Khovanov–Rozansky homology via a canopolis formalism

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In this paper, we describe a canopolis (ie categorified planar algebra) formalism for Khovanov and Rozansky's link homology theory. We show how this allows us to organize simplifications in the matrix factorizations appearing in their theory. In particular, it will put the equivalence of the original definition of Khovanov–Rozansky homology and the definition using Soergel bimodules in a more general context, allow us to give a new proof of the invariance of triply graded homology and give a new analysis of the behavior of triply graded homology under the Reidemeister IIb move.

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In [9; 10], Khovanov and Rozansky introduced a series of homology theories for links. These theories categorify the quantum invariants for $\mathfrak{sl}_n$, and the HOMFLYPT polynomial. Unfortunately, they remain very difficult to calculate, not least because of the complicated matrix factorizations used in their original combinatorial definition. Later work of I Frenkel, Khovanov and Stroppel [6; 8; 14; 15] has suggested a more systematic definition of these invariants and a connection between these theories and the structure of the BGG category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n$, but progress toward computational simplifications along these lines has been slow.

In this paper, we will show that these invariants can be understood, computed and in fact, defined in the context of canopolises. We hope that this approach will both lead to computational benefits and help the reader to understand the definition of Khovanov–Rozansky homology better. A *canopolis*[1] is a categorification of the notion of a planar algebra defined by Bar-Natan [2] (see Section 2).

Consider a disk in the plane with m disks removed from its interior (we call the places left by these removed disks "holes"). An *oriented planar arc diagram* (or "spaghetti-and-meatballs diagram") η on this disk is a collection of oriented simple curves with endpoints on the boundary of the disk (including the boundary of the holes), along with choice of a distinguished point on each boundary of a component (in diagrams, this point is distinguished by putting a star next to it), and an ordering of the holes of the diagram.

[1] Bar-Natan uses the term "canopoly" in the published version of [2], but the consensus choice now seems to be the more etymologically correct "canopolis."

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Let $\mathcal{Q}_i(\eta)$ be the set of planar arc diagrams ω such that the outer boundary of ω matches the boundary of the i–th hole of η. That is, there is the same number of endpoints, and if we order the endpoints, starting at our distinguished point, the orientations of the arcs match.

Any fixed planar arc diagram η with a distinguished hole defines an operation

$$\widetilde{\eta}\colon \mathcal{Q}_1(\eta)\times\cdots\times\mathcal{Q}_m(\eta)\to\mathcal{Q}_0(\eta),$$

by shrinking the given m–tuple of diagrams, and pasting it into the holes of η, as is shown in Figure 1. This "multiplication" is a particular instance of an algebraic structure called an *colored operad*.

The operad of planar arc diagrams acts on tangle diagrams in a disk as well. Phrased in the language of [2], the set of tangle diagrams $\mathcal{T}_{S,\epsilon}$ in a disk, with endpoints on the boundary and a marked point on the boundary, partitioned according to the orientation $\epsilon\colon S\to\{\pm 1\}$ of the endpoints, form *a planar algebra*, that is, a set on which the operad of planar arc diagrams acts. In fact, one can build any tangle diagram from single crossings in a disk and the action of a planar arc diagram. More generally, we will be interested in factoring a tangle as the action of a planar arc diagram on simpler tangle diagrams. We depict these operations in Figure 2.

One fruitful approach to the Jones polynomial and other quantum invariants is to regard each as a homomorphism of planar algebras. Thus, one can compute the Jones polynomial of a tangle once (recall that this is an element of a certain vector space over $\mathbb{C}(q)$, rather than just a polynomial), and then whenever one wishes to compute the Jones polynomial of a knot, one cuts it into a planar arc diagram acting on tangles whose Jones polynomial is known.

This approach is of more than theoretical value; if programmed skillfully, it can be extremely efficient. Bar-Natan [1; 2] presented a beautiful extension of this approach to Khovanov's original link homology (the $\mathfrak{sl}_2$–version of Khovanov–Rozansky), which at once gives a simple description of the knot homology and is extremely computationally efficient, allowing the computation by computer of Khovanov homology for knots of dozens of crossings.

Unfortunately, we do not know how to give a similar, matrix factorization–free description of Khovanov–Rozansky. Instead, we will show that Khovanov–Rozansky can be defined using certain homotopy categories of matrix factorizations which admit an action of planar arc diagrams, that is, a canopolis structure, which we can see as an analogue of Bar-Natan's geometric canopolis.

While this is an essentially formal construction, it allows us to simplify the KR-complex of a small tangle before or after we apply the action of a planar arc diagram

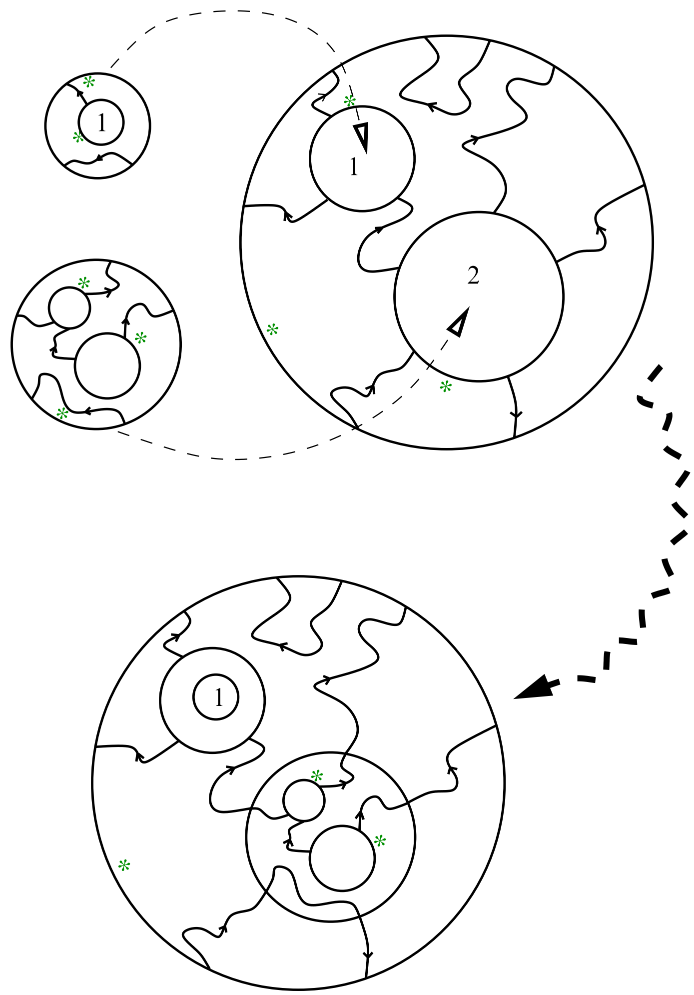

Figure 1: The action of planar arc diagrams

(where "simplify" has a very precise definition, given in Section 2.2), allowing us to organize computations of KR homology according to Bar-Natan's "divide and conquer" philosophy. In particular, it will give us a new understanding of the equivalence between KR homology and the homology defined by Soergel bimodules, shown by Khovanov in the HOMFLYPT case [8].

We will also apply this approach to the triply graded homology theory discussed in [8; 10] to give a new proof of invariance and show that the changes in triply graded homology when the diagram undergoes a second Reidemeister move is controlled by a certain spectral sequence.

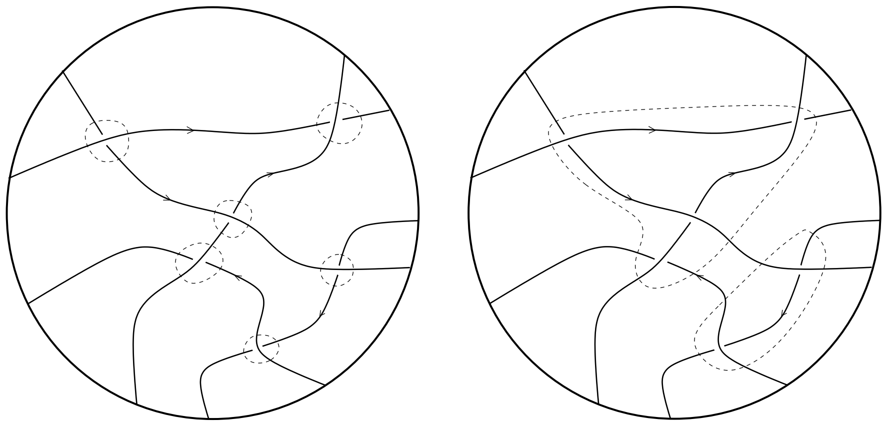

Figure 2: Factoring a tangle (a) as a product of crossings and a planar arc diagram or (b) as a more general factorization

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1 Matrix factorizations

1.1 Preliminaries on matrix factorizations

We will attempt to follow the notations and terminology of Rasmussen [11]. Let M be a $\mathbb{Z}$–graded module over a ring $\mathcal{S}$.

