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Even-dimensional l–monoids and L–theory

JÖRG SIXT

Surgery theory provides a method to classify n–dimensional manifolds up to diffeomorphism given their homotopy types and $n \geq 5$. In Kreck's modified version, it suffices to know the normal homotopy type of their $\frac{n}{2}$–skeletons. While the obstructions in the original theory live in Wall's L–groups, the modified obstructions are elements in certain monoids $l_n(\mathbf{Z}[\pi])$. Unlike the L–groups, the Kreck monoids are not well-understood.

We present three obstructions to help analyze $\theta \in l_{2k}(\Lambda)$ for a ring Λ. Firstly, if $\theta \in l_{2k}(\Lambda)$ is elementary (ie trivial), flip-isomorphisms must exist. In certain cases flip-isomorphisms are isometries of the linking forms of the manifolds one wishes to classify. Secondly, a further obstruction in the asymmetric Witt-group vanishes if θ is elementary. Alternatively, there is an obstruction in $L_{2k}(\Lambda)$ for certain flip-isomorphisms which is trivial if and only if θ is elementary.

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1 Introduction

Let $q \geq 2$ and $\epsilon = (-)^q$. Let Λ be a weakly finitering with an involution $x \longmapsto \bar{x}$, for example $\Lambda = \mathbb{Z}[\pi]$ with the usual involution. (Weakly finite means that the rank of a free module is well-defined; see also Cohn [1, page 143 ff].) All modules are left Λ–modules. "Free module" always means free based f.g. module and any isomorphism between them is a simple isomorphism. Let $I = [0,1]$.

1.1 Surgery theories

Surgery theory was developed by Browder, Kervaire, Milnor, Novikov, Sullivan, Wall and others in order to classify manifolds up to diffeomorphism, PL–isomorphism or homeomorphism. Given a homotopy equivalence $f\colon M_1 \xrightarrow{\simeq} M_0$ between two n–dimensional manifolds M_0 and M_1, the surgery programme can decide (in principle) if f is homotopic to a diffeomorphism or not. By analysing certain homotopy invariants of f, one can find all normal cobordisms of the type:

$$(e, \mathrm{id}, f)\colon (W, M_0, M_1) \to M_0 \times (I, \{0\}, \{1\})$$

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Each of them gives rise to an obstruction $\theta(W) \in L_n(\mathbb{Z}[\pi_1(M_0)])$. The groups $L_n(\Lambda)$ are Witt-groups of forms and formations and of a purely algebraic nature. They have been extensively studied and computed. If $n \geq 5$, an obstruction $\theta(W)$ vanishes if and only if W is normally cobordant rel ∂ to an s–cobordism. In this case, M_0 and M_1 are diffeomorphic by the s–cobordism theorem. The surgery machine was successfully applied to classify exotic spheres and finite free group actions on spheres (see also Wall [15] and Ranicki [14]). Under certain restrictions, surgery theory also works in the topological and PL–categories if $n = 4$ (see Freedman and Quinn [2]).

In applications, it is sometimes difficult to determine the (normal) homotopy type of a manifold. Therefore Kreck produced an enhanced surgery theory which classifies manifolds whose normal homotopy type is only known up to the middle dimension (see Kreck [7]). An embryonic version helped to compute the cobordism of automorphisms [6]. Later, it helped to classify complete intersections, 4–dimensional manifolds with finite fundamental groups up to homeomorphism and certain 7–dimensional homogenous spaces (see Hambleton, Kreck, Teichner and Stolz [3; 4; 5; 8; 9] and Kreck [7, Section 8]).

We briefly sketch the even-dimensional version of Kreck's theory. Let $B \to BO$ be a fibration. A *B–manifold* is an n–dimensional smooth manifold M together with a lift $\overline{\nu}$ of its normal bundle to B. It is called a *$(q-1)$–smoothing* if $\overline{\nu}$ is q–connected. The construction of Postnikov towers is a useful way to find a suitable (and in some sense universal) B for a given manifold (see [7, Section 2]).

Let $(M_0, \overline{\nu}_0)$ and $(M_1, \overline{\nu}_1)$ be two $(2q+1)$–dimensional $(q-1)$–smoothings. In principle, spectral sequences allow us to determine the cobordism groups of B–manifolds and, therefore, help to decide whether $(M_0, \overline{\nu}_0)$ and $(M_1, \overline{\nu}_1)$ are B–cobordant. We shall call a B–cobordism $(\overline{\nu}_W, \overline{\nu}_0, \overline{\nu}_1)\colon (W, M_0, M_1) \to B$ a *modified surgery problem over B*. Surgery below the middle dimension yields a new B–cobordism $(\overline{\nu}_{W'}, \overline{\nu}_0, \overline{\nu}_1)\colon (W', M_0, M_1) \to B$ such that $(W', \overline{\nu}_{W'})$ is a q–smoothing. A cobordism invariant $\theta(W) \in l_{2q+2}(\mathbb{Z}[\pi_1(B)])$ of W can then be defined as follows: let $l_{2q+2}(\Lambda)$ be the set of equivalence classes of *preformations*, ie, tuples $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ of homomorphisms γ and μ together with a $(-\epsilon)$–quadratic form $(G, \gamma^*\mu, \theta)$. The equivalence relation is given by isometries and stabilization with "hyperbolic" tuples (Definitions 2.4, 2.5). The obstruction $\theta(W)$ is represented by the preformation

$$(1) \qquad (H_{q+1}(W', M_0) \xleftarrow{\gamma} H_{q+2}(B, W') \xrightarrow{\mu} H_{q+1}(W', M_1), \psi_{W'})$$

where γ and μ are taken from the long exact sequences of the triads (B, W', M_i) and $\psi_{W'}$ is induced by the self-intersection form on W' (Corollary 2.10).

Theorem 1.1 [7, Theorem 3] *W is B–cobordant rel∂ to an s–cobordism if and only if $\theta(W)$ is elementary, ie, it allows a "generalized lagrangian" (see Definition 2.7).*

The lack of understanding of the l–monoids can be a serious obstacle when one tries to apply modified surgery theory. Therefore this paper aims to relate these monoids to the better understood L–groups.

1.2 Results

The l–monoids are the set of equivalence classes of ϵ–preformations. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ be a *regular ϵ–preformation*, ie, a tuple of free modules F and G, $\binom{\gamma}{\mu} \in \mathrm{Hom}_\Lambda(G, F \oplus F^*)$ together with an $(-\epsilon)$–quadratic form $(G, \gamma^*\mu, \theta)$. (It is interesting to observe that regular ϵ–preformations where $\binom{\gamma}{\mu}$ is a split injection of a half-rank direct summand (so-called *non-singular split formations*) are the building blocks of the Wall-groups $L_{2q-1}(\Lambda)$.)

A *flip-isomorphism* of z is a weak isomorphism (Definition 2.5) between z and its flip $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\theta)$. If z is the obstruction of a modified surgery problem (W, M_0, M_1), then z' is the obstruction of the "flipped" surgery problem $(-W, M_1, M_0)$.

Theorem 1 (Proposition 5.2) *z has a flip-isomorphism if it is elementary.*

In certain cases flip-isomorphisms can be interpreted as isometries of linking forms associated to z which in turn are related to the topological linking forms on M_i if $[z] = \theta(W)$ as in (1).

Theorem 2 (Proposition 9.4 for $\Lambda = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$) *Assume that γ and μ are $\mathbb{Q}$–isomorphisms. (Then z is called an S-ϵ–preformation.)*

(i) *Split $(-\epsilon)$–quadratic linking forms L_μ and L_γ can be defined on* $\operatorname{coker}\mu$ *and* $\operatorname{coker}\gamma$, *and their isometry classes are invariants of $[z] \in l_{2q+2}(\mathbb{Z})$.*

(ii) *Every flip-isomorphism induces an isometry $L_\mu \xrightarrow{\cong} L_\gamma$ and conversely any such isometry is induced by a stable flip-isomorphism of z.*

Theorem 3 (Theorem 9.7) *Let (W, M_0, M_1) be a modified surgery problem over B such that $\pi_1(B) = 0$, $\dim W = 2q+2 \geq 6$ and $|H_{q+1}(B, M_j)| < \infty$. Then the obstruction preformation is an S-ϵ–preformation, and L^γ and L^μ are the linking forms on $H_{q+1}(B, M_j)$ which are induced by the topological linking forms of M_j.*

For any choice of flip-isomorphism t, one can define an *asymmetric signature* $\sigma^*(z,t)$ in the Witt-group $WAsy(\Lambda)$ of non-singular asymmetric forms, ie, isomorphisms $\lambda\colon M \xrightarrow{\cong} M^*$ where no symmetry conditions are imposed (Definitions 6.1 and 6.6). If z is an S-ϵ–preformation, one can replace flip-isomorphisms by isometries $L_\mu \xrightarrow{\cong} L_\gamma$ (Theorem 9.5).

Theorem 4 (Theorem 6.8) *If $[z] \in l_{2q+2}(\Lambda)$ is elementary then $\sigma^*(z,t) = 0 \in WAsy(\Lambda)$ for all flip-isomorphisms t of z.*

The vanishing of all asymmetric signatures of a preformation does not ensure that it is elementary (Example 8.4). A complete set of obstructions is given by the *quadratic signatures*. The precondition for their definition is the existence of a special type of flip-isomorphism. In general, any flip-isomorphism induces an automorphism of a certain $2q$–dimensional quadratic Poincaré complex (Theorem 5.3). If there is a homotopy (Δ,η) between this map and the identity, the flip-isomorphism is called a *flip-isomorphism rel*$\,\partial$ (Definition 7.6) and one constructs the *quadratic signature* $\rho^*(z,t,\Delta,\eta) \in L_{2q+2}(\Lambda)$ (Definition 7.8).

Theorem 5 (Theorem 7.9) *$[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if there is a flip-isomorphism rel$\,\partial$ t of z' with $[z'] = [z] \in l_{2q+2}(\Lambda)$ and a homotopy $(\Delta,\eta)\colon (1,0) \simeq (h_t,\chi_t)$ such that $\rho^*(z',t,\Delta,\eta) = 0 \in L_{2q+2}(\Lambda)$.*

Asymmetric and quadratic signatures are related as one would expect.

Theorem 6 (Theorem 7.10) *The quadratic signatures are mapped to the asymmetric signatures under the canonical homomorphism*

$$L_{2q+2}(\Lambda) \longrightarrow WAsy(\Lambda), \quad (K,\psi) \longmapsto (K,\psi - \epsilon\psi^*)$$

The rather technical definition of flip-isomorphisms rel$\,\partial$ can be avoided if z is in fact a *formation*, ie, if $\binom{\gamma}{\mu}\colon G \to H_\epsilon(F)$ is the inclusion of a lagrangian. For example, assume that B is a $(2q+1)$–dimensional Poincaré space and the modified surgery problem $(W,M,M') \to B \times (I,\{0\},\{1\})$ is a normal degree 1 cobordism. Its modified surgery obstruction $z = \theta(W)$ is a formation and any flip-isomorphism is a flip-isomorphism rel$\,\partial$ (Theorem 8.1). In addition, it turns out that the asymmetric signatures do not depend on the choice of flip-isomorphism (Theorem 8.2). For simply-connected manifolds one can use these facts to show the following result.

Theorem 7 (Theorem 9.7) *Let (W,M_0,M_1) be a modified surgery problem such that $\pi_1(B) = 0$, q is odd,* $\dim W = 2q \geq 6$ *and $H_{q+1}(B,M_j)$ are finite. Assume that* *the induced linking forms on $H_{q+1}(B, M_j)$ are non-singular. Then W is cobordant $\mathrm{rel}\partial$ to an s–cobordism if and only if there is an isometry l of the linking forms on $H_{q+1}(B, M_j)$ such that its asymmetric signature $\sigma(\theta(W), l)$ vanish.*

1.3 The strategy of proof

The proofs of the theorems in this paper rely heavily on the algebraic theory of surgery due to A Ranicki (see [11]). Its objects are n–dimensional quadratic and symmetric Poincaré complexes and pairs. Symmetric Poincaré complexes are algebraic shadows of geometric Poincaré spaces, whereas their quadratic counterparts model normal maps $f\colon M \to X$ from a manifold M to a Poincaré space X. Symmetric and quadratic Poincaré pairs are the relative versions corresponding to geometric Poincaré pairs and normal cobordisms.

Topological notions of cobordism, s–cobordism, surgery, boundary, gluing, etc all have analogues in this algebraic world. The cobordism groups of quadratic Poincaré complexes can be identified with $L_n(\Lambda)$.

At the heart of this paper is Theorem 4.3. It assigns to any preformation z a $(2q+2)$–dimensional quadratic Poincaré pair

$$x = (g\colon D' \cup_h D \to E, (\delta\omega, \delta\psi' \cup_\chi \delta\psi))$$

whose boundary $c' \cup_{(h,\chi)} -c = (D' \cup_h D, \delta\psi' \cup_\chi \delta\psi)$ is the union of certain $(2q+1)$–dimensional quadratic Poincaré pairs $c = (f\colon C \to D, (\delta\psi, \psi))$ and $c' = (f'\colon C' \to D', (\delta\psi', \psi'))$ along an isomorphism $(h, \chi)\colon (C, \psi) \xrightarrow{\cong} (C', \psi')$ of the boundaries of c' and c. The Poincaré pair x is designed in such a way that it is cobordant $\mathrm{rel}\,\partial$ to an algebraic s–cobordism if and only if $[z] \in l_{2q+2}(\Lambda)$ is elementary. (Unlike in classical surgery theory, we don't know of any realization result in Kreck's theory, ie, it is not known whether any preformation z arises as an obstruction of some modified surgery problem. Theorem 4.3 mends matters by offering a kind of algebraic realization.)

If $[z] \in l_{2q+2}(\Lambda)$ is elementary (ie, x is cobordant $\mathrm{rel}\,\partial$ to an algebraic s–cobordism) the Poincaré pairs c and c' must be (homotopy) equivalent. It turns out that this is the same as a stable weak isomorphism between z and its flip, ie, a *flip-isomorphism* of z (Definition 5.1). Hence, any choice of flip-isomorphism t induces an equivalence $c \xrightarrow{\simeq} c'$ and thus enables us to transform the boundary $c' \cup_{(h,\chi)} -c = (D' \cup_h D, \delta\psi' \cup_\chi \delta\psi)$ of x into a twisted double $c \cup_{(h_t,\chi_t)} -c = (D \cup_{h_t} D, \delta\psi \cup_{\chi_t} \delta\psi)$. Here, (h_t, χ_t) is the composition of (h, χ) and an equivalence $(C', \psi') \xrightarrow{\cong} (C, \psi)$ induced by t (Theorem 5.3).

The new algebraic cobordism $x_t = (g\colon D \cup_{h_t} D \to E, (\delta\omega, \delta\psi \cup_{\chi_t} \delta\psi))$ is the algebraic analogue of a manifold W with a twisted double structure $\partial W = N \cup_g -N$ on the boundary. These manifolds have been studied by H E Winkelnkemper [16; 17], F Quinn [10] and others. They have shown that W is cobordant rel ∂ to a compatible twisted double if and only if a certain obstruction (its *asymmetric signature*) vanishes in the asymmetric Witt-group $WAsy(\Lambda) = LAsy^{2q+2}(\Lambda)$. There is a corresponding theorem in the world of algebraic surgery (Ranicki [13, Section 0B]) which can be applied to the Poincaré pair x_t. The asymmetric signature $\sigma(z,t) \in WAsy(\Lambda)$ from Definition 6.6 is nothing but the asymmetric signature for x_t. Since an (algebraic) s–cobordism is an (algebraic) twisted double, all asymmetric signatures for an elementary preformation must vanish (Theorem 6.8).

