Non-commutative multivariable Reidemester torsion and the Thurston norm

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Given a 3–manifold the second author defined functions $\delta_n\colon H^1(M;\mathbb{Z}) \to \mathbb{N}$, generalizing McMullen's Alexander norm, which give lower bounds on the Thurston norm. We reformulate these invariants in terms of Reidemeister torsion over a non-commutative multivariable Laurent polynomial ring. This allows us to show that these functions are semi-norms.

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1 Introduction

Let M be a 3–manifold. Throughout the paper we will assume that all 3–manifolds are compact, connected and orientable. Let $\phi \in H^1(M;\mathbb{Z})$. The *Thurston norm* of ϕ is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \phi\},$$

where for a surface S with connected components $S_1, \dots, S_k$ we write $\chi_-(S) = \sum_{i=1}^k \max\{0, -\chi(S_i)\}$. We refer to Thurston [18] for details.

Generalizing work of Cochran [1], the second author introduced in [7] a function

$$\delta_n\colon H^1(M;\mathbb{Z}) \to \mathbb{N}_0 \cup \{-\infty\}$$

for every $n \in \mathbb{N}$ and showed that δ_n gives a lower bound on the Thurston norm. These functions are invariants of the 3–manifold and generalize the Alexander norm defined by C McMullen in [11]. We point out that the definition we use in this paper differs slightly from the original definition when $n = 0$ and a few other special cases. We refer to Section 4.3 for details.

The relationship between the functions δ_n and the Thurston norm was further strengthened in Harvey [8] (cf also Cochran [1] and Friedl [4]) where it was shown that the

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δ_n give a never decreasing series of lower bounds on the Thurston norm, ie for any $\phi \in H^1(M;\mathbb{Z})$ we have

$$\delta_0(\phi) \le \delta_1(\phi) \le \delta_2(\phi) \le \cdots \le \|\phi\|_T.$$

Furthermore it was shown in Friedl–Kim [5] that under a mild assumption these inequalities are an equality modulo 2.

In his original paper [18], Thurston showed that $\|-\|_T$ is a semi-norm. It is therefore a natural question to ask whether the invariants δ_n are semi-norms as well. In [7] this was shown to be the case for $n=0$. The following theorem, which is a special case of the main theorem of this paper (cf Theorem 4.2), gives an affirmative answer to this question for $n \ge 1$.

Theorem 1.1 *Let M be a 3–manifold with empty or toroidal boundary. Assume that $\delta_n(\phi) \neq -\infty$ for some $\phi \in H^1(M;\mathbb{Z})$, then*

$$\delta_n\colon H^1(M;\mathbb{Z}) \to \mathbb{N}_0$$

is a semi-norm.

In particular, this allows us to show that the sequence $\{\delta_n\}$ is eventually constant. That is, there exists an $N \in \mathbb{N}$ such that $\delta_n = \delta_N$ for all $n \ge N$ (cf Proposition 4.4).

Before we address whether the δ_n are norms, we discuss a more algebraic problem. Recall that given a multivariable Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ over a commutative field $\mathbb{F}$ we can associate to any non–zero $f = \sum_{\alpha\in\mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{F}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ a semi-norm on $\hom(\mathbb{Z}^m,\mathbb{R})$ by

$$\|\phi\|_f := \sup\{\phi(\alpha)-\phi(\beta) \mid a_\alpha \neq 0, a_\beta \neq 0\}.$$

Thus, to any square matrix B over $\mathbb{F}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ with $\det(B) \neq 0$, we can associate a norm using $\det(B) \in \mathbb{F}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

Generalizing this idea to the non-commutative case, in Section 2.1 we introduce the notion of a *multivariable skew Laurent polynomial ring* $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ of rank m over a skew field $\mathbb{K}$. Given a square matrix B over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ we can study its Dieudonné determinant $\det(B)$ which is an element in the abelianization of the multiplicative group $\mathbb{K}(t_1,\dots,t_m)\setminus\{0\}$ where $\mathbb{K}(t_1,\dots,t_m)$ denotes the quotient field of $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. This determinant will in general not be represented by an element in $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. Our main technical result (Theorem 2.2) is that nonetheless there is a natural way to associate a norm to B that generalizes the commutative case.

Given a 3–manifold M and a 'compatible' representation

$$\pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$$

we will show in Section 3 that the corresponding Reidemeister torsion can be viewed as a matrix over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Moreover, we will show in Section 4.3 that for appropriate representations the norm that we can associate to this matrix that agrees with ρ_n, δ_n. This implies Theorem 1.1. We conclude this paper with examples of links for which we compute the Thurston norm using the results in this paper.

As a final remark we point out that the results in this paper completely generalize the results in [6]. Furthermore, the results can easily be extended to studying 2–complexes together with the Turaev norm which is modeled on the definition of the Thurston norm of a 3–manifold. We refer to Thurston [21] for details.

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2 The non-commutative Alexander norm

In this section we will introduce the notion of a multivariable skew Laurent polynomial ring and we will then show that matrices over such rings give rise to semi-norms.

2.1 Multivariable Laurent polynomials

Let $\mathcal{R}$ be a (non-commutative) domain and $\gamma\colon \mathcal{R} \to \mathcal{R}$ a ring homomorphism. We denote by $\mathcal{R}[s^{\pm 1}]$ the *one-variable skew Laurent polynomial ring over* $\mathcal{R}$. Specifically, the elements in $\mathcal{R}[s^{\pm 1}]$ are formal sums $\sum_{i=m}^{n} a_i s^i$ ($m \le n \in \mathbb{Z}$) with $a_i \in \mathcal{R}$. Addition is given by addition of the coefficients, and multiplication is defined using the rule $s^i a = \gamma^i(a) s^i$ for any $a \in \mathcal{R}$ (where $\gamma^i(a)$ stands for $(\gamma \circ \dots \circ \gamma)(a)$). We point out that any element $\sum_{i=m}^{n} a_i s^i \in \mathcal{R}[s^{\pm 1}]$ can also be written uniquely in the form $\sum_{i=m}^{n} s^i \tilde{a}_i$, indeed, $\tilde{a}_i = s^{-i} a_i s^i \in \mathcal{R}$.

In the following let $\mathbb{K}$ be a skew field. We then define a *multivariable skew Laurent polynomial ring of rank m over* $\mathbb{K}$ (in non-commuting variables) to be a ring R which is an algebra over $\mathbb{K}$ with unit (ie we can view $\mathbb{K}$ as a subring of R) together with a decomposition $R = \oplus_{\alpha \in \mathbb{Z}^m} V_\alpha$ such that the following hold:

(1) V_α is a one-dimensional $\mathbb{K}$–vector space,

(2) $V_\alpha \cdot V_\beta = V_{\alpha+\beta}$ and

(3) $V_{(0,\dots,0)} = \mathbb{K}$.

In particular R is $\mathbb{Z}^m$–graded. Note that these properties imply that any V_α is invariant under left and right multiplication by $\mathbb{K}$, that any element in $V_\alpha \setminus \{0\}$ is a unit, and that R is a (non-commutative) domain. The example that the reader should keep in mind is a commutative Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Indeed, let $t^\alpha := t_1^{\alpha_1} \cdot \dots \cdot t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$, then $V_\alpha = \mathbb{F}t^\alpha, \alpha \in \mathbb{Z}^m$ has the required properties.

Let R be a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$. To make our subsequent definitions and arguments easier to digest we will always pick $t^\alpha \in V_\alpha \setminus \{0\}$ for $\alpha \in \mathbb{Z}^m$. It is easy to see that we can in fact pick $t^\alpha, \alpha \in \mathbb{Z}^m$ such that $t^{n\alpha} = (t^\alpha)^n$ for all $\alpha \in \mathbb{Z}^m$ and $n \in \mathbb{Z}$. Note that this choice implies that $t^{(0,\dots,0)} = 1$. Using the above choices, the set of t^α for $\alpha \in \mathbb{Z}$ satisfies the following properties:

(1) $t^\alpha t^{\tilde{\alpha}} t^{-(\alpha+\tilde{\alpha})} \in \mathbb{K}^\times$ for all $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$ and

(2) $t^\alpha \mathbb{K} = \mathbb{K} t^\alpha$ for all α.