Definition 1 A *($\mathbb{Z}$–graded) matrix factorization* on M with potential $\varphi \in \mathcal{S}$ is a map $d = d_+ + d_-\colon M \to M$ with $d_\pm$ of graded degree ± 1 such that $d^2 = \varphi$.

Though this is not usual definition of a matrix factorization (where typically we only assume a $\mathbb{Z}/2$ grading), this richer structure is more useful from the perspective of knot theory.

Matrix factorizations over $\mathcal{S}$ with a fixed potential naturally form an abelian category, with morphisms given by maps commuting with d. We only assume that these maps are homogeneous with respect to the $\mathbb{Z}/2$–grading.

Even better, we can think of all matrix factorizations over all unital rings as a 2–category **MF** such that:

- The objects are given by a pairs of a unital ring and an element of that ring $(\mathcal{S},\varphi)$.
- The 1–morphisms between $(\mathcal{S}_1,\varphi_1)$ and $(\mathcal{S}_2,\varphi_2)$ are given by matrix factorizations over $\mathcal{S}_1\otimes\mathcal{S}_2$ with potential $\varphi_1\otimes 1-1\otimes\varphi_2$, and composition is given by tensor product of matrix factorizations of bimodules.
- The 2–morphisms between two matrix factorizations are given by morphisms in the usual sense.

While matrix factorizations may seem strange, in fact, they arise very naturally in homological algebra (see, for example, Eisenbud [4]). Consider a module N, and a ring element $\varphi\in\mathcal{S}$ which annihilates N. Fix a finite-length free resolution $\mathbf{M}^\bullet$ of N (by convention, we give this resolution the cohomological grading, ie the differential is of degree 1), and let M be the direct sum of all its components. Let $d_+\colon M\to M$ be the differential, and φ_M be the action of multiplication by φ on M. Since the induced map φ_N is 0, by standard homological algebra, φ_M is homotopic to 0, ie there exists a map $d_-\colon \mathbf{M}^\bullet\to\mathbf{M}^\bullet$ of degree 1 such that $d_+d_-+d_-d_+=\varphi$.

Now, assume that d_- also defines the structure of a chain complex on $\mathbf{M}^\bullet$, that is, $d_-^2=0$. Let $d=d_++d_-$. By the homotopy formula above, we have

$$d^2=d_+^2+d_+d_-+d_-d_++d_-^2=\varphi,$$

that is, d defines a matrix factorization with potential φ on M with the grading given by homological degree.

Recall for an ordered n–tuple $(x_1,\dots,x_n)\in R^n$ in a commutative ring R is called a *regular sequence* if the action of x_i is a non-zerodivisor (multiplication by it is injective) on $R/(x_1,\dots,x_{i-1})$ for all i.

In the case when N is the quotient $\mathcal{S}/(\mathbf{x})$ of $\mathcal{S}$ by the ideal generated by a regular sequence $\mathbf{x}=\{x_1,\dots,x_m\}$, the matrix factorizations constructed from N have a particularly nice interpretation, as described in slightly different language by Eisenbud in [5, Section 17.4]. Let Z_{x_i} denote the two-term complex $\mathcal{S}\xrightarrow{x_i}\mathcal{S}$. If $\mathcal{S}$ is graded and each x_i is homogeneous, then we can shift gradings so that this differential is of degree 2 (while this may seem like a peculiar choice, it the one which makes this grading match with the standard variable q in the quantum invariants which KR homology should categorify). This is clearly a free resolution of $\mathcal{S}/(x_i)$. Thus, the easiest possible guess for a free resolution of N is the Koszul complex

$$\mathcal{Z}_{\mathbf{x}} = \bigotimes_{i=1}^{m} Z_{x_i}.$$

This complex is indeed a resolution of N, since $\mathbf{x}$ is regular [5, Section 17.2] (otherwise, it might have higher cohomology).

Since $\varphi_N = 0$, we must have $\varphi = \sum_{i=1}^{m} y_i x_i$ for some sequence (not necessarily unique or regular) $\mathbf{y} = \{y_1, \dots, y_n\}$. Instead of taking the complex Z_{x_i}, consider the matrix factorization

$$\widetilde{Z}_{x_i,y_i} = \begin{tikzcd} \mathcal{S} \arrow[r, bend left, "x_i"] & \mathcal{S} \arrow[l, bend left, "y_i"] \end{tikzcd}.$$

By analogy with the Koszul resolution, we define the *Koszul matrix factorization* of the pair $(\mathbf{x}, \mathbf{y})$ to be the tensor product

$$\widetilde{\mathcal{Z}}_{\mathbf{x},\mathbf{y}} = \bigotimes_{i=1}^{m} \widetilde{Z}_{x_i,y_i}.$$

This is a matrix factorization with potential $\sum_i x_i y_i = \varphi$.

1.2 Near-isomorphisms

We would like to generalize the following standard fact of homological algebra:

Proposition 1.1 *Let $f\colon C \to C'$ be a chain map between complexes of $\mathcal{S}$ modules which induces an isomorphism on homology, that is, a quasi-isomorphism. Then for any complex of D of projective $\mathcal{S}$–modules, $f \otimes 1\colon C \otimes D \to C' \otimes D$ is also a quasi-isomorphism.*

Proof We use a spectral sequence on $C \otimes D$, associated to the filtration

$$F_n = \bigoplus_{i \le n} C_j \otimes D_i.$$

In this case $F_n/F_{n-1} \cong C \otimes D_n$. Since $f \otimes 1$ is filtered, it induces a map of spectral sequence, and since D_n is projective, the induced map on the E^0 term is an isomorphism. Thus it is an isomorphism on the E^∞ term as well, which is the associated graded module of $H^*(C \otimes D)$. Thus, f is a quasi-isomorphism. □

Note that this result depends heavily on the fact that D is projective (or more generally, flat).

How might such a fact be generalized to matrix factorizations? First, let us define a class of maps analogous to quasi-isomorphisms of complexes (unfortunately "quasi-isomorphism of matrix factorizations," as defined by Rasmussen [11] means something slightly different, and more analogous to a homotopy equivalence of complexes).

Definition 2 We call a chain map $f\colon M_+ \to M'_+$ a near-isomorphism if for each matrix factorization D on a projective $\mathcal{S}$–module of potential $-\varphi$, the map

$$H^t(M \otimes D) \to H^t(M' \otimes D)$$

is an isomorphism.

Unlike in the case where $\varphi = 0$, the hard part now will be identifying such morphisms. While we know of no explicit characterization, one example will be sufficient for our purposes

Let M be a matrix factorization of potential φ with $M^i = 0$ for all $i > 0$, and let $\mathcal{H}(M)$ be $H^0(M_+)$. Obviously, there is a chain map $\pi\colon M_+ \to \mathcal{H}(M)$, where $\mathcal{H}(M)$ is considered as a complex concentrated is degree 0. Note that $\mathcal{H}(M)$ must be annihilated by φ, since the image of d_+ in M^0 contains $\varphi \cdot M^0$.

Theorem 1.2 *If the natural map $\pi\colon M_+ \to \mathcal{H}(M)$ is a quasi-isomorphism of complexes, then it is also a near-isomorphism of matrix factorizations.*

Proof Let $K = \ker d_+ \subset M$. Fix a matrix factorization of projective modules D of potential $-\varphi$, and consider the following filtration on $M \otimes D$:

$$G^n = \Bigg(\bigoplus_{i<n, j\in\mathbb{Z}} M^i \otimes D^j\Bigg) \oplus \Bigg(\bigoplus_{j\in\mathbb{Z}} K^n \otimes D^j\Bigg)$$

This filtration is a mix of the standard choice on a tensor product of complexes, and that used by Rasmussen for any matrix factorization of potential zero [11, Lemma 5.11].

Since $d = d_+ + d_-$ preserves this filtration, there is an associated spectral sequence, converging to $H^t(M \otimes D)$.

The E^0 term of this spectral sequence is

$$G_{n+1}/G_n \cong \big(K^{n+1} \oplus M^n/K^n\big) \otimes D,$$

and as Rasmussen computed, the differential d_0 is given by the total differential on the above tensor product, when we make $K^{n+1} \oplus M^n/K^n$ into a matrix factorization by

$$K^{n+1} \underset{d_+^M}{\overset{d_-^M}{\rightleftarrows}} M^n/K^n.$$

Since we assumed that $H^i(M_+) = 0$ for all $i \neq 0$, the map d_+^M induces an isomorphism $M^i/K^i \cong K^{i+1}$ for all $n \neq 0$, the complex $(G_{n+1}/G_n)_+$ is exact and the homology of G_{n+1}/G_n is trivial if $n \neq 0$.

