemma 6.12 *The map* $\nu^K_n \colon K^{n+1}_\alpha \to K^n_\alpha$ *factors through* L^n_α.

Proof By Lemma 6.8, the following diagram is commutative:

$$\begin{tikzcd}
K_\alpha^{n+1} \arrow[r, "\nu_n^K"] \arrow[d, hook] & K_\alpha^n \arrow[r, "\nu_{n-1}^K"] \arrow[d, hook] & K_\alpha^{n-1} \arrow[d, hook] \\
\Lambda^{n+1} \otimes \mathrm{iso}_\alpha \arrow[r, "\nu_n"'] & \Lambda^n \otimes \mathrm{iso}_\alpha \arrow[r, "\nu_{n-1}"'] & \Lambda^{n-1} \otimes \mathrm{iso}_\alpha
\end{tikzcd}$$

Consequently, we deduce from Lemma 6.11, that $\nu_{n-1}^K \circ \nu_n^K = 0$. Hence, there is a morphism $\tilde{\nu}_n^K \colon K_\alpha^{n+1} \to L_\alpha^n$ making the following diagram commutative:

$$\begin{tikzcd}
 & K_\alpha^{n+1} \arrow[dl, dotted, "\tilde{\nu}_n^K"'] \arrow[d, "\nu_n^K"] \arrow[dr, "0"] & \\
L_\alpha^n = \mathrm{Ker}(\nu_{n-1}) \arrow[r] & K_\alpha^n \arrow[r, "\nu_{n-1}^K"'] & K_\alpha^{n-1}
\end{tikzcd}$$

□

Then, we have the following proposition.

Proposition 6.13 *For n a nonzero integer, there is a short exact sequence:*

$$0 \to L_\alpha^{n+1} \to K_\alpha^{n+1} \to L_\alpha^n \to 0.$$

Proof It is sufficient to prove that the natural map $\tilde{\nu}_n^K \colon K_\alpha^{n+1} \to L_\alpha^n$ constructed in the proof of the previous Lemma is an epimorphism of $\mathcal{F}_{\mathrm{quad}}$.

Let V be an object in $\mathcal{T}_q$ and let $v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h]$ be a generator of $L_\alpha^n(V)$. By definition of $L_\alpha^n(V)$ we have $B(v_i, h(x)) = 0$ for all i in $\{1, \ldots, n-1\}$. Since $h(x)$ is a nonzero element in the nondegenerate quadratic space V, there is an element v in V such that $B(v, h(x)) = 1$. Then, we prove that the element

$$v_1 \wedge \ldots \wedge v_{n-1} \wedge v \wedge h(x) \otimes [h] \text{ of } K_\alpha^{n+1}(V)$$

verifies

$$(\tilde{\nu}_n^K)_V(v_1 \wedge \ldots \wedge v_{n-1} \wedge v \wedge h(x) \otimes [h]) = v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h].$$

Hence, $\tilde{\nu}_n^K$ is surjective. □

Remark Proposition 6.13 is equivalent to the following statement: the complex

$$\ldots \to K_\alpha^{n+1} \xrightarrow{\nu_n^K} K_\alpha^n \xrightarrow{\nu_{n-1}^K} K_\alpha^{n-1} \to \ldots$$

is exact.

6.2 Simplicity of the functors L^n_α

In this section, we prove the following result, where the functors L^n_α are the subfunctors of $\Lambda^n \otimes \mathrm{iso}_\alpha$ defined in Definition 6.9.

Theorem 6.14 *The functors L^n_α are simple.*

To prove this theorem, we need the following fundamental lemma.

Lemma 6.15 *If J is a subfunctor of L^n_α, then for any object V in $\mathcal{T}_q$, either $J(V) = \{0\}$, or $J(V) = L^n_\alpha(V)$.*

Proof Let J be a subfunctor of L^n_α and V be an object in $\mathcal{T}_q$. Suppose that $J(V) \neq \{0\}$ and denote by y a nonzero element of $J(V)$. We have

$$y = \sum_{[h] \in \mathcal{G}(\mathrm{iso}_\alpha(V))} z_h \wedge h(x) \otimes [h] \tag{6–1}$$

where z_h is an element of $\Lambda^{n-1}(\mathrm{Vect}(h(x))^\perp)$ by Lemma 6.10 and $\mathcal{G}(\mathrm{iso}_\alpha(V))$ is the set of canonical generators of the space $\mathrm{iso}_\alpha(V)$.