This shows that the notion of multivariable skew Laurent polynomial ring of rank m is a generalization of the notion of twisted group ring of $\mathbb{Z}^m$ as defined in Passman [13, page 13]. If $m = 1$ then we have $t^{(n)} \in V_{(n)}$ such that $t^{(n)} = (t^{(1)})^n$ for any $n \in \mathbb{Z}$. We write $t^n = t^{(n)}$. In particular, when $m = 1$, R is a one-variable skew Laurent polynomial ring as above.

The argument of Dodziuk et al [3, Corollary 6.3] can be used to show that any such Laurent polynomial ring is a (left and right) Ore domain and in particular has a (skew) quotient field. We normally denote a multivariable skew Laurent polynomial ring of rank m over $\mathbb{K}$ suggestively by $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we denote the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ by $\mathbb{K}(t_1, \dots, t_m)$.

2.2 The Dieudonné determinant

In this section we recall several well-known definitions and facts about the Dieudonné determinant. Let $\mathcal{K}$ be a skew field; in our applications $\mathcal{K}$ will be the quotient field of a multivariable skew Laurent polynomial ring. First define $\mathrm{GL}(\mathcal{K}) := \varinjlim \mathrm{GL}(\mathcal{K}, n)$, where we have the maps $\mathrm{GL}(\mathcal{K}, n) \to \mathrm{GL}(\mathcal{K}, n+1)$ in the direct system, given by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then define $K_1(\mathcal{K}) = \mathrm{GL}(\mathcal{K})/[\mathrm{GL}(\mathcal{K}), \mathrm{GL}(\mathcal{K})]$. For details we refer the reader to Milnor [12] or Turaev [19].

Let A a square matrix over $\mathcal{K}$. After elementary row operations and destabilization we can arrange that in $K_1(\mathcal{K})$ the matrix A is represented by a 1×1–matrix (d). Then the Dieudonné determinant $\det(A) \in \mathcal{K}^{\times}_{ab} := \mathcal{K}^{\times}/[\mathcal{K}^{\times}, \mathcal{K}^{\times}]$ (where $\mathcal{K}^{\times} := \mathcal{K} \setminus \{0\}$) is defined to be d. It is well-known that the Dieudonné determinant induces an isomorphism $\det\colon K_1(\mathcal{K}) \to \mathcal{K}^{\times}_{ab}$. We refer to Rosenberg [14, Theorem 2.2.5 and Corollary 2.2.6] for more details.

2.3 Multivariable skew Laurent polynomial rings and semi-norms

In this section we show that matrices defined over a multivariable skew Laurent polynomial ring give rise to a semi-norm. We also relate this norm to degrees of one-variable polynomials.

Let $\mathbb{K}[s^{\pm 1}]$ be a one-variable skew Laurent polynomial ring and let $f \in \mathbb{K}[s^{\pm 1}]$. If $f = 0$ then we write $\deg(f) = -\infty$, otherwise, for $f = \sum_{i=m}^{n} a_i s^i \in \mathbb{K}[s^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f) := n - m$. This extends to a homomorphism $\deg\colon \mathbb{K}(t) \setminus \{0\} \to \mathbb{Z}$ via $\deg(fg^{-1}) = \deg(f) - \deg(g)$. Since deg is a homomorphism to an abelian group this induces a homomorphism $\deg\colon \mathbb{K}(t)^{\times}_{ab} \to \mathbb{Z}$. Note that throughout this paper we will apply the convention that $-\infty < a$ for any $a \in \mathbb{Z}$.

For the remainder of this section let $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m together with a choice of $t^{\alpha}, \alpha \in \mathbb{Z}^m$ as above. Let $f \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We can write $f = \sum_{\alpha \in \mathbb{Z}^m} a_{\alpha} t^{\alpha}$ for some $a_{\alpha} \in \mathbb{K}$. We associate a semi-norm $\|-\|_f$ on $\hom(\mathbb{R}^m, \mathbb{R})$ to f as follows. If $f = 0$, then we set $\|-\|_f := 0$. Otherwise we set

$$\|\phi\|_f := \sup\{\phi(\alpha) - \phi(\beta) \mid a_{\alpha} \neq 0, a_{\beta} \neq 0\}.$$

Clearly $\|-\|_f$ is a semi-norm and does not depend on the choice of t^{α}. This semi-norm should be viewed as a generalization of the degree function.

Now let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ and let $f_n, f_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^{\times}_{ab}$. Then define

$$\|\phi\|_{\tau} := \max\{0, \|\phi\|_{f_n} - \|\phi\|_{f_d}\}$$

for any $\phi \in \hom(\mathbb{R}^m, \mathbb{R})$. By the following proposition this function is well-defined.

Proposition 2.1 *Let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$. Let $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^{\times}_{ab}$. Then*

$$\|-\|_{f_n} - \|-\|_{f_d} = \|-\|_{g_n} - \|-\|_{g_d}.$$

We postpone the proof to Section 2.4.

Let B be a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. In general, it is not the case that $\det(B)$ can be represented by an element in $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. But we still have the following result which is the main technical result of this paper.

Theorem 2.2 *If $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, then $\|-\|_\tau$ defines a semi-norm on $\hom(\mathbb{R}^m, \mathbb{R})$.*

We postpone the proof to Section 2.5.

Now let $\phi\colon \mathbb{Z}^m \to \mathbb{Z}$ be a non-trivial homomorphism. We will show that $\|\phi\|_B$ can also be viewed as the degree of a polynomial associated to B and ϕ. We begin with some definitions. Consider

$$\mathbb{K}[\mathrm{Ker}(\phi)] := \bigoplus_{\alpha \in \mathrm{Ker}(\phi)} \mathbb{K}t^\alpha \subset \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

This clearly defines a subring of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and the argument of Dodziuk et al [3, Corollary 6.3] shows that $\mathbb{K}[\mathrm{Ker}(\phi)]$ is an Ore domain with skew field which we denote by $\mathbb{K}(\mathrm{Ker}(\phi))$.

Let $d \in \mathbb{Z}$ such that $\mathrm{Im}(\phi) = d\mathbb{Z}$ and pick $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{Z}^m$ such that $\phi(\beta) = d$. Let $\mu := t^\beta$. Then we can form one-variable Laurent polynomial rings $(\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm 1}]$ and $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm 1}]$ where $sk := \mu k \mu^{-1} s$ for all $k \in \mathbb{K}[\mathrm{Ker}(\phi)]$ respectively for all $k \in \mathbb{K}(\mathrm{Ker}(\phi))$. We get an isomorphism

$$\begin{array}{rcl} \gamma_\phi\colon \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] & \xrightarrow{\cong} & (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm 1}] \\ \sum_{\alpha \in \mathbb{Z}^m} k_\alpha t^\alpha & \mapsto & \sum_{\alpha \in \mathbb{Z}^m} k_\alpha t^\alpha \mu^{-\phi(\alpha)/d} s^{\phi(\alpha)/d}, \end{array}$$

where $k_\alpha \in \mathbb{K}$ for all $\alpha \in \mathbb{Z}^m$. Note that $k_\alpha t^\alpha \mu^{-\phi(\alpha)/d} \in \mathbb{K}[\mathrm{Ker}(\phi)]$. An easy computation shows that γ_ϕ is an isomorphism of rings. We also get an induced isomorphism $\mathbb{K}(t_1, \dots, t_m) \xrightarrow{\cong} (\mathbb{K}(\mathrm{Ker}(\phi)))(s)$.

Let B be a matrix over $\mathbb{K}(t_1, \dots, t_m)$. Define $\deg_\phi(B) := \deg(\det(\gamma_\phi(B)))$ where we view $\gamma(B)$ as a matrix over $\mathbb{K}(\mathrm{Ker}(\phi))(s)$.

