Some results on vector bundle monomorphisms

DACIBERG L GONÇALVES
ALICE K M LIBARDI
OZIRIDE MANZOLI NETO

In this paper we use the singularity method of Koschorke [2] to study the question of how many different nonstable homotopy classes of monomorphisms of vector bundles lie in a stable class and the percentage of stable monomorphisms which are not homotopic to stabilized nonstable monomorphisms. Particular attention is paid to tangent vector fields. This work complements some results of Koschorke [3; 4], Libardi–Rossini [7] and Libardi–do Nascimento–Rossini [6].

57R90; 57R25

1 Introduction

For $a < n$ let α^a and β^n be vector bundles of dimension a and n, respectively, over a closed smooth connected n-dimensional manifold M. For simplicity they are denoted by α and β, respectively. The following two related problems have been considered by several authors. The first one is to know if there is a stable (resp. nonstable) monomorphism between the vector bundles α and β. This is a quite general problem, which has been extensively studied. In particular it includes the problem of the span of a manifold (the maximal number of vector fields over a manifold which are linearly independent over every point). Although we are not particularly concerned with this problem here, we would like to point out the following relevant result by Koschorke [4, Theorem 1] for the problem above. He gives a complete answer to the question of existence of a stable (nonstable) monomorphism from α to $\beta(\alpha \oplus \varepsilon^l$ to $\beta \oplus \varepsilon^l, l > 0)$ in the so-called metastable range $n > 2a$, in spite of the fact that it is not an easy task to verify the conditions on which the answer is based, even for $a = 1, 2$. The second problem is: whenever there is a stable (resp. nonstable) vector bundle monomorphism from α to β, find how many stable (resp. nonstable) monomorphisms there are, and the relation between them. In order to study this problem, we recall Koschorke [2, Theorem 4.14], that there is a bijection between the set of homotopy classes of monomorphisms and the normal bordism group $\Omega_a(M \times P^\infty; \Phi)$ in the stable case (resp. $\Omega_a(P(\alpha); \Phi_1)$ in the nonstable case), for $n > 2a + 1$. Here $P(\alpha)$ is the projectification of the vector

Published: 20 June 2007 DOI: 10.2140/agt.2007.7.829

bundle α, and $\Phi = \lambda \otimes p_1^*(\beta - \alpha) - p_1^*\tau_M$, $\Phi_1 = \lambda \otimes p^*(\beta - \alpha) - p^*\tau_M$, which we denote simply by $\lambda \otimes (\beta - \alpha) - \tau_M$, are virtual vector bundles over $M \times P^\infty$ and $P(\alpha)$, respectively, where $p_1\colon M \times P^\infty \to M$ and $p\colon P(\alpha) \to M$ are the projections. The line bundle λ in the case of Φ is the pullback by p_2 of the canonical line bundle over P^∞ and in the case of Φ_1 denotes the canonical line bundle over $P(\alpha)$.

Then we have the stabilizing homomorphism

$$\mathrm{st}_a\colon \Omega_a(P(\alpha); \Phi_1) \longrightarrow \Omega_a(M \times P^\infty; \Phi),$$

where the cardinalities of the kernel and cokernel of this homomorphism measure, respectively, the number of different homotopy classes of nonstable monomorphisms, which stabilize to the same homotopy class of stable monomorphisms, and the percentage of stable monomorphisms which are not homotopic to (stabilized) nonstable monomorphisms. The second problem has been studied by Koschorke [4] for the case $a = 2$, n odd, by Libardi and Rossini [7], for the nonstable case $a = 2$, n even and $H_1(M; \mathbb{Z}) = 0$ and by Libardi, Nascimento and Rossini [6], for the stable case, $a = 2$ and n odd.

In this paper we study the kernel and the cokernel of st_a in the cases $a = 1, n > 3$ and $a = 2, n > 5$ and even, which complements some results of [4; 7] and [6].

Throughout this paper we will make use of the auxiliary virtual vector bundle $\Phi_0 = \lambda \otimes p_1^*\beta - p_1^*\tau_M$, which we denote simply by $\lambda \otimes \beta - \tau_M$, over $M \times P^\infty$, $\eta = \beta - \alpha - \tau_M$ and of its orientation line bundle ξ_η over M.

We denote $\rho\colon H_*(X; \widetilde{\mathbb{Z}}) \to H_*(X; \mathbb{Z}_2)$ the modulo two reduction homomorphism of the integral local coefficient $\widetilde{\mathbb{Z}}$ and we point out that for $a = 1$, $P(\alpha) = M$ and the virtual vector bundle Φ_1 over $P(\alpha) = M$ becomes $\alpha \otimes \beta - \tau_M$.

We state our main results.

Theorem 1.1 *Let $a = 1$, $n > 3$ odd and suppose there is a monomorphism $u_0\colon \alpha \hookrightarrow \beta$ over M. Then the cokernel coker st_1 is isomorphic to $\mathbb{Z}_2$ and the kernel $\ker \mathrm{st}_1$ is either zero or isomorphic to $\mathbb{Z}_2$.*

The $\ker \mathrm{st}_1$ is zero if one of the conditions below holds.

