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Connective Im(J)–theory for cyclic groups

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We study connective Im(J)–theory for the classifying space $B\mathbb{Z}/p^a$ of a finite cyclic p–group and compute the Im(J)–cohomology groups completely. We also compute the Im(J)–homology groups, with the exception of a finite range of dimensions.

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1 Introduction

Im(J)–theory A is a generalized homology theory appearing both in homotopy theory and algebraic K–theory. It is (sometimes) a fairly good first approximation to stable homotopy capturing v_1–periodic phenomena, and is closely related to the classical J–homomorphism and the e–invariant. Let p be an odd prime, k be an integer which generates $(\mathbb{Z}/p^2)^*$ and ψ^k be the (stable) Adams operation in p–local complex K–theory. Then the spectrum of nonconnected Im(J)–theory Ad may be defined by the cofiber sequence:

$$\to \mathrm{Ad} \xrightarrow{D} K \xrightarrow{\psi^k - 1} K \xrightarrow{\triangle} \Sigma\,\mathrm{Ad} \to$$

See Knapp [13] and Crabb and Knapp [5]. Connective Im(J)–theory A is then defined as the (-1)–connected cover of Ad. More importantly, A appears as the p–localization of the K–theory of a finite field. If k is chosen as a prime power, then

$$A \simeq (K\mathbb{F}_k)_{(p)}$$

where $K\mathbb{F}_k$ is the algebraic K–theory spectrum of the finite field $\mathbb{F}_k$.

Since Ad and A are not complex orientable, the computation of Im(J)–groups even of rather simple spaces is not obvious. But due to the close relationship of $K^*(BG)$ to the representation ring of G, the computation of the nonconnected Im(J)–groups of the classifying space of a finite group BG turned out to be surprisingly simple [12]. For example, for the cyclic group $G = \mathbb{Z}/p^a$ we have

$$\mathrm{Ad}_i(B\mathbb{Z}/p^a) = \begin{cases} (\mathbb{Z}_{p^\infty})^a & i = -1, -2, \\ \bigoplus_{j=1}^{a} \mathbb{Z}/p^{j+\nu_p(n)} & i = 2n-1, n \in \mathbb{Z} - \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

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and $$\mathrm{Ad}^i(B\mathbb{Z}/p^a) = \begin{cases} (\mathbb{Z}_p^\wedge)^a & i = 0, 1, \\ \bigoplus_{j=1}^a \mathbb{Z}/p^{j+\nu_p(n)} & i = 2n+1,\ n \in \mathbb{Z} - \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\nu_p(n)$ denotes the power of p in n and $\mathbb{Z}_p^\wedge$ are the p–adic integers.

The noticeable duality results form a universal coefficient formula which is satisfied by nonconnected $\mathrm{Im}(J)$–theory (but not by A) [13].

In this paper we compute the connective $\mathrm{Im}(J)$–groups of the classifying spaces of cyclic groups with the exception of a finite dimension range in homology. More precisely, we determine $A^*(B\mathbb{Z}/p^a)$ and $A_{2n}(B\mathbb{Z}/p^a)$ for $n \geq n_0(a)$ and p an odd prime. Here $n_0(a)$ is a constant which is roughly $(a+1)p^a$. The groups $A_{2n-1}(B\mathbb{Z}/p^a)$ for $n \geq n_0(a)$ are given by $\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^a)$ [12]. Determining $A_{2n-1}(B\mathbb{Z}/p^a)$ from $A_{2n-1}(P_\infty\mathbf{C})$ for $n < n_0(a)$ is carried out in [10], but the result is difficult to describe explicitly. The main results of this paper are as follows:

Theorem 1.1

$$A^j(B\mathbb{Z}/p^a) \cong \begin{cases} \mathrm{Ad}^j(B\mathbb{Z}/p^a) & j \leq 1, \\ \displaystyle\bigoplus_{i=0}^{a-2} \mathbb{Z}/p^{a-i+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)-a} & j = 2n+1 > 1, \\ & \nu_p(n) \geq a-1 \geq 0, \\ \displaystyle\bigoplus_{\substack{i=1 \\ i \neq a-\nu_p(n)}}^{a} \mathbb{Z}/p^{i+\nu_p(n)} & j = 2n+1 > 1, \\ & \nu_p(n) < a-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 1.2 $A_{2n-2}(B\mathbb{Z}/p^a) \cong A^{2n+1}(B\mathbb{Z}/p^a)$ *for* $n \geq n_0(a)$.

The case $a = 1$ was known before:

$$A_{2n-1}(B\mathbb{Z}/p) \cong \mathrm{Ad}_{2n-1}(B\mathbb{Z}/p) = \mathbb{Z}/p^{1+\nu_p(n)}$$
$$A_{2n-2}(B\mathbb{Z}/p) \cong \mathbb{Z}/p^{\nu_p(n)} \ (n > 0)$$

A direct computation of $A_{2n-2}(B\mathbb{Z}/p^a)$ via the K–theory of lens spaces seems to be inaccessible. Our method is therefore to find at first a lift of the problem to a torsion free situation. We achieve this by showing that $A_{2n-2}(B\mathbb{Z}/p^a)$ is isomorphic to $A^{2n+1}(B\mathbb{Z}/p^a)$ for $n \geq n_0(a)$. This isomorphism is a consequence of Anderson duality for connective $\mathrm{Im}(J)$–theory [14], but we give a direct approach via the universal coefficient formula for Ad–theory. Now by definition we have with $B = B\mathbb{Z}/p^a$

$$\begin{aligned} A^{2n+1}(B) &= \operatorname{Im} \operatorname{Ad}^{2n+1}(B, B^{2n}) \longrightarrow \operatorname{Ad}^{2n+1}(B, B^{2n-1}) \\ &= \operatorname{Im} \operatorname{Ad}^{2n+1}(\widetilde{T}(nH)) \longrightarrow \operatorname{Ad}^{2n+1}(T(nH)) \end{aligned}$$

where H is the Hopf line bundle on B, $T(nH)$ is the Thom space of nH and $\widetilde{T}(nH) = T(nH)/S^{2n}$ is the reduced Thom space. This is a lift to characteristic zero, since the group $\operatorname{Ad}^{2n+1}(\widetilde{T}(nH))$ is the cokernel of $\psi^k - 1$ on $K^{2n}(\widetilde{T}(nH)) \cong K^{2n}(B)$. Now the action of ψ^k on $K^{2n}(\widetilde{T}(nH))$ is much more complicated than the one on $K^{2n}(B)$, but nevertheless we have the following:

Theorem 1.3 *There exists a ψ^k–equivariant isomorphism $K^0(\widetilde{T}(nH)) \cong K^0(B)$.*

We prove this for n with $\nu_p(n) \geq a-1$ by an explicit construction, whereas for n with $\nu_p(n) < a-1$ we only have an existence proof. An immediate consequence is that the localizations of B and $\widetilde{T}(nH)$ with respect to K–theory are equivalent. This is not true for $T(nH)$ itself or reduced Thom spectra of other bundles (eg H^p on $B\mathbb{Z}/p^2$). The groups $A^{2n+1}(B)$ are related to $\operatorname{Ad}^{2n+1}(\widetilde{T}(nH))$ via the group extension

$$0 \to \mathbb{Z}/p^a \to \operatorname{Ad}^{2n+1}(\widetilde{T}(nH)) \to A^{2n+1}(B) \to 0 \,.$$

For n with $\nu_p(n) \geq a-1$ the explicit ψ^k–isomorphism $K^0(\widetilde{T}(nH)) \cong K^0(B)$ allows us to solve this extension problem, whereas for the other values of n we need to know which multiples of H are orientable with respect to $\operatorname{Ad}^*(-;\mathbb{F}_p)$.

The paper is organized as follows. In Section 2 we investigate the ψ^k–module $K^0(\widetilde{T}(nH))$ and construct for n with $\nu_p(n) \geq a-1$ a Thom class of nH with good invariance properties under the action of ψ^k. The existence proof for a ψ^k–isomorphism $K^0(\widetilde{T}(nH)) \cong K^0(B)$ for $n \not\equiv 0 \bmod p^{a-1}$ is postponed to Section 4. We first show that $K^0(\widetilde{T}(nH);\mathbb{F}_p)$ is a ψ^k–permutation representation, which is ψ^k–isomorphic to $K^0(B;\mathbb{F}_p)$, and then lift this to integral K–theory. In Section 3 we relate $A_{2n-2}(B)$ to $A^{2n+1}(B)$ and solve the extension problem which leads to the computation of $A^{2n+1}(B)$. A few applications are contained in Section 5. We give the proof for the equivalence of the K–localizations of B and $\widetilde{T}(nH)$, complete the calculation of $A_i(B\mathbb{Z}/p^2)$ in the range not covered by Theorem 1.2 and derive the formula of [15] for the group order of $J(L^n(p^a))$, the J–group of the lens space $L^n(p^a) = B^{2n+1}$.

We shall use the following notation throughout. We abbreviate $B\mathbb{Z}/p^a$ by B and its m–skeleton by B^m. H is the Hopf line bundle on B, $T(nH)$ is the Thom spectrum of nH, $n \in \mathbb{Z}$, and $\widetilde{T}(nH) = T(nH)/S^{2n}$ is the associated reduced Thom spectrum.

We assume that p is an odd prime and $k \in \mathbb{Z}$ is chosen to reduce to a generator of $(\mathbb{Z}/p^2)^*$.

All (co)homology theories are taken as reduced.

2 Nonconnective Im(J)–theory of $\widetilde{T}(nH)$

In this section we investigate the nonconnective Im(J)–groups $\mathrm{Ad}^*(\widetilde{T}(nH))$ of the reduced Thom spectra $\widetilde{T}(nH)$ of multiples of the Hopf line bundle H on $B = B\mathbb{Z}/p^a$. As explained in Section 1, these results will be used to compute $A^*(B)$ later on. In order to determine $\mathrm{Ad}^*(\widetilde{T}(nH))$ we must study $K^*(\widetilde{T}(nH))$ as a module over the Adams operation ψ^k. As preparation for this, we collect in the first part of this section some standard material and notation which we shall use throughout the paper.

We begin by recalling some standard cofiber sequences.

It is well known that for $n \geq 0$ the Thom space $T(nH)$ is homeomorphic to the stunted lens space B/B^{2n-1}, hence we have a cofiber sequence

$$B^{2n-1} \longrightarrow B \xrightarrow{j} T(nH) \longrightarrow \Sigma B^{2n-1} \tag{1}$$

which may be identified with the middle row in the commutative diagram of standard cofiber sequences:

$$\begin{array}{ccccccc}
S^{2n-1} & & & & S^{2n} & & \Sigma S^{2n-1} \\
\downarrow & & & & \downarrow & & \downarrow \\
B^{2n-1} & \longrightarrow & B & \longrightarrow & B/B^{2n-1} & \longrightarrow & \Sigma B^{2n-1} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
B^{2n} & \longrightarrow & B & \longrightarrow & B/B^{2n} & \longrightarrow & \Sigma B^{2n}
\end{array}$$

Simple degree considerations show that the map $S^{2n} \longrightarrow B/B^{2n-1} = T(nH)$ in this diagram represents the generator in $\pi_{2n}(T(nH))$, hence we may identify B/B^{2n} with $\widetilde{T}(nH) = T(nH)/S^{2n}$. The inclusion $B^{2n} \hookrightarrow B^{2n+1}$ then induces the commutative diagram

$$\begin{array}{ccccccc}
B^{2n} & \longrightarrow & B & \xrightarrow{j_1} & \widetilde{T}(nH) & \longrightarrow & \Sigma B^{2n} \\
\downarrow & & \| & & \downarrow j & & \downarrow \\
B^{2n+1} & \longrightarrow & B & \xrightarrow{j_2} & T((n+1)H) & \longrightarrow & \Sigma B^{2n+1} \\
\downarrow & & & & \downarrow \delta & & \downarrow \\
S^{2n+1} & & & & S^{2n+2} & \longrightarrow & \Sigma S^{2n+1}
\end{array} \tag{2}$$

giving the cofiber sequence

$$\widetilde{T}(nH) \xrightarrow{j} T((n+1)H) \xrightarrow{\delta} S^{2n+2}.$$

Since the degree of δ in dimension $2n+2$ is p^a we finally obtain the commutative diagram of cofiber sequences

$$\begin{array}{ccccccccc}
\longrightarrow & \widetilde{T}(nH) & \xrightarrow{\bar{j}} & \widetilde{T}((n+1)H) & \xrightarrow{\bar{\delta}} & \Sigma^{2n+2}M_a & \xrightarrow{\bar{h}} \\
 & \| & & \uparrow & & \uparrow & \\
\longrightarrow & \widetilde{T}(nH) & \xrightarrow{j} & T((n+1)H) & \xrightarrow{\delta} & S^{2n+2} & \xrightarrow{h} \\
 & & & \uparrow i & & \uparrow p^a & \\
 & & & S^{2n+2} & = & S^{2n+2} &
\end{array} \tag{3}$$

with $M_a = S^0 \cup_{p^a} e^1$ the Moore spectrum for $\mathbb{Z}/p^a$.