Theorem 2.3 *Let B be a matrix over $\mathbb{K}(t_1, \dots, t_m)$. Let $\phi \in \hom(\mathbb{Z}^m, \mathbb{Z})$ be non-trivial and let $d \in \mathbb{N}$ such that $\mathrm{Im}(\phi) = d\mathbb{Z}$. Then*

$$\|\phi\|_B = d \max\{0, \deg_\phi(B)\}.$$

In particular, this shows that $\deg_\phi(B)$ is independent of the choice of β. The above theorem is a generalization of [7, Proposition 5.12] to the non-commutative case.

Proof Since γ and deg are homomorphisms it is clearly enough to show that for any $g \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ we have

$$\|\phi\|_g = d \deg(\gamma_\phi(g)).$$

Write $g = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha$ with $a_\alpha \in \mathbb{K}$. Let d, β, μ and $\gamma\colon \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \xrightarrow{\cong} (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm 1}]$ as above. Note that $\mathrm{Ker}(\phi) \oplus \mathbb{Z}\beta = \mathbb{Z}^m$, hence

$$\begin{aligned} g &= \textstyle\sum_{i \in \mathbb{Z}} \sum_{\alpha \in \mathrm{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta}, \\ \gamma_\phi(g) &= \textstyle\sum_{i \in \mathbb{Z}} \left(\sum_{\alpha \in \mathrm{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \right) s^i. \end{aligned}$$

Note that $a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \subset \mathbb{K}t^\alpha$. Since $\mathbb{K}[\mathrm{Ker}(\phi)] = \oplus_{\alpha \in \mathrm{Ker}(\phi)} \mathbb{K}t^\alpha$ we get the following equivalences:

$$\begin{array}{rrcl} & \sum_{\alpha \in \mathrm{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} &=& 0 \\ \Leftrightarrow & a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} &=& 0 \ \text{ for all } \alpha \in \mathrm{Ker}(\phi) \\ \Leftrightarrow & a_{\alpha+i\beta} &=& 0 \ \text{ for all } \alpha \in \mathrm{Ker}(\phi). \end{array}$$

Therefore

$$\begin{aligned} \|\phi\|_g &= \quad d \max\nolimits_{i \in \mathbb{Z}} \{\text{there exists } \alpha \in \mathrm{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0\} \\ &\quad - d \min\nolimits_{i \in \mathbb{Z}} \{\text{there exists } \alpha \in \mathrm{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0\} \\ &= \quad d \max\nolimits_{i \in \mathbb{Z}} \{\textstyle\sum_{\alpha \in \mathrm{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \neq 0\} \\ &\quad - d \min\nolimits_{i \in \mathbb{Z}} \{\textstyle\sum_{\alpha \in \mathrm{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \neq 0\} \\ &= \quad d \deg(\gamma_\phi(g)). \end{aligned}$$

Thus the theorem is proved. □

2.4 Proof of Proposition 2.1

We start out with the following three basic lemmas.

Lemma 2.4 *Let $f, g \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$, then $\|-\|_{fg} = \|-\|_f + \|-\|_g$.*

This lemma is well-known. It follows from the fact that the Newton polytope of non-commutative multivariable polynomials fg is the Minkowski sum of the Newton polytopes of f and g. We refer to Sturmfels [16, page 31] for details.

Lemma 2.5 *Let $d \in \mathbb{K}(t_1, \dots, t_m)$ and let $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $d = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$. Then*

$$\|-\|_{f_n} - \|-\|_{f_d} = \|-\|_{g_n} - \|-\|_{g_d}.$$

In particular

$$\|-\|_d := \|-\|_{f_n} - \|-\|_{f_d}$$

is well-defined.

Proof Recall that by the definition of the Ore localization $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$ is equivalent to the existence of $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $f_n u = g_n v$ and $f_d u = g_d v$. The lemma now follows immediately from Lemma 2.4. □

Lemma 2.6 *Let* $d, e \in \mathbb{K}(t_1, \dots, t_m)$*, then*

$$\|-\|_{de} = \|-\|_d + \|-\|_e.$$

Proof Pick $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $f_n f_d^{-1} = d$ and $g_n g_d^{-1} = e$. By the Ore property there exist $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $g_n u = f_d v$. It follows that

$$f_n f_d^{-1} g_n g_d^{-1} = f_n v u^{-1} g_d^{-1} = (f_n v)(g_d u)^{-1}.$$

The lemma now follows immediately from Lemma 2.4. □

We can now give the proof of Proposition 2.1.

Proof of Proposition 2.1 Let B be a matrix defining an element $K_1(\mathbb{K}(t_1, \dots, t_m))$. Assume that we have $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $\det(B) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times_{ab}$. We can lift the equality $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times_{ab}$ to an equality

$$f_n f_d^{-1} = \prod_{i=1}^{r} [a_i, b_i] g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times \tag{1}$$

for some $a_i, b_i \in \mathbb{K}(t_1, \dots, t_m)$. It follows from Lemma 2.6 that $\|-\|_{[a_i, b_i]} = 0$. It then follows from Lemma 2.6 that $\|-\|_{f_n f_d^{-1}} = \|-\|_{g_n g_d^{-1}}$. □

2.5 Proof of Theorem 2.2

Let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ that can be represented by a matrix B defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We will show that $\|-\|_\tau = \|-\|_B$ defines a semi-norm on $\hom(\mathbb{R}^m, \mathbb{R})$. Because of the continuity and the $\mathbb{N}$–linearity of $\|-\|_B$ it is enough to show that for any two non-trivial homomorphisms $\phi, \tilde{\phi} \colon \mathbb{Z}^m \to \mathbb{Z}$ we have

$$\|\phi + \tilde{\phi}\|_B \le \|\phi\|_B + \|\tilde{\phi}\|_B.$$

Let $\phi, \widetilde{\phi}\colon \mathbb{Z}^m \to \mathbb{Z}$ be non-trivial homomorphisms. Let $d \in \mathbb{Z}$ such that $\mathrm{Im}(\phi) = d\mathbb{Z}$ and pick β with $\phi(\beta) = d$. We write $\mu = t^\beta$. As in Section 2.3 we can form $\mathbb{K}[\mathrm{Ker}(\phi)]$ and we also have an isomorphism $\gamma_\phi\colon \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \xrightarrow{\cong} (\mathbb{K}[\mathrm{Ker}(\phi)])[s^{\pm 1}]$.

Consider $\gamma_\phi(B)$, it is defined over the PID $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm 1}]$. Therefore we can use elementary row operations to turn $\gamma_\phi(B)$ into a diagonal matrix with entries in $\mathbb{K}(\mathrm{Ker}(\phi))[s^{\pm 1}]$. In particular we can find $a_i, b_i \in \mathbb{K}[\mathrm{Ker}(\phi)]$ such that

$$\det(\gamma_\phi(B)) = \sum_{i=r_1}^{r_2} s^i a_i b_i^{-1}.$$

Since $\mathbb{K}[\mathrm{Ker}(\phi)]$ is an Ore domain we can in fact find a common denominator for $a_i b_i^{-1}, i = r_1, \dots, r_2$. More precisely, we can find $c_{r_1}, \dots, c_{r_2} \in \mathbb{K}[\mathrm{Ker}(\phi)]$ and $d \in \mathbb{K}[\mathrm{Ker}(\phi)]$ such that $a_i b_i^{-1} = c_i d^{-1}$ for $i = r_1, \dots, r_2$. Now let $c = \sum_{i=r_1}^{r_2} s^i c_i$. Then

$$\det(\gamma_\phi(B)) = cd^{-1} \in \mathbb{K}(\mathrm{Ker}(\phi))(s)^\times_{ab}$$