A stronger obstruction can be obtained by gluing the Poincaré pair x along its boundary using a suitable flip-isomorphism t. If the resulting quadratic Poincaré complex is null-cobordant, then x is cobordant rel ∂ to an s–cobordism.

Unfortunately this gluing operation requires extra conditions on the flip-isomorphism. Let t be some flip-isomorphism and x_t the Poincaré pair with the twisted double structure on the boundary as before. If that twisted double is trivial (ie, (h_t, χ_t) is homotopic to the identity), it is possible to glue both ends of x_t together or, alternatively, stick an algebraic tube $(D \cup_C D \to D, (0, \delta\psi \cup_\psi \delta\psi))$ onto x_t. Hence, we require t to permit a homotopy $(\Delta, \eta)\colon (h_t, \chi_t) \simeq (1, 0)$. (Similarly, a tube $N \times I$ cannot generally be glued onto a manifold W with $\partial W = N \cup_g -N$ unless g is isotopic to the identity.) In this instance, t is called a *flip-isomorphism rel* ∂ (Definition 7.6). Gluing yields a $(2q+2)$–dimensional quadratic Poincaré complex. Its cobordism class in $L_{2q+2}(\Lambda)$ is the *quadratic signature* $\rho^*(z, t, \Delta, \eta)$ (Definition 7.8). The class vanishes for some flip-isomorphism t and choice of homotopy (Δ, η) if and only if x_t and therefore x is cobordant rel ∂ to an s–cobordism.

Unfortunately, the rel ∂–conditions for a flip-isomorphism are quite complicated. If, however, C and C' are contractible the problem disappears. This is the case if and only if z is a non-singular formation in the sense of [11, Section 2]. Whence quadratic signatures can be defined for any choice of flip-isomorphism (Theorem 8.2).

2 Surgery obstruction monoids and groups

2.1 Forms and $L_{2q}(\Lambda)$

Definition 2.1 [11, Section 2] Let M be a module.

(i) The canonical map $M \to M^{**}$ defines the *ϵ–duality involution map*

$$T_\epsilon\colon \mathrm{Hom}_\Lambda(M, M^*) \longrightarrow \mathrm{Hom}_\Lambda(M, M^*), \quad \phi \longmapsto (x \longmapsto \epsilon\overline{\phi(-)(x)})$$

and the abelian groups $Q^\epsilon(M) = \ker(1 - T_\epsilon)$ and $Q_\epsilon(M) = \mathrm{coker}(1 - T_\epsilon)$

(ii) An *ϵ–symmetric form* (M,λ) is tuple with $\lambda \in Q^{\epsilon}(M)$. It is *non-singular* if λ is an isomorphism. A *lagrangian* $j\colon L \hookrightarrow M$ is a free direct summand such that $0 \to L \xrightarrow{j} M \xrightarrow{j^*\lambda} L^* \to 0$ is exact.

(iii) An *ϵ–quadratic form* (M,λ,ν) is an ϵ–symmetric form (M,λ) together with a map $\nu\colon M \to Q_{\epsilon}(\Lambda)$ such that for all $x,y \in M$ and $a \in \Lambda$

(a) $\nu(x+y)-\nu(x)-\nu(y) = \lambda(x,y) \in Q_{\epsilon}(\Lambda)$

(b) $\nu(x)+\epsilon\overline{\nu(x)} = \lambda(x,x) \in Q^{\epsilon}(\Lambda)$

(c) $\nu(ax) = a\nu(x)\overline{a} \in Q_{\epsilon}(\Lambda)$

A *lagrangian* L of (M,λ,ν) is a lagrangian of (M,λ) such that $\nu|L = 0$.

Remark 2.2 [11, Section 2] If M is free, an ϵ–quadratic form (M,λ,ν) can also be thought of as an equivalence class of *split ϵ–quadratic forms*, ie, tuples $(M,\psi \in \mathrm{Hom}_{\Lambda}(M,M^*))$ with $(1+T_{\epsilon})\psi = \lambda$ and $\psi(x)(x) = \nu(x)$. Two split structures ψ and ψ' on M are equivalent if $[\psi] = [\psi'] \in Q_{\epsilon}(M)$.

Definition 2.3 For any free module L we define a *hyperbolic form*

$$H_{\epsilon}(L) = \Big(L \oplus L^*, \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \colon L \oplus L^* \longrightarrow \big(L \oplus L^*\big)^*\Big)$$

Two non-singular ϵ–quadratic forms are *stably isometric* if they are isometric after adding hyperbolic forms. $L_{2q}(\Lambda)$ is the group of stable isometry classes.

2.2 Preformations, $l_{2q+2}(\Lambda)$ and $L_{2q+1}(\Lambda)$

Definition 2.4 ([7], [11, Section 2])

(i) An *ϵ–preformation* $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is a tuple consisting of a free module F, a f.g. module G and $\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}\right) \in \mathrm{Hom}_{\Lambda}(G, F \oplus F^*)$ such that $(G, \gamma^*\mu)$ is a $(-\epsilon)$–symmetric form.

A *split ϵ–preformation* $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ is an ϵ–preformation and a map $\theta\colon G \to Q_{-\epsilon}(\Lambda)$ such that $(G,\gamma^*\mu,\theta)$ is a $(-\epsilon)$–quadratic form. z is *regular* if G is free. Then we interpret θ as a split quadratic structure $\theta \in \mathrm{Hom}_{\Lambda}(G,G^*)$ on G. Moreover, if $\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}\right)$ is the inclusion of a lagrangian for $H_{\epsilon}(F)$, it is called a *split ϵ–formation*.[1]

[1] An ϵ–formation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ is a non-singular ϵ–quadratic formation $(H_{\epsilon}(F), F, G)$ and a split ϵ–formation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ is a non-singular split ϵ–quadratic formation $\Big(F, \Big(\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}\right), \overline{\theta}\Big) G\Big)$ together with a choice of representative θ for $\overline{\theta} \in Q_{-\epsilon}(G)$. See also [11, page 127].

(ii) The *boundary of a $(-\epsilon)$–quadratic form* (K,θ) on a free module K is the split ϵ–formation $\partial(K,\theta)=(K \xleftarrow{1} K \xrightarrow{\theta-\epsilon\theta^*} K^*,\theta)$.

(iii) A *trivial formation* is a split ϵ–formation of the form $(P,P^*)=(P \xleftarrow{0} P \xrightarrow{1} P^*,0)$ with P a free module.

Surprisingly, the obstructions in both even-dimensional modified (Section 1.1) and odd-dimensional classical surgery theory are preformations. Let B be a $(2q+2)$–dimensional Poincaré space. The Wall surgery obstruction for a $(2q+1)$–dimensional normal map $f\colon M_0 \to B$ can be constructed as follows: Perform surgery on some set of generators of $K_q(M_0)$. Then the trace $(W', M_0, M_1) \to B \times (I, \{0\}, \{1\})$ is a $(q+1)$–connected modified surgery problem and (1) is a formation and the surgery obstruction of f (see [14, Section 12.2]). Although the obstruction preformations are the same in both surgery theories, the equivalence relations are quite different! Two B–diffeomorphic modified surgery problems induce *strongly isomorphic* obstruction preformations. If they are B–cobordant rel ∂, they will only differ by some connected sum of tori (see [7, Section 4]). Hence the surgery obstructions are isomorphic after adding by some "hyperbolic" elements. Therefore (1) lives in the the set $l_{2q+2}(\Lambda)$ of stable strong isomorphism classes.

However, in odd-dimensional classical surgery theory, an equivalence of q–connected $(2q+1)$–dimensional normal maps $M_0 \to B$ gives rise to a *stable weak isomorphism* of the obstruction (pre-)formations. The stable weak equivalence classes modulo all boundaries yield the classical surgery group $L_{2q+1}(\Lambda)$.

Definition 2.5 ([7, section 5], [11, Sections 2,5]) Let $z=(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*,\theta)$ and $z'=(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*,\theta')$ be two split ϵ–preformations.

(i) A *weak isomorphism*[2] $(\alpha,\beta,\nu,\kappa)$ between z and z' is a tuple consisting of isomorphisms $\alpha\in \mathrm{Hom}_\Lambda(F,F')$, $\beta\in\mathrm{Hom}_\Lambda(G,G')$ and maps $\nu\in\mathrm{Hom}_\Lambda(F^*,F)$ and $\kappa\in\mathrm{Hom}_\Lambda(G,G^*)$ such that
   (a) $\alpha\gamma+\alpha(\nu-\epsilon\nu^*)^*\mu=\gamma'\beta\in\mathrm{Hom}_\Lambda(G,F')$
   (b) $\alpha^{-*}\mu=\mu'\beta\in\mathrm{Hom}_\Lambda(G,F'^*)$
   (c) $\theta+\mu^*\nu\mu=\beta^*\theta'\beta+\kappa+\epsilon\kappa^*\in\mathrm{Hom}_\Lambda(G,G^*)$

(ii) A *stable weak isomorphism of z and z'* is a weak isomorphism $z\oplus t\cong z'\oplus t'$ for trivial formations t, t'. The *Witt-group* $L_{2q+1}(\Lambda)$ is the set of equivalence

[2] $(\alpha,\beta,\nu,\kappa)$ is a refinement of the isomorphism $(\alpha,\beta,[\nu])$ of the underlying non-singular split ϵ–quadratic formations [11, page 128].

classes of all (split)[3] ϵ–formations where $z \sim z'$ if there are boundaries b and b' such that $z \oplus b$ and $z' \oplus b'$ are stably weakly isomorphic.

(iii) A weak isomorphism $(\alpha, \beta, 0, 0)$ between z and z' is called a *strong isomorphism* (α, β).

(iv) A *stable strong isomorphism* between z and z' is a strong isomorphism $z \oplus \partial h \cong z' \oplus \partial h'$ for some hyperbolic forms h and h'. The *l–monoid* $l_{2q+2}(\Lambda)$ is the set of the equivalence classes.

Remark 2.6 Every stable strong isomorphism is also a stable weak isomorphism, for there exists a weak isomorphism

$$\left(1, \left(\begin{smallmatrix} 0 & 1 \\ -\epsilon & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ \epsilon & 0 \end{smallmatrix}\right), 0\right) \colon \partial H_{-\epsilon}(P) \xrightarrow{\cong} (P \oplus P^*, (P \oplus P^*)^*)$$

2.3 Elementary preformations

A modified surgery problem is B–cobordant rel ∂ to an s–cobordism if and only if its obstruction is *elementary* [7, Theorem 3]. Several alternative definitions for this central concept will be given. We will also show that testing a preformation for elementariness is equivalent to testing some related regular preformation. This is important since the secondary obstructions (the asymmetric and quadratic signatures in §) are only defined for regular preformations.

Definition 2.7 [7, page 730] A split ϵ–preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ is *elementary* if there is a free submodule $j \colon U \hookrightarrow G$ such that

(i) $j^*\gamma^*\mu j = 0$ and $\theta j = 0$,

(ii) γj and μj are split injections with images U_0 and U_1,

(iii) $R_1 = F^*/U_1 \to U_0^*$, $f \mapsto f|U_0$ is an isomorphism.

Such a U is called an *s–lagrangian*. An element in $l_{2q+2}(\Lambda)$ is *elementary* if it has an elementary representative.

Proposition 2.8 *For a split ϵ–preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ and a free submodule $j \colon U \hookrightarrow G$ the following statements are equivalent:*

(i) *z is elementary with s–lagrangian U.*

(ii) *$\theta|U = 0$ and $0 \longrightarrow U \xrightarrow{\mu j} F^* \xrightarrow{(\gamma j)^*} U^* \longrightarrow 0$ is exact.*

[3] The Witt-groups of split and non-split formations are isomorphic [14, 12.33].

(iii) $\theta|U = 0$ *and the two horizontal chain maps*

$$\begin{tikzcd} & U \arrow[d, "j"'] & \\ G \arrow[r, "1"] \arrow[d, "\gamma"'] & G \arrow[d, "j^*\gamma^*\mu"'] & G \arrow[l, "1"'] \arrow[d, "\mu"] \\ F \arrow[r, "-\epsilon j^*\mu^*"] & U^* & F^* \arrow[l, "j^*\gamma^*"'] \end{tikzcd}$$

are chain equivalences.

(iv) *The preformation is strongly isomorphic to a preformation of the form*

$$(U \oplus U^* \xleftarrow{\left(\begin{smallmatrix}1 & 0\\ 0 & \sigma\end{smallmatrix}\right)} U \oplus R \xrightarrow{\left(\begin{smallmatrix}0 & -\epsilon\sigma\\ 1 & \tau\end{smallmatrix}\right)} U^* \oplus U, \theta)$$

for a split ϵ*–preformation* $(U^* \xleftarrow{\sigma} R \xrightarrow{\tau} U, \theta')$ *such that*

$$\theta\colon U \oplus R \longrightarrow Q_{-\epsilon}(\Lambda), \quad (u, r) \longmapsto \theta'(r) - \epsilon\sigma(r)(u).$$

Proof The only difficult direction is (i) $\Rightarrow$ (iii): Let $\pi\colon F \to U_0 = \gamma(U)$ be the projection along some complement R_0. Decompose $G = U \oplus R$ with $R = \ker(\pi\gamma)$. Let $R_1 \subset F^*$ be some complement of $U_1 = \mu(U)$. Write

$$\begin{aligned} \gamma &= \left(\begin{smallmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{smallmatrix}\right)\colon U \oplus R \longrightarrow U_0 \oplus R_0 \\ \mu &= \left(\begin{smallmatrix}\mu_1 & \mu_2\\ \mu_3 & \mu_4\end{smallmatrix}\right)\colon U \oplus R \longrightarrow U_0^* \oplus R_0^* \\ \Phi &= \begin{pmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{pmatrix}\colon U_1 \oplus R_1 \longrightarrow U_0^* \oplus R_0^*, \qquad f \in F^* \longmapsto (f|U_0, f|R_0) \\ \mu' &= \begin{pmatrix}\mu_1' & \mu_2'\\ \mu_3' & \mu_4'\end{pmatrix}\colon U \oplus R \longrightarrow U_1 \oplus R_1, \qquad x \longmapsto \Phi^{-1}\mu(x). \end{aligned}$$

By assumption, γ_1 and μ_1' are isomorphisms and γ_3 and μ_3' vanish. We can apply the strong isomorphism $\left(1, \left(\begin{smallmatrix}\gamma_1 & \gamma_2\\ 0 & 1\end{smallmatrix}\right)\right)$ to achieve the simpler situation of $\gamma = \begin{pmatrix}1 & 0\\ 0 & \gamma_4\end{pmatrix}$ and $U_0 = U$. We compute $\gamma^*\mu$ and see that $\Phi_1 = 0$. From Definition 2.7, (iii) implies that Φ_2 is an isomorphism and therefore Φ_3 is bijective as well. We use these facts to see:

$$\mu = \Phi\mu' = \begin{pmatrix}0 & \Phi_2\mu_4'\\ \Phi_3\mu_1' & \Phi_3\mu_2' + \Phi_4\mu_4'\end{pmatrix} = \left(\begin{smallmatrix}0 & \mu_2\\ \mu_3 & \mu_4\end{smallmatrix}\right)$$

Therefore μ_3 must be an isomorphism. Because $\gamma^*\mu$ is $(-\epsilon)$–symmetric, $\mu_2 = -\epsilon\mu_3^*\gamma_4$. Finally, we apply the strong isomorphism $\left(\begin{pmatrix}1 & 0\\ 0 & \mu_3^*\end{pmatrix}, 1_G\right)$. $\square$

Lemma 2.9 *Let* $x = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ *and* $y = (F \xleftarrow{\sigma} H \xrightarrow{\tau} F^*, \psi)$ *be two split* ϵ*–preformations and* $\pi\colon G \twoheadrightarrow H$ *an epimorphism such that*

(2)
$$\begin{tikzcd} F & G \arrow[l, "\gamma"'] \arrow[r, "\mu"] \arrow[d, two heads, "\pi"'] & F^* \\ & H \arrow[ul, "\sigma"] \arrow[ur, "\tau"'] & \end{tikzcd}$$

commutes and $\theta = \psi\pi$. *Then* x *is elementary if and only if* y *is elementary.*

Corollary 2.10

(i) $\theta(W)$ *on [7, page 729] can be replaced by* (1).