Thus, all higher differentials have trivial source or trivial target, so our sequence collapses at E^1. Since $G^1/G^0 \cong \mathcal{H}(M) \otimes D$, we find that

$$H^t(M \otimes D) \cong H^t(\mathcal{H}(M) \otimes D),$$

with the isomorphism induced by the natural projection π. □

1.3 Khovanov–Rozansky matrix factorizations

Matrix factorizations appear in knot theory through the work of Khovanov and Rozansky: they associate to any oriented tangle diagram T and any polynomial p, which vanishes at 0, a complex of matrix factorizations we denote by $\mathcal{R}_p(T)$, defined as follows:

Consider the graph $\mathcal{G}(T)$ of T which has vertices corresponding to crossings or end points of components of T and edges corresponding to segments of diagram between crossings. The orientation of T induces an orientation on $\mathcal{G}(T)$. Let $\mathcal{F}(T)$ denote the set of flags of the graph $\mathcal{G}(T)$, that is, pairs (x, e) of adjacent edges and vertices. Let $\widetilde{\mathcal{S}}_T = \widetilde{\mathcal{S}}$ be polynomials over k in the variables $t_{x,e}$, where (x, e) ranges over $\mathcal{F}(T)$.

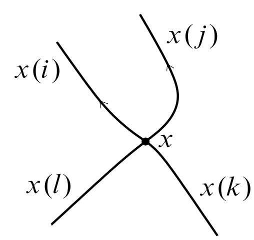

Figure 3: The labeling of edges around a crossing

For each crossing x, number the adjacent edges as shown in Figure 3. Let $t_h = t_{x,e_h}$, where $h = i, j, k, \ell$.

Let L'_x be a Koszul matrix factorization on the Koszul resolution of

$$\mathbf{t}'_x = (t_i + t_j - t_k - t_\ell, t_i t_j - t_k t_\ell)$$

with potential $\varphi_x = p(t_i) + p(t_j) - p(t_k) - p(t_\ell)$. Recall that we have fixed the grading on Koszul resolutions such that d_+ has graded degree 2 and the 0–th term has the same grading as the ring itself with no shift. Such a factorization exists, since $p(t_i) + p(t_j)$ is a polynomial in $t_i + t_j$ and $t_i t_j$, by the fundamental theorem of symmetric function theory. Its exact form will not be important for us at the moment. Since $\mathbf{t}'_x$ is regular, $(L'_x)_+$ is a free resolution of $\mathcal{H}(M) \cong \widetilde{\mathcal{S}}/(\mathbf{t}'_x)$, where $(\mathbf{t}'_x)$ denotes the ideal in $\widetilde{\mathcal{S}}$ generated by the elements of $\mathbf{t}'_x$.

Let L''_x be a Koszul matrix factorization with potential φ_x on the Koszul complex of $\mathbf{t}''_x = (t_i - t_\ell, t_j - t_k)$. As with $\mathbf{t}'_x$, this is a regular sequence, so $(L''_x)_+$ is a free resolution of $\mathcal{H}(M) \cong \widetilde{\mathcal{S}}/(\mathbf{t}''_x)$.

Khovanov and Rozansky define a two term complex L_x which depends on whether the crossing was positive or negative; if x is positive, it is a complex of the form $\rho^+_x\colon L'_x \to L''_x$ and if x is negative, it is of the form $\rho^-_x\colon L''_x \to L'_x\{-2\}$, where $\{a\}$ denotes grading shift by a, in each case with L''_x in homological degree 0. The exact form of these maps will not be important to us at the moment. We only note that after applying $\mathcal{H}(-)$:

- The induced map $\mathcal{H}(\rho^+_x)\colon \widetilde{\mathcal{S}}/(\mathbf{t}') \to \widetilde{\mathcal{S}}/(\mathbf{t}'')$ is the obvious projection.
- The induced map $\mathcal{H}(\rho^-_x)\colon \widetilde{\mathcal{S}}/(\mathbf{t}'') \to \widetilde{\mathcal{S}}/(\mathbf{t}')\{-2\}$ is that induced by multiplication by $t_i - t_k$.

It's worth noting that the complex $\mathcal{H}(L_x)$ is independent of p.

For each edge e, directed from x_a to x_b, we let L_e be the Koszul matrix factorization of the pair $\mathbf{x} = (t_a - t_b)$ and $\mathbf{y} = (p(t_a) - p(t_b))/(t_a - t_b)$, where $t_a = t_{x_a,e}$ and $t_b = t_{x_b,e}$, again with polynomial grading such that d_+ is of degree 2.

We define the Khovanov–Rozansky complex of the diagram T to be the complex of matrix factorizations given by the tensor product

$$\mathcal{R}_p(T) = \Big(\bigotimes_e L_e\Big) \otimes \Big(\bigotimes_x L_x\Big).$$

Since each matrix factorization in this complex is a direct sum of Koszul matrix factorizations, we can apply the functor $\mathcal{H}$ component-wise, to obtain a complex of modules over $\widetilde{\mathcal{S}}$, which we will refer to as the "naive Khovanov–Rozansky complex" $\mathcal{N}(T) = \mathcal{H}(\mathcal{R}_p(T))$. As with a single crossing, $\mathcal{N}(T)$ is independent of p, since $\mathcal{R}_p(T)_+$ is independent of p.

Let $\mathcal{L}(T) \subset \mathcal{F}(T)$ be the set of flags containing vertices of degree 1. There is only one such flag for each vertex of degree 1, and these correspond to the open ends of the tangle T. Let $\mathcal{S}_T$ denote the subring of $\widetilde{\mathcal{S}}$ generated by the variables corresponding to elements of $\mathcal{L}(T)$. Define $\epsilon\colon \mathcal{F}(T) \to \{\pm 1\}$ by:

$$\epsilon(x,e) = \begin{cases} 1 & e \text{ is directed into } x \\ -1 & e \text{ is directed out of } x \end{cases}$$

Then the potential of $\mathcal{R}_p(T)$ is given by the sum:

$$\varphi_T = \sum_{(\ell,e)\in\mathcal{L}(T)} \epsilon(\ell,e)\, p(t_{\ell,e})$$

In particular, we can consider $\mathcal{R}_p(T)$ as a complex of matrix factorizations over $\mathcal{S}_T$, a change which seems small, but will be key to simplifications we do later.

If T is the diagram of a link (ie a closed tangle), then $\mathcal{R}_p(T)$ has potential 0, and we can take the total homology of each matrix factorization. In this case, we obtain a complex of graded vector spaces, which we call $\mathcal{K}_p(T)$. The homology of this complex (as a bigraded vector space) is what is typically called *unreduced Khovanov–Rozansky homology*. We can obtain *reduced Khovanov–Rozansky* (for a knot) by quotienting out by the action of one of the generators of $\mathcal{S}_T$ on $\mathcal{K}_p(T)$, and then taking homology. If p is homogeneous (ie $p(x) = x^{N+1}$ for some N) then on both these homologies two gradings will survive (otherwise, we will only have the homological grading). We will not go into the details, as they are not of great importance to the rest of the paper, and are covered in great detail in [11, Section 2].

2 Canopolises

2.1 Canopolises of matrix factorizations

The theory of planar algebras originated with Vaughan Jones's theory of subfactors [7], and they have shown themselves to be a very useful formalism for dealing with knot invariants. In his reformulation of Khovanov homology, Bar-Natan [2] uses a categorification of a planar algebra, called a canopolis:

Definition 3 A (oriented) *canopolis* is an assignment of:

(1) A category $\mathcal{C}_\epsilon^X$ for each set totally ordered finite set X equipped with sign map $\epsilon\colon X \to \{+,-\}$. We think of this as being associated to a disk with signed marked points on the boundary, and with a distinguished marked point (so points are totally ordered, not just cyclically ordered).

(2) A functor $\widetilde{\eta}\colon \mathcal{C}^{X_1}_{\epsilon(1)} \times \cdots \times \mathcal{C}^{X_m}_{\epsilon(m)} \to \mathcal{C}^{X_0}_{\epsilon(0)}$, for each oriented planar arc diagram η, where X_j denotes the set of endpoints of arcs on the j–th boundary component, with the sign determined by the orientation of the arc at that point. The action of planar arc diagrams should commute with composition of functors.

Bar-Natan's original examples were for the most part very geometric, being modifications of various cobordism categories. Rather than attempt to do justice to his presentation, we refer the interested reader to his paper [2, p 31].

The geometric canopolis of interest to us will be as follows:

Let $\mathcal{C}^X_\epsilon$ be the category of oriented tangles in a thickening of the disk D^2 to a 3–ball $\Sigma D^2 = B^3$, with the endpoints of the tangles on the distinguished points X, and the orientation of the tangle matching the sign sequence ϵ. Morphisms between T and T' are oriented cobordisms embedded in $I \times B_3$, with boundary given by $T \times \{0\} \cup T' \times \{1\} \cup X \times I$. We denote this canopolis by $\mathcal{C}$.

As promised, for any polynomial p (vanishing at 0, as before), we will define an associated canopolis $\mathcal{M}_p$ of matrix factorizations, which is a natural home for KR homology.