where $c \in \mathbb{K}[\mathrm{Ker}(\phi)][s^{\pm 1}]$ and $d \in \mathbb{K}[\mathrm{Ker}(\phi)]$. Now let $f = \gamma_\phi^{-1}(c) \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $g = \gamma_\phi^{-1}(d) \in \mathbb{K}[\mathrm{Ker}(\phi)]$. Then $\det(B) = fg^{-1}$ and by Proposition 2.1 we have

$$\|-\|_B = \|-\|_f - \|-\|_g.$$

We now observe that $\|\phi\|_g = 0$ and $\|\phi + \widetilde{\phi}\|_g = \|\widetilde{\phi}\|_g$ since $g \in \mathbb{K}[\mathrm{Ker}(\phi)]$. Therefore it follows that

$$\begin{aligned}
\|\phi + \widetilde{\phi}\|_B &= \|\phi + \widetilde{\phi}\|_f - \|\phi + \widetilde{\phi}\|_g \\
&= \|\phi + \widetilde{\phi}\|_f - \|\widetilde{\phi}\|_g \\
&\le \|\phi\|_f + \|\widetilde{\phi}\|_f - \|\widetilde{\phi}\|_g \\
&= (\|\phi\|_f - \|\phi\|_g) + (\|\widetilde{\phi}\|_f - \|\widetilde{\phi}\|_g) \\
&= \|\phi\|_B + \|\widetilde{\phi}\|_B.
\end{aligned}$$

This concludes the proof of Theorem 2.2.

3 Applications to the Thurston norm

In this section we will show that the Reidemeister torsion corresponding to 'compatible' representations over a multivariable skew Laurent polynomial ring give rise to semi-norms that give lower bounds on the Thurston norm.

3.1 Reidemeister torsion

Let X be a finite connected CW–complex. Denote the universal cover of X by $\widetilde{X}$. We view $C_*(\widetilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$–module via deck transformations. Let R be a ring and let $\varphi\colon \pi_1(X) \to \mathrm{GL}(R,d)$ be a representation. This equips R^d with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the right R–module chain complex $C_*^{\varphi}(X;R^d) := C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. We denote its homology by $H_i^{\varphi}(X;R^d)$. If $H_*^{\varphi}(X;R^d)=0$, then we define the Reidemeister torsion $\tau(X,\varphi) \in K_1(R)/\pm\varphi(\pi_1(X))$ otherwise we write $\tau(X,\varphi):=0$. If the homomorphism φ is clear we may also write $\tau(X,R^d)$.

Let M be a manifold. Since Reidemeister torsion only depends on the homeomorphism type of the space we can define $\tau(M,\varphi)$ by picking any CW–structure for M. We refer to the excellent book of Turaev [19] for the details.

3.2 Compatible homomorphisms and the higher order Alexander norm

In the following let M be a 3–manifold with empty or toroidal boundary. Let $\psi\colon H_1(M) \to \mathbb{Z}^m$ be an epimorphism. Let $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m as in Section 2.1.

A representation $\varphi\colon \pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}],d)$ is called *ψ–compatible* if for any $g \in \pi_1(X)$ we have $\varphi(g) = At^{\psi(g)}$ for some $A \in \mathrm{GL}(\mathbb{K},d)$. This generalizes definitions in Turaev [20] and Friedl [4]. We denote the induced representation $\pi_1(M) \to \mathrm{GL}(\mathbb{K}(t_1,\dots,t_m),d)$ by φ as well and consider the corresponding Reidemeister torsion $\tau(M,\varphi) \in K_1(\mathbb{K}(t_1,\dots,t_m))/\pm\varphi(\pi_1(M)) \cup \{0\}$.

We say φ is a *commutative representation* if there exists a commutative subfield $\mathbb{F}$ of $\mathbb{K}$ such that for all g we have $\varphi(g) = At^{\psi(g)}$ with A defined over $\mathbb{F}$ and if $t^{\alpha}, t^{\widetilde{\alpha}}$ commute for any $\alpha, \widetilde{\alpha} \in \mathbb{Z}^m$. The following result is our main application of the purely algebraic results of Section 2.

Theorem 3.1 *Let M be a 3–manifold with an empty or toroidal boundary, let $\psi\colon H_1(M) \to \mathbb{Z}^m$ be an epimorphism and let $\varphi\colon \pi_1(M) \to GL(\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}],d)$ be a ψ–compatible representation such that $\tau(M,\varphi) \neq 0$. If one of the following holds:*

(1) *φ is commutative or*

(2) *there exists $g \in \mathrm{Ker}\{\pi_1(M) \to \mathbb{Z}^m\}$ such that $\varphi(g) - id$ is invertible over $\mathbb{K}$,*

then $\|-\|_{\tau(M,\varphi)}$ is a semi-norm on $\hom(\mathbb{R}^m,\mathbb{R})$ and for any $\phi\colon \mathbb{R}^m \to \mathbb{R}$ we have

$$\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M,\varphi)}.$$

We point out that if $g \in \operatorname{Ker}\{\pi_1(M) \to \mathbb{Z}^m\}$, then $\varphi(g) - \mathrm{id}$ is defined over $\mathbb{K}$ since φ is ψ–compatible. We refer to $\|-\|_{\tau(M,\varphi)}$ as the *higher-order Alexander norm*.

In the case that $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ equals $\mathbb{Q}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, the usual commutative Laurent polynomial ring, we recover McMullen's Alexander norm $\|-\|_A$ (cf McMullen [11]). The general commutative case is the main result in Friedl–Kim [6]. The proof we give here is different in its nature from the proofs in [11] and [6].

Proof In the case that $m = 1$ it is clear that $\|-\|_{\tau(M,\varphi)}$ is a semi-norm. The fact that it gives a lower bound on the Thurston norm was shown in [1; 7; 20; 4]. We therefore assume now that $m > 1$.

We first show that $\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M,\varphi)}$ for any ϕ: $\mathbb{R}^m \to \mathbb{R}$. Since both sides are $\mathbb{N}$–linear and continuous we only have to show that $\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M,\varphi)}$ for all epimorphisms ϕ: $\mathbb{Z}^m \to \mathbb{Z}$. So from now on, we will assume that ϕ: $\mathbb{Z}^m \to \mathbb{Z}$ is an epimorphism.

Pick $\mu \in \mathbb{Z}^m$ with $\phi(\mu) = 1$ as in the definition of $\deg_\phi(\tau(M,\varphi))$. We can then form the rings $\mathbb{K}[\operatorname{Ker}(\phi)][s^{\pm 1}]$ and $\mathbb{K}(\operatorname{Ker}(\phi))(s)$. First note that by Theorem 2.3

$$\|\phi\|_{\tau(M,\varphi)} = \deg_\phi(\tau(M,\varphi))$$

since ϕ is surjective. The representation

$$\pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \to \mathrm{GL}(\mathbb{K}(\operatorname{Ker}(\phi))[s^{\pm 1}], d)$$

is ϕ–compatible since $\pi_1(M) \to \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ is ψ–compatible. It now follows from Friedl [4, Theorem 1.2] that $\|\phi \circ \psi\|_T \geq \deg(\tau(M, \mathbb{K}(\operatorname{Ker}(\phi))(s))) = \deg_\phi(\tau(M,\varphi))$ (cf also Turaev [20]).

In the remainder of the proof we will show that if $m > 1$ then the Reidemeister torsion $\tau(M,\varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m))/\pm\varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. It then follows from Theorem 2.2 that $\|-\|_{\tau(M,\varphi)}$ is a semi-norm.

Consider the case that φ is a commutative representation and let $\mathbb{F}$ be the commutative subfield $\mathbb{F}$ in the definition of a commutative representation. Denote by $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ the ordinary Laurent polynomial ring. Then we have ψ–compatible representations $\pi_1(M) \to \mathrm{GL}(\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \hookrightarrow \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$. By [19, Proposition 3.6] we have

$$\tau(M, \mathbb{F}(t_1, \dots, t_m)) = \tau(M, \mathbb{K}(t_1, \dots, t_m)) \in K_1(\mathbb{K}(t_1, \dots, t_m))/\pm\varphi(\pi_1(M)).$$

Since $m > 1$ it follows from [19, Theorem 4.7] combined with [6, Lemmas 6.2 and 6.5] that $\det(\tau(M, \mathbb{F}(t_1, \dots, t_m))) \in \mathbb{F}(t_1, \dots, t_m)$ equals the twisted multivariable

Alexander polynomial, in particular it is defined over $\mathbb{F}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. This concludes the proof in the commutative case.