(ii) *Let* x *be a split* ϵ*–preformation. Then there is a regular split* ϵ*–preformation which is elementary if and only if* x *is elementary.*

3 Algebraic surgery theory

In this paper, algebraic surgery theory is the key to a better understanding of $l_{2q+2}(\Lambda)$. This section summarizes the main concepts from Ranicki [11; 12] and presents a new theory of surgery for Poincaré pairs.

3.1 A short introduction

Definition 3.1 [11; 12] Let C be a chain complex.

(i) The *duality involution* T is defined as:

$$T\colon \mathrm{Hom}_\Lambda(C^p, C_q) \longrightarrow \mathrm{Hom}_\Lambda(C^q, C_p), \quad \psi \longmapsto (-)^{pq}\psi^*$$

(ii) The chain complexes $W^{\%}(C)$ and $W_{\%}(C)$ are defined as:

$$\begin{aligned} W^{\%}(C)_n &= \{\phi_s\colon C^{n-r+s} \longrightarrow C_r \mid r \in \mathbb{Z}, s \geq 0\} \\ d^{\%}\colon W^{\%}(C)_n &\longrightarrow W^{\%}(C)_{n-1} \\ \{\phi_s\} &\longmapsto \{d\phi_s + (-)^r \phi_s d^* + (-)^{n+s-1}(\phi_{s-1} + (-)^s T\phi_{s-1})\colon \\ &\qquad C^{(n-1)-r+s} \longrightarrow C_r \mid r \in \mathbb{Z}, s \geq 0\}, \quad (\phi_{-1} := 0) \\ W_{\%}(C)_n &= \{\psi_s\colon C^{n-r-s} \longrightarrow C_r \mid r \in \mathbb{Z}, s \geq 0\} \\ d_{\%}\colon W_{\%}(C)_n &\longrightarrow W_{\%}(C)_{n-1} \\ \{\psi_s\} &\longmapsto \{d\psi_s + (-)^r \psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1} T\psi_{s+1})\colon \\ &\qquad C^{(n-1)-r-s} \longrightarrow C_r \mid r \in \mathbb{Z}, s \geq 0\} \end{aligned}$$

Their homology groups are the *symmetric Q–groups* $Q^n(C) = H_n(W^{\%}(C))$ and the *quadratic Q–groups* $Q_n(C) = H_n(W_{\%}(C))$.
They are related by the *symmetrization map*:

$$Q_n(C) \longrightarrow Q^n(C), \quad \{\psi_s\} \longmapsto \begin{cases} (1+T)\psi_0 & : \text{if } s = 0, \\ 0 & : \text{if } s \neq 0. \end{cases}$$

(iii) For $n \in \mathbb{N}$ define the chain complex C^{n-*} by

$$d_{C^{n-*}} = (-)^r d_C^*\colon (C^{n-*})_r = C^{n-r} = C^*_{n-r} \longrightarrow (C^{n-*})_{r-1}.$$

(iv) A *symmetric n–dimensional complex* (C, ϕ) is a tuple containing a cycle $\phi \in W^{\%}(C)_n$. It is *Poincaré* if $\phi_0\colon C^{n-*} \to C$ is a chain equivalence.

(v) A *quadratic n–dimensional complex* (C, ψ) is a tuple containing a cycle $\psi \in W_{\%}(C)_n$. It is *Poincaré* if $(1+T)\psi_0\colon C^{n-*} \to C$ is a chain equivalence.

(vi) A *morphism* $(f, \rho)\colon (C, \psi) \to (C', \psi')$ *of quadratic n–dimensional complexes* is a chain map $f\colon C \to C'$ together with an element $\rho \in W_{\%}(C')_{n+1}$ such that $\psi' - f\psi f^* = d_{\%}(\rho)$. The *composition* of two morphisms is defined to be $(f', \sigma') \circ (f, \sigma) = (f' f, \sigma' + f'\sigma f'^*)$.

Definition 3.2 [11; 12] Let $f\colon C \to D$ be a chain map.

(i) The chain complex $W_{\%}(f)$ is given by

$$\begin{aligned} W_{\%}(f)_{n+1} &= \{(\delta\psi, \psi) \in W_{\%}(D)_{n+1} \oplus W_{\%}(C)_n\} \\ d_{\%}\colon W_{\%}(f)_{n+1} &\to W_{\%}(f)_n \\ (\delta\psi, \psi) &\mapsto (d_{\%}(\delta\psi) + (-)^n f\psi f^*, d_{\%}(\psi)) \end{aligned}$$

The homology groups are the *relative quadratic Q–groups* $Q_n(f)$

(ii) An $(n+1)$*–dimensional quadratic pair* $(f\colon C \to D, (\delta\psi, \psi))$ is a tuple containing a cycle $(\delta\psi, \psi) \in W_{\%}(f)_{n+1}$. It is called a *Poincaré pair* or *cobordism* if $\begin{pmatrix} (1+T)\delta\psi_0 \\ (-)^{n+1-r}(1+T)\psi_0 f^* \end{pmatrix}\colon D^{n+1-r} \to \mathcal{C}(f)_r$ is a chain equivalence.

Remark 3.3 [11] Let M be an n–dimensional closed manifold. The diagonal approximation map produces an n–dimensional symmetric Poincaré complex $\sigma^*(M) = (C, \phi)$ where $C = C_*(\widetilde{M})$ is the $\mathbb{Z}[\pi_1(M)]$–chain complex of the universal cover of M and $\phi_0 = - \cap [M]$ the Poincaré duality map. If M has a boundary $i\colon \partial M \hookrightarrow M$, then a similar construction endows the chain map $\widetilde{i}_*\colon C_*(\widetilde{\partial M}) \to C_*(\widetilde{M})$ with a relative symmetric structure $(\delta\phi, \phi) \in Q^n(\widetilde{i}_*)$. The quadratic construction assigns an n–dimensional quadratic Poincaré complex to any normal map $M \to X$. Its symmetrization plus $\sigma^*(X)$ is $\sigma^*(M)$. There is also a relative version for normal cobordism.

Remark 3.4 [11; 12] There are several important constructions and concepts in algebraic surgery theory used in this paper.

(i) An *equivalence* of $(n+1)$–dimensional quadratic pairs

$$(g,h;k)\colon (f\colon C\to D,(\delta\psi,\psi)) \xrightarrow{\simeq} (f'\colon C'\to D',(\delta\psi',\psi'))$$

consists of chain equivalences $g\colon C\xrightarrow{\simeq} C'$, $h\colon D\xrightarrow{\simeq} D'$ and a chain homotopy $k\colon f'g\simeq hf$ such that $(g,h;k)_{\%}(\delta\psi,\psi)=(\delta\psi',\psi')\in Q_{n+1}(f')$. (See [11, page 140].)

(ii) For every n–dimensional quadratic Poincaré pair $c=(f\colon C\to D,(\delta\psi,\psi))$ one can endow the mapping cone $\mathcal{C}(f)$ with an n–dimensional quadratic structure: the *Thom complex* $(\mathcal{C}(f),\delta\psi/\psi)$ *of* c. Conversely, any n–dimensional quadratic complex (N,ζ) determines an n–dimensional quadratic Poincaré pair $(\partial N\to N^{n-*},(0,\partial\zeta))$: its *Thickening*. These operations establish inverse natural bijections between the equivalence classes of n–dimensional quadratic Poincaré pairs and connected n–dimensional quadratic complexes. (See [11, pages 141–144].)

(iii) One can glue two n–dimensional quadratic Poincaré pairs $c=(f\colon C\to D,(\delta\psi,\psi))$ and $c'=(f'\colon C\to D',(\delta\psi',\psi))$ along their common boundary [11, page 135]. Their *union* is the n–dimensional quadratic Poincaré complex $c\cup -c'=(D\cup_C D',\delta\psi\cup_\psi\delta\psi')$. A Poincaré pair $(f\colon C\oplus C\to D,(\delta\psi,\psi\oplus -\psi))$ with two identical boundary components can be glued together as well. (See [13, page 266].)

(iv) Two n–dimensional quadratic Poincaré complexes are *cobordant* if their sum is the boundary of a Poincaré pair. The set of cobordism classes form a group which is canonically isomorphic to $L_n(\Lambda)$. (See [11, Propositions 3.2, 4.3 and 5.2].)

(v) Given an n–dimensional quadratic pair $(f\colon C\to D,(\delta\psi,\psi))$ one can perform *algebraic surgery* on (C,ψ) killing $\mathrm{im}(f^*\colon H^*(D)\to H^*(C))$. The result is an n–dimensional quadratic complex (C',ψ'). (C,ψ) is Poincaré if and only if (C',ψ') is Poincaré. Two Poincaré complexes are cobordant if and only if one can be obtained from the other by finitely many algebraic surgeries and equivalences [11, Section 4].

Definition 3.5 [13, Definition 30.8] Let $c=(f\colon C\to D,(\delta\psi,\psi))$ be an $(n+1)$–dimensional quadratic Poincaré pair and $(g,\sigma)\colon (C',\psi')\xrightarrow{\simeq}(C,\psi)$ an equivalence. We define the $(n+1)$–dimensional quadratic Poincaré pair

$$(g,\sigma)_{\%}(c)=(fg\colon C'\to D,(\delta\psi+(-)^n f\sigma f^*,\psi'))$$

Let $c' = (f'\colon C \to D', (\delta\psi', \psi'))$ be another $(n+1)$–dimensional quadratic Poincaré pair. We define

$$c \cup_{(g,\sigma)} -c' = (D \cup_g D', \delta\psi \cup_\sigma \delta\psi') = (g,\sigma)_\%(c) \cup -c'$$

If $c = c'$ we call this union a *twisted double of* c *in respect to* (h,χ).

3.2 Cobordism of pairs, surgery inside a pair

For our purposes we have to extend the results and definitions for surgery and cobordism from quadratic Poincaré complexes to quadratic cobordisms.

Definition 3.6 Two $(n+1)$–dimensional quadratic Poincaré pairs c and c' with identical boundaries are *cobordant rel* ∂ if $c \cup -c' = 0 \in L_{n+1}(\Lambda)$.

Definition 3.7 Let $c = (f\colon C \to D, (\delta\psi, \psi))$ be an $(n+1)$–dimensional quadratic Poincaré pair and $d = (g\colon \mathcal{C}(f) \to B, (\delta\sigma, \delta\psi/\psi))$ an $(n+2)$–dimensional quadratic pair. Write $g = \begin{pmatrix} a & b \end{pmatrix}\colon D_r \oplus C_{r-1} \to B_r$. *The result of the surgery* d *on the inside of* c is the $(n+1)$–dimensional quadratic Poincaré pair $c' = (f'\colon C \to D', (\delta\psi', \psi))$ given by

$$d_{D'} = \begin{pmatrix} d_D & 0 & (-)^n(1+T)\delta\psi_0 a^* + (-)^n f(1+T)\psi_0 b^* \\ (-)^r a & d_B & (-)^r(1+T)\delta\sigma_0 + (-)^{n+1} b\psi_0 b^* \\ 0 & 0 & (-)^r d_B^* \end{pmatrix}\colon$$
$$D'_r = D_r \oplus B_{r+1} \oplus B^{n+2-r} \longrightarrow D'_{r-1}$$
$$f' = \begin{pmatrix} f \\ -b \\ 0 \end{pmatrix}\colon C_r \longrightarrow D'_r$$
$$\delta\psi'_0 = \begin{pmatrix} \delta\psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\colon D'^{n+1-r} \longrightarrow D'_r$$
$$\delta\psi'_s = \begin{pmatrix} \delta\psi_s & (-)^s T\delta\psi_{s-1}a^* - fT\psi_{s-1}b^* & 0 \\ 0 & (-)^{n-r-s+1}T\delta\sigma_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}\colon D'^{n+1-r-s} \longrightarrow D'_r \quad (s > 0)$$

These formulas are derived from standard procedures of algebraic surgery theory, namely, Thom complex, algebraic surgery and thickening:

Proposition 3.8 *The result of the surgery* $d = (g\colon \mathcal{C}(f) \to B, (\delta\sigma, \delta\psi/\psi))$ *on the Thom complex of* c *is isomorphic to the Thom complex of* c'.

Proof The isomorphisms

$$u_r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & (-)^{n-r}\psi_0 b^* \end{pmatrix}\colon M_r = (D_r \oplus C_{r-1}) \oplus B_{r+1} \oplus B^{n+2-r}$$

$$\xrightarrow{\cong} \mathcal{C}(f')_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1}$$

define an isomorphism $(u, 0)\colon (M, \tau) \xrightarrow{\cong} (\mathcal{C}(f'), \delta\psi'/\psi)$ between the result (M, τ) of the surgery d on $(\mathcal{C}(f), \delta\psi/\psi)$ and the Thom-complex of c'. □

Two manifolds are cobordant if and only if one is derived from the other by a finite sequence of surgeries and diffeomorphisms. The same statement holds for Poincaré complexes (by [11, Proposition 4.1]) and Poincaré pairs:

Proposition 3.9 *Two $(n+1)$–dimensional quadratic Poincaré pairs with identical boundaries are cobordant rel∂ if and only if one can be obtained from the other by a finite sequence of surgeries and equivalences of the type $(1, h; k)$.*

Proof Let $c = (f\colon C \to D, (\delta\psi, \psi))$ and $c' = (f'\colon C \to D', (\delta\psi', \psi))$ be two $(n+1)$–dimensional quadratic Poincaré pairs. Let $(1, h; k)\colon c \xrightarrow{\simeq} c'$ be an equivalence. There is a $(\delta\chi, \chi) \in W_{\%}(f')_{n+2}$ such that $(1, h; k)_{\%}(\delta\psi, \psi) - (\delta\psi', \psi') = d_{\%}(\delta\chi, \chi)$. The $(n+2)$–dimensional quadratic Poincaré pair $(b\colon D \cup_C D' \longrightarrow D', ((-)^n\delta\chi, \delta\psi \cup_{\psi} \delta\psi'))$ with $b = (h, (-)^{r-1}k, -1)\colon (D \cup_C D')_r = D_r \oplus C_{r-1} \oplus D'_{r-1} \to D'_r$ is a cobordism between c and c'.