First, associated to the sign sequence $\epsilon\colon X \to \{+,-\}$ is the category $\mathsf{MF}^{X,p}_\epsilon$ of matrix factorizations over a polynomial ring $k[X]$ with generators indexed by C of potential $\sum_{x\in X} \epsilon_x p(t_x)$.

As we discussed before, the most natural functors from $\mathsf{MF}^{X_1,p}_{\epsilon(1)} \times \cdots \times \mathsf{MF}^{X_m,p}_{\epsilon(m)}$ to $\mathsf{MF}^{X_0,p}_{\epsilon(0)}$ are those induced by tensor product with a matrix factorization over the ring $k[X_0,\dots,X_m]$ with potential:

$$\sum_{x\in X_0} \epsilon_x(0)p(t_x) - \sum_{j=1}^{m} \sum_{x\in X_j} \epsilon_x(j)p(t_x)$$

In fact, there is a clear choice in this category: Let $\mathcal{A}(\eta)$ be the set of arcs of η. Each arc $\alpha \in \mathcal{A}(\eta)$ has a head α_+ and a tail α_-.

Now define sequences $\mathbf{x}, \mathbf{y}$ by

$$\mathbf{x} = \big(t_{\alpha_+} - t_{\alpha_-}\big)_{\alpha\in\mathcal{A}(\eta)}$$
$$\mathbf{y} = \left(\frac{p(t_{\alpha_+}) - p(t_{\alpha_-})}{t_{\alpha_+} - t_{\alpha_-}}\right)_{\alpha\in\mathcal{A}(\eta)}$$

and let $\widetilde{\mathcal{Z}}_\eta$ be the Koszul matrix factorization of this pair. This is a matrix factorization over $k[X_0, \dots, X_m]$, and its potential is

$$\sum_{\alpha\in\mathcal{A}(\eta)} p(t_{\alpha_+}) - p(t_{\alpha_-}) = \sum_{x\in X_0} \epsilon_x(0)p(t_x) - \sum_{j=1}^{m}\sum_{x\in X_j} \epsilon_x(j)p(t_x)$$

since each marked point on any boundary of η is the endpoint of exactly one arc.

The canopolis functor $\widetilde{\eta}\colon \mathsf{MF}_{\epsilon(1)}^{X_1,p} \times \cdots \times \mathsf{MF}_{\epsilon(m)}^{X_m,p} \to \mathsf{MF}_{\epsilon(0)}^{X_0,p}$ will simply be tensor product with $\widetilde{\mathcal{Z}}_\eta$ over $k[X_1, \dots, X_m]$.

Note that this also induces a canopolis structure on the categories of complexes of matrix factorizations $\mathrm{Kom}(\mathsf{MF}_\epsilon^{X,p})$ and the homotopy category of complexes $\mathcal{K}(\mathsf{MF}_\epsilon^{X,p})$, since tensoring with $\widetilde{\mathcal{Z}}_\eta$ is exact.

Theorem 2.1 *These functors define a canopolis structure $\mathcal{M}_p$ on $\mathcal{K}(\mathsf{MF}_\epsilon^{X,p})$. Furthermore, $\mathcal{R}_p\colon \mathcal{C} \to \mathcal{M}_p$ is a functor of canopolises, ie the diagram*

$$\begin{tikzcd}
\prod_{j=1}^{m} \mathcal{T}_{X_j,\epsilon(j)} \arrow[r, "\widetilde{\eta}"] \arrow[d, "\mathcal{R}_p"'] & \mathcal{T}_{X_0,\epsilon(0)} \arrow[d, "\mathcal{R}_p"] \\
\prod_{j=1}^{m} \mathcal{K}(\mathsf{MF}_{\epsilon(j)}^{X_j,p}) \arrow[r, "\widetilde{\eta}"'] & \mathcal{K}(\mathsf{MF}_{\epsilon(0)}^{X_0,p})
\end{tikzcd}$$

is commutative. In particular, if $\widetilde{\eta}(T)$ is closed, then $\mathcal{K}_p(\widetilde{\eta}(T))$ and $H^*(\widetilde{\eta}_p(\mathcal{R}_p(T)))$ are isomorphic as complexes.

Note that while $\mathcal{R}_p$ is a $\mathbb{Z}$–graded matrix factorization, the maps associated to cobordisms typically only preserve the $\mathbb{Z}/2$–grading.

Proof Luckily, all the necessary computations were done by Khovanov and Rozansky in [9]. Checking that the composition of planar arc diagrams matches with composition of functors is simply applying [9, Proposition 15] at each pair of boundary points which are glued together.

The commutation with the functor $\mathcal{R}_p$ is simply rephrasing the original definition, after placing a mark on each connected pair of boundary points. □

In particular, though $\mathcal{R}_p(T)$ was first defined over a ring $\widetilde{\mathcal{S}}_T$ with variables correspond to all elements of $\mathcal{F}(T)$, this canopolis formalism shows that we need only remember the action of variables corresponding to endpoints, not to internal edges of $\mathcal{G}(T)$. Often after restricting to this smaller subring, we can identify trivial summands of the complex $\mathcal{R}_p(T)$.

2.2 Simplifications in $\mathcal{R}_p(T)$

For our purposes, a simplification of a matrix factorization will be a quotient such that the projection map is a near-isomorphism.

Consider a Koszul matrix factorization $M = \widetilde{\mathcal{Z}}_{\mathbf{x},\mathbf{y}}$. Then we expect M to have a great number of simplifications. For any $n < m$, we can rewrite M as a tensor product $M \cong M' \otimes_{\mathcal{S}} M''$, where $M' = \widetilde{\mathcal{Z}}_{\mathbf{x}',\mathbf{y}'}$ and $M'' = \widetilde{\mathcal{Z}}_{\mathbf{x}'',\mathbf{y}''}$, and

$$\mathbf{x}' = (x_1, \dots, x_n), \qquad \mathbf{y}' = (y_1, \dots, y_n),$$
$$\mathbf{x}'' = (x_{n+1}, \dots, x_m), \qquad \mathbf{y}'' = (y_{n+1}, \dots, y_m).$$

Now, assume $\mathbf{x}'$ is a regular sequence. In this case M'_+ is a free resolution of $\mathcal{H}(M') = \mathcal{S}/(\mathbf{x}')$. By Theorem 1.2, the natural map $\pi\colon M'_+ \to \mathcal{H}(M')$ is an near-isomorphism, and thus, $\pi \otimes 1\colon M \to \mathcal{H}(M') \otimes_{\mathcal{S}} M''$ is as well.

This is a very useful principle in Khovanov–Rozansky homology. For instance, it gives a new proof of the equivalence of Rasmussen's definition of KR homology with the original definition: simply apply the above construction the subsequence $(t_{e,\alpha(e)} - t_{e,\omega(e)})_{E \in \mathcal{G}(T)}$, which appears in the sequence for each term of the Khovanov–Rozansky complex.

We will concentrate on the dual approach of simplifying the matrix factorizations corresponding to crossings. This explains why we want a different notion of equivalence for matrix factorizations from Rasmussen's: his approach was adapted to keeping projective matrix factorizations on crossings, and tensoring them with nonprojective modules on edges, whereas ours is adapted to having nonprojective complexes on crossings, and projective matrix factorizations on edges.

First of all, we note that simplifications are preserved by the canopolis action.

Proposition 2.2 *If f is a near-isomorphism, then $\widetilde{\eta}(f)$ is also a near-isomorphism. Thus if f is a simplification, so is $\widetilde{\eta}(f)$.*

Proof For the first statement, note that $\widetilde{\mathcal{Z}}_\eta$ is a matrix factorization on a projective module, and thus tensor product with it preserves near-isomorphisms. For the second, we need only recall that tensor product is right exact. □

Thus, if we would like to calculate $\mathcal{K}_p(T)$ for some link diagram T, but do not know how to simplify $\mathcal{R}_p(T)$, then we might hope to factor T as $T = \widetilde{\eta}(T'_1, \dots, T'_m)$, where the T'_j are simpler tangles for which we *can* simplify $\mathcal{R}_p(T)$, and then apply the action of our canopolis.

For example, we have a simplification of $\mathcal{R}_p(T_\pm)$, where $T_\pm$ is a disk with a single positive or negative crossing. In this case, $\mathcal{R}_p(T_\pm)$ is simply the two term complex L_x corresponding to the single vertex in the graph of its projection.

Proposition 2.3 *The map $L_x \to \mathcal{H}(L_x)$ is a degree-wise near-isomorphism.*

Proof We noted in Section 1.3 that L'_x and L''_x are both matrix factorizations on free resolutions, and the map $L_x \to \mathcal{H}(L_x)$ is just the map to $\mathcal{H}$ applied degree-wise (in the homological grading). Thus, by Theorem 1.2, it is a degree-wise near-isomorphism. $\square$

If we factor a tangle into disks with a single crossing and a planar arc diagram η, as shown in Figure 2(a), then $\mathcal{R}_p(T) \cong \tilde{\eta}(\{L_x\})$ where x ranges over crossings, ordered according to the order chosen on the holes of η.