It therefore remains to consider the case when there exists $g \in \operatorname{Ker}\{\pi_1(M)\to\mathbb{Z}^m\}$ such that $\varphi(g)-\mathrm{id}$ is invertible. We first consider the case that M is a closed 3–manifold. The proof will use the special CW–structure from the next claim.

Claim There exists a CW–structure for M with one 0–cell and one 3–cell and such the closure of a 1–cell and the cocore of a 2–cell represent g.[1]

In order to prove the claim pick a Heegaard decomposition $M = G_0 \cup H_0$. We can add a handle g_0 (respectively h_0) to G_0 (respectively H_0) in H_0 (respectively G_0) so that the core of g_0 (respectively h_0) represents g. Adding further handles $h_1,\dots,h_r$ (respectively $g_1,\dots,g_r$) in $M\setminus G_0$ (respectively $M\setminus H_0$) we can assume that complement $H_0\setminus\big(\bigcup_{i=0}^r g_i\big)$ (respectively $G_0\setminus\big(\bigcup_{i=0}^s h_i\big)$) is again a handlebody. It follows that $G := \big(G_0\cup\bigcup_{i=0}^r g_i\big)\setminus\big(\bigcup_{i=0}^s h_i\big)$ and $H := \big(H_0\cup\bigcup_{i=0}^s h_i\big)\setminus\big(\bigcup_{i=0}^r g_i\big)$ are handlebodies and hence $M = G\cup H$ is a handlebody decomposition of M.

Now give M the CW structure as follows: take one 0–cell, attach 1–cells along a choice of cores of G such that g is represented by the closure of a 1–cell. Attach 2–cells along cocores of H such that one cocore represents g. Finally attach one 3–cell. This CW–structure clearly has the required properties to complete the claim.

Denote the number of 1–cells by n. Consider the chain complex of the universal cover $\widetilde{M}$:

$$0\to C_3(\widetilde{M})^1 \xrightarrow{\partial_3} C_2(\widetilde{M})^n \xrightarrow{\partial_2} C_1(\widetilde{M})^n \xrightarrow{\partial_1} C_0(\widetilde{M})^1 \to 0,$$

where the superscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$. Picking appropriate lifts of the cells of M to cells of $\widetilde{M}$ and picking an appropriate order we get bases for the $\mathbb{Z}[\pi_1(M)]$–modules $C_i(\widetilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 and A_3 are of the form

$$\begin{aligned} A_3 &= (1-g, 1-b_1,\dots,1-b_{n-1})^t,\\ A_1 &= (1-g, 1-a_1,\dots,1-a_{n-1}), \end{aligned}$$

for some $a_i, b_i \in \pi_1(M), i=1,\dots,n-1$. By assumption $\mathrm{id}-\varphi(g)$ is invertible over $\mathbb{K}$. Denote by B_2 the result of deleting the first column and the first row of A_2. Let $\tau := (\mathrm{id}-\varphi(g))^{-1}\varphi(B_2)(\mathrm{id}-\varphi(g))^{-1}$. Note that τ is defined over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. Since we assume that $\tau(M,\varphi)\neq 0$ it follows that $\varphi(B_2)$ is invertible over $\mathbb{K}(t_1,\dots,t_m)$ and $\tau(M,\varphi)=\tau\in K_1(\mathbb{K}(t_1,\dots,t_m))/\pm\varphi(\pi_1(M))$ (we refer to [19, Theorem 2.2]

[1] By cocore, we mean the element in π_1 given by taking a point on the 2–cell and connecting the two push-offs through an arc in the 3–cell.

for details). Therefore $\tau(M,\varphi) \in K_1(\mathbb{K}(t_1,\dots,t_m))/\pm\varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. This concludes the proof in the case that M is a closed 3–manifold.

In the case that M is a 3–manifold with non-empty toroidal boundary we can find a (simple) homotopy equivalence to a 2–complex X with $\chi(X)=0$. We can assume that the CW–structure has one 0–cell, n 1–cells and $n-1$ 2–cells, furthermore we can assume that the closure of a 1–cell represents an element $g \in \operatorname{Ker}\{\psi\colon \pi_1(X) \to \mathbb{Z}^m\}$ such that $\mathrm{id}-\varphi(g)$ is invertible. We get a chain complex

$$0 \to C_2(\widetilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\widetilde{X})^n \xrightarrow{\partial_1} C_0(\widetilde{X})^1 \to 0.$$

Picking appropriate lifts of the cells of X to cells of $\widetilde{X}$ we get bases for the $\mathbb{Z}[\pi_1(X)]$–modules $C_i(\widetilde{X})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 is of the form

$$A_1 = (1-g, 1-a_1, \dots, 1-a_{n-1}),$$

for some $a_i \in \pi_1(M)$. Now denote by B_2 the result of deleting the first row of A_2. Then $\tau := \varphi(B_2)(\mathrm{id}-\varphi(g))^{-1}$ is again defined over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ and the proof continues as in the case of a closed 3–manifold. □

Remark It follows from [4] that if M is closed, or if M has toroidal boundary, then $\tau(M,\varphi) \neq 0$ is equivalent to $H_1(M;\mathbb{K}(t_1,\dots,t_m))=0$, or equivalently, that $H_1(M;\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}])$ has rank zero over $\mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

Remark The computation of polynomials

$$f_d \in \mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}] \text{ and } f_n \in \mathbb{K}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$$

such that $\det(\tau(M,\varphi)) = f_n f_d^{-1}$ is computationally equivalent to the computation of $\deg_\phi(\tau(M,\varphi))$ for some $\phi\colon H_1(M) \to \mathbb{Z}$. Put differently we get the perhaps surprising fact that computing the higher-order Alexander norm does not take longer than computing a single higher-order, one-variable Alexander polynomial.

4 Examples of ψ–compatible homomorphisms

Following [1] and [7] we will use the rational derived series to give examples of ψ–compatible homomorphisms. For a given a 3–manifold, we will show that these give rise to a never decreasing, eventually constant sequence of semi-norms all of which give lower bounds on the Thurston norm.

4.1 Skew fields of group rings

A group G is called *locally indicable* if for every finitely generated non-trivial subgroup $U \subset G$ there exists a non-trivial homomorphism $U \to \mathbb{Z}$. We recall the following well-known theorem.

Theorem 4.1 *Let G be a locally indicable and amenable group and let R be a subring of $\mathbb{C}$. Then $R[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.*

Higman [9] showed that $R[G]$ has no zero divisors. The theorem now follows from [17] or [3, Corollary 6.3].

We recall that a group G is called *poly-torsion-free-abelian* (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well-known that PTFA groups are amenable and locally indicable (cf [15]). The group rings of PTFA groups played an important role in Cochran–Orr–Teichner [2], Cochran [1] and Harvey [7].

4.2 Admissible pairs and multivariable skew Laurent polynomial rings

We slightly generalize a definition from Harvey [8].

Definition Let π be a group and let $\psi\colon \pi \to \mathbb{Z}^m$ be an epimorphism and let $\varphi\colon \pi \to G$ be an epimorphism to a locally indicable and amenable group G such that there exists a map $G \to \mathbb{Z}^m$ (which we also denote by ψ) such that

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & G \arrow[d, "\psi"] \\ & \mathbb{Z}^m \end{tikzcd}$$

commutes. Following [8, Definition 1.4] we call (φ, ψ) an *admissible pair* for π.