Now let c, c' and d as in Definition 3.7. Let $(V, \sigma) = c \cup -c$. Define a connected $(n+2)$–dimensional quadratic pair $\widetilde{d} = (\widetilde{g}\colon V \to B, (\delta\sigma, \tau))$ by $\widetilde{g} = \begin{pmatrix} a & b & 0 \end{pmatrix}\colon V_r = D_r \oplus C_{r-1} \oplus D_r \to B_r$. The result $(\widetilde{V}, \widetilde{\tau})$ of this surgery is isomorphic to $c' \cup -c$ via

$$u_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & (-)^{n-r}\psi_0 b^* \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\colon \widetilde{V}_r = (D_r \oplus C_{r-1} \oplus D_r) \oplus B_{r+1} \oplus B^{n+2-r}$$
$$\longrightarrow (D' \cup_C D)_r = (D_r \oplus B_{r+1} \oplus B^{n+2-r}) \oplus C_{r-1} \oplus D_r$$

Clearly (V, σ) is null-cobordant and so is $c' \cup -c$ by [11, Proposition 4.1]. Conversely, let $c = (f\colon C \to D, (\delta\psi, \psi))$ and $c' = (f'\colon C \to D', (\delta\psi', \psi))$ be two cobordant $(n+1)$–dimensional quadratic Poincaré pairs, ie, there is an $(n+2)$–dimensional quadratic Poincaré pair

$$e = (h\colon D \cup_C D' \longrightarrow E, (\delta\omega, \omega = \delta\psi \cup_{\psi} -\delta\psi'))$$

Write $h = \begin{pmatrix} j_0 & k & j_1 \end{pmatrix}\colon D_r \oplus C_{r-1} \oplus D'_r \to E_r$. We define the connected $(n+2)$–dimensional quadratic pair

$$d = (g\colon \mathcal{C}(f) \longrightarrow B = \mathcal{C}(j_1), (\delta\sigma, \delta\psi/\psi))$$
$$g = \begin{pmatrix} j_0 & k \\ 0 & -f \end{pmatrix}\colon \mathcal{C}(f)_r = D_r \oplus C_{r-1} \longrightarrow B_r = E_r \oplus D'_{r-1}$$

$$\delta\sigma_s = \begin{pmatrix} \delta\omega_s & 0 \\ (-)^{n-r-1}(\delta\psi'_s j_1^* + (-)^s f'\psi_s k^*) & (-)^{n-r-s} T\delta\psi'_{s+1} \end{pmatrix} :$$
$$B^{n+2-r-s} = E^{n+2-r-s} \oplus D'^{n+1-r-s} \longrightarrow B_r = E_r \oplus D'_{r-1}$$

The result of the surgery d inside of c is the $(n+1)$–dimensional quadratic Poincaré pair $c'' = (f''\colon C \to D'', (\delta\psi'', \psi))$. The maps

$$m_r = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \delta\psi'_0 \end{pmatrix} \colon D''_r \longrightarrow D'_r$$

define an equivalence $(1, m; 0)\colon c'' \xrightarrow{\simeq} c'$. □

4 From preformations to quadratic complexes

Given a regular split ϵ–preformation z, we will construct a Poincaré pair x such that x is cobordant rel ∂ to an s–cobordism if and only if $[z] \in l_{2q+2}(\Lambda)$ is elementary. This algebraic "realization" result allows us to apply algebraic surgery techniques to preformations.

Proposition 4.1 [11, Propositions 2.3, 2.5] *Let $\mathcal{C}$ be the category of $(2q+1)$–dimensional quadratic complexes concentrated in dimension q and $q+1$ with isomorphisms as morphisms. Let $\mathcal{P}$ be the category of regular split ϵ–preformations and weak isomorphisms. There is an equivalence* $\mathbf{F}$: $\mathcal{P} \xrightarrow{\simeq} \mathcal{C}$ *mapping* $(\alpha, \beta, \nu, \kappa)\colon (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta) \xrightarrow{\cong} (F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \theta')$ *to a morphism* $(e, \rho)\colon (N, \zeta) \xrightarrow{\cong} (N', \zeta')$ *given by:*

$$\begin{array}{ccc} N_{q+1} = F & \xrightarrow{e_{q+1}=\alpha} & N'_{q+1} = F' \\ \mu^* \downarrow & & \downarrow \mu'^* \\ N_q = G^* & \xrightarrow{e_q=\beta^{-*}} & N'_q = G'^* \end{array}$$

$$\zeta_0 = \gamma\colon N^q \longrightarrow N_{q+1} \qquad \zeta_1 = \epsilon\theta\colon N^q \longrightarrow N_q$$
$$\rho_0 = \alpha\nu\alpha^*\colon N'^{q+1} \longrightarrow N'_{q+1} \qquad \rho_1 = \mu'^*\alpha\nu\alpha^*\colon N'^{q+1} \longrightarrow N'_q$$
$$\rho_2 = -\beta^{-*}\kappa^*\beta^{-1}\colon N'^q \longrightarrow N'_q$$

$\mathbf{F}$ *induces a bijection between the equivalence classes in* $\mathrm{Obj}(\mathcal{C})$ *and the stable weak isomorphism classes in* $\mathrm{Obj}(\mathcal{P})$.

Definition 4.2 Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ be a split ϵ–preformation. Its *Flip* is the split ϵ–preformation $z' = (F^* \xleftarrow{\epsilon\mu} G \xrightarrow{\gamma} F, -\theta)$.

The flip of the surgery obstruction (1) is the surgery obstruction of the "reverse" surgery problem $(-W, M_1, M_0) \to B$.

Theorem 4.3 *Given a regular split ϵ–preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$, there exists a $(2q+2)$–dimensional quadratic Poincaré pair $x = (g\colon \partial E \to E, (0,\omega))$ and $(2q+1)$–dimensional quadratic Poincaré pairs*

$$c = (f\colon C \longrightarrow D, (\delta\psi, \psi)), \qquad c' = (f'\colon C' \longrightarrow D', (\delta\psi', \psi'))$$

together with an isomorphism $(h,\chi)\colon (C,\psi) \xrightarrow{\cong} (C',\psi')$ such that

(i) *c and c' are the thickenings of $\mathbf{F}(z)$ and $\mathbf{F}(z')$ where z' is the flip of z.*

(ii) *$(\partial E, \omega) = c' \cup_{(h,\chi)} -c$.*

(iii) *$[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if x is cobordant rel∂ to an s–cobordism, ie, a Poincaré pair $\big((j_0' \;\; k' \;\; j_1')\colon \partial E \to E', (\delta\omega', \omega)\big)$ such that $j_0'\colon D \xrightarrow{\simeq} E'$ and $j_1'\colon D' \xrightarrow{\simeq} E'$ are chain equivalences.*

Proof Let c and c' be the thickening of $\mathbf{F}(z)$. Then (h,χ) is given by:

$$\begin{aligned}
h_{q+1} &= 1\colon C_{q+1} = G \xrightarrow{\cong} C'_{q+1} = G \\
h_q &= \left(\begin{smallmatrix} 0 & \epsilon \\ 1 & 0 \end{smallmatrix}\right)\colon C_q = F \oplus F^* \xrightarrow{\cong} C'_q = F^* \oplus F \\
h_{q-1} &= 1\colon C_{q-1} = G^* \xrightarrow{\cong} C'_{q-1} = G^* \\
\chi_1 &= \left(\begin{smallmatrix} 0 & -\epsilon \\ 0 & 0 \end{smallmatrix}\right)\colon C'^q = F \oplus F^* \longrightarrow C'_q = F^* \oplus F \\
\chi_2 &= \left(\begin{smallmatrix} -\mu \\ 0 \end{smallmatrix}\right)\colon C'^{q-1} = G \longrightarrow C'_q = F^* \oplus F \\
\chi_3 &= \theta\colon C'^{q-1} = G \longrightarrow C'_{q-1} = G^*
\end{aligned} \tag{3}$$

The $(-\epsilon)$–quadratic form (G,θ) gives rise to the $(2q+2)$–dimensional quadratic Poincaré pair $y = (p\colon A \to E, (0,\tau))$ given by $p = 1\colon A_{q+1} = G \to E_{q+1} = G$, $E_i = 0$ $(i \neq q+1)$ and $(A,\tau) = \mathbf{F}(\partial(G,\theta))$. There is an equivalence $(a,\kappa)\colon (\partial E, \omega) := c' \cup_{(h,\chi)} -c \xrightarrow{\simeq} (A,\tau)$ given by:

$$\begin{aligned}
a_q &= \big(\epsilon\mu^* \;\; -1 \;\; \gamma^*\big)\colon \partial E_q = F \oplus G^* \oplus F^* \longrightarrow A_q = G^* \\
\kappa_2 &= \epsilon\theta\colon A^q = G \longrightarrow A_q = G^*
\end{aligned} \tag{4}$$

Then we set $x = (a,\kappa)_\%(y)$.

Now we assume that z is elementary. An s–lagrangian $i\colon U \hookrightarrow G$ defines a $(2q+3)$–dimensional quadratic pair:

$$d = (m\colon \mathcal{C}(g) \longrightarrow B = S^{q+1}U^*, (\delta\sigma, \sigma))$$

$$\begin{aligned} m &= \begin{pmatrix} a & b \end{pmatrix} : \mathcal{C}(g)_{q+1} = G \oplus (F \oplus G^* \oplus F^*) \to B_{q+1} = U^* \\ a &= -i^*\gamma^*\mu, \quad b = \begin{pmatrix} -\epsilon i^*\mu^* & i^* & -i^*\gamma^* \end{pmatrix} \end{aligned}$$

The result of the surgery d on the inside of x is the $(2q+2)$–dimensional quadratic Poincaré pair $x' = (g'\colon \partial E \to E', (\delta\omega', \omega))$ given by:

$$\begin{tikzcd}
\partial E_{q+2} = 0 \oplus G \oplus 0 \arrow[r, "0"] \arrow[d, "-\epsilon\begin{pmatrix} 1 \\ \gamma \\ \mu \\ 1 \end{pmatrix}"'] & E'_{q+2} = 0 \oplus 0 \oplus U \arrow[d, "-i"] \\
\partial E_{q+1} = G \oplus (F \oplus F^*) \oplus G \arrow[r, "(1\ 0\ 0\ {-1})"] \arrow[d, "\begin{pmatrix} -\epsilon\gamma & \epsilon & 0 & 0 \\ 0 & \mu^* & \epsilon\gamma^* & 0 \\ 0 & 0 & \epsilon & -\epsilon\mu \end{pmatrix}"'] & E'_{q+1} = G \oplus 0 \oplus 0 \arrow[d, "\epsilon i^*\gamma^*\mu"] \\
\partial E_q = F \oplus G^* \oplus F^* \arrow[r, "(\epsilon i^*\mu^*\ -i^*\ i^*\gamma^*)"] & E'_q = 0 \oplus U^* \oplus 0
\end{tikzcd}$$

Applying Proposition 2.8 (iii) to $g' \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \colon D' \oplus D \to E'$ shows that x' is an algebraic s–cobordism.

Finally, we prove the converse. Let x be cobordant rel∂ to an s–cobordism $x' = (g'\colon \partial E \to E', (\delta\omega', \omega))$. In order to simplify our calculations we use the equivalence (a, κ) from (4). Let $y' = (p'\colon A \to E', (\delta\tau', \tau))$ be the $(2q+2)$–dimensional quadratic Poincaré pair such that $x' \simeq (a, \kappa)_\%(y')$. The proof of Proposition 3.9 allows us to assume that y' is the result of a surgery $d = (m\colon \mathcal{C}(p) \to B, (\delta\sigma, \sigma = \partial\tau/\tau))$ inside of y with:

$$\begin{tikzcd}
\mathcal{C}(p)_{q+2} = G \arrow[r, "m_{q+2}=b_{q+2}"] \arrow[d, "\begin{pmatrix} -\epsilon \\ -\epsilon\gamma^*\mu \end{pmatrix}"'] & B_{q+2} \arrow[d, "d"] \\
\mathcal{C}(p)_{q+1} = G \oplus G^* \arrow[r, "m_{q+1}=(a_{q+1}\ b_{q+1})"] & B_{q+1}
\end{tikzcd}$$

For $r \geq q+3$ or $r \leq q$ the complex E' is given by:

$$d'_r = \begin{pmatrix} d & (-)^r(1+T)\delta\sigma_0 \\ 0 & (-)^r d^* \end{pmatrix} \colon E'_r = B_{r+1} \oplus B^{2q+3-r} \longrightarrow E'_{r-1} = B_r \oplus B^{2q+4-r}$$

The top differentials are dual to those on the bottom:

$$\begin{pmatrix} 0 & (-)^r \\ 1 & 0 \end{pmatrix} d'^*_r \begin{pmatrix} 0 & (-)^{r-1} \\ 1 & 0 \end{pmatrix} = d'_{2q+3-r}$$

for $r \geq q+3$ and $r \leq q$. Because of $E' \simeq D$, the homology groups $H_r(E')$ vanish for $r \neq q+1, q$. Hence there is a stably free submodule $X \subset E'_q$ such that

$\ker d'_q \oplus X = E'_q$. It follows that $E'_{q+2}/\ker d'_{q+2} = \operatorname{coker} d'_{q+3} = U$ is stably free and $U^* = \ker d'_q = \operatorname{im} d'_{q+1}$. Thus, we are allowed to cut off the top and bottom parts of E' and find a chain equivalence $l\colon E' \xrightarrow{\simeq} E''$.

(5)
$$\begin{tikzcd}
E'_{q+2} = B_{q+3} \oplus B^{q+1} \arrow[r, "\mathrm{proj}_X"] \arrow[d, "d'_{q+2}"'] & E''_{q+2} = U \arrow[d, "i"] \\
E'_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ b_{q+2} & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "d'_{q+1}"'] & E''_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} \arrow[d, "p"] \\
E'_q = B_{q+1} \oplus B^{q+3} \arrow[r, "\left[\left(\begin{smallmatrix} 0 & -1 \\ -\epsilon & 0 \end{smallmatrix}\right)\right]"] & E''_q = U^*
\end{tikzcd}$$

with:

$$d'_{q+2} = \begin{pmatrix} 0 & -b^*_{q+1} \\ d & \epsilon(1+T)\delta\sigma_0 + b_{q+2}b^*_{q+1} \\ 0 & \epsilon d^* \end{pmatrix} \qquad i = \left[\begin{pmatrix} 0 & -b^*_{q+1} \\ d & \epsilon(1+T)\delta\sigma_0 \\ 0 & \epsilon d^* \end{pmatrix}\right]$$

$$d'_{q+1} = \begin{pmatrix} -\epsilon a_{q+1} & d & -\epsilon(1+T)\delta\sigma_0 \\ 0 & 0 & -\epsilon d^* \end{pmatrix} \qquad p = \left[\begin{pmatrix} 0 & 0 & \epsilon d^* \\ -b_{q+1}\gamma^*\mu & -\epsilon d & (1+T)\delta\sigma_0 \end{pmatrix}\right]$$

We define a regular split ϵ–preformation z' by:

$$(F' \xleftarrow{\gamma'} G' \xrightarrow{\mu'} F'^*, \theta') = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta) \oplus \partial\left(B_{q+2} \oplus B^{q+2}, \left(\begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix}\right)\right)$$

Clearly $[z] = [z'] \in l_{2q+2}(\Lambda)$. Additionally, one observes that $p = i^*\gamma'^*\mu'$. The formulas for surgery (Definition 3.7) describe the map $g'\colon A \to E'$:

$$\begin{tikzcd}
A_{q+1} = G \arrow[r, "\left(\begin{smallmatrix} 1 \\ -b_{q+2} \\ 0 \end{smallmatrix}\right)"] \arrow[d, "(\gamma^*\mu)^*"'] & E'_{q+1} = G \oplus B_{q+2} \oplus B^{q+2} \arrow[d, "d'_{q+1}"] \\
A_q = G^* \arrow[r, "\left(\begin{smallmatrix} -b_{q+1} \\ 0 \end{smallmatrix}\right)"] & E'_q = B_{q+1} \oplus B^{q+3}
\end{tikzcd}$$

x' is equivalent to a cobordism $(m \circ g' \circ a\colon \partial E \to E'', (\delta\omega'', \omega))$ where the boundary remains fixed. Applying Proposition 2.8 (iii) proves that z' is elementary. □

5 Flip-isomorphisms and twisted doubles

The existence of flip-isomorphisms is our first obstruction to elementariness. One can detect them via linking forms (see Section 9). The secondary obstructions (asymmetric and quadratic signatures) will depend on a choice of flip-isomorphism. In regard to Theorem 4.3, a flip-isomorphism is an isomorphism of the "boundaries" c and c' of x. Therefore, we will use flip-isomorphisms to transform $\partial E = c' \cup -c$ into a twisted double $\partial E_t = c \cup_{(h_t,\chi_t)} -c$.