The action of ψ^k on $K^0(T(nH))$ is the one on $K^0(B_+)$ but twisted by the Bott characteristic class ρ^k which describes the action of ψ^k on the Thom class. We therefore recall some simple properties of ρ^k and identify the kernel of the map $j^*\colon K^0(T(nH)) \to K^0(B)$. If n is divisible by p^{a-1} this will be sufficient to identify the ψ^k–module $K^0(T(nH))$.

Let $\mathrm{reg} := \sum_{i=0}^{p^a-1} H^i \in K^0(B_+)$ be the element defined by the regular representation of $\mathbb{Z}/p^a$. Then the ideal generated by reg is the kernel of the map $K^0(B_+) \to K^0(B)$ defined by multiplication with $x = H-1$. Since multiplication by x is injective on $K^0(B)$, (reg) is also the kernel of multiplication by x^n or by any power series in x beginning with x^n. Moreover $\mathrm{reg}\cdot z = 0$ for any $z \in K^0(B)$, since any such z is divisible by x.

Lemma 2.1 *For $n \geq 0$ the kernel of $j^*\colon K^0(T(nH)) \to K^0(B)$ is generated by $\mathrm{reg}\cdot U$, where U is any Thom class for nH.*

Proof The commutative diagram

$$\begin{array}{ccc}
K^0(T(nH)) & \xrightarrow{j^*} & K^0(B) \\
\phi\uparrow\cong & & \| \\
K^0(B_+) & \xrightarrow{\cdot e(nH)} & K^0(B)
\end{array}$$

with ϕ the Thom isomorphism defined by U and $e(nH)$ the Euler class belonging to U shows that $\ker j^*$ is generated by $\mathrm{reg}\cdot U$, since $e(nH) = x^n + \cdots$. $\square$

Let now x_M be any complex orientation of p–local complex K–theory. This defines a multiplicative Thom class $U_M(E)$ for complex vector bundles E [2]. Associated to this is an exponential Bott class $\rho^k_M(E) \in K^0(X_+)$ defined by the equation

$$\psi^k U_M(E) = \rho^k_M(E) \cdot U_M(E).$$

Lemma 2.2 $\psi^l \rho^r_M(E) \cdot \rho^l_M(E) = \rho^{r \cdot l}_M(E)$.

Proof This follows directly from the definition writing $\psi^{r \cdot l} = \psi^l \circ \psi^r$. □

Corollary 2.3 $\rho^k_M(E + \psi^k E + \cdots + \psi^{k^{r-1}} E) = \rho^{k^r}_M(E)$.

Let $U_K(E)$ denote the standard K–theory Thom class with Euler class $L-1$ for $E = L$ a complex line bundle. The exponential Bott class associated to this choice of complex orientation will simply be denoted by ρ^k. It satisfies in addition $\rho^l(L) = 1 + L + L^2 + \cdots + L^{l-1}$ for a line bundle L.

The Adams summand G of p–local K–theory will be used in several places, we therefore recall the splitting of p–local K–theory (see Adams [1] and Jankowski [8]).

Let $\omega = (k, k^p, k^{p^2}, \ldots) \in \mathbb{Z}_p^\wedge = \varprojlim \mathbb{Z}/p^i$, then ω is a primitive $(p-1)$–root of unity in $\mathbb{Z}_p^\wedge$ satisfying $\omega \equiv k^{p^{a-1}} \bmod p^a$. The idempotents

$$\Phi_i := \frac{1}{p-1} \sum_{r=1}^{p-1} \omega^{-r \cdot i} \psi^{\omega^r} \qquad \text{for } 0 \le i \le p-2$$

with ψ^ω a p–adic Adams operation are defined p–locally and split $M = K^0(X)$ into $p-1$ pieces:

$$M = \Phi_0(M) \oplus \Phi_1(M) \oplus \cdots \oplus \Phi_{p-2}(M)\ .$$

Then $\Phi_0(M) = G^0(X)$ and from $\Phi_i(u \cdot x) = u \cdot \Phi_{i-1}(x)$ we get $u^{-i} \Phi_i(M) = G^{2i}(X)$. Moreover

$$K \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} G$$

with $G_*(S^0) = \mathbb{Z}_{(p)}[v_1, v_1^{-1}],\ v_1 = u^{p-1}$.

Since the splitting maps commute with ψ^k, ψ^k restricts to a self map of G and $\mathrm{Im}(J)$–theory may equally well be defined by replacing K in the definition of Ad by G [13]:

$$\to \mathrm{Ad} \xrightarrow{D} G \xrightarrow{\psi^k - 1} G \xrightarrow{\triangle} \Sigma\, \mathrm{Ad} \to$$

The Adams summand G is a multiplicative complex orientable cohomology theory. We now choose a fixed complex orientation x_G for G. This defines a multiplicative Thom class $U_G(\xi)$ for complex vector bundles ξ on X and an exponential Bott class $\rho^k_G(\xi) \in G^0(X_+)$ as above.

For convenience we recall now the computation of $\mathrm{Ad}^*(B_+)$ [12; 11]. The key observation is that $K^0(B_+)$ is a permutation representation of ψ^k which results in an easy computation of $\mathrm{Ad}^*(B_+)$. The different ψ^k–orbits in $K^0(B_+)$ are generated by H^{p^i} with $i \in \{0, 1, \dots, a-1\}$. Let $L = H^{p^i}$ and $s_i := (p-1)p^{a-i-1}$ then $\psi^{k^{s_i}} L = L$ and the ψ^k invariant subspace $W_i := \langle L, \psi^k L, \psi^{k^2} L, \dots, \psi^{k^{s_i-1}} L\rangle$ of $K^0(B_+)$ has rank s_i. It is then elementary to determine $\ker(\psi^k - 1)$ and $\mathrm{coker}(\psi^k - 1)$ on W_i. In dimension 0 we have $\mathrm{coker}(\psi^k-1) = \mathbb{Z}^\wedge_p$ generated by $\triangle(L) = \triangle(\psi^{k^j} L)$ and $\ker \psi^k - 1 = \mathbb{Z}^\wedge_p$ generated by $\sum_{j=0}^{s_i-1} \psi^{k^j} L$. In dimension $2n \neq 0$ the map $k^{-n}\psi^k_0 - 1$ (which is the stable Adams operation $\psi^k - 1$ on $K^{2n}(B)$ by definition, with ψ^k_0 the classical Adams operation on $K^0(B)$) has determinant $k^{n\cdot s_i} - 1$ on W_i, and since $\mathrm{coker}(\psi^k - 1)_{|W_i}$ must again be cyclic we get

$$\ker(\psi^k - 1)_{|W_i} = 0 \quad \text{and} \quad \mathrm{coker}\,(\psi^k - 1)_{|W_i} = \mathbb{Z}^\wedge_p/(k^{n\cdot s_i} - 1)\cdot \triangle(L).$$

Note that $\nu_p(k^{n\cdot s_i} - 1) = a - i + \nu_p(n)$. Hence we conclude the following:

Proposition 2.4

$$\mathrm{Ad}^i(B) = \begin{cases} (\mathbb{Z}^\wedge_p)^a & i = 0, 1, \\ \bigoplus_{j=1}^a \mathbb{Z}/p^{j+\nu_p(n)} & i = 2n+1,\ n \in \mathbb{Z} - \{0\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Using $\mathbb{Z}/p^b$ with $b = a + \nu_p(n)$ as coefficients, we see that $p^i \cdot \sum_{j=0}^{s_i-1} k^{-nj} L^{k^j}$ is in $\ker(k^{-n}\psi^k_0 - 1)$ on $W_i \otimes \mathbb{Z}/p^b$. This describes the elements in $\mathrm{Ad}^{2n+1}(B) \cong \mathrm{Ad}^{2n}(B; \mathbb{Z}/p^b)$ as elements in $\ker \psi^k - 1 \subset K^{2n}(B; \mathbb{Z}/p^b)$.

After these preparations we now turn to $\mathrm{Ad}^*(\widetilde{T}(nH))$.

Now the simplest method to compute $\mathrm{Ad}^*(T(nH))$ would be to use a Thom isomorphism. The orientability conditions for $\mathrm{Im}(J)$–theory are well known [5, Section 5]: A complex vector bundle ξ^n on X has a Thom class $U_A(\xi) \in \mathrm{Ad}^{2n}(T(nH))$, or equivalently a ψ^k–invariant Thom class in K–theory, if and only if $\triangle([\xi]) = 0$ in $\mathrm{Ad}^1(X)$. Here $[\xi] \in K^0(X)$ is the class defined by ξ and $\triangle\colon K^0(X) \to \mathrm{Ad}^1(X)$ the map appearing in the definition of $\mathrm{Im}(J)$–theory. As we have seen above $\triangle(H-1) \in \mathrm{Ad}^1(B)$ is of infinite order, hence no multiple of H can be Ad^*–orientable.

Our main goal in this section will be to derive a ψ^k–equivariant isomorphism

$$g\colon K^0(\widetilde{T}(nH)) \longrightarrow K^0(B) \tag{4}$$

allowing a computation of $\mathrm{Ad}^*(\widetilde{T}(nH))$ along the lines as for $\mathrm{Ad}^i(B)$ above. In particular it follows that $K^0(\widetilde{T}(nH))$ is a ψ^k–permutation representation.

Although there is no Thom isomorphism between $\mathrm{Ad}^*(T(nH))$ and $\mathrm{Ad}^*(B_+)$ the map g is related to a relative Thom isomorphism as follows.

For n of the form $n = m(p-1)p^{a-1}$ we shall explicitly construct a Thom class $U'(nH) \in K^0(T(nH))$ which is nearly invariant under ψ^k, giving g as in (4) directly. The rest of the argument is much more technical. For n still divisible by p^{a-1} but not necessarily by $p-1$, this is extended by using the Adams summand G. For the other values of n we only have an existence proof which is postponed to Section 4.

Case 1 We assume $n = m \cdot (p-1)p^{a-1}$. To construct this nearly ψ^k–invariant Thom class, observe first that for $i \not\equiv 0 \bmod p$ the map given by $x \longmapsto x^{\otimes i}$ on fibers defines a p–local fiber homotopy equivalence between H and H^i. The join of such maps gives a p–local fiber homotopy equivalence f between $m \cdot \xi$ and $n \cdot H$ where $\xi = H + H^k + \cdots + H^{k^{s_0 - 1}}$ and $s_i = (p-1)p^{a-1-i}$. Define

$$U'(nH) := d^{-1} f^* U_K(m\xi) \in K^0(T(nH)) \tag{5}$$

where $U_K(E)$ is the standard K–theory Thom class and d is the degree of f on fibers. Since two Thom classes for the same bundle differ by a unit e we may write $U'(nH) = e \cdot U_K(nH)$ with $e \in K^0(B_+)$. It is not hard to see

$$e = \prod_{i=0}^{s_0-1} \rho^{k^i}(mH - m)\ .$$

Proposition 2.5 *For the Thom class U' defined in* (5) *we have*

$$\psi^k U' = U' + c \cdot \mathrm{reg} \cdot U'$$

in $K^0(T(nH))$, ie U' is invariant mod $\ker j^*$.

Proof By definition, naturality and Corollary 2.3,

$$\psi^k U' = \rho^k(m\xi) \cdot U' = \rho^{k^{s_0}}(mH) \cdot U'\ .$$

But $k^{(p-1)p^{a-1}} - 1 = c_1 p^a$ and $\rho^r(H) = 1 + H + \cdots + H^{r-1}$ immediately give

$$\rho^{k^{s_0}}(H) = 1 + c_1 \cdot \mathrm{reg} \qquad \text{and} \qquad \rho^{k^{s_0}}(mH) = (1 + c_1 \cdot \mathrm{reg})^m = 1 + c \cdot \mathrm{reg}\ . \quad \square$$

If we restrict now the Thom isomorphism with U' to classes z of virtual dimension zero, ie to $K^0(B) \subset K^0(B_+)$, we have

$$\psi^k(z \cdot U') = \psi^k z \cdot \psi^k U' = \psi^k z \cdot (U' + c \cdot \mathrm{reg} \cdot U') = (\psi^k z) \cdot U'$$

since $\psi^k z \cdot \mathrm{reg} = 0$. Hence we have the following theorem.