Clearly $G_\psi := \operatorname{Ker}\{G \to \mathbb{Z}^m\}$ is locally indicable and amenable. It follows now from Passman [13, Lemma 3.5 (ii)] that $(\mathbb{Z}[G], \mathbb{Z}[G_\psi] \setminus \{0\})$ satisfies the Ore property. Now pick elements $t^\alpha \in G, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{n\alpha} = (t^\alpha)^n$ for any $\alpha \in \mathbb{Z}^m, n \in \mathbb{Z}$.

Clearly $\mathbb{Z}[G](\mathbb{Z}[G_\psi] \setminus \{0\})^{-1} = \sum_{\alpha \in \mathbb{Z}^m} \mathbb{K}(G_\psi) t^\alpha$ is a multivariable skew Laurent polynomial ring of rank m over the field $\mathbb{K}(G_\psi)$ as defined in Section 2.1. We denote this ring by $\mathbb{K}(G_\psi)[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. Note that $\mathbb{Z}[\pi]\to\mathbb{Z}[G]\to\mathbb{K}(G_\psi)[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ is a ψ–compatible homomorphism and that $\mathbb{K}(G_\psi)(t_1,\dots,t_m)$ is canonically isomorphic to $\mathbb{K}(G)$.

A family of examples of admissible pairs is provided by the rational derived series of a group π introduced by the second author (cf [7, Section 3]). Let $\pi_r^{(0)} := \pi$ and define inductively

$$\pi_r^{(n)} := \big\{g\in\pi_r^{(n-1)} | g^d\in\big[\pi_r^{(n-1)},\pi_r^{(n-1)}\big] \text{ for some } d\in\mathbb{Z}\setminus\{0\}\big\}.$$

Note that $\pi_r^{(n-1)}/\pi_r^{(n)}\cong\big(\pi_r^{(n-1)}/\big[\pi_r^{(n-1)},\pi_r^{(n-1)}\big]\big)/\mathbb{Z}$–torsion. By [7, Corollary 3.6] the quotients $\pi/\pi_r^{(n)}$ are PTFA groups for any π and any n. If $\psi\colon\pi\to\mathbb{Z}^m$ is an epimorphism, then $(\pi\to\pi/\pi_r^{(n)},\psi)$ is an admissible pair for π for any $n>0$.

4.3 Admissible pairs and semi-norms

Let M be a 3–manifold with empty or toroidal boundary. Let

$$(\varphi\colon \pi_1(M)\to G, \psi\colon \pi_1(M)\to\mathbb{Z}^m)$$

be an admissible pair for $\pi_1(M)$. We denote the induced map

$$\mathbb{Z}[\pi_1(M)]\to\mathbb{K}(G_\psi)(t_1,\dots,t_m)$$

by φ as well.

Let $\phi\colon\mathbb{Z}^m\to\mathbb{Z}$ be a non-trivial homomorphism. We denote the induced homomorphism $G\to\mathbb{Z}^m\to\mathbb{Z}$ by ϕ as well. We write $G_\phi := \mathrm{Ker}\{G\to\mathbb{Z}\}$. Pick $\mu\in G$ such that $\phi(\mu)\mathbb{Z}=\mathrm{Im}(\phi)$. We define $\mathbb{Z}[G_\phi][u^{\pm 1}]$ via $uf=\mu f\mu^{-1}u$. Note that we get an isomorphism $\mathbb{K}(G_\phi)(u)\cong\mathbb{K}(G)$. If $\tau(M,\varphi)\neq 0$, then we define

$$\delta_G(\phi) := \max\{0,\deg(\tau(M,\mathbb{K}(G_\phi)(u)))\},$$

otherwise we write $\delta_G(\phi)=-\infty$. We will adopt the convention that $-\infty<a$ for any $a\in\mathbb{Z}$. By [4] this agrees with the definition in [8, Definition 1.6] if $\delta_G(\phi)\neq-\infty$ and if $\varphi\colon G\to\mathbb{Z}^m$ is not an isomorphism or if $m>1$. In the case that $\varphi\colon G\to\mathbb{Z}$ is an isomorphism and $M\neq S^1\times D^2, S^1\times S^2$, this definition differs from [8, Definition 1.6] by the term $1+b_3(M)$. In the case that $\varphi\colon\pi\to\pi/\pi_r^{(n+1)}$ then we also write $\delta_n(\phi)=\delta_{\pi/\pi_r^{(n+1)}}(\phi)$.

The following theorem implies Theorem 1.1.

Theorem 4.2 *Let M be a 3–manifold with an empty or toroidal boundary. Let $(\varphi\colon \pi_1(M)\to G,\ \psi\colon\pi_1(M)\to\mathbb{Z}^m)$ be an admissible pair for $\pi_1(M)$ such that*

$\tau(M,\varphi) \neq 0$. Then for any $\phi\colon \mathbb{Z}^m \to \mathbb{Z}$ we have $\|\phi\|_{\tau(M,\varphi)} = \delta_G(\phi)$ and $\phi \mapsto \max\{0, \delta_G(\phi)\}$ defines a semi-norm which is a lower bound on the Thurston norm.

Proof Let $\phi\colon \mathbb{Z}^m \to \mathbb{Z}$ be a non-trivial homomorphism. As in Section 2.1 we can form $\mathbb{K}(G_\phi)[s^{\pm 1}]$ and $\mathbb{K}(G_\psi)(\mathrm{Ker}(\phi))[s^{\pm 1}]$. Note that these rings are canonically isomorphic Laurent polynomial rings. If $\psi\colon G \to \mathbb{Z}^m$ is an isomorphism, then φ is commutative. Otherwise we can find a non-trivial $g \in \mathrm{Ker}(\psi)$, so clearly $1-\varphi(g) \neq 0 \in \mathbb{K}(G)$. This shows that we can apply Theorem 3.1 which then concludes the proof. □

In the case that $\varphi\colon \pi \to \pi/\pi_r^{(n+1)}$ we denote the semi-norm $\phi \mapsto \max\{0, \delta_n(\phi)\}$ by $\|-\|_n$. Note that in the case $n=0$ this was shown by the second author [7, Proposition 5.12] to be equal to McMullen's Alexander norm [11].

4.4 Admissible triple

We now slightly extend a definition from [8].

Definition Let π be a group and $\psi\colon \pi \to \mathbb{Z}^m$ an epimorphism. Furthermore let $\varphi_1\colon \pi \to G_1$ and $\varphi_2\colon \pi \to G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2. We call $(\varphi_1, \varphi_2, \psi)$ an *admissible triple* for π if there exist epimorphisms $\Phi\colon G_1 \to G_2$ and $\psi_2\colon G_2 \to \mathbb{Z}^m$ such that $\varphi_2 = \Phi\circ\varphi_1$, and $\psi = \psi_2\circ\varphi_2$.

Note that $(\varphi_i, \psi), i = 1, 2$ are admissible pairs for π. Combining Theorem 4.2 with [4, Theorem 1.3] (cf also [8]) we get the following result.

Theorem 4.3 *Let M be a 3–manifold with empty or toroidal boundary. If $(\varphi_1, \varphi_2, \psi)$ is an admissible triple for $\pi_1(M)$ such that $\tau(M, \varphi_2) \neq 0$, then we have the following inequalities of semi-norms:*

$$\|-\|_{\tau(M,\varphi_2)} \le \|-\|_{\tau(M,\varphi_1)} \le \|-\|_T.$$

In particular we have

$$\|-\|_0 \le \|-\|_1 \le \cdots \le \|-\|_T.$$

Let M be a 3–manifold with empty or toroidal boundary and let $\phi \in H^1(M;\mathbb{Z})$. Since $\delta_n(\phi) \in \mathbb{N}$ for all n it follows immediately from Theorem 4.3 that there exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \ge N$. But we can in fact prove a slightly stronger statement, namely that there exists such an N independent of the choice of $\phi \in H^1(M;\mathbb{Z})$.