Definition 5.1 A *flip-isomorphism of a regular split ϵ–preformations z* is a weak isomorphism with its flip. A *stable flip-isomorphism of z* is a flip-isomorphism of a preformation z' with $[z]=[z'] \in l_{2q+2}(\Lambda)$.

Proposition 5.2 *Every elementary regular (split) ϵ–preformation has a flip-isomorphism.*

Proof Let z be a regular split ϵ–preformation of the form described in Proposition 2.8 (iv). Then there is a flip-isomorphism $\left(\left(\begin{smallmatrix} 0 & -1 \\ -\epsilon & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -\tau \\ 0 & 1 \end{smallmatrix}\right), 0, \left(\begin{smallmatrix} 0 & 0 \\ 0 & \theta' \end{smallmatrix}\right)\right)$. □

Theorem 5.3 *With the notation from Theorem 4.3: A flip-isomorphism t of z induces isomorphisms $(h_t,\chi_t)\colon (C,\psi) \xrightarrow{\cong} (C,\psi)$ and $(a_t,\sigma_t)\colon (\partial E,\omega) \xrightarrow{\cong} (\partial E_t,\omega_t) := c \cup_{(h_t,\chi_t)} -c$. x is cobordant rel∂ to an s–cobordism if and only if this is true for $x_t = (a_t,\sigma_t)_\%(x)$.*

Proof By Proposition 4.1, $t=(\alpha,\beta,\nu,\kappa)$ induces an isomorphism $(e_t,\rho_t)=\mathbf{F}(t)\colon \mathbf{F}(z) \xrightarrow{\cong} \mathbf{F}(z')$ of $(2q+1)$–dimensional quadratic complexes. Since the Poincaré pairs c and c' are thickenings of $\mathbf{F}(z)$ and $\mathbf{F}(z')$ the isomorphism (e_t,ρ_t) leads to an equivalence $(\partial e_t, e_t^{-*};0)\colon c \xrightarrow{\simeq} c'$ [11, Proposition 3.4]. Define an automorphism of (C,ψ) by $(h_t,\chi_t)=(h,\chi)^{-1}\circ(\partial e_t,\partial\rho_t)$. Then there is an isomorphism $(a_t,\sigma_t)\colon (\partial E,\omega)=c'\cup_{(h,\chi)} -c \xrightarrow{\cong} (\partial E_t,\omega_t)$ given by:

$$a_{t,q+2}=\beta\colon \partial E_{t,q+2}=G \longrightarrow \partial E_{q+2}=G$$

$$a_{t,q+1}=\begin{pmatrix} 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \alpha^{-*} \\ 0 & 0 & \epsilon\alpha & \epsilon\alpha(\nu^*-\epsilon\nu) \\ 1 & 0 & 0 & 0 \end{pmatrix}\colon$$

$$\partial E_{t,q+1}=G\oplus G\oplus(F\oplus F^*) \longrightarrow \partial E_{q+1}=G\oplus(F\oplus F^*)\oplus G$$

$$a_{t,q}=\begin{pmatrix} 0 & \alpha^{-*} & 0 \\ 0 & 0 & \beta^{-*} \\ 1 & 0 & 0 \end{pmatrix}\colon \partial E_{t,q}=F^*\oplus F^*\oplus G^* \longrightarrow \partial E_q=F\oplus G^*\oplus F^*$$

$$\sigma_{t,0}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\nu\alpha^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\colon \partial E^{q+1} \longrightarrow \partial E_{q+1}$$

$$\sigma_{t,0}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\colon \partial E^{q+2} \longrightarrow \partial E_q$$

$$\sigma_{t,1}=\begin{pmatrix} 0 & 0 & \epsilon & 0 \\ -1 & -\gamma^* & -\mu^*+\epsilon\gamma^*\alpha\nu\alpha^* & -1 \\ 0 & 1 & \epsilon\alpha\nu\alpha^* & 0 \end{pmatrix}\colon \partial E^{q+1} \longrightarrow \partial E_q$$

$$\sigma_{t,2} = \begin{pmatrix} 0 & 0 & 0 \\ -\gamma^* & -\epsilon\beta^{-*}\kappa\beta^{-1} & 0 \\ -1 & -\epsilon\mu-\epsilon\alpha\nu\alpha^*\gamma & -\epsilon\alpha\nu\alpha^* \end{pmatrix} : \partial E^q \longrightarrow \partial E_q$$

Define the $(2q+2)$–dimensional quadratic Poincaré pair $x_t = (a_t, \sigma_t)_\%(x) = (g_t\colon \partial E_t \longrightarrow E, (0, \omega_t))$. The last claim follows from the fact that a_t maps each copy of D in ∂E_t onto a copy of D and D' in ∂E. □

6 Asymmetric signatures

Let W be a $(2q+2)$–dimensional manifold where the boundary is a twisted double $M \cup_h -M$. An obstruction $\sigma^*(W)$ in the asymmetric Witt-group $WAsy(\Lambda)$ vanishes if and only if W is cobordant rel∂ to a manifold which carries a twisted double structure compatible with the boundary (Ranicki [13, Section 30], Winkelnkemper [17] and Quinn [10]). An s–cobordism is a twisted double. Hence, the asymmetric signature is also an (incomplete) obstruction for $(W, M \cup_h -M)$ to be cobordant rel∂ to an s–cobordism. Analogous results and constructions hold in the realm of algebraic surgery theory. They will be applied to x_t (Theorem 5.3) in order to obtain the asymmetric signature of a flip-isomorphism.

6.1 Asymmetric L–theory

Definition 6.1 [13, Sections 28F, 30B]

(i) A *(non-singular) asymmetric form* (M, λ) is a free module M together with an isomorphism $\lambda\colon M \xrightarrow{\cong} M^*$. It is *metabolic* if there is a free direct summand $j\colon L \hookrightarrow M$ such that $0 \to L \xrightarrow{j} M \xrightarrow{j^*\lambda} L^* \to 0$ is exact.

(ii) Two asymmetric forms are stably isometric if they are isometric after addition of metabolic forms. The set of stable isometry classes is the *asymmetric Witt-group* $WAsy(\Lambda)$.

(iii) An *n–dimensional asymmetric Poincaré complex* (C, λ) is a chain complex C together with a chain equivalence $\lambda\colon C^{n-*} \xrightarrow{\simeq} C$.

(iv) An *equivalence* $f\colon (C, \lambda) \xrightarrow{\simeq} (C', \lambda')$ *of n–dimensional asymmetric Poincaré complexes* is a chain equivalence $f\colon C \xrightarrow{\simeq} C'$ such that there is a chain homotopy $\lambda' \simeq f\lambda f^*$.

(v) An *$(n+1)$–dimensional asymmetric cobordism* $(f\colon C \to D, (\delta\lambda, \lambda))$ is an n–dimensional asymmetric Poincaré complex (C, λ), a chain map $f\colon C \to D$ and a chain homotopy $\delta\lambda\colon f\lambda f^* \simeq 0\colon D^{n-*} \to D$ such that

$$\begin{pmatrix} \delta\lambda \\ (-)^{r+1}\lambda f^* \end{pmatrix} : D^{n+1-r} \longrightarrow \mathcal{C}(f)_r = D_r \oplus C_{r-1}$$

$$\begin{pmatrix} \delta\lambda & (-)^n f\lambda \end{pmatrix} : \mathcal{C}(f)^{n+1-r} = D^{n+1-r} \oplus C^{n-r} \longrightarrow D_r$$

induce chain equivalences.

(vi) The *asymmetric L–group $LAsy^n(\Lambda)$* is defined to be the cobordism group of n–dimensional asymmetric Poincaré complexes.

Remark 6.2 [13, Propositions 28.31 and 28.34]

$$LAsy^{2n}(\Lambda) \cong WAsy(\Lambda), \qquad LAsy^{2n+1}(\Lambda) = 0$$

Definition 6.3 [13, Definition 30.10] Let $x = (g\colon \partial E \to E, (\theta, \partial\theta))$ be an $(n+1)$–dimensional symmetric Poincaré pair such that the boundary $(\partial E, \partial\theta) = c \cup_{(h,\chi)} -c$ is a twisted double of an n–dimensional symmetric Poincaré pair $c = (f\colon C \to D, (\delta\phi, \phi))$ with respect to a self-equivalence $(h, \chi)\colon (C, \phi) \xrightarrow{\simeq} (C, \phi)$. We write $g = \begin{pmatrix} j_0 & j_1 & k \end{pmatrix} \colon \partial E_r = D_r \oplus D_r \oplus C_{r-1} \longrightarrow E_r$. The *asymmetric signature $\sigma^*(x) = [(B, \lambda)] \in LAsy^{n+1}(\Lambda)$ of x* is given by $B = \mathcal{C}(j_0 - j_1\colon D \longrightarrow \mathcal{C}(j_0 f\colon C \to E))$ and a chain equivalence which fits into the chain homotopy commutative diagrams of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & C^{n-*} \arrow[r, "\binom{0}{1}"] \arrow[d, "\pm\phi_0 h^*"', "\simeq"] & \mathcal{C}(j_0 f)^{n+1-*} \arrow[r, "(1\ 0)"] \arrow[d, "\kappa"', "\simeq"] & E^{n+1-r} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & C_{*-1} \arrow[r, "\begin{pmatrix} k \\ -fh \\ -f \end{pmatrix}"] & \mathcal{C}(j_0, j_1) \arrow[r, "\binom{1}{0}"] & \mathcal{C}(g) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & D^{n-*} \arrow[r, "\binom{0}{1}"] \arrow[d, "\simeq"] & B^{n+1-*} \arrow[r, "(1\ 0)"] \arrow[d, "T\lambda"', "\simeq"] & \mathcal{C}(j_0 f)^{n+1-*} \arrow[r] \arrow[d, "\kappa"', "\simeq"] & 0 \\
0 \arrow[r] & \mathcal{C}(f) \arrow[r, "\begin{pmatrix} j_0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & B \arrow[r, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 1 \\ 0 & 0 & -1 \end{pmatrix}"] & \mathcal{C}(j_0, j_1) \arrow[r] & 0
\end{tikzcd}$$

One can give an explicit formula for the asymmetric complex (B, λ).

Proposition 6.4 *λ is given (up to chain homotopy) by*

$$T\lambda_r = \begin{pmatrix} \theta_0 & (-)^{n-1} j_0 f \chi_0 + (-)^{n-r} k\phi_0 h^* & j_1 \delta\phi_0 \\ (-)^{n-r}\phi_0 k^* & (-)^{n-r+1}\phi_0(1+h^*) & (-)^{n-r}\phi_0 f^* \\ (-)^{n-r+1}(\delta\phi_0 j_0^* + f\phi_0 k^*) & (-)^{n-r} f\phi_0 h^* & (-)^{n-r+1} f\phi_0 f^* \end{pmatrix} :$$
$$B^{n+1-r} = E^{n+1-r} \oplus C^{n-r} \oplus D^{n-r} \longrightarrow B_r = E_r \oplus C_{r-1} \oplus D_{r-1}$$

Corollary 6.5 *We use the notation of Definition 6.3.*

(i) *If $\partial E = 0$ then $(B, \lambda) \simeq (E, \theta_0)$ as asymmetric complexes.*

(ii) *Let $C = 0$. Let (V, σ) be the n–dimensional symmetric Poincaré complex obtained by gluing the n–dimensional symmetric Poincaré pair $((j_0, j_1)\colon D \oplus D \to E, (\theta, \delta\phi \oplus -\delta\phi))$ along its boundary. Then $(B, \lambda) = (V, \sigma_0) \in LAsy^{n+1}(\Lambda)$.*

6.2 Constructing asymmetric signatures of flip-isomorphisms

Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ be a regular split ϵ–preformation and $t = (\alpha, \beta, \nu, \kappa)$ be a flip-isomorphism. Let $\sigma = (\nu - \epsilon\nu^*)^*$. We apply the asymmetric signature construction from Proposition 6.4 to the symmetrization of x_t. The result is a $(2q+2)$–dimensional asymmetric complex (B, λ). There is an equivalence $b\colon (B, \lambda) \xrightarrow{\simeq} (B', \lambda')$ to a smaller asymmetric complex (B', λ') given by

$$\begin{aligned}
b_{q+2} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \colon B_{q+2} = G \oplus G \longrightarrow B'_{q+2} = G \\
b_{q+1} &= -\epsilon \begin{pmatrix} \gamma & -1 & 0 & -\alpha^{-*} \\ 0 & 0 & 0 & 1 \\ 0\mu-\epsilon\alpha\gamma & \epsilon\alpha & -1 & \epsilon\alpha\alpha^{-*}-1-\epsilon\alpha\sigma \end{pmatrix} \colon \\
&\qquad B_{q+1} = G \oplus F \oplus F^* \oplus F^* \longrightarrow B'_{q+1} = F \oplus F^* \oplus F^* \\
b_q &= 1\colon B_q = G^* \longrightarrow B'_q = G^* \\
d'_{q+2} &= \begin{pmatrix} \gamma \\ \mu \\ 0 \end{pmatrix} \colon B'_{q+2} \longrightarrow B'_{q+1} \\
d'_{q+1} &= \begin{pmatrix} \epsilon(1+\beta^{-*})\mu^* & (1+\beta^{-*})\gamma^* & \gamma^* \end{pmatrix} \colon B'_{q+1} \longrightarrow B'_q \\
\lambda'_{q+2} &= \epsilon\colon B'^{q} \longrightarrow B'_{q+2} \\
\lambda'_{q+1} &= \begin{pmatrix} 0 & 0 & 1 \\ \epsilon & 0 & -\alpha^* \\ -\epsilon & \epsilon\alpha & \alpha^*-\epsilon\alpha+\epsilon\alpha\sigma\alpha^* \end{pmatrix} \colon B'^{q+1} \longrightarrow B'_{q+1} \\
\lambda'_q &= -\beta^{-*}\colon B'^{q+2} \longrightarrow B'_q
\end{aligned}$$

Now we apply [13, (Errata) 28.34] and compute a highly-connected $(2q+2)$–dimensional asymmetric complex (B'', λ'') which is cobordant to the asymmetric complex (B', λ'). The module automorphisms

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & -\alpha\gamma\beta \\ 0 & \epsilon\alpha^{-*} & 0 & 0 & -\epsilon\alpha^{-*}\mu\beta \\ 0 & 1 & 1 & 0 & -\mu\beta \\ \epsilon(1+\beta^{-*})\mu^* & (1+\beta^{-*})\gamma^* & \gamma^* & -\epsilon\beta^{-*} & 0 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix} \colon$$
$$B''_{q+1} = F \oplus F^* \oplus F^* \oplus G^* \oplus G \xrightarrow{\cong} B''_{q+1}$$

transform (B'', λ'') into another asymmetric complex from which we read off the non-singular asymmetric form:

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha\sigma\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \longrightarrow M^*$$

It represents the image of $\sigma^*(x_t)$ under $LAsy^{2q+2}(\Lambda) \xrightarrow{\cong} WAsy(\Lambda)$.