The proof is divided into three steps; in the first one we prove that there exists a generator $[h]$ of $\mathrm{iso}_\alpha(V)$ such that $z_h \wedge h(x) \otimes [h] \in J(V)$. We deduce, in the second part, that for all other generator $[h']$ of $\mathrm{iso}_\alpha(V)$ we have a nonzero element of the form $z' \wedge h'(x) \otimes [h']$ in $J(V)$. Finally, we prove that for each element of the form $v_1 \wedge \ldots \wedge v_{n-1}$ in $\Lambda^{n-1}(\mathrm{Vect}(h(x))^\perp)$ and each canonical generator $[h]$ of $\mathrm{iso}_\alpha(V)$, the element $v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h]$ belongs to $J(V)$. This will prove that the two spaces $J(V)$ and $L^n_\alpha(V)$ are isomorphic, by the characterization of the space $L^n_\alpha(V)$ given in Lemma 6.10.

(1) Let $[h]$ be a canonical generator of $\mathrm{iso}_\alpha(V)$ such that, in the decomposition of y given in (6–1) the element $z_h \wedge h(x) \otimes [h]$ is nonzero. Since the space V is nondegenerate, there exists an element v in V such that $B(v, h(x)) = 1$. We deduce a symplectic decomposition of V of the form

$$V = \mathrm{Vect}(h(x), v) \perp \mathrm{Vect}(v_1, w_1) \perp \ldots \perp \mathrm{Vect}(v_m, w_m) = \mathrm{Vect}(h(x), v) \perp V'. \tag{6–2}$$

We consider the morphism of $\mathcal{E}_q$, where we denote by $\{a_0^k, b_0^k\}$ a symplectic basis of the k–th copy of H_0 in $(H_0)^{\perp(2m+1)}$:

$$\begin{array}{rlcl} f\colon & V & \longrightarrow & V \perp (H_0)^{\perp(2m+1)} \\ & h(x) & \longmapsto & h(x) \\ & v & \longmapsto & v + a_0^1 \\ & v_k & \longmapsto & v_k + a_0^{2k} \\ & w_k & \longmapsto & w_k + a_0^{2k+1} \end{array}$$

for k an integer between 1 and m, which allows us to define the following morphism in $\mathcal{T}_q$: $T=[V\xrightarrow{f}V\perp(H_0)^{\perp(2m+1)}\xhookleftarrow{i}V]$. We deduce from the two cartesian diagrams below:

$$\begin{tikzcd}
(x,\alpha) \arrow[rr] \arrow[d, "\mathrm{Id}"'] & & V \arrow[d, hook, "i"] \\
(x,\alpha) \arrow[r, "h"'] & V \arrow[r, "f"'] & V\perp(H_0)^{\perp(2m+1)}
\end{tikzcd}$$

and, for $h_i\neq h$,

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[d, "\mathrm{Id}"'] & & V \arrow[d, hook, "i"] \\
(x,\alpha) \arrow[r, "h_i"'] & V \arrow[r, "f"'] & V\perp(H_0)^{\perp(2m+1)}
\end{tikzcd}$$

that $J(T)(y)=z_h\wedge h(x)\otimes[h]\in J(V)$, since $\epsilon(T)=\mathrm{Id}_V$.