Proposition 4.4 *Let M be a 3–manifold with empty or toroidal boundary. There exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \geq N$ and all $\phi \in H^1(M;\mathbb{R})$.*

Proof Write $\pi = \pi_1(M), \pi_n = \pi/\pi_r^{(n+1)}$ and $m = b_1(M)$. Let $\psi\colon \pi \to \mathbb{Z}^m$ be an epimorphism. Write $(\pi_n)_\psi = \mathrm{Ker}\{\psi\colon \pi_n \to \mathbb{Z}^m\}$. Now pick elements $t^\alpha \in \pi_n, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{k\alpha} = (t^\alpha)^k$ for any $\alpha \in \mathbb{Z}^m, k \in \mathbb{Z}$. Consider the map $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi_n] \to \mathbb{K}((\pi_n)_\psi)(t_1,\dots,t_m)$. We write $\tau_n = \tau(M, \mathbb{K}((\pi_n)_\psi)(t_1,\dots,t_m))$. We can find $f_n, g_n \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1},\dots,t_m^{\pm 1}]$ such that $\tau_n = f_n g_n^{-1}$.

Now let H be a real vector space and $C \subset H$ a convex subset. Then C defines a dual convex subset $d(C) \subset H^* = \hom(H,\mathbb{R})$. Under the canonical identification $(H^*)^* = H$ we have $d(d(C)) = C$. We use ψ to identify $H_1(M;\mathbb{R})$ with $\mathbb{R}^m$. Let $f = \sum_{\alpha\in\mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1},\dots,t_m^{\pm 1}]$ and denote by $N(f)$ its Newton polytope, ie $N(f)$ is the convex hull of $\{\alpha | a_\alpha \neq 0\} \subset H_1(M;\mathbb{R})$. Clearly $d(N(f)) \subset (H_1(M;\mathbb{R}))^* = H^1(M;\mathbb{R})$ equals the norm ball of $\|{-}\|_f$. By the above discussion we see that $d(\|{-}\|_f) = N(f)$, in particular $d(\|{-}\|_f)$ has only integral vertices.

By the definition of $\delta_n = \|{-}\|_{\tau_n} = \|{-}\|_{f_g g_n^{-1}}$ it follows that

$$d(\delta_n) + d(g_n) = d(\tau_n) + d(g_n) = d(f_n)$$

where "$+$" denotes the Minkowski sum of convex sets. It is easy to see that this implies that $d(\delta_n)$ has only integral vertices.

Theorem 4.3 implies that there is a sequence of inclusions

$$d(\delta_0) \subset d(\delta_1) \subset \cdots \subset d(\|{-}\|_T).$$

Since $d(\|{-}\|_T)$ is compact and since $d(\delta_n)$ has integral vertices for all n it follows immediately that there exists $N \in \mathbb{N}$ such that $d(\delta_n) = d(\delta_N)$ for all $n \geq N$. This completes the proof of the proposition. □

5 Examples

Before we discuss the Thurston norm of a family of links we first need to introduce some notation for knots. Let K be a knot. We denote the knot complement by $X(K)$. Let $\phi\colon H_1(X(K)) \to \mathbb{Z}$ be an isomorphism. We write $\delta_n(K) := \delta_n(\phi)$. This agrees with the original definition of Cochran [1] for $n > 0$ and if $\Delta_K(t) = 1$, and it is one less than Cochran's definition otherwise.

In the following let $L = L_1 \cup \cdots \cup L_m$ be any ordered oriented m–component link. Let $i \in \{1,\dots,m\}$. Let K be an oriented knot with $\Delta_K(t) \neq 1$ which is separated from L by a sphere S. We pick a path from a point on K to a point on L_i and denote by $L\#_i K$ the link given by performing the connected sum of L_i with K (cf Figure 1). Note that this connected sum is well-defined, ie independent of the choice of the path. We will study the Thurston norm of $L\#_i K$.

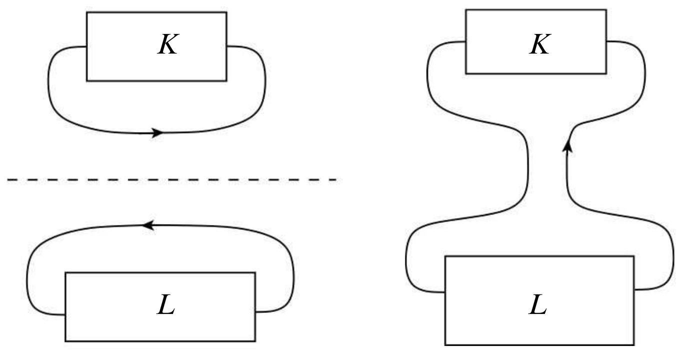

Figure 1: The link $L\#_i K$.

Now assume that L is a non–split link with at least two components and such that $\|-\|_0 = \|-\|_T$. Many examples of such links are known (cf [11]). For the link $L\#_i K$ denote its meridians by $\mu_i, i = 1, \dots, m$. Let $\psi\colon H_1(X(L\#_i K)) \to \mathbb{Z}^m$ be the isomorphism given by $\psi(\mu_i) = e_i$, where e_i is the i th vector of the standard basis of $\mathbb{Z}^m$.

We write $\pi := \pi_1(X(L\#_i K))$. For all $\alpha \in \mathbb{Z}^m$ we pick $t^\alpha \in \pi/\pi_r^{(n+1)}$ with $\psi(t^\alpha) = \alpha$ and such that $t^{l\alpha} = (t^\alpha)^l$ for all $\alpha \in \mathbb{Z}^m$ and $l \in \mathbb{Z}$. Furthermore write $t_i := t^{e_i}$.

Proposition 5.1 *Consider the natural map*

$$\varphi\colon \pi \to \mathbb{K}(\pi/\pi_r^{(n+1)}) = \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m).$$

where π is as defined above. There exists

$$f(t_i) \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}] \subset \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$$

such that $\deg(f(t_i)) = \delta_n(K)+1$, *and there exists a* $d = d(t_1, \dots, t_m) \in \mathbb{K}(t_1, \dots, t_m)$ *with* $\|-\|_d = \|-\|_0$, *such that*

(2)
$$\tau(X(L\#_i K), \varphi) = d(t_1, \dots, t_m) f(t_i) \in K_1(\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m))/\pm\varphi(\pi).$$

Furthermore, if $\delta_n(K) = 2\text{genus(K)} - 1$, *then*

$$\|-\|_{\tau(X(L\#_i K),\varphi)} = \|-\|_T.$$

Proof Let S be the embedded sphere in S^3 coming from the definition of the connected sum operation (cf Figure 1). Let D be the annulus $S \cap X(L\#_i K)$ and we denote by P the closure of the component of $X(L\#_i K) \setminus D$ corresponding to K. We denote the closure of the other component by P' (see Figure 2 below). Note that P is homeomorphic to $X(K)$ and P' is homeomorphic to $X(L)$. Denote the induced

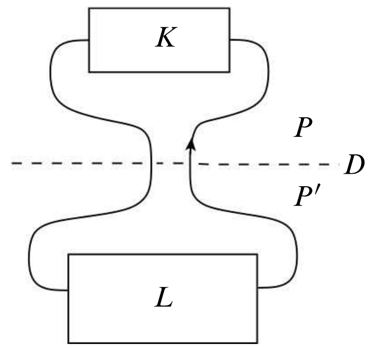

Figure 2: The link complement of $L\#_i K$ cut along the annulus D.

maps to $(K) := \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m)$ by φ as well. We get an exact sequence

$$0 \to C_*^\varphi(D;(K)) \to C_*^\varphi(P;(K)) \oplus C_*^\varphi(P';(K)) \to C_*^\varphi(X(L\#_i K);(K)) \to 0$$

of chain complexes. It follows from [19, Theorem 3.4] that

(3) $\quad \tau(P,\varphi)\tau(P',\varphi) = \tau(D,\varphi)\tau(X(L_i\#K),\varphi) \in \big(K_1((K))/\pm\varphi(\pi)\big) \cup \{0\}.$

First note that D is homotopy equivalent to a circle and that $\mathrm{Im}\{\psi\colon \pi_1(D) \to \mathbb{Z}^m\} = \mathbb{Z}e_i$. It is now easy to see that $\tau(D,\varphi) = (1-at_i)^{-1}$ for some $a \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)}) \setminus \{0\}$. Next note that $\mathrm{Im}\{\psi\colon \pi_1(P) \to \mathbb{Z}^m\} = \mathbb{Z}e_i$. In particular $\tau(P,\varphi)$ is defined over the one-variable Laurent polynomial ring $\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ which is a PID. Recall that we can therefore assume that its Dieudonné determinant $f(t_i)$ lies in $\mathbb{K}\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ as well.