Definition 6.6 Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ be a split ϵ–preformation. The *asymmetric signature* $\sigma^*(z,t) \in WAsy(\Lambda)$ *of a flip-isomorphism* $t = (\alpha, \beta, \nu, \kappa)$ *of* z is given by

$$\rho = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & -\epsilon \\ 0 & 1 & \epsilon\alpha(\nu^* - \epsilon\nu)\alpha^* \end{pmatrix} : M = F \oplus F^* \oplus F \xrightarrow{\cong} M^*$$

Remark 6.7 The asymmetric signature only depends on the flip-isomorphism $(\alpha, \beta, \nu^* - \epsilon\nu)$ of the underlying non-split ϵ–preformation $(F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$.

6.3 Asymmetric signatures and elementariness

Theorem 6.8 *If* $[z] \in l_{2q+2}(\Lambda)$ *is elementary and regular then* $\sigma^*(z,t) = 0 \in WAsy(\Lambda)$ *for all flip-isomorphisms* t.

The proof follows from Theorem 5.3 and the following Proposition.

Proposition 6.9 *Let* $x = (g\colon \partial E \to E, (\theta, \partial\theta))$ *and* $x' = (g'\colon \partial E \to E', (\theta', \partial\theta))$ *be two* $(n+1)$*–dimensional symmetric Poincaré pairs such that the boundary* $(\partial E, \partial\theta)$ *is a twisted double of an* n*–dimensional symmetric Poincaré pair* $(f\colon C \to D, (\delta\phi, \phi))$ *with respect to a homotopy self-equivalence* (h, χ).

(i) *If* x *and* x' *are cobordant rel*∂*, then* $\sigma^*(x) = \sigma^*(x') \in LAsy^{n+1}(\Lambda)$.

(ii) $\sigma^*(x) - \sigma^*(x') = \sigma^*(x \cup -x') \in LAsy^{n+1}(\Lambda)$.

(iii) *If* x *is an* s*–cobordism then* $\sigma^*(x) = 0 \in LAsy^{n+1}(\Lambda)$.

Proof The first statement is a special case of [13, Proposition 30.11(iii)], so only the third claim requires a proof. Let (B, λ) be the asymmetric complex of x from Proposition 6.4. Then there is an $(n+2)$–dimensional asymmetric cobordism $\big(\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} : B \longrightarrow D_{*-1}, (\pm\delta\phi_0, \lambda)\big)$. □

7 Quadratic signatures

A manifold W with a twisted double structure $(M \cup_h -M)$ on the boundary cannot be glued together along its boundary unless $h\colon \partial M \xrightarrow{\cong} \partial M$ is isotopic to the identity. Similarly, the Poincaré pair x_t of Theorem 5.3 can generally not be glued together (or attached to an s–cobordism with the same boundary). This is possible, however, if the equivalence (h_t, χ_t) from Theorem 5.3 is homotopic to $(1, 0)$. Choices of flip-isomorphisms t for which this is the case will be called *flip-isomorphisms rel* ∂ t. Then the result of gluing x_t in $L_{2q+2}(\Lambda)$ will be the *quadratic signature*.

7.1 Homotopy and twisted doubles

The following rather technical section extends [11, Proposition 1.1(i)] to a thorough theory of homotopies of morphisms of quadratic complexes. At the end, Lemma 7.5 proves that homotopic self-equivalences yield equivalent twisted doubles.

Definition 7.1 Let $\Delta\colon f \simeq f'\colon C \to C'$ be a chain homotopy of two chain maps. Let $\psi \in W_{\%}(C')_n$. Define $\Delta_{\%}\psi \in W_{\%}(C')_{n+1}$ by

$$(\Delta_{\%}\psi)_s = -\Delta\psi_s f^* + (-)^{r+1}(f'\psi_s + (-)^n \Delta T\psi_{s+1})\Delta^*\colon C'^{n+1-r-s} \longrightarrow C'_r$$

Lemma 7.2 *Let $\Delta\colon f \simeq f'\colon C \to C'$ be a chain homotopy of two chain maps.*

(i) *Let $\psi \in W_{\%}(C)_n$. Then $d(\Delta_{\%}\psi) = -\Delta_{\%}(d\psi) + f\psi f^* - f'\psi f'^*$*

(ii) *If $(f, \chi)\colon (C, \psi) \to (C', \psi')$ is a morphism of n–dimensional quadratic complexes, then $(f', \chi + \Delta_{\%}\psi)\colon (C, \psi) \to (C', \psi')$ is one as well.*

Definition 7.3 A *homotopy* (Δ, η) *of two morphisms of n–dimensional quadratic complexes* $(f, \chi), (f', \chi')\colon (C, \psi) \to (C', \psi')$ is a chain homotopy $\Delta\colon f \simeq f'\colon C \to C'$ and an element $\eta \in W_{\%}(C')_{n+2}$ such that

$$\chi' - \chi = \Delta_{\%}\psi + d(\eta) \in W_{\%}(C')_{n+1}.$$

Lemma 7.4 *Let (C, ψ) and (C', ψ') be n–dimensional quadratic complexes. Then homotopy is an equivalence relation on all morphisms $(C, \psi) \to (C', \psi')$.*

Proof Let $(\Delta, \eta)\colon (f, \chi) \simeq (f', \chi')$ be a homotopy. Then $(-\Delta, \widetilde{\eta})\colon (f', \chi') \simeq (f, \chi)$ is also a homotopy where $\widetilde{\eta}_s = -\eta_s + (-)^{r+1}\Delta\psi_s\Delta^*\colon C'^{n+2-r-s} \to C'_r$. Let $(\Delta', \eta')\colon (f', \chi') \simeq (f'', \chi'')$ be another homotopy. Then $(\Delta + \Delta', \eta'')\colon (f, \chi) \simeq (f'', \chi'')$ is a homotopy with $\eta''_s = \eta_s + \eta'_s + (-)^r \Delta'\psi_s\Delta^*\colon C'^{n+2-r-s} \to C'_r$. $\square$

Lemma 7.5 *Let $c=(f\colon C\to D,(\delta\psi,\psi))$ be an n–dimensional quadratic Poincaré pair. Let $(\Delta,\eta)\colon (h,\chi)\simeq(h',\chi')\colon (C,\psi)\xrightarrow{\cong}(C,\psi)$ be a homotopy of self-equivalences. There is an isomorphism $(a,\sigma)\colon (c\cup_{(h,\chi)}-c)\xrightarrow{\cong}(c\cup_{(h',\chi')}-c)$ where:*

$$a_r=\begin{pmatrix} & (-)^r f\Delta & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\colon (D\cup_h D)_r=D_r\oplus C_{r-1}\oplus D_r\longrightarrow (D\cup_{h'}D)_r$$
$$\sigma_s=\begin{pmatrix}(-)^{n-1}f\eta_s f^* & 0 & 0\\ (-)^n\psi_s\Delta^* f^* & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\colon (D\cup_{h'}D)^{n+1-r-s}\longrightarrow (D\cup_{h'}D)_r$$

7.2 Flip-isomorphisms rel ∂

Definition 7.6 A *flip-isomorphism t rel∂ of a regular split ϵ–preformation* $z=(F\xleftarrow{\gamma}G\xrightarrow{\mu}F^*,\theta)$ is a flip-isomorphism $t=(\alpha,\beta,\nu,\kappa)$ of z such that $(1,0)\simeq(h_t,\chi_t)\colon (C,\psi)\xrightarrow{\cong}(C,\psi)$ with (h_t,χ_t) as defined in Theorem 5.3.

Proposition 7.7 *Every elementary preformation has a flip-isomorphism rel∂.*

Proof Let z be of the form described in Proposition 2.8 (iv). Then the flip-isomorphism defined in Proposition 5.2 is a flip-isomorphism rel ∂ with a homotopy $(\Delta,\eta)\colon (1,0)\simeq(h_t,\chi_t)\colon (C,\psi)\xrightarrow{\cong}(C,\psi)$ given by:

$$\Delta_{q+1}=\begin{pmatrix}\epsilon&0&0&\epsilon\\0&0&0&0\end{pmatrix}\colon C_q=(U\oplus U^*)\oplus(U^*\oplus U)\longrightarrow C_{q+1}=U\oplus R$$
$$\Delta_q=\begin{pmatrix}0&0\\-1&0\\-\epsilon&0\\0&0\end{pmatrix}\colon C_{q-1}=U^*\oplus R^*\longrightarrow C_q=(U\oplus U^*)\oplus(U^*\oplus U)$$
$$\eta_1=\begin{pmatrix}0&0&1&0\\0&0&0&0\end{pmatrix}\colon C^q=(U^*\oplus U)\oplus(U\oplus U^*)\longrightarrow C_{q+1}=U\oplus R$$
$$\eta_1=\begin{pmatrix}0&0\\\epsilon&0\\1&0\\0&0\end{pmatrix}\colon C^{q+1}=U^*\oplus R^*\longrightarrow C_q=(U\oplus U^*)\oplus(U^*\oplus U)$$
$$\eta_2=\begin{pmatrix}1&0\\0&0\end{pmatrix}\colon C^{q-1}=U\oplus R\longrightarrow C_{q+1}=U\oplus R$$
$$\eta_2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&\epsilon&0\end{pmatrix}\colon C^q=(U^*\oplus U)\oplus(U\oplus U^*)\longrightarrow C_q$$
$$\eta_3=\begin{pmatrix}-\epsilon&0\\0&0\\0&0\\0&0\end{pmatrix}\colon C^{q-1}=U\oplus R\longrightarrow C_q=(U\oplus U^*)\oplus(U^*\oplus U)$$
$$\eta_4=\begin{pmatrix}0&0\\-\sigma^*&0\end{pmatrix}\colon C^{q-1}=U\oplus R\longrightarrow C_{q-1}=U^*\oplus R^*$$
□

7.3 Construction and properties of the quadratic signature

Let $t=(\alpha,\beta,\nu,\kappa)$ be a flip-isomorphism rel∂ of a regular split ϵ–preformation $z=(F\xleftarrow{\gamma}G\xrightarrow{\mu}F^*,\theta)$, ie, a homotopy $(\Delta,\eta)\colon(1,0)\simeq(h_t,\chi_t)$ exists. Write $\Delta_{q+1}=\begin{pmatrix}R & S\end{pmatrix}\colon C_q=F\oplus F^*\to C_{q+1}=G$ and $\Delta_q=\left(\begin{smallmatrix}U\\V\end{smallmatrix}\right)\colon C_{q-1}=G^*\to C_q=F\oplus F^*$. We recall the construction of the quadratic Poincaré pairs x_t, c, (h_t,χ_t), etc from Theorem 5.3. Lemma 7.5 provides us with an isomorphism $(a,\sigma)\colon(\partial E',\omega')\colon= c\cup -c\xrightarrow{\cong}(\partial E_t,\omega_t)$ that gives rise to the $(2q+2)$–dimensional quadratic Poincaré pair:

$$w_t=(a,\sigma)_\%(x_t)=(g'_t\colon\partial E'\longrightarrow E,(\delta\omega',\omega')) \tag{6}$$
$$g'_{t,q+1}=\begin{pmatrix}1 & -\epsilon R & -\epsilon S & -\beta\end{pmatrix}\colon\partial E'_{q+1}=G\oplus(F\oplus F^*)\oplus G\longrightarrow E_{q+1}=G$$
$$\delta\omega'_{t,0}=-\eta_0\colon E^{q+1}=G^*\longrightarrow E_{q+1}=G$$

In the next step w_t is stuck onto the algebraic s–cobordism $y=(m\colon\partial E'\to D,(0,\omega'))$ with $m_r=\begin{pmatrix}-1 & 0 & 1\end{pmatrix}\colon\partial E'_r=D_r\oplus C_{r-1}\oplus D_r\to D_r$. Let the result be the $(2q+2)$–dimensional quadratic Poincaré complex $(V,\tau)=w_t\cup -y$. There is an equivalence $l\colon V\xrightarrow{\simeq}V'$ to a smaller complex:

$$\begin{tikzcd}
V_{q+2}=G\oplus F\oplus F^*\oplus G \arrow[r, "(-\gamma\ 1\ 0\ 0)"] \arrow[d] & V'_{q+2}=F \arrow[d, "\left(\begin{smallmatrix}R\\ \mu^*\end{smallmatrix}\right)"] \\
V_{q+1}=G\oplus F^*\oplus G^*\oplus F^*\oplus G \arrow[r, "\left(\begin{smallmatrix}1 & -\epsilon S & 0 & 0 & \beta\\ 0 & -\gamma^* & 1 & 0 & 0\end{smallmatrix}\right)"] & V'_{q+1}=G\oplus G^*
\end{tikzcd}$$

Applying the instant surgery obstruction of [11, Proposition 4.3] to $(V',l_\%\tau)$ we obtain a non-singular $(-\epsilon)$–quadratic form (M,ξ)

$$\xi=\begin{pmatrix}-\eta_0 & \beta & 0\\ 0 & \theta & 0\\ R^* & \mu & 0\end{pmatrix}\colon M=G^*\oplus G\oplus F\xrightarrow{\cong}M^* \tag{7}$$

Definition 7.8 $\rho^*(z,t,\Delta,\eta)=[(M,\xi)]\in L_{2q+2}(\Lambda)$ is the *quadratic signature of the regular split ϵ–preformation z and the flip-isomorphism rel∂ t.*

Theorem 7.9 *$[z]\in l_{2q+2}(\Lambda)$ is elementary if and only if there is a flip-isomorphism rel∂ t of a z' where $[z']=[z]\in l_{2q+2}(\Lambda)$ and a homotopy $(\Delta,\eta)\colon(1,0)\simeq(h_t,\chi_t)$ such that $\rho^*(z',t,\Delta,\eta)=0$*

Proof If z is elementary then use the flip-isomorphism constructed in the proof of Proposition 7.7. If, on the other hand, one quadratic signature vanishes then $(V,\tau)=$

$y \cup -w_t$ constructed in the previous section is null-cobordant. Hence w_t and the s–cobordism y are cobordant rel∂. We easily conclude that the Poincaré pair x from Theorem 4.3 is cobordant rel∂ to an s–cobordism and therefore $[z] \in l_{2q+2}(\Lambda)$ is elementary. □

7.4 The relationship between quadratic and asymmetric signatures

Theorem 7.10 *Let z, t, Δ and η as in Definition 7.8. Then $\rho^*(z,t,\Delta,\eta)$ is mapped to the asymmetric signature $\sigma^*(z,t)$ under the canonical homomorphism $L_{2q+2}(\Lambda) \longrightarrow WAsy(\Lambda)$, $(K,\psi) \longmapsto (K, \psi - \epsilon\psi^*)$*

Proof By construction, the quadratic signature $\rho^*(z,t,\Delta,\eta) = (V,\tau) = w_t \cup -y$ is the union of the $(2q+2)$–dimensional quadratic Poincaré pairs defined in Section 7.3. By Proposition 6.9 the image of (V,τ) in $WAsy(\Lambda)$ is the difference of the asymmetric signatures of the Poincaré pairs w_t and y. Since y is an s–cobordism its asymmetric signature vanishes. The asymmetric signature of w_t is the asymmetric signature of x_t by the following general fact. □

Lemma 7.11 *Let $c = (f\colon C \to D, (\delta\phi,\phi))$ be an n–dimensional symmetric Poincaré pair. Let $(\Delta,\eta)\colon (h,\chi) \simeq (h',\chi')\colon (C,\phi) \xrightarrow{\simeq} (C,\phi)$ be a homotopy of self-equivalences. Then there is an isomorphism*

$$(a,\sigma)\colon (\partial E,\theta) = c \cup_{(h,\chi)} -c \xrightarrow{\cong} (\partial E',\theta') = c \cup_{(h',\chi')} -c$$

by Lemma 7.5. Let $x' = (g'\colon \partial E' \to E, (\delta\theta',\theta'))$ be an $(n+1)$–dimensional symmetric Poincaré pair. Then $\sigma^*((a,\sigma)^{\%}(x')) = \sigma^*(x') \in LAsy^{n+1}(\Lambda)$.