Let $\widehat{\mathcal{R}}(T)$ be the complex $\tilde{\eta}(\{\mathcal{H}(L_x)\}) \cong \left(\bigotimes_x \mathcal{H}(L_x)\right) \otimes \left(\bigotimes_e L_e\right)$. Combining Proposition 2.2 and Proposition 2.3, we obtain the following:

Proposition 2.4 *The natural map $\mathcal{R}_p(T) \to \widehat{\mathcal{R}}_p(T)$ is a degree-wise near-isomorphism.*

Thus, we have a new complex of matrix factorizations which is the tensor product of a complex of modules and a single regular Koszul matrix factorization, and whose homology is $\mathcal{K}_p(T)$.

Of course, we may hope that we can simplify more general tangles than single crossings using this philosophy. For each tangle, we have a map $\pi_T\colon \mathcal{R}_p(T) \to \mathcal{N}(T)$ from the honest KR complex to the naive one. In Khovanov–Rozansky homology, as in life, things would be easier if we could just be naive, but if we aren't savvy often enough, we can run into trouble. While Rasmussen's results show we can be "savvy" about crossings and "naive" about edges, and Proposition 2.4 shows we can be "naive" about crossings and "savvy" about edges, we will lose too much information if try to take the naive Khovanov–Rozansky of an entire knot. After all, the naive Khovanov–Rozansky homology does not depend on the polynomial p, but the honest Khovanov–Rozansky homology does, since different choices of p categorify $\mathfrak{sl}_n$ invariants for all n.

Thus, we would like to find a class of tangles about which we can be naive, while still recovering honest KR homology.

2.3 Acyclic tangles

Definition 4 We call an oriented tangle diagram *acyclic* if the graph $\mathcal{G}(T)$ has no oriented cycles.

Obviously a single crossing is acyclic. Also, the tangles inside of the dashed circles in Figure 2(b) are acyclic, whereas the entire tangle is not. Note that a tangle with a closed component is never acyclic, whereas a braid always is.

Theorem 2.5 *If T is acyclic, then $\pi_T\colon \mathcal{R}_p(T) \to \mathcal{N}(T)$ is an near-isomorphism.*

Proof Since it is irrelevant to question at hand, we forget about the chain complex structure on $\mathcal{R}_p(T)$ and consider it simply as a matrix factorization. By Theorem 1.2, we need only show that $H_i(\mathcal{R}_p(T)_+) = 0$, for any $i \neq 0$.

We induct on the number of crossings. Since T is acyclic, the direction of edges induces a partial ordering on vertices. Take any maximal element x. Using the conventions shown in Figure 3, $i(x)$ and $j(x)$ must be leaves or the adjacent vertex would be higher than x in our partial order. We will assume for simplicity that $k(x)$ and $\ell(x)$ are not leaves. The case where they are follows from the same arguments we present below.

Let T' be T with the crossing x removed. Thus, $\mathcal{S}_T \cong \mathcal{S}_{T'} \otimes k[x_i, x_j, x_k, x_\ell]$ where, as before, we let $t_h = t_{x,h(x)}$ where $h = i, j, k, \ell$, and we have:

$$\mathcal{R}_p(T)_+ \cong \mathcal{R}_p(T')_+ \otimes_k (L_x)_+ \otimes_k (L_{\ell(x)})_+ \otimes_k (L_{k(x)})_+$$

Since $\mathcal{R}_p(T')_+ \otimes_k (L_x)_+$ is projective as a $\mathcal{S}_T$ module, we can project $(L_{\ell(x)})_+$ and $(L_{k(x)})_+$ to their cohomology without changing the cohomology of $\mathcal{R}_p(T)$. This simplification is isomorphic to $\mathcal{R}_p(T')_+ \otimes_{k[t_\ell,t_k]} (L_x)_+$ where we let $k[t_\ell, t_k]$ act on $\mathcal{R}_p(T)_+$ by the variable $t_{\ell'}, t_{k'}$ corresponding to the other end of the edges $\ell(x), k(x)$.

Since both $\mathcal{R}_p(T')_+$ and $(L_x)_+$ are projective resolutions, the cohomology of this complex is $\mathrm{Tor}^i_{k[x_\ell,t_k]}(\mathcal{N}(T), \mathcal{H}(L_x))$.

Both $\mathcal{H}(L'_x)$ and $\mathcal{H}(L''_x)$ are free as $k[x_\ell, t_k]$–modules, as was proved by Soergel [13]. Thus, all higher Tor's vanish, and we are done. □

Corollary 2.6 *Let T be a tangle diagram which can be factored as the action of η on a set of acyclic tangles $\{T_i\}$, then the natural map $\mathcal{R}_p(T) \cong \tilde{\eta}(\{\mathcal{R}_p(T_i)\}) \to \tilde{\eta}(\{\mathcal{N}(T_i)\})$ is a near-isomorphism. If T is a link diagram, then $\mathcal{K}_p(T) \cong H^*(\tilde{\eta}(\{\mathcal{N}(T_i)\}))$.*

While this may not look like an impressive simplification, it does have a significant advantage: as a complex of modules, it is much easier to identify trivial summands of the naive complex $\mathcal{N}(T_i)$ in a way that was not at all clear in the matrix factorization picture.

Remark 1 For instance, this allows us to replace Khovanov and Rozansky's exhaustive computations for invariance under Reidemeister moves with simple computations in Soergel bimodules (done by Rouquier [12]) for all Reidemeister moves except type I and type IIb. This is because we can go between any two projections by

- applying Vogel's algorithm which uses only moves of type IIb and passing strands through infinity on the 2–sphere (which doesn't change Khovanov–Rozansky, since it doesn't change the topology of the knot projection) to take both projections to braid-like ones;
- applying type I moves and identities in the braid group (covered by Rouquier) to move from one braid projection to the other, which is possible by Markov's theorem.

2.4 Braids

Braids have an important role to play here, especially when we wish to consider HOMFLYPT homology, as we will in Section 3. As we noted, one of the best examples of a complicated acyclic tangle is a braid. Thus, Proposition 2.2 and Theorem 2.5 show that if our diagram L is the closure of a d–strand braid σ (all Seifert circles are nested), then $\mathcal{R}_p(L)$ is near-isomorphic to $\widetilde{\gamma}_d(\mathcal{N}(\sigma))$, where γ_d is the planar arc diagram shown in Figure 4.

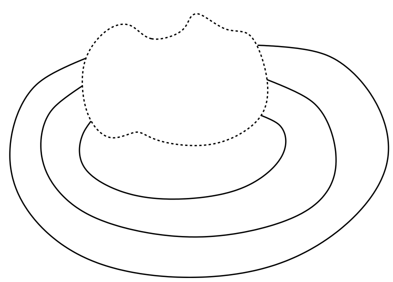

Figure 4: The braid closure arc diagram for 3 strands, γ_3

In this case, the naive KR complex $\mathcal{N}(\sigma)$ can be considered a complex of bimodules over a polynomial ring $\mathcal{S} = k[x_1, \dots, x_d]$ with generators x_i indexed by strands of our braid, and composition of braids (which can also be written as the action of a planar arc diagram) passes to tensor product of complexes, so

$$\mathcal{N}(\sigma\sigma') \cong \mathcal{N}(\sigma) \otimes_{\mathcal{S}} \mathcal{N}(\sigma').$$

Thus, we can consider $\mathcal{N}$ as a categorification of the braid group. In fact, this is precisely the categorification of the braid group described by Rouquier [12]. The bimodules which appear in this complex are so-called *Soergel bimodules*, which appeared in Soergel's research on category $\mathcal{O}$. Furthermore, $\widetilde{\gamma}_d$ is simply the Koszul matrix factorization $\widetilde{\mathcal{Z}}_p$ of the sequences

$$\{x_i \otimes 1 - 1 \otimes x_i\} \quad \text{and} \quad \left\{ \frac{p(x_i \otimes 1) - p(1 \otimes x_i)}{x_i \otimes 1 - 1 \otimes x_i} \right\}.$$

This is an $\mathfrak{sl}_n$ version of the result of Khovanov relating the Rouquier complex and Khovanov homology:

Theorem 2.7 *The complex* $H^t\left(\mathcal{N}(\sigma) \otimes \widetilde{\mathcal{Z}}_p\right)$ *is isomorphic to* $\mathcal{K}_p(\overline{\sigma})$.