Claim

$$\deg(\tau(P,\varphi\colon \pi_1(P) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) = \delta_n(K).$$

Proof First recall that there exists a homeomorphism $P \cong X(K)$. We also have an inclusion $X(L\#_i K) \to X(L_i\#K)$. Combining with the degree one map $X(L_i\#K) \to X(K)$ we get a factorization of an automorphism of $\pi_1(X(K))$ as follows:

$$\pi_1(X(K)) \cong \pi_1(P) \to \pi_1(X(L\#_i K)) \to \pi_1(X(L_i\#K)) \to \pi_1(X(K)).$$

Since the rational derived series is functorial (cf [7]) we in fact get that

$$\begin{aligned}\pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} &\cong \pi_1(P)/\pi_1(P)_r^{(n+1)} \\ &\to \pi_1(X(L_i\#K))/\pi_1(X(L_i\#K))_r^{(n+1)} \\ &\to \pi_1(X(K))/\pi_1(X(K))_r^{(n+1)}\end{aligned}$$

is an isomorphism. In particular

$$\pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} \to \pi_1(X(L\#_i K))/\pi_1(X(L\#_i K))_r^{(n+1)}$$

is injective, and the induced map on Ore localizations is injective as well. Finally note that $\mathrm{Ker}\{\pi_1(X(K)) \to \pi_1(P) \xrightarrow{\psi} \mathbb{Z}^m\} = \mathrm{Ker}(\phi)$ where $\phi\colon \pi_1(X(K)) \to \mathbb{Z}$ is the abelianization map. It now follows that

$$\begin{aligned}\delta_n(K) &= \deg(\tau(X(K), \pi_1(X(K)) \to \mathbb{K}(\pi_1(X(K))_\phi/\pi_1(X(K))_r^{(n+1)})(t_i)) \\ &= \deg(\tau(X(K), \pi_1(X(K)) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) \\ &= \deg(\tau(P, \pi_1(P) \to \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)).\end{aligned}$$

Note that the second equality follows from the functoriality of torsion (cf [19, Proposition 3.6]) and the fact that going to a supfield does not change the degree of a rational function. This concludes the proof of the claim. □

Claim We have the following equality of norms on $H^1(X(L);\mathbb{Z})$:

$$\|{-}\|_{\tau(P',\varphi)} = \|{-}\|_T.$$

Proof First recall that P' is homeomorphic to $X(L)$. The claim now follows immediately from Theorem 4.3 applied to φ and to the abelianization map of $\pi_1(P')$, and from the assumption that $\|{-}\|_0 = \|{-}\|_T$ on $H^1(X(L);\mathbb{Z})$. □

Putting these computations together and using Equation (3) we now get a proof of Equation (2).

Now assume that $\delta_n(K) = 2\mathrm{genus(K)}-1$. Let S_i be a Seifert surface of K with minimal genus. Let $\phi\colon \mathbb{Z}^m \to \mathbb{Z}$ be an epimorphism and let $l = \phi(\mu_i) \in \mathbb{Z}$. We first view ϕ as an element in $\hom(H_1(X(L);\mathbb{Z})$. A standard argument shows that ϕ is dual to a (possibly disconnected) surface S which intersects the tubular neighborhood of L_i in exactly l disjoint curves. Then the connected sum S' of S with l copies of S_i gives a surface in $X(L\#_i K)$ which is dual to ϕ viewed as an element in $\hom(H_1(X(L\#_i K);\mathbb{Z})$. A standard argument shows that S' is Thurston norm minimizing (cf eg [10, page 18]).

Clearly $\chi(S') = \chi(S) + l(\chi(S_i) - 1)$. A straightforward argument shows that furthermore $\chi_-(S') = \chi_-(S) + l(\chi_-(S_i) + 1)$ since L is not a split link and since K is non-trivial.

We now compute

$$\begin{aligned} \|\phi\|_T &= \chi_-(S') \\ &= \chi_-(S) - n(\chi(S_i) - 1) \\ &= \|\phi\|_T + 2l\,\mathrm{genus(K)} \\ &= \|\phi\|_d + 2(\delta_n(K) + 1) \\ &= \|\phi\|_d + 2\deg(f(t_i)) \\ &= \|\phi\|_{\tau(X(L\#_i K),\varphi)}. \end{aligned}$$

By the $\mathbb{R}$–linearity and the continuity of the norms it follows that

$$\|\phi\|_{\tau(X(L\#_i K),\varphi)} = \|\phi\|_T$$

for all $\phi\colon \mathbb{Z}^m \to \mathbb{R}$. □

We now combine Proposition 5.1 with results of [1] to give explicit examples of the sequence of semi-norms $\|-\|_n$.

Denote by $\Diamond(n,m)$ the convex polytope given by the vertices $(\pm\frac{1}{n},0)$ and $(0,\pm\frac{1}{m})$. Let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be never decreasing sequences of odd positive numbers which are eventually constant, ie there exists an N such that $n_i = n_N$ for all $i \geq N$ and $m_i = m_N$ for all $i \geq N$. According to [1] we can find knots K_1 and K_2 such that $\delta_i(K_1) = n_i$ for any i, $\delta_N(K_1) = 2\,\mathrm{genus(K_1)} - 1$ and $\delta_i(K_2) = m_i$ for any i and $\delta_N(K_2) = 2\,\mathrm{genus(K_2)} - 1$.

Let $H(K_1,K_2)$ be the link formed by adding the two knots K_1 and K_2 from above to the Hopf link (cf Figure 3). Recall that the Thurston norm ball of the Hopf link is given by $\Diamond(1,1)$. Let $\pi := \pi_1(X(L))$. It follows immediately from applying Proposition 5.1

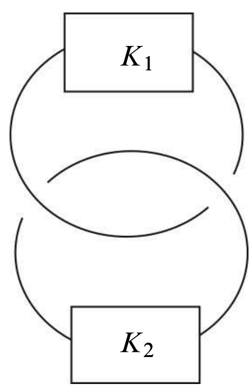

Figure 3: $H(K_1,K_2)$ is obtained by tying K_1 and K_2 into the Hopf link

twice that the norm ball of $\|-\|_i$ equals $\diamondsuit(n_i+1, m_i+1)$ and that $\|-\|_N = \|-\|_T$. The following result is now an immediate consequence of Proposition 5.1.

Corollary 5.2 *We have the following sequence of inequalities of semi-norms*

$$\|-\|_A = \|-\|_0 \le \|-\|_1 \le \|-\|_2 \le \cdots \le \|-\|_N = \|-\|_T.$$

In [7] the second author gave examples of 3–manifolds M such that

$$\|-\|_A = \|-\|_0 \le \|-\|_1 \le \|-\|_2 \le \cdots$$

but in that case it was not known whether the sequence of norms $\|-\|_i$ eventually agrees with $\|-\|_T$.

It is an interesting question to determine which 3–manifolds satisfy $\|-\|_T = \|-\|_n$ for large enough n. We conclude this paper with the following conjecture.

Conjecture 5.3 *If $\pi_1(M)_r^{(\omega)} := \bigcap_{n\in\mathbb{N}} \pi_1(M)_r^{(n)} = \{1\}$, then there exists $n \in \mathbb{N}$ such that $\|-\|_T = \|-\|_n$.*

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