Proposition 7.12 *Let $\Lambda = \mathbb{Z}$, $q = 2k-1$ and z a regular split ϵ–preformation.*

(i) *$[z] \in l_{4k}(\mathbb{Z})$ is elementary if and only if there is a flip-isomorphism rel∂ t such that $\sigma^*(z,t) = 0 \in WAsy(\mathbb{Z})$.*

(ii) *The quadratic signature $\rho^*(z,t,\Delta,\eta) \in L_{4k}(\mathbb{Z})$ only depends on z and t.*

Proof The canonical homomorphism $L_{4k}(\mathbb{Z}) \to WAsy(\mathbb{Z})$ is an injection. □

8 Formations

If the chain complex C in the constructions of Theorems 4.3 and 5.3 was contractible (ie, the preformation z is in fact a formation), all flip-isomorphisms would automatically be rel∂. Additionally, we will show that asymmetric signatures of formations do not depend on the choice of flip-isomorphism.

8.1 Quadratic signatures of formations

An ϵ–formation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ is a split ϵ–preformation such that $\binom{\gamma}{\mu}\colon G \to H_\epsilon(F)$ is the inclusion of a lagrangian. By [11, Proposition 2.2], this map extends to an isomorphism $\left(f = \begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \widetilde{\gamma}^*\mu & \widetilde{\theta} \end{pmatrix}\right)\colon H_\epsilon(G) \xrightarrow{\cong} H_\epsilon(F)$ of hyperbolic ϵ–quadratic forms. For any $\tau\colon G^* \to G$ the maps $\widetilde{\gamma}' = \widetilde{\gamma} + \gamma(\tau - \epsilon\tau^*)$, $\widetilde{\mu}' = \widetilde{\mu} + \mu(\tau - \epsilon\tau^*)$, $\widetilde{\theta}' = \widetilde{\theta} + (\tau - \epsilon\tau^*)^*\theta(\tau - \epsilon\tau^*) + \widetilde{\gamma}^*\mu(\tau - \epsilon\tau^*)^* - \epsilon\tau$ define another extension. Any other choice of $\widetilde{\gamma}'$, $\widetilde{\mu}'$ or $\widetilde{\theta}'$ emerges in this way.

Theorem 8.1 *Let $t = (\alpha, \beta, \nu, \kappa)$ be a flip-isomorphism of z.*

(i) *t is a flip-isomorphism rel∂.*

(ii) *A choice of $\widetilde{\gamma}$, $\widetilde{\mu}$ and $\widetilde{\theta}$ defines a homotopy $(\Delta, \rho)\colon (1,0) \simeq (h_t, \chi_t)$. The quadratic signature $\widetilde{\rho}^*(z, t, \widetilde{\gamma}, \widetilde{\mu}, \widetilde{\theta}) = \rho^*(z, t, \Delta, \rho)$ is given by:*

$$\begin{pmatrix} \widetilde{\gamma}^*\widetilde{\mu} + \widetilde{\gamma}^*\alpha\nu\alpha^*\widetilde{\gamma} & -\widetilde{\gamma}^*\alpha\gamma & 0 \\ 0 & \epsilon\theta^* & 0 \\ \epsilon(\alpha^*\widetilde{\gamma} - \widetilde{\mu}) & -\mu & 0 \end{pmatrix}\colon M = G^* \oplus G \oplus F^* \longrightarrow M^* \tag{8}$$

(iii) *$[z'] \in l_{2q+2}(\Lambda)$ is elementary if and only if for some representative $z \in [z']$, a flip-isomorphism t of z and choices for $\widetilde{\gamma}$, $\widetilde{\mu}$ and $\widetilde{\theta}$ as above*

$$\widetilde{\rho}^*(z, t, \widetilde{\gamma}, \widetilde{\mu}, \widetilde{\theta}) = 0 \in L_{2q+2}(\Lambda)$$

Proof

(i) A choice of $\widetilde{\gamma}$, $\widetilde{\mu}$ and $\widetilde{\theta}$ leads to homotopies $\Delta_C\colon 1 \simeq 0\colon C \to C$ and $(\Delta, \eta)\colon (1,0) \simeq (h_t, \chi_t)\colon (C, \psi) \to (C, \psi)$ given by

$$\Delta_{C,q+1} = \begin{pmatrix} \epsilon\widetilde{\mu}^* & \widetilde{\gamma}^* \end{pmatrix}\colon C_q = F \oplus F^* \longrightarrow C_{q+1} = G$$

$$\Delta_{C,q} = -\epsilon \begin{pmatrix} \widetilde{\gamma} \\ \widetilde{\mu} \end{pmatrix}\colon C_{q-1} = G^* \longrightarrow C_q = F \oplus F^*$$

$$(\Delta, \eta) = (\Delta_C(1 - h_t), \Delta_{C\%}(\chi_t - \Delta_{\%}\psi))$$

(ii) Let Δ and η as in (i). Transforming (7) via the isomorphism

$$f = \begin{pmatrix} 1 & 0 & -\widetilde{\gamma}^*\alpha(\nu^* - \epsilon\nu) - \widetilde{\mu}^*\alpha^{-*} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\colon M^* \xrightarrow{\cong} M^*$$

yields the alternative representative.

(iii) Let z be elementary. We assume it is of the form described in Proposition 2.8 (iv). Then $\binom{\sigma}{\tau}\colon R \to U \oplus U^*$ is the inclusion of a lagrangian. Again this map can be extended to an isometry:

$$\left(\begin{pmatrix} \sigma & \widetilde{\sigma} \\ \tau & \widetilde{\tau} \end{pmatrix}, \begin{pmatrix} \theta' & 0 \\ \widetilde{\sigma}^*\tau & \widetilde{\theta}' \end{pmatrix}\right)\colon H_\epsilon(R) \longrightarrow H_\epsilon(U^*)$$

The maps

$$\widetilde{\gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\sigma} \end{pmatrix} \colon G^* = U^* \oplus R^* \longrightarrow F = U \oplus U^*$$
$$\widetilde{\mu} = \begin{pmatrix} 1 & -\epsilon\widetilde{\sigma} \\ 0 & \widetilde{\tau} \end{pmatrix} \colon G^* = U^* \oplus R^* \longrightarrow F^* = U^* \oplus U$$
$$\widetilde{\theta} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\theta}' \end{pmatrix} \colon G^* = U^* \oplus R^* \longrightarrow G = U \oplus R$$

complete $\left(\begin{smallmatrix} \gamma \\ \mu \end{smallmatrix}\right)$ to an isometry of hyperbolic forms. Let $t = (\alpha, \beta, \nu, \kappa)$ be the flip-isomorphism from Proposition 5.2. Then the associated $(-\epsilon)$–quadratic form (8) has a lagrangian:

$$i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \epsilon\widetilde{\sigma} & 0 \end{pmatrix} \colon U^* \oplus R^* \oplus U \longrightarrow M = U^* \oplus R^* \oplus U \oplus R \oplus U^* \oplus U \qquad \square$$

8.2 Asymmetric signatures of formations

Theorem 8.2 *The asymmetric signature of (split) ϵ–formations is independent of the choice of flip-isomorphism.*

We apply this theorem to the image of the map $\partial\colon L_{2q+2}(\Lambda) \hookrightarrow l_{2q+2}(\Lambda)$.

Corollary 8.3 *Let (K, θ) be a $(-\epsilon)$–quadratic form and $z = \partial(K, \theta)$.*

(i) *z has a stable flip-isomorphism if and only if (K, θ) is non-singular. Then $[z] \in l_{2q+2}(\Lambda)$ is elementary if and only if $(K, \theta) = 0 \in L_{2q+2}(\Lambda)$.*

(ii) *If (K, θ) is non-singular, $\sigma^*(z, t) = [(K, \theta - \epsilon\theta^*)] \in WAsy(\Lambda)$ for any stable flip-isomorphism t.*

Proof Let $\lambda = \theta - \epsilon\theta^*$. If (K, θ) is non-singular then $t = (\lambda^*, 1, \epsilon\lambda^{-1}, 0)$ is a flip-isomorphism of z. Let (M, ρ) be the asymmetric form of Definition 6.6. Then $\rho \oplus -\lambda$ has the lagrangian $\begin{pmatrix} \epsilon & 1 \\ 0 & -\epsilon\lambda \\ 1 & 0 \\ -\epsilon & 1 \end{pmatrix}$. $\square$

Example 8.4 By Corollary 8.3, $\partial\left(\mathbb{Z}^2, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$ has stable flip-isomorphisms and all asymmetric signatures vanish but it is not stably elementary.

Finally, a proof of Theorem 8.2 can be given. We recall that in Section 6.2 the asymmetric signature $\sigma^*(z,t) \in WAsy(\Lambda)$ was defined as the asymmetric signature of the $(2q+2)$–dimensional symmetric Poincaré pair x_t. In our case C is contractible and $D \oplus D$ and ∂E_t are chain equivalent. The following lemma treats this situation in general.

Lemma 8.5 *Let $c = (f\colon C \to D, (\delta\phi, \phi))$ be an n–dimensional symmetric Poincaré pair and $(h,\chi)\colon (C,\phi) \xrightarrow{\simeq} (C,\phi)$ a self-equivalence. Let $(\partial E, \partial\theta) = c \cup_{(h,\chi)} -c$ be the twisted double of c in respect to (h,χ). Assume that C is contractible with $\Delta\colon 1 \simeq 0\colon C \to C$. Define $\nu = \delta\phi + (-)^{n-1} f\Delta^{\%}\phi f^*$ and $\overline{\rho} = \Delta^{\%}(\Delta^{\%}\phi - \chi - h\Delta^{\%}\phi h^*)$.*

(i) *There is an equivalence $(a,\sigma)\colon (D,\nu) \oplus (D,-\nu) \xrightarrow{\simeq} (\partial E, \partial\theta)$ of n–dimensional symmetric Poincaré complexes given by:*

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\colon D_r \oplus D_r \longrightarrow \partial E_r = D_r \oplus D_r \oplus C_{r-1}$$

$$\sigma_s = \begin{pmatrix} (-)^n f\overline{\rho}_s f^* & 0 & 0 \\ 0 & 0 & (-)^{s-1} f\Delta^{\%}\phi_s \\ (-)^{n+1-r}\Delta^{\%}\phi_s h^* f^* & 0 & (-)^{n+1-r+s} T\Delta^{\%}\phi_{s-1} \end{pmatrix}\colon \partial E^{n+1-r+s} \longrightarrow \partial E_r$$

(ii) *Let $x = (g\colon \partial E \to E, (\theta, \partial\theta))$ be an $(n+1)$–dimensional symmetric Poincaré pair. Write $g = \begin{pmatrix} j_0 & j_1 & k \end{pmatrix}\colon \partial E_r = D_r \oplus D_r \oplus C_{r-1} \longrightarrow E_r$.*
Let $x' = (a,\sigma)^{\%}(x)$. Let (B,λ) be the asymmetric complex of x and (B',λ') the asymmetric complex of x'. Then there is an equivalence $(b,\xi)\colon (B, T\lambda) \xrightarrow{\simeq} (B', T\lambda')$ of $(n+1)$–dimensional asymmetric complexes given by:

$$b = \begin{pmatrix} 1 & (-)^r j_0 f\Delta & 0 \\ 0 & 0 & 1 \end{pmatrix}\colon B_r = E_r \oplus C_{r-1} \oplus D_r \longrightarrow B'_r = E_r \oplus D_r$$

$$\xi = \begin{pmatrix} (-k\Delta\phi_0 h^* + (-)^{r+1} j_0 f\overline{\rho}_0)\Delta^* f^* j_0^* & 0 \\ f\Delta\phi_0((-)^{n-r}k^* - h^*\Delta^* f^* j_0^*) & (-)^{n-r} f\Delta\phi_0 f^* \end{pmatrix}\colon$$

$$B'^{n+2-r} = E^{n+2-r} \oplus D^{n+1-r} \longrightarrow B'_r = E_r \oplus D_r$$

Proof of Theorem 8.2 By Lemma 8.5, $\sigma^*(z,t) \in WAsy(\Lambda)$ is the asymmetric signature of the $(2q+2)$–dimensional symmetric Poincaré pair:

$$\begin{aligned} x'^t &= (g'^t\colon D \oplus D \longrightarrow E, (\delta\theta', \nu \oplus -\nu)) \\ g'^t_{q+1} &= \begin{pmatrix} 1 & -\beta \end{pmatrix}\colon D_{q+1} \oplus D_{q+1} = G \oplus G \longrightarrow E_{q+1} = G \\ \delta\theta'_0 &= -\epsilon Y\colon E^{q+1} = G^* \longrightarrow E_{q+1} = G \\ \nu_0 &= -\widetilde{\mu}^*\colon D^q = F \longrightarrow D_{q+1} = G \\ \nu_0 &= -\widetilde{\mu}\colon D^{q+1} = G^* \longrightarrow D_q = F^* \end{aligned}$$

By Corollary 6.5, $\sigma^*(x'^t)$ is the image of the union of x'^t in $LAsy^{2q+2}(\Lambda)$. In order to construct x'^t in a different way we consider the $(2q+2)$–dimensional quadratic Poincaré pair $\widetilde{x} = (\widetilde{g}\colon D \oplus D' \longrightarrow E, (0, \nu \oplus -\nu'))$ by:

$$\begin{aligned}
\widetilde{g} = \begin{pmatrix} 1 & -1 \end{pmatrix} &\colon D_{q+1} \oplus D_{q+1} = G \oplus G \longrightarrow E_{q+1} = G \\
\nu'_0 = -\widetilde{\gamma}^* &\colon D'^q = F^* \longrightarrow D'_{q+1} = G \\
\nu'_0 = -\widetilde{\gamma} &\colon D'^{q+1} = G^* \longrightarrow D'_q = F
\end{aligned}$$

and the isomorphism $(\overline{e}_t, \overline{\chi}_t)\colon (D, \nu) \xrightarrow{\cong} (D', \nu')$ given by:

$$\begin{aligned}
\overline{e}_{t,q+1} = \beta &\colon D_{q+1} = G \longrightarrow D'_{q+1} = G \\
\overline{e}_{t,q} = \alpha^{-*} &\colon D_q = F^* \longrightarrow D'_q = F \\
\overline{\chi}_{t,0} = -\epsilon Y &\colon D'^{q+1} = G^* \longrightarrow D'_{q+1} = G
\end{aligned}$$

The isomorphism can be used to replace the "boundary component" (D', ν') by (D, ν). The result will be x'^t. Gluing both ends (ie, D and D') of $\widetilde{x}$ together using $(\overline{e}_t, \overline{\chi}_t)$ yields the union of x'^t. Hence all unions of x'^t for different choices of t are in the same *algebraic Schneiden-und-Kleben-cobordism* class. Due to [13, 30.30(ii)], their images in $LAsy^{2q+2}(\Lambda)$ coincide. Those images are precisely the asymmetric signatures $\sigma^*(x'^t) = \sigma^*(z, t)$. □

9 Preformations and linking forms

Let (W, M_0, M_1) be a modified surgery problem over a simply-connected B such that $H_{q+1}(B, M_j)$ are finite. Its surgery obstruction will be a preformation $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ over $\mathbb{Z}$ where γ and μ have finite cokernel. Flip-isomorphisms of such z are basically just isometries of linking forms induced on those cokernels. The asymmetric signature turns out to be well-defined on those isometries. Next, we will prove even more general statements using localization: Let $S^{-1}\Lambda$ be the localization of Λ away from the central and multiplicative subset $S \subset \Lambda$. We repeat some concepts from [12, Sections 3.1, 3.4]:

Definition 9.1

(i) A (Λ, S)–*module* M is a module M such that there is an exact sequence of modules $0 \to P \xrightarrow{d} Q \to M \to 0$ where P and Q are free and d is an S–isomorphism.