Theorem 2.6 *For $n = m \cdot (p-1)p^{a-1}$, multiplication by U' defines a ψ^k–equivariant relative Thom isomorphism*

$$\phi\colon K^0(B) \cong K^0(\widetilde{T}(nH)) .$$

Corollary 2.7 *For $r, t \in \mathbb{Z}$, multiplication by U' gives a ψ^k–equivariant Thom isomorphism*

$$K^0(\widetilde{T}((t + rs_0)H)) \cong K^0(\widetilde{T}(tH)) .$$

By the five lemma there is an induced isomorphism in $\mathrm{Im}(J)$–theory:

Corollary 2.8 *For $n = m \cdot (p-1)p^{a-1}$,*

$$\mathrm{Ad}^i(\widetilde{T}(nH)) \cong \mathrm{Ad}^i(B),$$

and if $l \equiv j \mod s_0$,

$$\mathrm{Ad}^i(\widetilde{T}(lH)) \cong \mathrm{Ad}^i(\widetilde{T}(jH)).$$

This shows that the groups $\mathrm{Ad}^i(\widetilde{T}(jH))$ are periodic in j, so, in principle, we are left with finitely many cases.

Note however that there is no dimension shift as it would be in case of a standard Thom isomorphism. The reason is that we view U' as an element in $K^0(T(nH))$ and not in $K^{2n}(T(nH))$, and so the action of ψ^k is different. We shall call this the modified Thom isomorphism.

Case 2 We next turn to $n = m \cdot p^{a-1}$, that is without the factor $p-1$. In this case we shall choose the Thom class U' in $G^*(T(nH))$. This is less explicit than for $n \equiv 0 \mod (p-1)p^{a-1}$, but it will be sufficient to obtain an explicit description for the elements in $\mathrm{Ad}^*(\widetilde{T}(nH))$. This will be important later on in Section 3 for determining the group extension between $\mathrm{Ad}^*(\widetilde{T}(nH))$ and $\mathrm{Ad}^*(T(nH))$. To determine the action of ψ^k on U' we shall use Lemma 2.1. We therefore first collect some information on $G^{2j}(B)$ and the action of ψ^k on these groups.

Setting $M = K^0(B_+)$, the minimal polynomial p_ψ of ψ^k is

$$p_\psi(t) = t^{(p-1)p^{a-1}} - 1$$

(since $k^{(p-1)p^{a-1}}-1\equiv 0 \bmod p^a$ and $H^{p^a}=1$). This splits as

$$p_\psi(t)=(t^{p^{a-1}}-1)(t^{p^{a-1}}-\omega)\cdots(t^{p^{a-1}}-\omega^{p-2})$$

with ω as above. To describe $G^{2j}(B)$ we can proceed ψ^k–orbit wise. Let $L:=H^{p^b}$, $0\le b\le a-1$, and $W=\langle L,\psi^k L,\dots\rangle$ the orbit generated by L. Then $\psi^\omega L=L^{k^{p^c}}$ with $c=a-1-b$. Hence

$$L_i:=\Phi_i(L)=\frac{1}{p-1}\sum_{r=1}^{p-1}\omega^{-r\cdot i}L^{k^{p^c\cdot r}}\ .$$

Then

(a) $\psi^{k^{p^c}}L_i=\omega^i L_i$, and

(b) $E_i:=\left\{L_i,\psi^k L_i,\dots,\psi^{k^{p^c-1}}L_i\right\}$ gives a cyclic basis for $\Phi_i(W)$.

The first claim follows directly from the definition of Φ_i. For (b) one uses

$$L_0+\omega L_1+\cdots+\omega^{p-2}L_{p-2}=L$$

to see that $E_0\cup E_1\cup\ldots\cup E_{p-2}$ is a generating set for W. Since it is a minimal generating set it must be a basis. On $\Phi_i(M)$ the Adams operation $\psi^{k^{p^{a-1}}}$ acts as multiplication by ω^i. Thus the Adams splitting of $K^0(B)$ is nothing but the splitting into generalized eigenspaces or the primary decomposition for ψ^k. From $G^{2i}(B)=u^{-i}\Phi_i(M)$ we see that

(6) $\qquad \psi^{k^{p^{a-1}}}$ acts on $G^{2i}(B)$ as multiplication by $(\omega/k)^i$.

For later use we digress for a moment and work out the Jordan decomposition of ψ^k on $K^0(B)\otimes\mathbb{F}_p$. Since $p_\psi\equiv(t-1)^{p^{a-1}}(t-\omega)^{p^{a-1}}\cdots(t-\omega^{p-2})^{p^{a-1}} \bmod p$, the minimal polynomial p_ψ of ψ^k on $M\otimes\mathbb{F}_p$ splits into linear factors and ψ^k has a Jordan canonical form. Note that $\omega\equiv k \bmod p$. Recall that if $v,\psi^k v,\dots,\psi^{k^{m-1}}v$ with $\psi^{k^m}v=\lambda v$ is a cyclic basis, then $z^{(m)}=v$, $z^{(j)}=(\psi^k-\lambda)^{m-j}(v)$, $z^{(1)}=(\psi^k-\lambda)^{m-1}(v)$ will be a Jordan basis. Hence the basis sets E_i described above give that on $\Phi_i(W)$ the matrix of ψ^k is just the Jordan block $J_{\omega^i}(p^c)$.

Corollary 2.9 *$G^{2i}(B;\mathbb{F}_p)\cong G^0(B;\mathbb{F}_p)$ as ψ^k–modules.*

Proof With $G^{2i}(B)=u^{-i}\Phi_i(M)$ and $(\omega/k)^i=1 \bmod p$ we see that the Jordan matrix of ψ^k on $G^{2i}(B;\mathbb{F}_p)$ has the Jordan blocks

$$J_1(p^0),\ J_1(p),\dots,J_1(p^{a-1})$$

independently of i. Here $J_\lambda(m)$ denotes an $m\times m$–Jordan block associated to the eigenvalue λ. □

Returning to $K^0(\widetilde{T}(nH))$, let now $n = mp^{a-1}\geq 0$, set

$$\xi = H + \psi^k H + \cdots + \psi^{k^{(p^{a-1}-1)}} H$$

and let $U' := f^* U_G(m\xi)$ with f the obvious p–local fiber homotopy equivalence between $m\xi$ and nH. Then

$$\rho^k_G(m\xi) = \rho^{k^{p^{a-1}}}_G(mH)$$

as in Corollary 2.3. Hence

$$\psi^k U' = \rho^k_G(m\xi)U' = \rho^{k^{p^{a-1}}}_G(mH)U'$$

and
$$\begin{aligned}\psi^{k^{p^{a-1}}} U_G(mH) &= \rho^{k^{p^{a-1}}}_G(mH)\cdot U_G(mH)\\ &\equiv \left(\frac{\omega}{k^{p^{a-1}}}\right)^m U_G(mH) = \left(\frac{\omega}{k}\right)^n U_G(mH)\end{aligned}$$

mod $\ker j^*\colon G^{2n}(T(nH))\longrightarrow G^{2n}(B)$ by (6). Therefore

$$\psi^k U' = \left(\frac{\omega}{k}\right)^n U' \bmod \ker j^* \quad \text{in } G^{2n}(T(nH))\subset K^{2n}(T(nH)). \tag{7}$$

As above we have that $\mathrm{reg}\cdot U'$ generates $\ker j^*$ and $z\cdot\mathrm{reg} = 0$ for z of dimension 0. Multiplication by k^n puts the Thom class U' into $K^0(T(nH))$ and shows

Proposition 2.10 *For $n = mp^{a-1}$ the ψ^k–module $K^0(\widetilde{T}(nH))$ is a permutation representation of ψ^k with permutation basis*

$$\left\{(H^{p^c\cdot k^i}-1)\omega^{n\cdot i}\cdot U' \;\middle|\; 0\leq c\leq a-1,\ 0\leq i\leq s_c = (p-1)p^{a-c-1}\right\}$$

and is thus ψ^k–equivariantly isomorphic to $K^0(B)$.

Since $\omega^n\equiv 1 \bmod p^a$ we can get a ψ^k–equivariant relative Thom isomorphism by introducing $\mathbb{Z}/p^a$ coefficients (if $n\not\equiv 0 \bmod (p-1)$). Note that Proposition 2.10 gives explicit formulas for the elements in $\mathrm{Ad}^*(\widetilde{T}(nH))$. For example, $\mathrm{Ad}^*(\widetilde{T}(nH);\mathbb{Z}/p^b)$ for $b = a+\nu_p(n)$ is generated by the elements

$$x_c = p^c\cdot\sum_{i=0}^{s_c-1}\left(\frac{\omega}{k}\right)^{n\cdot i}((H^{p^c\cdot k^i}-1)\cdot U'), \qquad 0\leq c\leq a-1. \tag{8}$$

Case 3 We now return to general values for $n \geq 0$ and discuss first what can be said about $\mathrm{Ad}^*(\widetilde{T}(nH))$ without knowing that $K^0(\widetilde{T}(nH))$ is a ψ^k–permutation representation.

The short exact sequence

$$0 \to K^0(\widetilde{T}(nH)) \to K^0(B) \to K^0(B^{2n}) \to 0 \tag{9}$$

shows that this is true rationally, ie $K^0(\widetilde{T}(nH)) \otimes \mathbb{Q}$ is a permutation representation of ψ^k isomorphic to $K^0(B) \otimes \mathbb{Q}$, simply since $K^0(B^{2n})$ is finite. As a consequence we know the rank of $\mathrm{Ad}^{2i}(\widetilde{T}(nH)) = \ker \psi^k - 1$. This must be the same as the rank of $\mathrm{Ad}^{2i}(B)$. Hence

$$\mathrm{Ad}^{2i}(\widetilde{T}(nH)) \cong \begin{cases} 0 & i \neq 0, \\ (\mathbb{Z}_p^\wedge)^a & i = 0. \end{cases} \tag{10}$$

The number of elements in $\mathrm{Ad}^{2i+1}(\widetilde{T}(nH))$, $i \neq 0$, can be computed from (9). The cofiber sequence $B^{2n} \to B \to \widetilde{T}(nH)$ induces the exact sequence of finite groups

$$0 \to \mathrm{Ad}^{2i}(B^{2n}) \to \mathrm{Ad}^{2i+1}(\widetilde{T}(nH)) \to \mathrm{Ad}^{2i+1}(B) \to \mathrm{Ad}^{2i+1}(B^{2n}) \to 0$$

for $i \neq 0$. The groups $\mathrm{Ad}^{2i+1}(\widetilde{T}(nH))$ and $\mathrm{Ad}^{2i+1}(B)$ will have the same number of elements if this is true for $\mathrm{Ad}^{2i}(B^{2n})$ and $\mathrm{Ad}^{2i+1}(B^{2n})$. But this follows from the exact sequence

$$0 \to \mathrm{Ad}^{2i}(B^{2n}) \to K^{2i}(B^{2n}) \xrightarrow{\psi^k - 1} K^{2i}(B^{2n}) \to \mathrm{Ad}^{2i+1}(B^{2n}) \to 0$$

and the fact that $K^0(B^{2n})$ is finite. Hence for all n,

$$\nu_p \left| \mathrm{Ad}^{2i+1}(\widetilde{T}(nH)) \right| = \nu_p \left| \mathrm{Ad}^{2i+1}(B) \right| .$$

Also the exponent of $\mathrm{Ad}^{2i+1}(\widetilde{T}(nH))$ can be determined quite easily. From

$$\Big(\sum_{j=0}^{s_0 - 1} \psi^{k^j} \Big) \circ (\psi^k - 1) = \psi^{k^{s_0}} - 1,$$

we have $\psi^{k^{s_0}} = k^{i \cdot s_0}$ on $K^{2i}(\widetilde{T}(nH)) \subset K^{2i}(\widetilde{T}(nH)) \otimes \mathbb{Q}_p^\wedge \cong K^0(B) \otimes \mathbb{Q}_p^\wedge$ and $\nu_p(k^{i \cdot s_0} - 1) = a + \nu_p(i)$ it follows that for any $z \in K^{2i}(\widetilde{T}(nH))$ we must have $\triangle(p^{a+\nu_p(i)} \cdot z) = 0$. Hence

$$p^{a+\nu_p(i)} \cdot \mathrm{Ad}^{2i+1}(\widetilde{T}(nH)) = 0 .$$

It is also possible to show that the number of cyclic summands in $\mathrm{Ad}^{2i+1}(\widetilde{T}(nH))$ is a. We indicate only the main steps. The first step is to show that $\mathrm{Ad}^1(\widetilde{T}(nH))$ is torsion free (and hence isomorphic to $(\mathbb{Z}_p^\wedge)^a$). This can be done by showing that the boundary map in the exact sequence

$$(11)\quad 0 \to \mathrm{Ad}^0(\widetilde{T}(nH)) \to \mathrm{Ad}^0(B) \to \mathrm{Ad}^0(B^{2n}) \xrightarrow{\partial} \mathrm{Ad}^1(\widetilde{T}(nH)) \to \mathrm{Ad}^1(B) \to$$

is always zero. This is proved by showing that the restriction maps $\mathrm{Ad}^0(B^{2n}) \to \mathrm{Ad}^0(B^{2n-2})$ are all onto and using that $\partial = 0$ for $n = m \cdot p^{a-1}$ which follows from Corollary 2.8. Then one knows $\mathrm{Ad}^i(\widetilde{T}(nH); \mathbb{F}_p) \cong (\mathbb{F}_p)^a$ for all n and all $i \equiv 0, 1 \bmod 2(p-1)$ by Adams periodicity. Using the Thom isomorphism for $p^a H$ in $\mathrm{Ad}^*(-; \mathbb{F}_p)$ (see Section 2, Section 3 and the proof of Corollary 4.5), one gets the same conclusion for the other values of i.