The functor $H^t(- \otimes \widetilde{\mathcal{Z}}_p)$ is defined for any bimodule over $k[x_1, \cdots x_d]$, and it would very interesting to interpret it in terms of more familiar homological algebra. As is, there is a spectral sequence

$$HH^*(-) \Rightarrow H^t(- \otimes \widetilde{\mathcal{Z}}_p),$$

where $HH^*(-) = \mathrm{Tor}^*_{\mathcal{S}\otimes\mathcal{S}^{\mathrm{op}}}(-, \mathcal{S})$ denotes Hochschild homology, since

$$HH^i(-) \cong H^i\big(- \otimes (\widetilde{\mathcal{Z}}_p)_+\big).$$

Since there are, in all, $d!$ different indecomposable Soergel bimodules, the complex $\mathcal{N}(\sigma)$ typically has a very large number of redundant summands. In fact, the complex $\mathcal{N}(\sigma_i)$ for a braid generator splits after tensor product with exactly half of these modules, which alone leads to huge number of trivial summands in $\mathcal{N}(\sigma)$ for any large braid. This is discussed by Khovanov in the last section of [8], and will be covered in more detail in future work by the author.

It would be even better if we could implement these cancellations for more general acyclic tangle diagrams, since typically, a braid representative of a given knot has many more crossings than the smallest planar diagram of the same knot, which slows down computation if we have to use braid diagrams.

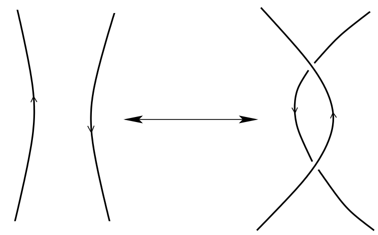

Figure 5: The Reidemeister IIb move

2.5 The IIb move

Another application which illustrates the power of this approach is the IIb Reidemeister move, which creates trouble in HOMFLY homology.

We only need to understand the local picture involving the tangles T and T' as shown in Figure 5. The tangle T is acyclic ,and thus quite easy to understand, but T' is not. Thus, we will cut open the oriented cycle, consider the naive complex of the resulting tangle T'', and then act with a planar diagram η to get $\mathcal{R}_p(T)$, as shown in Figure 6. Let $R' = k[a,b,c,d]$.

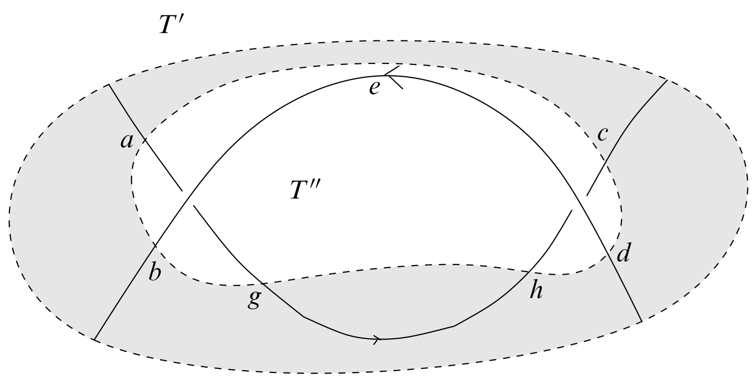

Figure 6: A decomposition of the tangle T'

Proposition 2.8 *The complex $\mathcal{R}_p(T')$ has a simplification which is a two-term complex of modules $N_1 \xrightarrow{\pi} N_2$ where*

$$N_1 = R'\{-2\}/(a-b+c-d, (b-d)(c-d))$$
$$N_2 = R'\{-2\}/(a-b, c-d)$$

and π is the natural projection.

Note that the kernel of this surjection is isomorphic to $R'/(b-d, a-c)$, which is, in turn near-isomorphic to $\mathcal{R}_p(T)$. Thus, if this surjection were to split, we would have shown the invariance of KR homology under the second Reidemeister move. However, it does not; N_1 is indecomposable as a graded module over R', since it is generated by a single element in minimal grade. However, as we shall see, this calculation can still give us interesting information about this Reidemeister move.

Proof Using the labels on edges above, and removing the variables g and h using the identity $a+e-g-b = c+e-h-d = 0$, we get that the naive complex of T'' is the 3 term complex

$$\begin{tikzcd} & M_2 \arrow[dr, "e-b"] & \\ M_1 \arrow[ur, "1"] \arrow[dr, "e-b"'] & \oplus & M_4 \\ & M_3 \arrow[ur, "1"'] & \end{tikzcd} \tag{1}$$

where $R = k[a,b,c,d,e]$ and

$$\begin{aligned} M_1 &= R/(a-b, (c-d)(e-d)) \\ M_2 &= R\{-2\}/(a-b, c-d) \\ M_3 &= R/((a-b)(e-b), (c-d)(e-d)) \\ M_4 &= R\{-2\}/((a-b)(e-b), c-d). \end{aligned}$$

Since the variable e corresponds to an edge that is closed, we will only be concerned with the structure of $\mathcal{N}(T'')$ as an R' module, but the purposes of calculations, it will be useful to remember the action of e.

Note that M_1 has a decomposition as an R'–module into $M_1' = R' \cdot 1$ and $M_1'' = R \cdot (e-d)$, and M_4 into $M_4' = R' \cdot 1$ and $M_4'' = R \cdot (e-b) = \operatorname{im} M_2$.

The module M_3 also has a decomposition along these lines, but a slightly more subtle one. We let $M_3' = R' \cdot 1 + R' \cdot e$, and $M_3'' = R \cdot (e-b)(e-d)$. Clearly, $M_3 = M_3' + M_3''$, since we write any expression with e^n for $n > 1$ appearing can be rewritten as the sum of an element of M_1'' and a expression with a lower degree in e. On the other hand, the intersection of these submodules is trivial, so they give a direct sum decomposition.

Thus, we can rewrite (1) as:

$$\begin{tikzcd} & M_3' \arrow[dr, "1"] \arrow[dddr, "\chi"] & \\ M_1' \arrow[ur, "e-b"] \arrow[dr, "1"] & \oplus & M_4' \\ \oplus & M_2 \arrow[dr, "1"] & \oplus \\ M_1'' \arrow[ur, "e-b"] \arrow[dr, "1"'] & \oplus & M_4'' \\ & M_3'' \arrow[ur, "e-b"'] & \end{tikzcd} \tag{2}$$

Each module in (2) is generated by a single element over R or R', so most maps are induced by multiplication by a ring element, and thus these have been denoted by the corresponding element. The single exception is the map χ, which is the natural (nonsplit) projection $\chi\colon M_3' \to M_3'/(R'\cdot 1)$, composed with the natural isomorphism of the target with the submodule of M_4'' generated over R' by $e-b$.

Now, the maps above from M_1'' to M_3'' and from M_2 to M_4'' are isomorphisms. Let ξ be the homotopy on this complex given by the inverse of these maps, killing all other components.

One can easily calculate that $\xi\partial+\partial\xi$ is the identity on M_1'', M_2, M_3'' and M_4'', and 0 on M_1', M_3' and M_4'. Thus, the complex $\mathcal{N}(T'')$ is homotopic to (2) with the lower diamond removed:

$$M_1' \xrightarrow{e-b} M_3' \xrightarrow{1} M_4' \tag{3}$$

Let $s=a-b-c+d$. Then acting with the planar diagram η to connect the ends corresponds on the matrix factorization side to tensor product with a two-term matrix factorization, whose positive complex is just $R' \xrightarrow{s} R'$.

Thus, we are interested in the vertical homology of the following complex:

$$\begin{tikzcd} M_1' \arrow[r] & M_3' \arrow[r] & M_4' \\ M_1' \arrow[r] \arrow[u, "s"] & M_3' \arrow[r] \arrow[u, "s"] & M_4' \arrow[u, "s"] \end{tikzcd} \tag{4}$$

Since $M_1' \cong R'/(a-b)$ and $M_4' \cong R'/(c-d)$, the element s is manifestly not a zero-divisor on either of these modules.

For M_3', we need only note that M_3' has a basis of the form $B_1 \cup B_2$ where

$$B_1=\{a^\alpha b^\beta c^\gamma d^\delta\} \quad\text{and}\quad B_2=\{b^\beta c^\gamma\cdot(e-b)\}.$$

Note that $(c-d)(e-b)=(c-d)(b-d)$. Thus, $s\cdot B_2$ is a linearly independent subset of $\operatorname{span}(B_1)$ with no a's appearing, whereas any k–linear combination of $s\cdot B_1$ has leading term containing an a. Thus, $s\cdot(B_1\cup B_2)$ is linearly independent, and s is not a zero-divisor, so (4) has no vertical homology.

Applying Theorem 1.2 again, we obtain that $\mathcal{R}_p(T')$ is near-isomorphic to

$$\widetilde{M}_1 \xrightarrow{e-b} \widetilde{M}_3 \xrightarrow{1} \widetilde{M}_4$$

where $\widetilde{M}_i = M_i/sM_i$. Unlike the situation before we quotiented by the action of s, the image of $\widetilde{M}_1$ in $\widetilde{M}_2$ is now complementary to $R'\cdot 1 \subset \widetilde{M}_3$. Thus, we can do another reducing homotopy, and see that $\mathcal{R}_p(T')$ is, in fact, near-isomorphic to the two term complex $N_1 = \widetilde{M}_3/\widetilde{M}_1 \to N_2 = \widetilde{M}_4$ where the map is the obvious surjection. $\square$

3 HOMFLYPT homology

Thus far, we have not discussed the triply graded theory of Khovanov–Rozansky defined in [10; 8] which categorifies the HOMFLYPT polynomial. Unfortunately, this theory lacks many of the good properties of the finite Khovanov–Rozansky theories. Most notably, it is unclear at the moment how it changes when the IIb move is applied to the diagram T, and it is not known whether or how it is functorial with respect to embedded cobordisms.