(ii) A homomorphism $f \in \mathrm{Hom}_\Lambda(P, Q)$ is an S–*isomorphism* if its induced homomorphism $S^{-1} f \in \mathrm{Hom}_{S^{-1}\Lambda}(S^{-1}P, S^{-1}Q)$ is an isomorphism.

(iii) An *ϵ–symmetric linking form* (M, λ) *over* (Λ, S) is a (Λ, S)–module M together with a pairing $\lambda\colon M \times M \to S^{-1}\Lambda/\Lambda$ such that $\lambda(x, -)\colon M \to S^{-1}\Lambda/\Lambda$ is Λ–linear for all $x \in M$ and $\lambda(x, y) = \epsilon\overline{\lambda(y, x)}$ for all $x, y \in M$.

(iv) A *split ϵ–quadratic linking form* (M, λ, ν) *over* (Λ, S) is an ϵ–symmetric linking form (M, λ) over (Λ, S) together with a map $\nu\colon M \to Q_\epsilon(S^{-1}\Lambda/\Lambda)$ such that for all $x, y \in M$ and $a \in \Lambda$

(a) $\nu(ax) = a\nu(x)\overline{a} \in Q_\epsilon(S^{-1}\Lambda/\Lambda)$

(b) $\nu(x + y) - \nu(x) - \nu(y) = \lambda(x, y) \in Q_\epsilon(S^{-1}\Lambda/\Lambda)$

(c) $(1 + T_\epsilon)\nu(x) = \lambda(x, x) \in Q^\epsilon(S^{-1}\Lambda/\Lambda)$

9.1 Flip-isomorphisms, asymmetric signatures and linking forms

Definition 9.2 A *split S-ϵ–preformation* $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ is a regular split ϵ–preformation such that γ and μ are S–isomorphisms. Then a split $(-\epsilon)$–quadratic linking form $L_\mu = (\operatorname{coker}\mu, \lambda_\mu, \nu_\mu)$ over (Λ, S) is given by:

$$\lambda_\mu\colon \operatorname{coker}\mu \times \operatorname{coker}\mu \longrightarrow S^{-1}\Lambda/\Lambda, \qquad (x, y) \longmapsto \tfrac{1}{s}\gamma^*(x)(g)$$
$$\nu_\mu\colon \operatorname{coker}\mu \longrightarrow Q_{-\epsilon}(S^{-1}\Lambda/\Lambda), \qquad y \longmapsto \tfrac{1}{s}\theta(g)(g)\tfrac{1}{\overline{s}}$$

for $x, y \in F^*$, $g \in G$, $s \in S$ such that $sy = \mu(g)$. Similarly, we can define the split $(-\epsilon)$–quadratic linking form L_γ on $\operatorname{coker}\gamma$. We denote the associated $(-\epsilon)$–symmetric linking forms by $L^\mu = (\operatorname{coker}\mu, \lambda_\mu)$, etc.

Remark 9.3 A split S-ϵ–preformation is a refinement of a *split ϵ–quadratic S–formation* [12, page 240]. The definitions of the linking forms are taken from the proof of [12, Proposition 3.4.3] which establishes a bijection between weak isomorphism classes of S–formations and linking forms up to isometry. Under this correspondence z is mapped to L_μ and its flip to L_γ.

Proposition 9.4 *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*, \theta)$ be a split S-ϵ–preformation.*

(i) *All flip-isomorphisms $(\alpha, \beta, \chi, \kappa)$ induce isometries $[\alpha^{-*}]\colon L_\mu \xrightarrow{\cong} L_\gamma$.*

(ii) *Every isometry $l\colon L_\mu \xrightarrow{\cong} L_\gamma$ is induced by a stable flip-isomorphism.*

(iii) *If $[z] \in l_{2q+2}(\Lambda)$ is elementary then $L_\mu \cong L_\gamma$.*

Proof The first statement is clear. The last claim follows from Proposition 2.8 (iv) and [12, Proposition 3.4.6(ii)]. It remains to prove the second statement. According to Remark 9.3, we can apply [12, Proposition 3.4.3] to z and its flip z'. The proof demonstrates that a stable isomorphism of split ϵ–quadratic S–formations between z and z' exists. Using Remark 2.6 it is not difficult to find a stable flip-isomorphism of z. □

Theorem 9.5 *Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be an S-ϵ–preformation. Two flip-isomorphisms $t = (\alpha, \beta, \chi)$ and $t' = (\alpha', \beta', \chi')$ of z which induce the same isometry $L^\mu \xrightarrow{\cong} L^\gamma$ have the same asymmetric signature. Hence, we can define the asymmetric signature $\sigma^*(z, l)$ of an isometry $l\colon L^\mu \xrightarrow{\cong} L^\gamma$ to be $\sigma^*(z, s)$ for any flip-isomorphism s that induces l.*

Proof In Section 6.2, $\sigma^*(z, t)$ is defined as the asymmetric signature of the symmetrization of the $(2q+2)$–dimensional quadratic Poincaré pair x_t which will be denoted by $x^t = (g^t\colon \partial E_t \to E, (0, \theta_t))$. Its boundary is a twisted double of the symmetrization of the quadratic Poincaré pair c (denoted by $(f\colon C \to D, (0, \phi)))$ in respect to the automorphism $(h_t, 0)$ of (C, ϕ) where $\phi = (1+T)\psi$. We will show that t and t' lead to homotopic $(h_t, 0) \simeq (h_{t'}, 0)$. Then the asymmetric signatures of x^t and $x^{t'}$ are the same by Lemma 7.11. By Proposition 4.1, t and t' induce two isomorphisms

$$(e, \rho) = \mathbf{F}(t), (e', \rho') = \mathbf{F}(t')\colon (N, \zeta) = \mathbf{F}(z) \xrightarrow{\cong} (N', \zeta') = \mathbf{F}(z')$$

where z' is the flip of z. The fact that t and t' induce the same isometries translates into $e^* = e'^*\colon H^*(N') \xrightarrow{\cong} H^*(N)$. Since N and N' are 1–dimensional, e and e' are chain homotopic. Let $\Delta\colon e \simeq e'$ be a chain homotopy. Due to the proof of [11, Proposition 3.4], (e, ρ) and (e', ρ') induce isomorphisms

$$(\partial e, 0), (\partial e', 0)\colon (C, \phi) = (\partial N, (1+T)\partial\zeta) \xrightarrow{\cong} (C', \phi') = (\partial N', (1+T)\partial\zeta').$$

Using the fact that N and N' are 1–dimensional and S–acyclic, one can show that there is a chain equivalence $(\partial\Delta, 0)\colon (\partial e, 0) \simeq (\partial e', 0)\colon (C, \phi) \to (C', \phi')$:

$$\partial\Delta_{q+1} = \begin{pmatrix} 0 & \epsilon\beta'\Delta^*\alpha^{-*} \end{pmatrix}\colon C_q = F \oplus F^* \longrightarrow C'_{q+1} = G^*$$
$$\partial\Delta_q = \begin{pmatrix} \Delta \\ 0 \end{pmatrix}\colon C_{q-1} = G^* \longrightarrow C'_q = F^* \oplus F$$

As explained in the proof of Theorem 5.3, we compose ∂e with the inverse of $(h, 0)\colon (C, \phi) \xrightarrow{\cong} (C', \phi')$ from (3) in order to obtain the automorphism $(h_t, 0)\colon (C, \phi) \xrightarrow{\cong} (C, \phi)$. Using Lemma 7.2, one finds a homotopy

$$(h^{-1}\partial\Delta, 0)\colon (h_t, 0) \simeq (h_{t'}, 0)\colon (C, \phi) \xrightarrow{\cong} (C, \phi)$$

which can be fed into Lemma 7.11. Hence, $\sigma^*(z,t') = \sigma^*(x^{t'}) = \sigma^*(x'') \in WAsy(\Lambda)$ with $x'' = (g''\colon \partial E_t \to E, (0,\theta_t))$ given by

$$g'' = \begin{pmatrix} 1 & \beta' & 0 - \beta'\Delta^*\alpha^{-*} \end{pmatrix}\colon \partial E_{t,q+1} = G \oplus G \oplus F \oplus F^* \longrightarrow E_{q+1} = G.$$

Finally, there is a homotopy equivalence $(1,1;l)\colon x'' \xrightarrow{\simeq} x^t$ given by

$$l = \begin{pmatrix} 0 & \epsilon\beta'\Delta^*\alpha^{-*} & 0 \end{pmatrix}\colon \partial E_{t,q} = F^* \oplus F^* \oplus G^* \longrightarrow E_{q+1} = G.$$

Clearly $\sigma^*(x'') = \sigma^*(x^t)$ by Proposition 3.9 and 6.9. □

9.2 Asymmetric signatures of certain surgery problems

Let $\Lambda = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$. Let $z = (F \xleftarrow{\gamma} G \xrightarrow{\mu} F^*)$ be an ϵ–preformation such that $\operatorname{coker}\gamma$ and $\operatorname{coker}\mu$ are finite. The regular ϵ–preformation $z' = (F \xleftarrow{[\gamma]} G/\ker\gamma \xrightarrow{[\mu]} F^*, \theta')$ can be used to extend Theorem 9.5 to z.

Definition 9.6 [14, Example 12.44] Let $M \to B$ be a $(2q+1)$–dimensional B–manifold. The *B-linking form* $(TH_{q+1}(B,M), l_M^B)$ on the torsion subgroup of $H_{q+1}(B,M)$ is the linking form induced by the topological linking form

$$l_M\colon TH_q(M) \times TH_q(M,\partial M) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad ([x],[y]) \longmapsto \frac{1}{s}\langle z, y\rangle$$

with $z \in C^q(M,\partial M)$ and $s \in \mathbb{Z}\setminus\{0\}$ such that $sx = d(z \cap [M]) \in C_q(M)$.

Theorem 9.7 *Let (W, M_0, M_1) be a modified surgery problem such that $\pi_1(B) = 0$, $\dim W = 2q \geq 6$ and $H_{q+1}(B, M_j)$ are finite. Let z be its surgery obstruction.*

(i) $L^\gamma = -l_{M_0}^B$ and $L^\mu = -l_{M_1}^B$.

(ii) *If W is cobordant rel∂ to an s–cobordism, then isometries $l\colon l_{M_1}^B \xrightarrow{\cong} l_{M_0}^B$ exist and all asymmetric signatures vanish.*

(iii) *Assume q is odd and $l_{M_0}^B$ is non-singular. Then W is cobordant rel∂ to an s–cobordism if and only if there is an isometry $l\colon l_{M_1}^B \xrightarrow{\cong} l_{M_0}^B$ such that its asymmetric signature vanish.*

Proof The complex $\widetilde{C}_{q+2} = H_{q+2}(B,W) \xrightarrow{\gamma} \widetilde{C}_{q+1} = H_{q+1}(W,M_0)$ has homology $H_i(\widetilde{C}) = H_i(B,M_0)$ $(i = q+1, q+2)$. There is a chain equivalence $m\colon \widetilde{C} \xrightarrow{\simeq} C(B,M_0)$ and there is a chain map $C(B,M_0) \to C_{*-1}(M_0)$ which induces the connecting homomorphism $\partial_*\colon H_*(B,M_0)\to H_{*-1}(M_0)$. Both maps together yield a chain map:

$$\begin{array}{ccc} C_{q+1}(M_0) & \xrightarrow{d} & C_q(M_0) \\ {\scriptstyle p}\uparrow & & {\scriptstyle p}\uparrow \\ \widetilde{C}_{q+2} & \xrightarrow[\gamma]{} & \widetilde{C}_{q+1} \end{array}$$

which induces the connecting map $p\colon H_{q+1}(B,M_0)\to H_q(M_0)$. Let $a,b\in\operatorname{coker}\gamma = H_{q+1}(B,M_0)=H_{q+1}(\widetilde{C})$. Represent both homology classes by chains $\overline{a},\overline{b}\in\widetilde{C}_{q+1}$. Then there is a $g\in\widetilde{C}_{q+2}$ and an $s\in\mathbb{Z}\backslash\{0\}$ such that $s\overline{a}=\gamma(g)$. Let $z\in C^q(M_0,\partial M_0)$ such that $p(g)=z\cap[M_0]$. Then $sp(\overline{a})=d(z\cap[M_0])$. Hence $l^B_{M_0}(a,b)=\frac{1}{s}\langle z,p(\overline{b})\rangle$. Let $b'\in H^{q+1}(W,M_0{}')$ such that $b'\cap[W]=\overline{b}$. Then $l^B_{M_0}(a,b)=\frac{1}{s}\langle p^*z,b'\cap[W]\rangle = -\epsilon\frac{1}{s}\langle b',p^*(z)\cap[W]\rangle$. Since p is a connecting homomorphism $p^*(z)\cap[W]=-\epsilon i(z\cap[M_0])=-\epsilon ip(g)=-\epsilon\mu(g)$. Hence $l^B_{M_0}(a,b)=\frac{1}{s}\langle b',\mu(g)\rangle=-\epsilon\frac{1}{s}\mu^*(b)(g)=-L^\gamma(a,b)$.

The last statement follows from Proposition 7.12. □

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Universität Heidelberg, Mathematisches Institut
Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

`sixt@mathi.uni-heidelberg.de`

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