The Thom isomorphism transforms (9) into a relative Gysin sequence of nH

$$0 \to K^0(B) \xrightarrow{x^n} K^0(B) \longrightarrow K^0(B^{2n}) \to 0$$

with $x^n = (H-1)^n$ the Euler class of nH. Hence to find a ψ^k–permutation basis for $K^0(\widetilde{T}(nH))$ is the same as to find one for $x^n \cdot K^0(B) \subset K^0(B)$. This is possible for small values of n, but becomes seemingly intractable for larger n.

Example 2.11 $(n = 1)$ For simplicity we take $a = 2$, the general case is similar. Let $A = (H-1)^2$ and $B = (H^p - 1) - \sum_{i=0}^{p-1} (H^{k^{(p-1)i}} - 1)$, then

$$\{A, \psi^k A, \dots, \psi^{k^{(p-1)p-1}} A\} \cup \{B, \psi^k B, \dots, \psi^{k^{p-2}} B\}$$

is a permutation basis of $x \cdot K^0(B)$.

Instead of trying to construct an explicit permutation basis of $K^0(\widetilde{T}(nH))$ for the values of n not divisible by p^{a-1} we shall give in Section 4 only an existence proof. Using this, we have:

Theorem 2.12 *There exists a* ψ^k*–equivariant isomorphism*

$$g\colon K^0(\widetilde{T}(nH)) \longrightarrow K^0(B)\,.$$

Corollary 2.13 *For all* j *and* n,

$$\mathrm{Ad}^j(\widetilde{T}(nH)) \cong \mathrm{Ad}^j(B)\,.$$

In particular, $$\mathrm{Ad}^{2n+1}(\widetilde{T}(nH)) \cong \oplus_{i=1}^a \mathbb{Z}/p^{i+\nu_p(n)} \quad (n \neq 0).$$

Remark The paper [7] contains a very short computation of the $\mathrm{Im}(J)$–groups $\mathrm{Ad}^{2m+1}(B/B^{2n}) = \mathrm{Ad}^{2m+1}(\widetilde{T}(nH))$, but the proof given there is, at least for $a > 2$, in contradiction with Corollary 3.9.

3 The connective Im(J)–groups of $B\mathbb{Z}/p^a$

In this section we compute the connective Im(J)–groups of $B = B\mathbb{Z}/p^a$ using the results of Section 2 and Section 4.

Let k denote the p–local connective complex K–theory spectrum, then for $p \neq 2$ connective Im(J)–theory A may also be defined by the cofiber sequence

$$\to A \xrightarrow{D} k \xrightarrow{Q} \Sigma^2 k \xrightarrow{\triangle} \Sigma A \to \tag{12}$$

where Q satisfies $u \cdot Q = \psi^k - 1$. This gives $A_*(B)$ as kernel and cokernel of Q:

$$0 \to A_{2n-1}(B) \xrightarrow{D} k_{2n-1}(B) \xrightarrow{Q} k_{2n-3}(B) \xrightarrow{D} A_{2n-2}(B) \to 0 \tag{13}$$

The groups $k_{2n-1}(B) \cong K_1(B^{2n}) \cong K^0(B^{2n-1})$ are known, even a set of (relation free) generators [15]. Nevertheless the action of Q on these generators seems to be too involved for a direct computation (except for $p = 2$ and $a = 2$ [16]).

Define the spectrum E as the fiber of $(\psi^k - 1)\colon k \to k$, then the resulting commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \to & A_{2n-1}(B) & \to & k_{2n-1}(B) & \xrightarrow{Q} & k_{2n-3}(B) & \to & A_{2n-2}(B) & \to & 0 \\
 & & \downarrow\cong & & \| & & \downarrow u & & \downarrow & & \\
0 & \to & E_{2n-1}(B) & \longrightarrow & k_{2n-1}(B) & \xrightarrow{\psi^k - 1} & k_{2n-1}(B) & \to & E_{2n-2}(B) & \to & 0 \\
 & & & & & & \downarrow ch_0 & & \downarrow & & \\
 & & & & & & H_{2n-1}(B) & = & H_{2n-1}(B) & &
\end{array}$$

shows $|E_{2n-1}(B)| = |E_{2n-2}(B)|$ since $k_{2n-1}(B)$ is finite. Because ch_0 is onto we therefore get:

Proposition 3.1 $\nu_p\,|A_{2n-2}(B)| = \nu_p\,|A_{2n-1}(B)| - a$.

The facts that one may in (12) replace k by l, the Adams summand of connective p–local K–theory, and that $l_{2n-1}(B\mathbb{Z}/p)$ is cyclic, give the computation for $a = 1$:

$$A_{2n-2}(B\mathbb{Z}/p) \cong \mathbb{Z}/p^{\nu_p(n)}$$

(from $A_{2n-1}(B\mathbb{Z}/p) \cong \mathbb{Z}/p^{1+\nu_p(n)}$).

Instead of a direct computation via (13) we reduce the determination of $A_*(B)$ to a computation of nonconnected Im(J)–groups. The nonconnected Im(J)–groups of

BG, G a finite group, are known [12; 13]. The case of a cyclic group is particularly simple. Since $K_1(B)$ is a ψ^k–permutation representation, the computation of $\mathrm{Ad}^*(B)$ in Section 2 may be copied to give

$$\mathrm{Ad}_i(B) = \begin{cases} (\mathbb{Z}_{p^\infty})^a & i = -1, -2, \\ \bigoplus_{j=1}^a \mathbb{Z}/p^{j+\nu_p(n)} & i = 2n-1,\ n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively one may use duality; see (15).

In [12] it is shown that for every finite group G the canonical map

$$d\colon A_{2n-1}(BG) \longrightarrow \mathrm{Ad}_{2n-1}(BG)$$

is onto for $n \geq n_0(G)$, with the constant $n_0(G)$ depending only on G. Since for a cyclic group $G = \mathbb{Z}/p^a$ the map d is always injective, we have

$$A_{2n-1}(B\mathbb{Z}/p^a) \cong \mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^a) \qquad \text{for } n \geq n_0(\mathbb{Z}/p^a).$$

A bound for $n_0(a) := n_0(B\mathbb{Z}/p^a)$, which is easy to write down is $(a+1)\cdot p^a$, this can be proved using the results of [10, Appendix]. A way to determine $A_{2n-1}(B)$ in the range $n < n_0(a)$ using results on $A_{2n-1}(P_\infty\mathbb{C})$ is given in [10]. For the relation to $P_\infty\mathbb{C}$ see also [12].

In the range $n \geq n_0(a)$ we shall use the following approach to $A_{2n-2}(B)$, which describes this group as the cokernel of $\psi^k - 1$ on a group of infinite order:

By definition of A_* as the (-1)–connected cover of Ad_* we have

$$A_{2n-2}(B) = \mathrm{Im}(i_*\colon \mathrm{Ad}_{2n-2}(B^{2n-2}) \to \mathrm{Ad}_{2n-2}(B^{2n-1}))\ .$$

The Ad_*–theory exact sequence of the pair (B^{2n-1}, B^{2n-2}) then shows

$$A_{2n-2}(B) \cong \mathrm{tor}\,\mathrm{Ad}_{2n-2}(B^{2n-1}) \tag{14}$$

(and $\mathrm{Ad}_{2n-2}(B^{2n-1}) = \mathbb{Z}_{(p)} \oplus A_{2n-2}(B)$).

Simply for convenience reasons we now pass to Ad–cohomology at this point. The universal coefficient formula of $\mathrm{Im}(J)$–theory [13]

$$\begin{aligned} 0 \to \mathrm{Ext}(\,\mathrm{Ad}_{2n-2}(B^{2n-1}), \mathbb{Z}_{(p)}) & \\ \to \mathrm{Ad}^{2n}(B^{2n-1}) \to \mathrm{Hom}(\mathrm{Ad}_{2n-1}(B^{2n-1}), \mathbb{Z}_{(p)}) \to 0 & \end{aligned} \tag{15}$$

then gives

$$A_{2n-2}(B) \cong \mathrm{tor}\,\mathrm{Ad}^{2n}(B^{2n-1})\ . \tag{16}$$

Using the exact sequence induced by the cofiber sequence (1)

$$(17)\qquad \begin{aligned} 0 \to \mathrm{Ad}^{2n}(B^{2n-1}) \to \mathrm{Ad}^{2n+1}(T(nH)) \\ \xrightarrow{j^*} \mathrm{Ad}^{2n+1}(B) \xrightarrow{i^*} \mathrm{Ad}^{2n+1}(B^{2n-1}) \to 0 \end{aligned}$$

and the easily verified fact that $i^*\colon \mathrm{Ad}^{2n+1}(B) \longrightarrow \mathrm{Ad}^{2n+1}(B^{2n-1})$ is injective if and only if $d\colon A_{2n-1}(B) \longrightarrow \mathrm{Ad}_{2n-1}(B)$ is onto, we arrive at the following:

Proposition 3.2 *$A_{2n-2}(B) \cong \mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH))$ for $n \geq n_0(a)$.*

The advantage of this description is that $\mathrm{Ad}^{2n+1}(T(nH))$ is the cokernel of $\psi^k - 1$ on $K^{2n}(T(nH)) \cong K^0(B_+)$, ie we have a lift of our problem to a torsion free situation.

Remark A variant of the above is obtained by S-duality D. We have

$$\begin{aligned} A_{2n-2}(B) \cong A_{2n-2}(B^{2n-1}) \cong A_{2n-2}(B_+^{2n-1}) \\ \overset{D}{\cong} A^1((B^{2n-1})^{\tilde{\nu}}) \cong \mathrm{tor}\,\mathrm{Ad}^1((B^{2n-1})^{\tilde{\nu}}) \end{aligned}$$

with $\tilde{\nu} = -n(H-1)$ the stable normal bundle of the manifold B^{2n-1}. Again for $n \geq n_0(a)$ we get

$$A_{2n-2}(B) \cong \mathrm{tor}\,\mathrm{Ad}^1((B^{2n-1})^{\tilde{\nu}}) \cong \mathrm{tor}\,\mathrm{Ad}^{1-2n}(T(-nH)).$$

As already mentioned the reduced Thom spectra $\tilde{T}(nH) = T(nH)/S^{2n}$ have simpler ψ^k–modules than $T(nH))$, so we divide the computation of $\mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH))$ into two parts. We first determine $\mathrm{Ad}^*(\tilde{T}(nH))$ and then solve the extension which leads to $\mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH))$. From the exact sequence

$$\begin{aligned} \mathrm{Ad}^{2n}(T(nH)) \xrightarrow{i^*} \mathrm{Ad}^{2n}(S^{2n}) \to \mathrm{Ad}^{2n+1}(\tilde{T}(nH)) \\ \to \mathrm{Ad}^{2n+1}(T(nH)) \to \mathrm{Ad}^{2n+1}(S^{2n}) \end{aligned}$$

and degree$(i^*) = p^a$ (to be proved below) we arrive at

$$(18)\qquad 0 \to \mathbb{Z}/p^a \to \mathrm{Ad}^{2n+1}(\tilde{T}(nH)) \to \mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH)) \to 0\ .$$

The results of Section 2 and Section 4 give $\mathrm{Ad}^{2n+1}(\tilde{T}(nH))$, so we are left with determining the group extension.

Before doing this, we pause to discuss the $\mathrm{Im}(J)$–cohomology groups of B. By definition we have

$$A^i(B) = \mathrm{Im}(\mathrm{Ad}^i(B, B^{i-1}) \to \mathrm{Ad}^i(B, B^{i-2}))$$

hence for $n > 0$,

$$A^{2n+1}(B) = \mathrm{Im}(\mathrm{Ad}^{2n+1}(\widetilde{T}(nH)) \to \mathrm{Ad}^{2n+1}(T(nH))) = \mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH))$$

and
$$A^{2n}(B) = \mathrm{Im}(\mathrm{Ad}^{2n}(T(nH)) \to \mathrm{Ad}^{2n}(\widetilde{T}((n-1)H))) = 0$$

by (10), whereas for $n \le 1$,

$$A^n(B) = \mathrm{Ad}^n(B)$$

by connectivity reasons. So the computation of $\mathrm{tor}\,\mathrm{Ad}^{2n+1}(T(nH))$ will also give $A^{2n+1}(B)$.

Corollary 3.3 *For* $n \ge n_0(a)$, $A^{2n+1}(B) \cong A_{2n-2}(B)$.