Typically, because of the issues surrounding the IIb move, HOMFLYPT homology is defined only using a braid representation of the knot. We would rather take the perspective that it is an invariant of tangle diagrams with good properties under Reidemeister moves. We will briefly discuss how our computational schema extends to HOMFLYPT homology, and use this to obtain some information about how this homology theory reacts to the IIb move.

Having already built up our machinery around $\mathcal{R}_p$, defining HOMFLYPT homology is simple: we consider $\mathcal{R}_0$, that is, the bicomplex given by considering the positive complex of each matrix factorization in $\mathcal{R}_p$. As before, we take homology first in the "matrix factorization" direction, and then take homology of the resulting chain complexes, and apply a grading shift of $-w+b$ in the polynomial grading, $w+b-1$ in the "matrix factorization" grading and $w-b+1$ in the "cohomological" grading.

We will draw our complexes with the "matrix factorization" direction being horizontal and the "cohomological" direction being vertical, and henceforth use these terms to describe them. We denote the i–th "horizontal" homology (remember this is itself a single chain complex) $H_h^i(C)$ and denote "horizontal then vertical" homology by $H_{hv}^{i,j}(C) = H_v^j(H_h^i(C))$ for any bicomplex C.

In line with our previous notation we denote $H_h^i(\mathcal{R}_0(T))$ by $\mathcal{K}_0^i(T)$. The direct sum of these over i is the (now bi-graded) complex $\mathcal{K}_0(T)$

Since each group $H_{hv}^{i,j}(\mathcal{R}_0(T))$ is still a graded module over $\mathcal{S}_T$, we have three gradings, one inherited from the polynomial ring $\mathcal{S}_T$, and two cohomological gradings.

Most of our previous theorems remain true, and in fact, are much easier to prove, since we no longer need to consider matrix factorizations. We have lost invariance under Reidemeister moves, but if we consider $\mathcal{R}_0$ only as an invariant of tangle diagrams, essentially everything works as before.

Theorem 3.1 *For a sequence of tangles $\{T_i\}$ and compatible oriented planar arc diagram η, we have*
$$\mathcal{R}_0(\widetilde{\eta}(\{T_i\})) \cong \widetilde{\eta}(\{\mathcal{R}_0(T_i)\}).$$
If $\mathcal{R}_0(T_i) \to \mathcal{N}(T_i)$ is a quasi-isomorphism, term-wise, then $\mathcal{R}_0(\widetilde{\eta}(\{T_i\})) \to \widetilde{\eta}(\mathcal{N}(T_i))$ is also a quasi-isomorphism.

In fact, since complexes are easier to deal with than matrix factorizations, in this case, our invariants can be understood in terms of standard homological algebra.

For instance, if we factor a closed diagram T into a planar arc diagram η acting on a set $\{T_i\}$ of acyclic tangles (as shown in Figure 2), then we can partition the set X of endpoints of the tangles T_i into pairs α_+, α_- where α ranges over $\mathcal{A}(\eta)$, the set of arcs of η. Let $\mathcal{S} = \bigotimes_i \mathcal{S}_{T_i} = k[X]$ be the ring over which the bicomplex $\bigotimes_i \mathcal{N}(T_i)$ is defined, and $\mathcal{U} = k[\mathcal{A}(\eta)]$. In this case, $\mathcal{S}$ can be written as a tensor product $\mathcal{U} \otimes \mathcal{U}^{\mathrm{op}}$ with the left action of t_e being t_{e_+} and its right action being t_{e_-}. Thus, we can also consider $\bigotimes_i \mathcal{N}(T_i)$ as a complex of bimodules over $\mathcal{U}$.

Proposition 3.2 *The horizontal homology complex $\mathcal{K}_0^i(T)$ is naturally isomorphic to the Hochschild homology $HH_i^{\mathcal{U}}\left(\bigotimes_i \mathcal{N}(T_i)\right)$ where $HH_i^{\mathcal{U}}(-)$ is applied term-wise.*

Proof By Theorem 3.1, $\mathcal{K}_0^i(T) \cong H_h^i(\widetilde{\eta}(\{\mathcal{N}(T_i)\}))$. The bicomplex $\widetilde{\eta}(\{\mathcal{N}(T_i)\})$ is simply the tensor product of $\otimes_i \mathcal{N}(T_i)$ (thought of as a vertical bicomplex) with the horizontal bicomplex which is the Koszul complex of $(t_e \otimes 1 - 1 \otimes t_e)$ over $\mathcal{S}$. This complex is a free resolution of $\mathcal{U}$ as a module over $\mathcal{S}$ (ie as a bimodules over itself), so the homology of the tensor product of this complex with a bimodule over $\mathcal{U}$ is precisely the Hochschild homology of that bimodule. □

Note that if T_1 is a braid, and T its closure, this theorem reduces precisely to [8, Theorem 1].

As this result suggests, we can use a stronger notion of equivalence in this HOMFLY case. Recall that $D^b(\mathcal{S}_T)$, the derived category of $\mathcal{S}_T$–modules, is the category of complexes of $\mathcal{S}_T$–modules, with a formal inverse to each quasi-isomorphism added. Since this category is additive, the homotopy category $\mathcal{K}(\mathcal{S}_T)$ of complexes in $D^b(\mathcal{S}_T)$ is well defined.

Of course, the operation of $H_h^*(-)$ is still well defined over $\mathcal{K}(\mathcal{S}_T)$, and results in a series of chain complexes of $\mathcal{S}_T$–modules. Furthermore, any nullhomotopic map in $\mathcal{K}(\mathcal{S}_T)$ induces nullhomotopic maps on these complexes. Thus the functor $H_{hv}^{*,*}(-)\colon \mathcal{K}(\mathcal{S}_T) \to \mathcal{S}_T\text{–}\mathsf{mod}$ is well-defined. Furthermore, using Proposition 1.1, we see that tensor product with a bicomplex of $\mathcal{S}_T \otimes \mathcal{S}_{T'}$–modules which is projective as a $\mathcal{S}_T$–module D defines a functor $D \otimes -\colon \mathcal{K}(\mathcal{S}_T) \to \mathcal{K}(\mathcal{S}'_T)$.

Thus, we have the following proposition.

Proposition 3.3 *The bigraded complex $\mathcal{K}_0(T)$ only depends (up to homotopy) on the class of $\mathcal{R}_0(T)$ in $\mathcal{K}(\mathcal{S}_T)$. Furthermore, the canopolis $\mathcal{M}_0$ defined in Section 2.1 descends to a canopolis structure $\mathcal{M}'_0$ on the categories $\mathcal{K}(\mathcal{S}_T)$.*

To show the power of this approach, let us give a proof of the invariance of HOMFLYPT using it. Recall the famous theorem of Markov:

Theorem 3.4 (Markov [3]) *Two closed braid projections represent the same knot if and only if they are related by isotopy, identities in the braid group, and type I Reidemeister moves.*

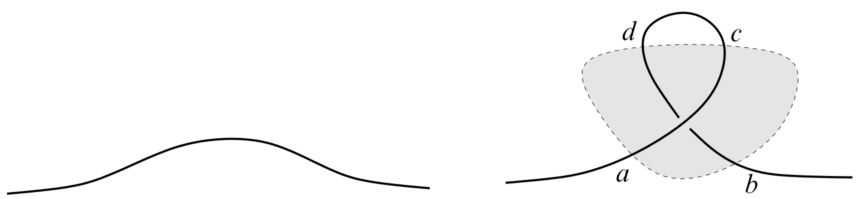

Figure 7: The Reidemeister I move

Proof of invariance of HOMFLYPT homology Isotopy does not affect the structure of the KR complex, and relations in the braid group have been dealt with by Rouquier [12]. Thus, we need only consider type I moves. In both cases, we will only obtain invariance with respect to a grading shift. This is accounted for in the global grading shift, which also depends on the diagram.

The left side of a type I move is simply the bicomplex which is $k[a,b]/(a-b)$ in cohomological grading $(0,0)$.