(2) Let $[h']$ be a canonical generator of $\mathrm{iso}_\alpha(V)$ different from $[h]$. We have the equality $q(h'(x))=q((h(x))=\alpha$, hence the linear isomorphism denoted by $\underline{f}$, from $(h(x),\alpha)$ to $(h'(x),\alpha)$ is a morphism in $\mathcal{E}_q^{\deg}$. So, we can apply Theorem 1.3 to obtain the existence of a morphism f of $\mathrm{Hom}_{\mathcal{E}_q}(V,V)$ making the following diagram commutative:

$$\begin{tikzcd}
V \arrow[r, "f"] & V \\
(h(x),\alpha) \arrow[u, hook] \arrow[r, "\underline{f}"'] & (h'(x),\alpha) \arrow[u, hook]
\end{tikzcd}$$

We deduce the following cartesian diagram:

$$\begin{tikzcd}
(x,\alpha) \arrow[rr, "h'"] \arrow[d, "\mathrm{Id}"'] & & V \arrow[d, "\mathrm{Id}"] \\
(x,\alpha) \arrow[r, "h"'] & V \arrow[r, "f"'] & V
\end{tikzcd}$$

Consequently, by the consideration of the morphism $T=[V\xrightarrow{f}V\xleftarrow{\mathrm{Id}}V]$, we obtain

$$J(T)(z_h\wedge h(x)\otimes[h])=\Lambda^{n-1}(f)(z_h)\wedge h'(x)\otimes[h']\in J(V).$$

(3) By the point (1) of the proof, there is a nonzero element of the form $z\wedge h(x)\otimes[h]$ in $J(V)$. We want to prove that, for each element of the form $v_1\wedge\ldots\wedge v_{n-1}$ in $\Lambda^{n-1}(\mathrm{Vect}(h(x))^\perp)$ and each canonical generator $[h]$ of $\mathrm{iso}_\alpha(V)$, the element

$v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h]$ belongs to $J(V)$. According to the proof of Lemma 6.10, it is sufficient to prove that the element $v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h]$ belongs to $J(V)$ for $v_1 \wedge \ldots \wedge v_{n-1}$ in $\Lambda^{n-1}(V')$ where V' is the space considered in the decomposition (6–2). By simplicity of the functor Λ^{n-1} in $\mathcal{F}$, we have the existence of an endomorphism g of $\epsilon(V')$ such that

$$\Lambda^{n-1}(g)(z) = v_1 \wedge \ldots \wedge v_{n-1}.$$

We deduce that

$$J(\mathrm{Id}_{\mathcal{T}_q}(\mathrm{Vect}(h(x), v)) \perp t_g)(z \wedge h(x) \otimes [h]) = v_1 \wedge \ldots \wedge v_{n-1} \wedge h(x) \otimes [h]$$

where t_g is an antecedent of $g \in \mathrm{End}_{\mathcal{E}^f}(\epsilon(V'))$ by the forgetful functor $\epsilon\colon \mathcal{T}_q \to \mathcal{E}^f$ which is full by [12, Proposition 3.5] and $\mathrm{Id}_{\mathcal{T}_q}(\mathrm{Vect}(h(x), v)) \perp t_g$ is the orthogonal sum of the morphisms of $\mathcal{T}_q$. □

Proof of Theorem 6.14 Let J be a nonzero subfunctor of L^n_α and V be an object in $\mathcal{T}_q$ such that the space $J(V)$ is nonzero. By Lemma 6.15, we have $J(V) = L^n_\alpha(V)$. We prove, in the following, that for all object W in $\mathcal{T}_q$, $J(W) = L^n_\alpha(W)$.

Let W be a fixed object in $\mathcal{T}_q$. The proof is divided into two parts.

(1) Let us prove that $J(V \perp W) \simeq L^n_\alpha(V \perp W)$.

Let i be the canonical inclusion from V to $V \perp W$ and $T = [V \xrightarrow{i} V \perp W \xleftarrow{\mathrm{Id}} V \perp W]$ be the morphism of $\mathcal{T}_q$. We have the following cartesian diagram:

$$\begin{tikzcd}
(x,\alpha) \arrow[rr, "i \circ h"] \arrow[d, "\mathrm{Id}"'] & & V \perp W \arrow[d, "\mathrm{Id}"] \\
(x,\alpha) \arrow[r, "h"'] & V \arrow[r, "i"'] & V \perp W
\end{tikzcd}$$

Since
$$[V \perp W \xrightarrow{\mathrm{Id}} V \perp W \xleftarrow{i} V] \circ [V \xrightarrow{i} V \perp W \xleftarrow{\mathrm{Id}} V \perp W] = \mathrm{Id}_V$$

we deduce that the space $J(V \perp W)$ is nonzero. Hence, by Lemma 6.15, we have $J(V \perp W) \simeq L^n_\alpha(V \perp W)$.