Note also that for $n \ge n_0(a)$ the canonical map

$$d\colon A^{2n+1}(B) \longrightarrow \mathrm{Ad}^{2n+1}(B)$$

is zero, since it is induced by the homomorphism j^* of (17), and this vanishes. This also shows that the complex e–invariant of Adams (defined with periodic K–theory) is zero on $\pi_S^{2n+1}(B)$ (see Crabb and Knapp [5] for the relation between the e–invariant and Im(J)–theory). But, of course, $\pi_S^{2n+1}(B) = 0$ for $n \ge 0$ by the Segal conjecture. Hence this is merely an example for the different behavior of Im(J)–theory and stable cohomotopy on spaces like B. With respect to stable homotopy Im(J)–homology is more interesting.

We first compute the degree of $i^*\colon \mathrm{Ad}^{2n}(T(nH)) \longrightarrow \mathrm{Ad}^{2n}(S^{2n})$.

Proposition 3.4 $i^*\colon \mathrm{Ad}^{2n}(T(nH)) \longrightarrow \mathrm{Ad}^{2n}(S^{2n})$ *is a multiplication by* p^a.

Proof Let z be a generator of $\mathrm{Ad}^{2n}(T(nH)) \cong \mathbb{Z}_{(p)}$. Then $\mathrm{Ad}^{2n}(B) = 0$ implies that z is in the image of $\delta\colon \mathrm{Ad}^{2n-1}(B^{2n-1}) \to \mathrm{Ad}^{2n}(T(nH))$, where we are using (1). Hence $D(z)$ is in $\ker j^*\colon K^{2n}(T(nH)) \to K^{2n}(B)$. From Lemma 2.1 we then have $D(z) = \gamma_1 \cdot \mathrm{reg} \cdot U_K(nH)$. On the other hand $\psi^k(\mathrm{reg} \cdot U_K(nH)) = \mathrm{reg} \cdot U_K(nH)$, hence $\mathrm{reg} \cdot U_K(nH) = \gamma_2 \cdot D(z)$ for suitable constants γ_1 and γ_2. Thus $\mathrm{reg} \cdot U_K(nH)$ generates $\mathrm{Im}(D\colon \mathrm{Ad}^{2n}(T(nH)) \to K^{2n}(T(nH)))$ and $i^*(\mathrm{reg} \cdot U_K(nH)) = p^a$ implies the statement. □

We now solve the group extension (18) for n with $\nu_p(n) \ge a-1$. Let $U' \in K^{2n}(T(nH))$ be the Thom class constructed in equation (7) of Section 2 which satisfies $\psi^k U' = (\omega/k)^n U' \bmod \ker j^*$ and set $\Omega = (\omega/k)^n$, $b = a + \nu_p(n)$. Then by (8) the elements

$$x_c = p^c \cdot \sum_{i=0}^{s_c-1} \Omega^i((H^{p^c \cdot k^i} - 1) \cdot U'), \qquad c \in \{0, 1, \ldots, a-1\}$$

generate cyclic subgroups whose direct sum is $\mathrm{Ad}^{2n}(\widetilde{T}(nH);\mathbb{Z}/p^b)$. We change this basis by changing the generator x_{a-1} into

$$\overline{x}_{a-1} = x_{a-1} + p\cdot x_{a-2} + \cdots + p^{a-1}\cdot x_0 \ .$$

Note that

$$p^{\nu_p(n)-a+1}\cdot \overline{x}_{a-1} = p^{\nu_p(n)}\cdot \sum_{j=1}^{p^a-1}(H^j-1)\cdot U' = p^{\nu_p(n)}\widetilde{\mathrm{reg}}\cdot U' \tag{19}$$

with $\widetilde{\mathrm{reg}} = \sum_{j=1}^{p^a-1}(H^j-1)$ since $\Omega \equiv 1 \mod p^a$.

Consider now the commutative diagram of exact sequences built up by Bockstein and cofiber sequences ($\widetilde{T}_n = \widetilde{T}(nH)$, $T_n = T(nH)$)

$$\begin{tikzcd}
\mathrm{Ad}^{2n}(\widetilde{T}_n;\mathbb{Z}/p^b) \arrow[r, "j^*"] & \mathrm{Ad}^{2n}(T_n;\mathbb{Z}/p^b) \arrow[r] & \mathrm{Ad}^{2n}(S^{2n};\mathbb{Z}/p^b) & \\
\mathrm{Ad}^{2n}(\widetilde{T}_n) \arrow[u] \arrow[r] & \mathrm{Ad}^{2n}(T_n) \arrow[u, "\mathrm{red}"] \arrow[r, "i^*"] & \mathrm{Ad}^{2n}(S^{2n}) \arrow[u] \arrow[r] & \mathrm{Ad}^{1+2n}(\widetilde{T}_n) \\
\arrow[u] & \arrow[u] \arrow[r] & \mathrm{Ad}^{2n}(S^{2n}) \arrow[u, "p^b"] \arrow[r, "\delta"] & \mathrm{Ad}^{1+2n}(\widetilde{T}_n) \arrow[u] \\
& & \arrow[u] & \mathrm{Ad}^{2n}(\widetilde{T}_n;\mathbb{Z}/p^b) \arrow[u, "\beta"]
\end{tikzcd}$$

Well known relations between the two connecting homomorphisms δ and β give

$$\delta(1) = \beta(w)$$

with $w = (j^*)^{-1}\circ \mathrm{red}\circ (i^*)^{-1}(p^b\cdot 1) \in \mathrm{Ad}^{2n}(\widetilde{T}(nH);\mathbb{Z}/p^b)$ (for a proof see Crabb and Knapp [5, Section 7]). But we have $(i^*)^{-1}(p^b\cdot 1) = p^{b-a}\,\mathrm{reg}\cdot U'$ in $\mathrm{Ad}^{2n}(T(nH)) \subset K^{2n}(T(nH))$. Hence

$$w = (j^*)^{-1}p^{b-a}\,\mathrm{reg}\cdot U' = p^{b-a}\widetilde{\mathrm{reg}}\cdot U'$$

with $\widetilde{\mathrm{reg}} = \mathrm{reg} - p^a$ in $\mathrm{Ad}^{2n}(\widetilde{T}(nH);\mathbb{Z}/p^b)$. From (19) we see that w is a multiple of $\overline{x}_{a-1}$, the generator of the cyclic summand of order $p^{1+\nu_p(n)}$. This shows that j^* maps the cyclic summands generated by x_i, $0\le i\le a-2$, injectively into $\mathrm{Ad}^{1+2n}(T(nH))$ whereas $j^*(\overline{x}_{a-1})$ has only order $p^{1+\nu_p(n)-a}$. Hence we have:

Theorem 3.5 *For $n > n_0(a)$ with $\nu_p(n) \ge a-1 \ge 1$ we have*

$$A_{2n-2}(B) \cong \bigoplus_{i=0}^{a-2}\mathbb{Z}/p^{a-i+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)-a}\ .$$

Theorem 3.6 *For $n > 0$ with $\nu_p(n) \geq a - 1 \geq 1$ we have*

$$A^{2n+1}(B) \cong \bigoplus_{i=0}^{a-2} \mathbb{Z}/p^{a-i+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)-a} .$$

Note that the elements generating $A_{2n-2}(B)$ in Theorem 3.5 are explicitly given. This will be different for the values of n with $\nu_p(n) < a - 1$, to which we turn now. Despite the fact that Corollary 2.13 gives $\mathrm{Ad}^{1+2n}(\widetilde{T}(nH))$ only abstractly, it is possible to determine the group extension (18). For this we need to know which complex vector bundles on B are orientable with respect to $\mathrm{Ad}^*(-;\mathbb{F}_p)$. Note also, that by simple connectivity reasons, any $\mathrm{Ad}^*(-;\mathbb{F}_p)$–Thom class also gives a Thom class for connective $\mathrm{Im}(J)$–theory $A^*(-;\mathbb{F}_p)$ with mod p coefficients.

The fundamental relation for $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientability is given by the following proposition.

Proposition 3.7 *The virtual bundle $p\widetilde{\xi}-\psi^p\widetilde{\xi}$ on X is $\mathrm{Ad}^*(\,;\mathbb{F}_p)$–orientable for any complex vector bundle ξ, in particular, $\psi^p\widetilde{\xi}$ is $\mathrm{Ad}^*(\,;\mathbb{F}_p)$–orientable if and only if $p\widetilde{\xi}$ is $\mathrm{Ad}^*(\,;\mathbb{F}_p)$–orientable.*

Proof Let $U = U_K(p\widetilde{\xi} - \psi^p\widetilde{\xi})$, $\widetilde{\xi} = \xi - \dim\xi$, then $\psi^k U = \rho^k(p\widetilde{\xi} - \psi^p\widetilde{\xi})U$ with

$$\rho^k(p\widetilde{\xi} - \psi^p\widetilde{\xi}) = \rho^k(p\widetilde{\xi})/\rho^k(\psi^p\widetilde{\xi}) \equiv \rho^k(\widetilde{\xi})^p/\psi^p\rho^k(\widetilde{\xi}) \equiv 1$$

since $\psi^p y = y^p \bmod p$. Therefore $U = D(U')$ for some $U' \in \mathrm{Ad}^0(T(p\widetilde{\xi}-\psi^p\widetilde{\xi});\mathbb{F}_p)$ and U' is an $\mathrm{Ad}^*(-;\mathbb{F}_p)$–Thom class for $p\widetilde{\xi} - \psi^p\widetilde{\xi}$. □

Therefore $p^a H$ has an $\mathrm{Ad}^*(-;\mathbb{F}_p)$–Thom class on B, since $H^{p^a} = 1$. We also need the reverse statement, which is harder to obtain. From [10] we shall use:

Theorem 3.8 *The vector bundle H^{p^r} is $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable on the complex projective space $P_n\mathbb{C}$ (of real dimension $2n$) if an only if $n < p^{r+1} - 1$.*

Corollary 3.9 *The line bundles $H, H^p, \ldots, H^{p^{a-1}}$ are not $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable on $B = B\mathbb{Z}/p^a$.*

Proof We have $K^0(B;\mathbb{F}_p) = \mathbb{F}_p\langle x, x^2, \ldots, x^{p^a-1}\rangle$ with $x = H - 1$ since $x^{p^a} \equiv 0$ mod p. Hence with $m = p^a - 1$,

$$K^0(B;\mathbb{F}_p) \cong K^0(B^{2m+1};\mathbb{F}_p) \underset{\cong}{\xleftarrow{\pi^*}} K^0(P_m\mathbb{C};\mathbb{F}_p) .$$

But as in the integral case one can reformulate the orientability condition as follows: The vector bundle ξ is $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable if and only if there exists a unit $e \in K^0(X_+;\mathbb{F}_p)$ with

$$\rho^k(\widetilde{\xi}) = \frac{\psi^k e}{e} \qquad \text{in } K^0(X_+;\mathbb{F}_p).$$

Given such e, then $e^{-1}\cdot U_K(\widetilde{\xi})$ will be ψ^k–invariant and vice versa. Hence the $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientability condition for $H^{p^{a-1}}$ on B, B^{2m+1} or $P_m\mathbb{C}$ are all equivalent. □

Corollary 3.10 *nH on B is $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable if an only if $\nu_p(n) \geq a$.*

Proof Write $n = n_1 p^c$ with $n_1 \not\equiv 0 \bmod p$, assume $c < a$ and choose s, b with $s\cdot n_1 + p^{a-c}b = 1$, then $s\cdot n_1 H^{p^c} + b\cdot p^{a-c}H^{p^c} = H^{p^c}$. If nH is orientable, then H^{p^c} is orientable, hence $c \geq a$, contradicting $c < a$. □

Assume now $\nu_p(n) < a$ and consider the commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Ad}^{2n}(T(nH)) & \longrightarrow & \mathrm{Ad}^{2n}(S^{2n}) & \xrightarrow{\delta} & \mathrm{Ad}^{1+2n}(\widetilde{T}(nH)) & \to \\
\downarrow & & \downarrow & & \downarrow\mathrm{red} & \\
\mathrm{Ad}^{2n}(T(nH);\mathbb{F}_p) & \xrightarrow{i^*} & \mathrm{Ad}^{2n}(S^{2n};\mathbb{F}_p) & \longrightarrow & \mathrm{Ad}^{1+2n}(\widetilde{T}(nH);\mathbb{F}_p) & \to
\end{array}$$

The restriction map $i^*\colon \mathrm{Ad}^{2n}(T(nH);\mathbb{F}_p) \to \mathrm{Ad}^{2n}(S^{2n};\mathbb{F}_p)$ is zero, since nH is not $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable. Hence $\delta(1)$, with $1 \in \mathrm{Ad}^{2n}(S^{2n})$ as generator, is not divisible by p in $\mathrm{Ad}^{1+2n}(\widetilde{T}(nH))$. Also tor $\mathrm{Ad}^{1+2n}(T(nH))$ has only $a-1$ cyclic summands. From Corollary 2.13 we have

$$\mathrm{Ad}^{1+2n}(\widetilde{T}(nH)) \cong \bigoplus_{i=1}^{a} \mathbb{Z}/p^{i+\nu_p(n)}\,.$$

Clearly we then may take $\delta(1)$ as a generator for the cyclic summand with $i = a-\nu_p(n)$. Hence the following theorems hold.