The right side is the tensor product of two 2–term bicomplexes, one being the naive complex of the crossing, and the other corresponding to the diagram closing one end of the crossing. Let

$$\begin{aligned} \mathcal{S} &= k[a,b,c,d] \\ M_0 &= \mathcal{S}/(a-b, c-d) \\ M_1 &= \mathcal{S}/\left(a+c-b-d, (a-b)(a-c)\right). \end{aligned}$$

Note that multiplication by $a-c$ defines a map $\rho\colon M_0\{2\}\to M_1$, which is injective. Let M_1'' denote the image of ρ. As a $k[a,b]$–submodule, we have a decomposition of the form $M_1''=\oplus_{i=0}^{\infty}k[a,b]\cdot c^i(a-c)$, and this image has a complement $M_1'=k[a,b]\cdot 1$ ie $M_1\cong M_1'\oplus M_1''$ (of course, M_1' is not a $k[a,b,c]$–submodule).

For the negative move, the bicomplex $\mathcal{R}_0(T_+)$ is:

$$\begin{tikzcd}
M_0 \arrow[d, "\rho"] & M_0 \arrow[d, "\rho"'] \\
M_1''\{-2\} \oplus M_1'\{-2\} \arrow[r, "a-b"'] & M_1''\{-2\} \oplus M_1'\{-2\}
\end{tikzcd}$$

Since the vertical map induces an isomorphism from M_0 to M_1', after removing the nullhomotopic summand, $\mathcal{R}_0(T_+)$ is just the horizontal complex $M_1'\xrightarrow{a-b}M_1'$, which is, in turn, quasi-isomorphic to $k[a,b]/(a-b)\{-2\}$ with a vertical shift of 1.

For the positive move, we must be a bit more subtle. We start with the following complex:

$$\begin{tikzcd}
M_1' \oplus M_1'' \arrow[r, "a-b"] \arrow[d, "1", "a-c"'] & M_1' \oplus M_1'' \arrow[d, "1", "a-c"'] \\
M_0 & M_0
\end{tikzcd}$$

Note that by the decomposition mentioned earlier $M_0\cong M_0\{2\}\oplus k[a,b]/(a-b)$. In the derived category, we can replace $k[a,b]/(a-b)$ by the complex $M_1'\xrightarrow{a-b}M_1'$. Thus, we can write another representative of this complex in the derived category:

$$\begin{tikzcd}
M_1' \oplus M_1'' \arrow[r, "a-b"] \arrow[d, "1"] \arrow[dd, bend right, "1", "a-c"'] & M_1' \oplus M_1'' \arrow[d, "1"] \arrow[dd, bend left, "1"] \\
M_1' \arrow[r, "a-b"'] & M_1' \\
\oplus\ M_0 & \oplus\ M_0\{2\}
\end{tikzcd}$$

The the top row of M_1''s and its image in the bottom row form a nullhomotopic subcomplex, as does the far right column. Canceling these off, we obtain a split injection (by the same decomposition we used before), with cokernel $k[a,b]/(a-b)$. Thus, $\mathcal{R}_0(T_-)$ is equivalent to a single copy of $k[a,b]/(a-b)$ but now with a horizontal shift of -1. $\square$

3.1 IIb again

Let us return to the IIb move. As we mentioned earlier, the behavior of HOMFLYPT homology under this Reidemeister move is not well understood. Using the results of the previous sections, we will make some headway toward understanding HOMFLYPT homology for general diagrams.

As we showed in Section 2.5, the homology $\mathcal{R}_0(T')$ is near-isomorphic (or more precisely, derived homotopic) to a two term complex $N_1 \xrightarrow{\pi} N_2$, which is quasi-isomorphic but not homotopic to $\mathcal{R}_0(T)$.

Thus, if K is any link diagram, with T a subdiagram which is isotopic to that on the left side of the IIb move, and K' the diagram which results after a IIb move, we can construct representatives in $\mathcal{K}(\mathcal{S}_K)$ of $\mathcal{R}_0(K)$ and $\mathcal{R}_0(K')$ such that there is an injective map $\iota\colon \mathcal{R}_0(K) \to \mathcal{R}_0(K')$, with the cokernel of ι given by the tensor product $\mathcal{R}_0(K \backslash T) \otimes \big(N_2 \xrightarrow{\mathrm{id}} N_2\big)$.

Of course, we are interested in understanding $\ker\big(H^{i,j}_{hv}(\iota)\big)$ and $\operatorname{coker}\big(H^{i,j}_{hv}(\iota)\big)$.

Proposition 3.5 *There is a spectral sequence $E_n^{i,j}$ such that:*

$$E_2^{i,j} = \begin{cases} \ker\,(H^{i,j}_{hv}\,(\iota)) & i = 3k \\ \operatorname{coker}\,(H^{i,j}_{hv}\,(\iota)) & i = 3k+1 \Rightarrow E_\infty^{i,j} = 0 \\ 0 & i = 3k+2 \end{cases}$$

Furthermore, we have a complex

$$\cdots \longrightarrow H^{i,j}_{hv}(K) \xrightarrow{H^{i,j}_{hv}(\iota)} H^{i,j}_{hv}(K') \xrightarrow{\alpha_i} H^{i-1,j-1}_{hv}(K) \xrightarrow{H^{i-1,j-1}_{hv}(\iota)} H^{i-1,j-1}_{hv}(K') \longrightarrow \cdots$$

which exact if and only if the spectral sequence above collapses at the E_3–term (ie $d^i = 0$ for $i \geq 3$).

Thus, if this spectral sequence collapses early on, the IIb move does relatively little damage; part of the HOMFLY homology shifts by one in horizontal and vertical grading, and part of it does not. Unfortunately, as of the moment, we have not able to obtain any real control over the higher differentials.

In the author's view, the most optimistic hope is that the spectral sequence does collapse at E_3, and that $H^{i,j}_{hv}\,(\iota)$ is a isomorphism if the crossing strands of the IIb move lie on the different Seifert circles and 0 (and thus α_i is an isomorphism) if they lie on the same Seifert circle. This would imply that with the grading shifts described earlier, HOMFLY homology was independent of the diagram chosen.

Another weaker possibility is that the spectral sequence collapses, and there is some good description of the differentials less clean than the hope above.

Weaker still is the hope that $d^{3n} = 0$ for all $n \geq 1$. This would at least imply that the total rank does not change under the IIb move, and in fact that each piece of the homology shifts by a vector lying in one of two affine rays in $\mathbb{Z}^3$.

Proof By the usual yoga, since we have a short exact sequence of bi-complexes (though of a vertical complexes of horizontal chain complexes) we obtain a long exact sequence of vertical complexes:

$$\cdots \to H_h^i(\mathcal{R}_0(K)) \to H_h^i\left(\mathcal{R}_0(K')\right) \to H_h^i(\operatorname{coker}\iota) \to H_h^{i-1}(\mathcal{R}_0(K)) \to \cdots$$

Now, think of this long exact sequence itself as a bicomplex. Then we have a pair of spectral sequences converging from "vertical then horizontal" homology to total homology and from "horizontal then vertical" homology to total homology. Since this is an exact sequence of complexes, the horizontal homology, and thus total homology is trivial. On the other hand, taking vertical homology first, we obtain a spectral sequence converging to 0 with E^1 page:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & H_{hv}^{i,j}(\mathcal{R}_0(K)) \arrow[r, "H_{hv}^{i,j}(\iota)"] & H_{hv}^{i,j}(\mathcal{R}_0(K')) \arrow[r] & 0 \arrow[r] & H_{hv}^{i-1,j}\mathcal{R}_0(K) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_{hv}^{i,j-1}(\mathcal{R}_0(K)) \arrow[r, "H_{hv}^{i,j-1}(\iota)"] & H_{hv}^{i,j-1}(\mathcal{R}_0(K')) \arrow[r] & 0 \arrow[r] & H_{hv}^{i-1,j-1}(\mathcal{R}_0(K)) \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Taking homology, we see that the E^2 page of the same spectral sequence is:

$$\begin{tikzcd}
 & \vdots \arrow[drr] & \vdots \arrow[drr] & \vdots & \vdots & \\
\cdots \arrow[drr] & \ker H_{hv}^{i,j}(\iota) \arrow[drr] & \operatorname{coker} H_{hv}^{i,j}(\iota) \arrow[drr] & 0 \arrow[drr] & \ker H_{hv}^{i-1,j}(\iota) & \cdots \\
\cdots & \ker H_{hv}^{i,j-1}(\iota) \arrow[drr] & \operatorname{coker} H_{hv}^{i,j-1}(\iota) \arrow[drr] & 0 & \ker H_{hv}^{i-1,j-1}(\iota) & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

This is the desired spectral sequence. Furthermore, by composition with the projection to $\operatorname{coker} H_{hv}^{i,j}(\iota)$ and the inclusion of $\ker H_{hv}^{i-1,j-1}(\iota)$, d^2 defines a map $\alpha_i\colon H_{hv}^{i,j}(\mathcal{R}_0(K')) \to H_{hv}^{i-1,j-1}(\mathcal{R}_0(K))$. These maps define a complex by definition, and this complex is exact if and only if d^2 is an isomorphism. Since this spectral sequence converges to 0, it collapses at E^n if and only if d^{n-1} is an isomorphism. $\square$

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