(2) Let us prove that $J(W) \simeq L^n_\alpha(W)$.

According to Lemma 6.15, it is sufficient to prove that if $L^n_\alpha(W)$ is nontrivial then $J(W)$ is not zero. Let j be the canonical inclusion from W to $V \perp W$.

We deduce from

$$[V \perp W \xrightarrow{\mathrm{Id}} V \perp W \xleftarrow{j} W] \circ [W \xrightarrow{j} V \perp W \xleftarrow{\mathrm{Id}} V \perp W] = \mathrm{Id}_W$$

the existence of the surjection $L^n_\alpha(V\perp W) \twoheadrightarrow L^n_\alpha(W)$. Furthermore, by the first point of the proof, we have $J(V\perp W) = L^n_\alpha(V\perp W)$. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
J(V\perp W) \arrow[r] \arrow[d, "\simeq"'] & J(W) \arrow[d, hook] \\
L^n_\alpha(V\perp W) \arrow[r, two heads] & L^n_\alpha(W)
\end{tikzcd}$$

Consequently, by a diagram chasing argument, we obtain that if $L^n_\alpha(W)$ is nonzero then $J(W)$ is nonzero. $\square$

We prove in the following proposition that this gives rise to two families of non isomorphic functors.

Proposition 6.16 *The functors in the union of the family $\{L^n_0 \mid n \in \mathbb{N}\}$ and the family $\{L^n_1 \mid n \in \mathbb{N}\}$ are pairwise nonisomorphic.*

Proof For a fixed α, there exists, for any integer n, a minimal integer $d(n)$ such that

$$L^n_\alpha(H_0^{\perp d(n)}) \neq 0.$$

Let k be an integer different from n, if $|n-k| \geq 2$, the integers $d(n)$ and $d(k)$ allow us to distinguish the simple functors L^n_α and L^k_α, in the contrary case, we prove that the dimensions of the spaces $L^n_\alpha(H_0^{\perp d(n)})$ and $L^k_\alpha(H_0^{\perp d(n)})$ are different. This proves that the simple functors L^n_α and L^k_α are not isomorphic.

Furthermore, two simple functors S_1 and S_2 in $\mathcal{F}_{\text{quad}}$ are not isomorphic if there exists a morphism T in $\mathcal{T}_q$ such that $S_1(T) = 0$ and $S_2(T) \neq 0$. Moreover, the morphisms T constructed in the first point of the proof of Lemma 6.15 verify

$$L^n_\alpha(T) \neq 0 \quad \text{and} \quad L^k_{(\alpha+1)}(T) = 0$$

where $(\alpha+1)$ is the reduction mod 2 of $\alpha+1$.

More precisely, if we consider a nonzero element $z_p \wedge h(x) \otimes [h]$ of $L^n_\alpha(V)$ and the morphism $T = [V \xrightarrow{f} V\perp(H_0)^{\perp(2m+1)} \xleftarrow{i} V]$ where

$$\begin{array}{rlll}
f: & V & \longrightarrow & V\perp(H_0)^{\perp(2m+1)} \\
& h(x) & \longmapsto & h(x) \\
& v & \longmapsto & v + a_0^1 \\
& v_k & \longmapsto & v_k + a_0^{2k} \\
& w_k & \longmapsto & w_k + a_0^{2k+1}
\end{array}$$

for k an integer between 1 and m. We deduce from following cartesian diagram

$$\begin{tikzcd}
(x,\alpha) \arrow[rr] \arrow[d, "\mathrm{Id}"'] & & V \arrow[d, hook, "i"] \\
(x,\alpha) \arrow[r, "h"'] & V \arrow[r, "f"'] & V\perp(H_0)^{\perp(2m+1)}
\end{tikzcd}$$

that $L^n_\alpha(T) \neq 0$.