Theorem 3.11 *For n with $\nu_p(n) < a$ and $n \geq n_0(a)$ we have*

$$A_{2n-2}(B\mathbb{Z}/p^a) \cong \bigoplus_{\substack{i=1\\ i\neq a-\nu_p(n)}}^{a} \mathbb{Z}/p^{i+\nu_p(n)}\,.$$

Theorem 3.12 *For $n > 0$ with $\nu_p(n) < a$ we have*

$$A^{2n+1}(B\mathbb{Z}/p^a) \cong \bigoplus_{\substack{i=1 \\ i \neq a-\nu_p(n)}}^{a} \mathbb{Z}/p^{i+\nu_p(n)} .$$

As already mentioned the Im(J)–cohomology groups in positive dimensions are uninteresting with respect to stable cohomotopy. For homology the situation is as roughly as follows. For any a there is a constant $n_1(a) \geq n_0(a)$ such that the Hurewicz map

$$h_A\colon \pi^S_{2n-1}(B) \longrightarrow A_{2n-1}(B)$$

is split onto [12]. In the range $n < n_1(a)$ there are usually (for $a > 2$) some elements in $\operatorname{coker} h_A$. In even dimensions there is no such bound. The case $a = 1$ is investigated in [9].

4 The ψ^k–module $K^0(\widetilde{T}(nH); \mathbb{F}_p)$

In this section we prove Theorem 2.12, that is show that there exists a ψ^k–equivariant isomorphism $h\colon K^0(B) \to K^0(\widetilde{T}(nH))$. This is done in three steps. We first show that $G^0(\widetilde{T}(nH); \mathbb{F}_p)$ is isomorphic to $G^0(B; \mathbb{F}_p)$ as ψ^k–module, extend this isomorphism to $K^0(\widetilde{T}(nH); \mathbb{F}_p) \cong K^0(B; \mathbb{F}_p)$ and lift this ψ^k–isomorphism in the last step to a ψ^k–isomorphism $h\colon K^0(B) \to K^0(\widetilde{T}(nH))$. In virtue of Corollary 2.7 we shall assume $n \geq 0$.

Step 1 The ψ^k–isomorphism $G^0(\widetilde{T}(nH); \mathbb{F}_p) \cong G^0(B; \mathbb{F}_p)$

Theorem 4.1 *There is a ψ^k–equivariant isomorphism $G^0(\widetilde{T}(nH); \mathbb{F}_p) \cong G^0(B; \mathbb{F}_p)$.*

We shall prove this inductively using the commutative diagram of exact sequences induced by (3). Abbreviate $\widetilde{T}(mH)$ by $\widetilde{T}_m$ and $T(mH)$ by T_m.

$$\begin{array}{cccccc}
 & & & G^0(S^{2n+2}; \mathbb{F}_p) & = & G^0(S^{2n+2}; \mathbb{F}_p) \\
 & & & \uparrow i^* & & \uparrow p^a \\
0 & \leftarrow & G^0(\widetilde{T}_n; \mathbb{F}_p) & \xleftarrow{j^*} G^0(T_{n+1}; \mathbb{F}_p) & \xleftarrow{\delta^*} & G^0(S^{2n+2}; \mathbb{F}_p) \\
 & & \| & \uparrow & & \uparrow \cong \\
0 \leftarrow G^{-2n-1}(M_a; \mathbb{F}_p) & \xleftarrow{\bar{h}^*} & G^0(\widetilde{T}_n; \mathbb{F}_p) & \xleftarrow{\bar{j}^*} G^0(\widetilde{T}_{n+1}; \mathbb{F}_p) & \leftarrow & G^{-2n-2}(M_a; \mathbb{F}_p) \leftarrow 0
\end{array} \tag{20}$$

Note first the following elementary facts.

(1) We may assume $n + 1 \equiv 0 \mod p - 1$, since otherwise $\bar{j}^*$ is an isomorphism.

(2) The composition

$$\overline{h}^* \circ j^* \circ (i^*)^{-1}\colon G^0(S^{2n+2};\mathbb{F}_p) \longrightarrow G^{-2n-1}(M_a;\mathbb{F}_p)$$

is an isomorphism, ie $\overline{h}^*(z) \neq 0$ for $z \in G^0(\widetilde{T}_n;\mathbb{F}_p)$ implies that $(j^*)^{-1}(z)$ is a Thom class for $(n+1)H$.

(3) If U is a Thom class for $(n+1)H$ and $u^{n+1} \in G^0(S^{2n+2};\mathbb{F}_p)$ a generator, then $\delta^*(u^{n+1}) = c_0 \cdot \mathrm{reg} \cdot U$ with $c_0 \neq 0$ in $\mathbb{F}_p$. This follows from Lemma 2.1 since the kernel of $j^*\colon G^0(T_{n+1};\mathbb{F}_p) \longrightarrow G^0(\widetilde{T}_n;\mathbb{F}_p)$ is the same as the kernel of $j_2^*\colon G^{2n+2}(\widetilde{T}_{n+1};\mathbb{F}_p) \longrightarrow G^{2n+2}(B;\mathbb{F}_p)$ because $j_1\colon B \to \widetilde{T}_n$ induces a monomorphism and $j \circ j_1 = j_2$ (see (2)).

(4) We shall abbreviate $c_0 \cdot \mathrm{reg} \cdot U = \delta^*(u^{n+1})$ by w_0. Clearly $(\psi^k - 1)(w_0) = 0$.

(5) We have $\psi^{k^{p^{a-1}}} = 1$ on $G^0(\widetilde{T}_m;\mathbb{F}_p)$ for all m. This follows from the corresponding statement (6) for $G^0(B)$ using the monomorphism $j_1^*\colon G^{2i}(\widetilde{T}_m) \to G^{2i}(B)$.

Denote $\psi^k - 1$ by T. Then by (5) T acts nilpotently on $G^0(\widetilde{T}_m;\mathbb{F}_p)$. We shall determine the Jordan canonical form of T inductively using (20).

Let $S(m)$ abbreviate the statement:

There are elements $b_0, b_1, \dots, b_{a-1}$ in $G^0(\widetilde{T}_m;\mathbb{F}_p)$ with b_i generating a string of length p^i under T, ie $T^{p^i} b_i = 0$, $T^{p^i-1} b_i \neq 0$ such that $\{T^j b_i\}$ is a Jordan basis for T on $G^0(\widetilde{T}_m;\mathbb{F}_p)$.

Note that $S(0)$ was proved in Corollary 2.9. Assume now $n+1 \equiv 0 \bmod p-1$, $\nu_p(n+1) = b < a$ and set $c = a-1-b$. We shall prove below:

Proposition 4.2 *There is a Thom class U for $(n+1)H$ with*

$$T^{p^c} U = c_0 \cdot w_0$$

where $c_0 \not\equiv 0$ mod p and w_0 as in (4).

Proposition 4.3 *For any Thom class U_1 of $(n+1)H$ in $G^0(T_{n+1};\mathbb{F}_p)$ we have $T^{p^c} U_1 \neq 0$, in particular, there is no Thom class U_2 for $(n+1)H$ with $T^j U_2 = 0$ and $j \leq p^c$.*

Assume $S(n)$ is true and choose elements b_i as in $S(n)$. We now construct elements $\overline{b}_i \in G^0(\widetilde{T}_{n+1};\mathbb{F}_p)$ with the properties stated in $S(n+1)$. We begin with $G^0(T_{n+1};\mathbb{F}_p)$. In this group the Thom class U provided by Proposition 4.2 generates a string of length p^c+1. Since $T^{p^c} U = c_0 \cdot w_0 \in \ker j^*$, the element $j^*(U)$ generates a string of length p^c in $G^0(\widetilde{T}_n;\mathbb{F}_p)$. Write $j^*(U)$ as a linear combination of the Jordan basis in $G^0(\widetilde{T}_n;\mathbb{F}_p)$:

$$j^*(U) = \alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_c b_c + \text{ elements in im}(T) \ .$$

Then $$0 \neq \overline{h}^* \circ j^*(U) = \alpha_0 \overline{h}^*(b_0) + \alpha_1 \overline{h}^*(b_1) + \cdots + \alpha_c \overline{h}^*(b_c) \ ,$$

since $\overline{h}^* \circ T = 0$. If $\overline{h}^*(b_i) \neq 0$ for $i < c$ than any preimage z of b_i under j^* satisfies $i^* z \neq 0$ by (1) and therefore z is a Thom class for $(n+1)H$. But then $T^{p^i+1}(z) = 0$ with $p^i + 1 \leq p^c$ contradicting Proposition 4.3. Hence $\overline{h}^*(b_i) = 0$ for $i < c$, $\overline{h}^*(b_c) \neq 0$ and $\alpha_c \neq 0$. We now change the basis in $G^0(\widetilde{T}_n;\mathbb{F}_p)$ by replacing b_c with $j^*(U)$. Since $\overline{h}^*(b_i) = 0$ for $i < c$, we can choose preimages $\overline{b_i}$ for b_i under $\overline{j}^*$. For $\overline{j}^{*-1}(b_c)$ we take $T(U)$. If $i > c$ then $\overline{h}^*(b_i)$ may be nonzero. In such a case we change b_i into b_i' by adding a suitable multiple of $j^*(U)$ in order to get $\overline{h}^*(b_i') = 0$. Then b_i' has the same string length as b_i and we can choose $\overline{b_i}$ with $\overline{j}^*(\overline{b_i}) = b_i'$.

It is clear that the elements $\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_{a-1}$ generate $G^0(\widetilde{T}_{n+1};\mathbb{F}_p)$ as T–module. We now check their string length. For $\overline{b}_c = T(U)$ we have $T^{p^c}(\overline{b}_c) = 0$ and $T^{p^c-1}(\overline{b}_c) = w_0$ (up to a nonzero constant). If $T^{p^i}(\overline{b}_i) = \gamma_i w_0$ for $i < c$ with $\gamma_i \neq 0$ we change $\overline{b}_i$ by adding a suitable multiple of $T^{p^c-p^i-1}(\overline{b}_c)$ in order to achieve $T^{p^i}(\overline{b}_i) = 0$. For $j > c$ we must have $T^{p^j}(\overline{b}_j) = 0$. If not, then $T^{p^j}(\overline{b}_j) = w_0$ (up to a nonzero constant) and we may change in $G^0(T_{n+1};\mathbb{F}_p)$ the original Thom class U by adding $T^{p^j-p^c}(\overline{b}_j)$ to get a Thom class U_1 with $T^{p^c}(U_1) = 0$. This contradicts Proposition 4.3. Hence we may assume that for all j the element $\overline{b}_j$ generates a string of length p^j and $\{T^{p^i}(\overline{b}_j)\}$ will be a Jordan basis for $G^0(\widetilde{T}_{n+1};\mathbb{F}_p)$. This proves $S(n+1)$ and shows that $S(n)$ is true for all n. But this means that T has the same Jordan canonical form on $G^0(\widetilde{T}(nH);\mathbb{F}_p)$ for all n, and this implies that $G^0(\widetilde{T}(nH);\mathbb{F}_p)$ and $G^0(B;\mathbb{F}_p)$ are ψ^k–equivariantly isomorphic, proving Theorem 4.1.

We now prove Proposition 4.2 and Proposition 4.3 (recall the assumption $n+1 \equiv 0$ mod $p-1$, $\nu_p(n+1) = b < a$ and $c = a-1-b$):

We begin with the case $n+1 = s\cdot(p-1)p^{a-1}$, $s \not\equiv 0$ mod p. Set $U = \Phi_0(U') \in G^0(T((n+1)H);\mathbb{F}_p)$, where U' is the Thom class from (5). From Proposition 2.5 we have

$$T(U) = c_0 \cdot w_0 \qquad \text{for some } c_0 \in \mathbb{F}_p \ .$$

Let U_1 be any Thom class for $(n+1)H$. We must have $T(U_1) \neq 0$, since $(n+1)H$ is not $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable (Corollary 3.10). In particular $c_0 \neq 0$. This proves Proposition 4.2 and Proposition 4.3 for $n+1$ with $\nu_p(n+1) = a-1$.

Let now $n+1=m\cdot p^b$, $m\equiv 0 \bmod (p-1)$, $m\not\equiv 0 \bmod p$, $b\leq a-1$, $c:=a-1-b$ set $L=H^{k^c}$, $L^0=H$ and consider

$$\xi = m(L^0+L^1+L^2+\cdots+L^{p^b-1}) .$$

Then ξ is p–locally fiber homotopy equivalent to $(n+1)H$.

Lemma 4.4 $\rho_G^{k^c}(m(L^0+L^1+L^2+\cdots+L^{p^b-1}))=\rho_G^{k^{a-1}}(mH)$.

Proof This follows from

$$\rho_G^{r^i}(H)=\rho_G^{r^{i-1}}(H)\cdot\psi^{r^{i-1}}(\rho_G^r(H))=\rho_G^{r^{i-1}}(H)\cdot\rho_G^r(H^{r^{i-1}}),$$

the exponential property of ρ_G and induction as in Corollary 2.3. □

Define now $U=f^*U_G(\xi)$ with f a suitable fiber homotopy equivalence between ξ and $(n+1)H$. Then

$$\psi^{k^{p^c}}(U)=f^*\psi^{k^{p^c}}(U_G(\xi))=\rho_G^{k^{p^c}}(\xi)\cdot f^*(U_G(\xi))=\rho_G^{k^{p^{a-1}}}(mH)\cdot U .$$

But $\rho_G^{k^{p^{a-1}}}(mH)-1=c'\cdot\mathrm{reg}$ since $\psi^{k^{p^{a-1}}}=1$ on $G^0(B)$ implies $\psi^{k^{p^{a-1}}}U_G(mH)-U_G(mH)\in\ker j_1^*\colon G^0(T(mH))\to G^0(B)$ and $\ker j_1^*$ is generated by $\mathrm{reg}\cdot U_G(mH)$ Lemma 2.1. To see $c'\not\equiv 0 \bmod p$, we may use Lemma 4.4 with $c=0$ in the reverse direction: If $\rho_G^{k^{p^{a-1}}}(mH)=1$ then

$$\rho_G^{k^{p^{a-1}}}(mH)=\rho_G^k(m(H+H^k+\cdots+H^{k^{p^{a-1}-1}}))=1 ,$$

but then $m(H+H^k+\cdots+H^{k^{p^{a-1}-1}})$, which is p–locally J–equivalent to the line bundle $(p^{a-1}m)H$, would have a ψ^k–invariant Thom class in $G^0(-;\mathbb{F}_p)$, contradicting the fact that $p^{a-1}mH$ is not $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable. This proves Proposition 4.2.

Let now U_2 be any Thom class for $\omega=(n+1)H$ with m, b and c as above. Then $U_2(\omega)=e\cdot U_G(\omega)$ for a unit $e\in G^0(B_+)$. Define $\rho_2^k(\omega)$ by the equation $\psi^kU_2=\rho_2^k\cdot U_2$. Then

$$\rho_2^{k^i}(\omega)=(\psi^{k^i}(e)/e)\cdot\rho_G^{k^i}(\omega)$$

(and $\rho_2^{k^{p^c}}(\omega)=(\psi^{k^{p^c}}(e)/e)\cdot\rho_G^{k^{p^c}}(\omega)$). Assume $\psi^{k^{p^c}}U_2(\omega)=U_2(\omega)$, ie $\rho_2^{k^{p^c}}(\omega)=1$. We derive a contradiction by showing that there exists a Thom class U_3 for $\xi=\omega+\psi^k\omega+\cdots+\psi^{k^{p^c-1}}\omega$ with $\psi^kU_3=U_3$. But ξ is p–locally fiber homotopy equivalent to $p^c\omega=p^{a-1}mH$, hence ξ can not have a mod p ψ^k–invariant Thom class by Corollary 3.10.

Note that $U_2(\psi^{k^i}\omega) := \psi^{k^i}U_2(\omega)$ is a Thom class for $\psi^{k^i}\omega$ since ψ^{k^i} may be induced by the map $m_{k^i}\colon B \to B$ which represents multiplication by k^i in the H–space structure of B. Then

$$U_3 := U_2(\omega) \cup U_2(\psi^k\omega) \cup \ldots \cup U_2(\psi^{k^{p^c-1}}\omega)$$

is a mod p Thom class for $\xi = \omega + \psi^k\omega + \cdots + \psi^{k^{p^c-1}}\omega$ and we have

$$\begin{aligned}
\psi^k U_3 &= \rho_2^k(\omega)\cdot\rho_2^k(\psi^k\omega)\cdots\rho_2^k(\psi^{k^{p^c-1}}\omega)\cdot U_3\\
&= \rho_2^k(\omega)\cdot\psi^k(\rho_2^k(\omega))\cdots\psi^{k^{p^c-1}}(\rho_2^k(\omega))\cdot U_3\\
&= \tfrac{\psi^k e}{e}\cdot\rho_G^k(\omega)\cdot\tfrac{\psi^{k^2}e}{\psi^k e}\cdot\psi^k(\rho_G^k(\omega))\cdots\tfrac{\psi^{k^{p^c}}e}{\psi^{k^{p^c-1}}e}\cdot\psi^{k^{p^c-1}}(\rho_G^k(\omega))\cdot U_3\\
&= \tfrac{\psi^{k^{p^c}}e}{e}\cdot\rho_G^k(\omega)\cdot\psi^k(\rho_G^k(\omega))\cdots\psi^{k^{p^c-1}}(\rho_G^k(\omega))\cdot U_3\\
&= \tfrac{\psi^{k^{p^c}}e}{e}\cdot\rho_G^{k^{p^c}}(\omega)\cdot U_3 \qquad \text{by Corollary 2.3 for } \rho_G\\
&= \rho_2^{k^{p^c}}(\omega)\cdot U_3 = U_3.
\end{aligned}$$

This finishes the proof of Proposition 4.3.

Step 2 The ψ^k–equivariant isomorphism $K^0(\widetilde{T}(nH);\mathbb{F}_p) \cong K^0(B;\mathbb{F}_p)$

Corollary 4.5 *For all n there is a ψ^k–equivariant isomorphism*

$$K^0(\widetilde{T}(nH);\mathbb{F}_p) \cong K^0(B;\mathbb{F}_p).$$

Proof Consider $G^{2i}(\widetilde{T}(mH);\mathbb{F}_p)$ for $i \not\equiv 0$ mod p and write $i = tp^a + s(p-1)$. Then

$$G^{2i}(\widetilde{T}(mH);\mathbb{F}_p) \overset{\phi}{\cong} G^{2(i-tp^a)}(\widetilde{T}((m-tp^a)H);\mathbb{F}_p) \overset{B^s}{\cong} G^0(\widetilde{T}((m-tp^a)H);\mathbb{F}_p)$$

as ψ–modules. Here ϕ is the Thom isomorphism with a mod p ψ^k–invariant Thom class of tp^aH (see Proposition 3.7) and B^s is Adams periodicity, ie multiplication by v_1^s, which is ψ^k–invariant mod p. From Theorem 4.1, with $n = m - tp^a$ and the Jordan decomposition of ψ^k on $G^0(B;\mathbb{F}_p)$ (Corollary 2.9) we get that the Jordan canonical form of ψ^k on $G^{2i}(\widetilde{T}(mH);\mathbb{F}_p)$ has exactly one Jordan block $J_1(p^i)$ for $i = 0,\ldots,a-1$. Going backwards from the decomposition $\bigoplus_{i=0}^{p-2} u^i G^{2i}(\widetilde{T}(mH);\mathbb{F}_p)$ to $K^0(\widetilde{T}(mH);\mathbb{F}_p)$ (see Section 2), we see that ψ^k has the same Jordan canonical form on both of $K^0(\widetilde{T}(mH);\mathbb{F}_p)$ and $K^0(B;\mathbb{F}_p)$. □

Remark Since $p^i H - H^{p^i}$ is $\mathrm{Ad}^*(-;\mathbb{F}_p)$–orientable (Proposition 3.7), we have a ψ^k–equivariant relative Thom isomorphism

$$K^0(\widetilde{T}\Big(\bigoplus_{i=0}^{r} n_i H^{p^i}\Big);\mathbb{F}_p) \cong K^0(\widetilde{T}\Big(\bigoplus_{i=0}^{r} n_i p^i H\Big);\mathbb{F}_p)\ ,$$

hence the statement of Corollary 4.5 is true for every complex vector bundle ξ on B.

Step 3 Lifting mod p isomorphisms

In the rest of this section we show how to lift a ψ^k–map $K^0(B;\mathbb{F}_p) \to K^0(\widetilde{T}(nH);\mathbb{F}_p)$ to integral K–theory. Denote $K^0(B)$ by M and $K^0(\widetilde{T}(mH))$ by M'. Additively $M = (\mathbb{Z}_p^\wedge)^{p^a-1}$, $M' = (\mathbb{Z}_p^\wedge)^{p^a-1}$. Then $M/p = M \otimes \mathbb{F}_p = K^0(B;\mathbb{F}_p)$ and similarly for M'.

Let now M_1, M_2 be two $\mathbb{Z}_p^\wedge$–modules with ψ^k–action. Denote by $\mathrm{Hom}_\Gamma(M_1,M_2)$ the $\mathbb{Z}_p^\wedge$–submodule of ψ^k–commuting homomorphisms in $\mathrm{Hom}_{\mathbb{Z}_p^\wedge}(M_1,M_2)$. If M_1, M_2 are free and of finite rank, then $\mathrm{Hom}_\Gamma(M_1,M_2) \subset \mathrm{Hom}_{\mathbb{Z}_p^\wedge}(M_1,M_2)$ is also free and of finite rank. Define

$$b(M_1,M_2) = \mathrm{rank}\ \mathrm{Hom}_\Gamma(M_1,M_2)\ .$$

Reducing mod p we consider the $\mathbb{F}_p$–vector spaces M_1/p, M_2/p and define

$$c(M_1,M_2) = \dim_{\mathbb{F}_p} \mathrm{Hom}_\Gamma(M_1/p, M_2/p)\ .$$

There is a canonical map

$$\phi\colon \frac{\mathrm{Hom}_\Gamma(M_1,M_2)}{p\cdot \mathrm{Hom}_\Gamma(M_1,M_2)} \longrightarrow \mathrm{Hom}_\Gamma(M_1/p, M_2/p) \tag{21}$$

which is trivially injective for M_1, M_2 as above. From the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_\Gamma(M_1,M_2) \cong (\mathbb{Z}_p^\wedge)^{b(M_1,M_2)} & & \\ \downarrow p & & \\ \mathrm{Hom}_\Gamma(M_1,M_2) \cong (\mathbb{Z}_p^\wedge)^{b(M_1,M_2)} & & \\ \downarrow & & \\ \frac{\mathrm{Hom}_\Gamma(M_1,M_2)}{p\cdot\mathrm{Hom}_\Gamma(M_1,M_2)} \cong (\mathbb{F}_p)^{b(M_1,M_2)} & \xrightarrow{\phi} & \mathrm{Hom}_\Gamma(M_1/p, M_2/p) \end{array} \tag{22}$$

it is clear that $b(M_1,M_2) \le c(M_1,M_2)$. There are examples where this inequality is strict, however it is an equality in the following circumstance.

Proposition 4.6 *Suppose $N = M_1 = M_2 = (\mathbb{Z}_p^\wedge)^b$ is a permutation representation of ψ^k, then $b(N,N) = c(N,N)$, ie the canonical map*

$$\phi\colon \frac{\mathrm{Hom}_\Gamma(N,N)}{p\cdot \mathrm{Hom}_\Gamma(N,N)} \longrightarrow \mathrm{Hom}_\Gamma(N/p, N/p)$$

is an isomorphism.