In the other hand, for any nonzero element $z_{p_i} \wedge h_i(x) \otimes [h_i]$ of $L^k_{(\alpha+1)}(V)$, we have $h_i \neq h$ since $\alpha \neq (\alpha+1)$. So we deduce from the following cartesian diagram

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[d, "\mathrm{Id}"'] & & V \arrow[d, hook, "i"] \\
(x,(\alpha+1)) \arrow[r, "h_i"'] & V \arrow[r, "f"'] & V\perp(H_0)^{\perp(2m+1)}
\end{tikzcd}$$

that $L^k_{(\alpha+1)}(T) = 0$. □

7 The composition factors of the functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$

We prove in this section that the functors $\mathrm{Mix}_{0,1}$ and $\mathrm{Mix}_{1,1}$ are uniserial (ie the lattice of its subfunctors is totally ordered) and the composition factors of the functor $\mathrm{Mix}_{0,1}$ are the functors L^n_0 and those of $\mathrm{Mix}_{1,1}$ are the functors L^n_1. For that, we identify the subquotient $k_d\Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1}$ of the filtration of the functor $\Sigma_{\alpha,1}$ by the functors $k_d\Sigma_{\alpha,1}$ introduced in Proposition 5.2 with the functor L^{d+1}_α defined in Definition 6.9. This gives rise to the following result.

Theorem 7.1 *The functor $\Sigma_{\alpha,1}$ is uniserial and its unique composition series is given by the decreasing filtration by the functors $k_d\Sigma_{\alpha,1}$*

$$\ldots \subset k_d\Sigma_{\alpha,1} \subset \ldots \subset k_1\Sigma_{\alpha,1} \subset k_0\Sigma_{\alpha,1} = \Sigma_{\alpha,1}$$

which verifies

$$k_d\Sigma_{\alpha,1}/k_{d+1}\Sigma_{\alpha,1} \simeq L^{d+1}_\alpha.$$

Remark Remark that the strategy of the following proof of the uniseriality of $\Sigma_{\alpha,1}$ is close to the proof of Lemma 6.15.

Proof To prove that the functor $\Sigma_{\alpha,1}$ is uniserial, it is sufficient to prove that if J is a nonzero subfunctor of $\Sigma_{\alpha,1}$ then there exists an integer d such that $J = k_d\Sigma_{\alpha,1}$.

Let V be an object in $\mathcal{T}_q$ such that $J(V) \neq 0$ and v be a nonzero element of $J(V)$, we have

$$v = \sum_{\{x,y\}\in A_V} \alpha_{\{x,y\}}(v)[\{x,y\}]$$

where $A_V = \{\{x,y\} | x \in V, y \in V, q(x+y) = \alpha, B(x,y) = 1\}$. One verifies easily that for $\{x,y\} \in A_V$ and $l \in V$, $\{x+l, y+l\} \in A_V$ if and only if $l \in (\mathrm{Vect}(x,y))^{\perp}$ or $l = x+y+l'$ where $l' \in (\mathrm{Vect}(x,y))^{\perp}$. Since $\{x+(x+y+l'), y+(x+y+l')\} = \{y+l', x+l'\} = \{x+l', y+l'\}$, after reordering we obtain

$$v = \sum_{i=1}^{n} \sum_{l\in\mathcal{L}_i} [\{x_i+l, y_i+l\}] \tag{7–1}$$

where for all i $\{x_i, y_i\} \in A_V$, for $i \neq j$, $x_i + y_i \neq x_j + y_j$ and $\mathcal{L}_i$ is a subvector space of $(\mathrm{Vect}(x_i, y_i))^{\perp}$ of dimension r_i. We consider a vector space V', an element v' of $J(V')$ and an element $\{x', y'\}$ in $A_{V'}$ such that the dimension r' of $\mathcal{L}'$ is minimal. We deduce from the decomposition (7–1) that $J \subset k_{r'}\Sigma_{\alpha,1}$.