Proof If we let ψ^k act on $\mathrm{Hom}_{\mathbb{Z}_p^\wedge}(M_1, M_2)$ by $\psi^k(f) = \psi^k \circ f \circ (\psi^k)^{-1}$, then $\mathrm{Hom}_\Gamma(M_1, M_2)$ is the fixed submodule $\mathrm{Hom}_{\mathbb{Z}_p^\wedge}(M_1, M_2)^{\psi^k}$. For M_1, M_2 as above, we have $\mathrm{Hom}_{\mathbb{Z}_p^\wedge}(M_1, M_2) \cong M_1^* \otimes M_2$ and therefore

$$\mathrm{Hom}_\Gamma(M_1, M_2) \cong \ker(\psi^k - 1\colon M_1^* \otimes M_2 \to M_1^* \otimes M_2).$$

Similarly

$$\mathrm{Hom}_\Gamma(M_1/p, M_2/p) \cong \ker(\psi^k - 1\colon M_1^*/p \otimes M_2/p \to M_1^*/p \otimes M_2/p).$$

If N is a permutation representation of ψ^k, then N^* and $N^* \otimes N$ are permutation representations of ψ^k too. For a permutation representation V of ψ^k which is free over $\mathbb{Z}_p^\wedge$ of finite rank, V/p is a ψ^k–permutation representation over $\mathbb{F}_p$ and we have

$$\dim_{\mathbb{F}_p} \ker(\psi^k - 1\colon V/p \longrightarrow V/p) = \mathrm{rank}\ \ker(\psi^k - 1\colon V \longrightarrow V)$$

as one easily sees by considering each ψ^k–orbit separately. With $V = N^* \otimes N$ the conclusion follows. Note that the equality above is equivalent to the statement that $\mathrm{coker}(\psi^k - 1\colon V \longrightarrow V)$ is torsion free. □

Consider now the ψ^k–modules $M = K^0(B)$ and $M' = K^0(\widetilde{T}(mH))$. On both M and M' the operation of ψ^k satisfies $(\psi^k)^{(p-1)p^{a-1}} - 1 = 0$, hence the action of ψ^k factorizes through the action of the finite group $\mathbb{Z}/(p-1)p^{a-1}$, ie we can let $\Gamma = \mathbb{Z}/(p-1)p^{a-1}$. From (9) we know that there exists a ψ^k–equivariant isomorphism $M \otimes \mathbb{Q}_p^\wedge \cong M' \otimes \mathbb{Q}_p^\wedge$. Hence

$$\begin{aligned} b(M,M') &= \mathrm{rank}\,\mathrm{Hom}_\Gamma(M,M') \\ &= \dim_{\mathbb{Q}_p^\wedge} \mathrm{Hom}_\Gamma(M,M') \otimes \mathbb{Q}_p^\wedge \\ &= \dim_{\mathbb{Q}_p^\wedge} \mathrm{Hom}_\Gamma(M \otimes \mathbb{Q}_p^\wedge, M' \otimes \mathbb{Q}_p^\wedge) \\ &= \dim_{\mathbb{Q}_p^\wedge} \mathrm{Hom}_\Gamma(M \otimes \mathbb{Q}_p^\wedge, M \otimes \mathbb{Q}_p^\wedge) \\ &= b(M,M). \end{aligned}$$

From Corollary 4.5 we have a ψ^k–equivariant isomorphism $h'\colon M/p \cong M'/p$. Hence

$$c(M,M') = \dim_{\mathbb{F}_p} \mathrm{Hom}_\Gamma(M/p, M'/p) = \dim_{\mathbb{F}_p} \mathrm{Hom}_\Gamma(M/p, M/p) = c(M,M).$$

Therefore $b(M, M') = c(M, M')$ ie the canonical map ϕ is onto. By (22) the mod p isomorphism $h'\colon M/p \to M'/p$ can be lifted to a ψ^k–equivariant homomorphism

$$h\colon M \longrightarrow M' .$$

This map is automatically an isomorphism since its mod p reduction h' is so. Observe that any basis of M/p can be lifted to a basis of M by lifting the corresponding idempotents [6]. We have proved the following.

Theorem 4.7 *For $n \in \mathbb{Z}$ there is a ψ^k–equivariant isomorphism*

$$h\colon K^0(B) \longrightarrow K^0(\widetilde{T}(nH)) .$$

5 Applications

In this section we discuss some applications of the results of Section 2–Section 4.

Let $L_K X$ denote the localization of the spectrum X with respect to p–local periodic K–theory [3]. For spectra like B or $\widetilde{T}(nH)$ with trivial rational homology we simply have

$$L_K X \simeq X \wedge \mathrm{Ad} .$$

The results in Section 2–Section 4 mean that $\pi_i(L_K B) \cong \pi_i(L_K \widetilde{T}(nH))$, but more is true:

Corollary 5.1 *The K–localizations of the spectra B and $\widetilde{T}(nH)$ are equivalent:*

$$L_K B \simeq L_K \widetilde{T}(nH) .$$

Proof This is a simple application of the results of [4]. Since $K_0(X) = 0$ for $X = B, \widetilde{T}(nH)$ both spectra are generalized Moore spectra in the sense of [4]. For such spectra the Adams spectral sequence based on G_*–theory

$$\mathrm{Ext}^{s,t}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH))) \Rightarrow \big[L_K B, L_K \widetilde{T}(nH)\big]_*$$

is trivial and there is a canonical isomorphism

$$\big[L_K B, L_K \widetilde{T}(nH)\big]_0 \cong \bigoplus_{s=0}^{2} \mathrm{Ext}^{s,s}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH)))$$

See Bousfield [4, Section 9]. Now $\mathrm{Ext}^{1,1}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH))) = 0$ because of degree reasons and the group $\mathrm{Ext}^{2,2}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH)))$ vanishes because $G_*(\widetilde{T}(nH))$ is an injective abelian group [4, 7.8], hence

$$\big[L_K B, L_K \widetilde{T}(nH)\big]_0 \cong \mathrm{Hom}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH))) .$$

With $\mathrm{Hom}_{\mathcal{B}}(\ ,\ )$ denoting the morphisms in the category of ψ^k–modules $\mathcal{B}$ introduced in [4], we have

$$\mathrm{Hom}_{G_*(G)}(G_*(B), G_*(\widetilde{T}(nH)) \cong \bigoplus_{i=0}^{2p-3} \mathrm{Hom}_{\mathcal{B}}(G_i(B), G_i(\widetilde{T}(nH)))$$

and $\mathrm{Hom}_{\mathcal{B}}(A, B)$ is the kernel of the map $d_1\colon \mathrm{Hom}_{\mathbb{Z}_{(p)}}(A, B) \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(A, B)$ where $d_1(f) = f \circ \psi^k - \psi^k \circ f$. Thus any set of $\mathbb{Z}_{(p)}$–homomorphisms $g_i\colon G_i(B) \to G_i(\widetilde{T}(nH))$, $i = 0, ..., 2p-3$, which commute with the action of ψ^k is induced by a map $\overline{g}\colon L_K B \to L_K \widetilde{T}(nH)$.

For $X \in \{B, \widetilde{T}(nH)\}$ the universal coefficient formula in K–theory gives

$$K_1(X) \cong \varinjlim \mathrm{Hom}(K^0(X^{(n)}); \mathbb{Q}/\mathbb{Z}) =: \mathrm{Hom}_c(K^0(X) : \mathbb{Q}/\mathbb{Z}) .$$

Now let $g''\colon K^0(\widetilde{T}(nH)) \to K^0(B)$ be a ψ^k–equivariant isomorphism which exists by Theorem 2.12. Since the p–adic topology on $K^0(X)$ for $X = B$, $\widetilde{T}(nH)$ is the same as the one given by the skeletal filtration, g'' will induce a continuous map between $K^0(\widetilde{T}(nH))$ and $K^0(B)$, ie we get

$$g'\colon K_1(B) \longrightarrow K_1(\widetilde{T}(nH))$$

by duality. Clearly g' is a ψ^k–equivariant isomorphism too. Using the splitting of p–local K–theory, g' will induce a set of ψ^k–equivariant isomorphisms

$$g_i\colon G_i(B) \longrightarrow G_i(\widetilde{T}(nH)) .$$

The discussion above then shows that g is induced by a map

$$\overline{g}\colon L_K B \longrightarrow L_K \widetilde{T}(nH) ,$$

which is an equivalence since it induces an isomorphism in homotopy groups. □

The computation of $A_{2n-2}(B\mathbb{Z}/p^a)$ was only done for $n \geq n_0(a)$ in Section 2–Section 4. For $a = 2$ it is not hard to complete the computation. Instead of giving all the details of the computation, we only explain the method and state the result. First of all the constant $n_0(2)$ is smaller than $3p^2$. Next the order of $A_{2n-1}(B\mathbb{Z}/p^2)$ as a subgroup of $\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^2) = \mathbb{Z}/p^{2+\nu_p(n)} \oplus \mathbb{Z}/p^{1+\nu_p(n)}$ has to be computed. The subgroup $\mathbb{Z}/p^{2+\nu_p(n)}$ of $\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^2)$ can always be chosen to be in $A_{2n-1}(B\mathbb{Z}/p^2)$ [12]. So it is only necessary to compute the skeleton filtration of the elements in the second summand $\mathbb{Z}/p^{1+\nu_p(n)}$. The necessary information for doing this in [10, Appendix]. The result is that the skeleton filtration of an element of order p^{1+b} in this summand is $2(rp - 1 + bp(p-1))$ where $n = r + t(p-1)$ with $1 \leq r \leq p-1$. This gives the groups $A_{2n-1}(B\mathbb{Z}/p^2)$ and is carried out in detail by Weth [17].

The order of $A_{2n-2}(B\mathbb{Z}/p^2)$ is given by $\nu_p\left|A_{2n-1}(B\mathbb{Z}/p^2)\right|-2$. For $\nu_p(n)\le 1$ the groups $A_{2n-1}(B\mathbb{Z}/p^2)$ are cyclic, hence we are done. Only the cases $n=p^2$ and $2p^2$ remain. From [12] we have that the transfer map

$$\mathrm{tr}\colon A_{2n-3}(P_\infty\mathbb{C}_+)\to A_{2n-2}(B\mathbb{Z}/p^2_+)$$

is onto. The groups $A_{2n-3}(P_\infty\mathbb{C}_+)$ for $n=sp^r, s<p$, are computed in [10]. It turns out that for $n=p^2,2p^2$ the transfer map is an isomorphism, since in these cases $|A_{2n-3}(P_\infty\mathbb{C}_+)|=|A_{2n-2}(B\mathbb{Z}/p^2_+)|$.

Proposition 5.2

n	$A_{2n-1}(B\mathbb{Z}/p^2)$	$A_{2n-2}(B\mathbb{Z}/p^2)$
$2p^2$	$\mathbb{Z}/p^4+\mathbb{Z}/p^2$	$\mathbb{Z}/p^3+\mathbb{Z}/p$
p^2	$\mathbb{Z}/p^4+\mathbb{Z}/p$	$\mathbb{Z}/p^2+\mathbb{Z}/p$
$p,2p,\dots,(p-1)p$	$\mathbb{Z}/p^3$	$\mathbb{Z}/p$
$(p+1)p,\dots,(2p-2)p$	$\mathbb{Z}/p^3+\mathbb{Z}/p$	$\mathbb{Z}/p^2$
$n\ne ip-j\not\equiv 0\ (p)$ for $2\le i\le 3p,\ 1\le j<i$	$\mathbb{Z}/p^2$	0

For the other values of $n>0$ we have $A_{2n-1}(B\mathbb{Z}/p^2)=\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^2)$ and the group $A_{2n-2}(B\mathbb{Z}/p^2)\cong A^{2n+1}(B\mathbb{Z}/p^2)$ is given by Theorem 1.1.

Our last application is a proof for the formula for the order of the J–group of lens spaces.

Corollary 5.3 [15] *The order of the J–group of the lens space $L^n(p^a)$ is given by:*

$$\nu_p|J(L^n(p^a))|=\sum_{s=0}^{a-1}\left(\frac{n}{p^s(p-1)}\right)$$

Proof Consider the long exact sequence of the pair (B^{2m},B^{2m-2}) :

$$\begin{aligned}0\longrightarrow \mathrm{Ad}^0(B^{2m}/B^{2m-2})\longrightarrow \mathrm{Ad}^0(B^{2m})\longrightarrow \mathrm{Ad}^0(B^{2m-2})&\\ \xrightarrow{\partial} \mathrm{Ad}^1(B^{2m}/B^{2m-2})\longrightarrow \mathrm{Ad}^1(B^{2m})\longrightarrow \mathrm{Ad}^1(B^{2m-2})\longrightarrow 0&\end{aligned}$$

From the commutative diagram

$$\begin{array}{ccc} \mathrm{Ad}^0(B^{2m-2}) & \xrightarrow{\partial} & \mathrm{Ad}^1(B^{2m}/B^{2m-2}) \\ \| & & \uparrow i^* \\ \mathrm{Ad}^0(B^{2m-2}) & \xrightarrow{\partial_1} & \mathrm{Ad}^1(B/B^{2m-2}) \end{array}$$

and $\partial_1 = 0$, since $\mathrm{Ad}^1(B/B^{2m-2}) = \mathrm{Ad}^1(\widetilde{T}((n-1)H)) \cong \mathrm{Ad}^1(B)$ by Corollary 2.13 is torsion free, we get $\partial = 0$. Hence

$$\nu_p\left|\mathrm{Ad}^1(B^{2m})\right| = \sum_{i=1}^{m} \nu_p\left|\mathrm{Ad}^1(B^{2i}/B^{2i-2})\right| .$$

Now $B^{2i}/B^{2i-2} = S^{2i-1} \cup_{p^a} e^{2i}$ is a Moore space and an easy computation gives $\sum_{s=0}^{a-1}(n/p^s(p-1))$ as value for this sum. From [5] we have $\mathrm{Ad}^1(B^{2m}) \cong J(B^{2m})$ and this is isomorphic to $J(L^m(p^a))$. □

Remark Since for $n = s(p-1)p^{a-1}$ the generators in $\mathrm{Ad}^0(\widetilde{T}(nH))$ are explicitly given Theorem 2.6 one may actually compute $\mathrm{Ad}^0(B^{2n})$ and $\mathrm{Ad}^1(B^{2n}) = J(L^n(p^a))$ using (11).

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