To prove that $k_{r'}\Sigma_{\alpha,1} \subset J$, we consider the following decomposition of $v' \in J(V')$:

$$v' = \sum_{l'\in\mathcal{L}'} [\{x'+l', y'+l'\}] + \sum_{j=1}^{p} \Big(\sum_{l\in\mathcal{L}_j} [\{x_j+l, y_j+l\}] \Big)$$

where for all j $x_j + y_j \neq x' + y'$. By definition of $\Sigma_{\alpha,1}$ and $\mathrm{Mix}_{\alpha,1}$, we have a natural map $\sigma\colon J \to P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha$ such that

$$\sigma_{V'}(v') = \sum_{l'\in\mathcal{L}'} \big([x'+l'] + [y'+l']\big) \otimes [x'+y'] + \sum_{j=1}^{p} \sum_{l\in\mathcal{L}_j} \big([x_j+l] + [y_j+l]\big) \otimes [x_j+y_j]$$

where, by abuse, we denote by v the linear map $\mathbb{F}_2 \to V$ determined by v. Let $f\colon \epsilon(V') \to \epsilon(V')$ be the linear map such that $f(x'+y') = x'+y'$ and $f(x_j+y_j) = x_j+y_j+m$ for m a nonzero element of $\mathcal{L}_j$. Since ϵ is full by [12, Proposition 3.5], we obtain the existence of a morphism T in $\mathrm{Hom}_{\mathcal{T}_q}(V', V')$ such that $\epsilon(T) = f$. We deduce that

$$(P_{\mathbb{F}} \otimes \mathrm{iso}_\alpha)(T)\sigma_V(v') = \sum_{l'\in\mathcal{L}'} \big([x'+l'] + [y'+l']\big) \otimes [x'+y']$$

and, consequently

$$J(T)(v') = \sum_{l' \in \mathcal{L}'} [\{x' + l', y' + l'\}] \in J(V')$$

where $\mathcal{L}'$ is a subvector space of $(\mathrm{Vect}(x', y'))^{\perp}$ of dimension r'.

Let W be an object of $\mathcal{T}_q$, $\sum_{l \in \mathcal{L}} [\{w_1 + l, w_2 + l\}]$, where $\mathcal{L}$ is a subvector space of $\mathrm{Vect}(w_1, w_2)^{\perp}$ of dimension r', be a generator of $k_{r'} \Sigma_{\alpha,1}(W)$ and $g\colon \epsilon(V') \to \epsilon(W)$ be the linear map such that $g(x' + y') = w_1 + w_2$ and g send a basis of $\mathcal{L}'$ to a basis of $\mathcal{L}$. By the fullness of ϵ we obtain a morphism T' in $\mathrm{Hom}_{\mathcal{T}_q}(V', W)$ such that

$$J(T')\Big(\sum_{l' \in \mathcal{L}'} [\{x' + l', y' + l'\}] \Big) = \sum_{l \in \mathcal{L}} [\{w_1 + l, w_2 + l\}] \in J(W).$$

Hence $J = k_{r'} \Sigma_{\alpha,1}$.

By Lemma 5.6, we have

$$(g_d \otimes \mathrm{iso}_{\alpha}) \circ i_d (k_d \Sigma_{\alpha,1}) \subset L_{\alpha}^{d+1}.$$

Consequently, by the commutative diagram (5–1) given in the proof of the second point of Proposition 5.5 we have the natural map

$$\sigma\colon k_d \Sigma_{\alpha,1} / k_{d+1} \Sigma_{\alpha,1} \to L_{\alpha}^{d+1},$$

which is nontrivial by Lemma 5.6. Since the quotients $k_d \Sigma_{\alpha,1} / k_{d+1} \Sigma_{\alpha,1}$ are nonzero by Proposition 5.7 and the functors L_{α}^{d+1} are simple by Theorem 6.14, the natural map σ is an equivalence. □

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