(a_1) $w_2(\Phi_1)[\rho(H_2(M; \widetilde{\mathbb{Z}}_\eta))] \neq 0$,

(a_2) $w_2(\Phi_1) = 0$ *and* $n \equiv 1(4)$ *and* $w_1(\alpha) = w_1(\beta)$,

(a_3) $w_2(\Phi_1) = 0$ *and* $n \equiv 3(4), w_1(M) = 0$ *and* $(w_1(\alpha) = 0$ *or* $w_1(\beta) = 0)$.

The $\ker \mathrm{st}_1 \simeq \mathbb{Z}_2$ if one of the conditions below holds.

(b_1) *$w_2(\Phi_1)=0$ and $n\equiv 1(4)$ and $w_1(\alpha)\neq w_1(\beta)$,*

(b_2) *$w_2(\Phi_1)=0$ and $n\equiv 3(4)$ and $w_1(M)\neq 0$.*

Denote by $|X|$ the cardinality of the set X and $\eta_0=\beta-\tau_M$ a virtual vector bundle over M. For $|H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})|$ finite, we define $k=k(\alpha,\beta)$, such that $|H_1(M;\mathbb{Z}_2)|\cdot|\ker \mathrm{st}_1|=k\cdot|H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})|$. Therefore $|\ker \mathrm{st}_1|$ is determined by the number k. We state the next result in terms of k.

Theorem 1.2 *Let $a=1$, $n>3$ be even and suppose there is a monomorphism $u_0\colon \alpha\hookrightarrow\beta$ over M.*

Then, $\operatorname{coker}\mathrm{st}_1=0$ *if $w_1(\beta)=w_1(M)$ and* $\operatorname{coker}\mathrm{st}_1\simeq\mathbb{Z}_2$ *if $w_1(\beta)\neq w_1(M)$.*

- (a) *If $|H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})|$ is infinite, $|\ker \mathrm{st}_1|$ is infinite.*
- (b) *If $|H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})|$ is finite then $|\ker \mathrm{st}_1|$ is determined by k as follows:*
  - (b_1) *For $w_2(\Phi_1)[\rho(H_2(M;\widetilde{\mathbb{Z}}_\eta))]\neq 0$, $k=1$ if $w_1(\beta)=w_1(M)$ and $k=2$ if $w_1(\beta)\neq w_1(M)$,*
  - (b_2) *For $w_2(\Phi_1)=0$ and $n\equiv 0(4)$, $k=2$ if $w_1(\beta)=w_1(M)$ and $k=4$ if $w_1(\beta)\neq w_1(M)$,*
  - (b_3) *For $w_2(\Phi_1)=0$ and $n\equiv 2(4)$, $k=1$ if $w_1(\beta)=w_1(M)$ and $(w_1(M))^2=0=w_1(\alpha)$, $k=2$ if $w_1(\beta)\neq w_1(M)$ and $w_1(\beta)w_1(M)=0=w_1(\alpha)$, $k=2$ if $w_1(\beta)=w_1(M)$ and $(w_1(M))^2\neq(w_1(\alpha))^2$, $k=4$ if $w_1(\beta)\neq w_1(M)$ and $(w_1(\alpha))^2\neq w_1(\beta)w_1(M)$.*

In the special case of the tangent vector fields, ie, when α is the trivial line bundle and β is the tangent bundle τ_M of M, the second Stiefel–Whitney class $w_2(\Phi_1)$ is zero. Then we obtain the following consequences from the two theorems above.

For n odd, we have:

(a) $\operatorname{coker}\mathrm{st}_1\simeq\mathbb{Z}_2$,

(b_1) if $w_1(M)=0$, then $\ker\mathrm{st}_1=0$ and

(b_2) if $w_1(M)\neq 0$ then $\ker\mathrm{st}_1\simeq\mathbb{Z}_2$.

For n even, we have:

(a) $\operatorname{coker}\mathrm{st}_1=0$,

(b) if $|H_1(M;\mathbb{Z}_M)|$ is infinite, where $\mathbb{Z}_M$ is the $\mathbb{Z}$-local system given by the orientation of the manifold M, $|\ker\mathrm{st}_1|$ is infinite, otherwise

(b_1) $k=2$ if $n\equiv 0(4)$ or ($n\equiv 2(4)$ and $(w_1(M))^2\neq 0$) and

(b_2) $k=1$ if $n \equiv 2(4)$ and $(w_1(M))^2 = 0$.

Now let $a=2$ and n even. Recall that the case n odd has been studied in [4].

Theorem 1.3 *Let $a=2$, $n>5$ be even and suppose there is a monomorphism $u_0 : \alpha \hookrightarrow \beta$ over M^n. Assume that one of conditions below is valid.*

$w_2(\eta_0)[\rho(H_2(M;\widetilde{\mathbb{Z}}_{\eta_0}))] \neq 0$ *or* $(w_1(\beta) \neq 0$ *for* $n \equiv 0(4)$ *and* $w_1(M) \neq 0$ *for* $n \equiv 2(4))$.

Let A be the subset of $H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})$ given by

$$\{z \mid \forall y \in H^1(M;\mathbb{Z}_2), y(\rho(z)) = (yw_2(\alpha))\rho(c_3) + yw_1(\beta-\tau_M)(c_2), c_3 \in H_3(M;\widetilde{\mathbb{Z}}_\eta), c_2 \in H_2(M;\mathbb{Z}_2)\},$$

then

(a_1) $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}_2$ *if also* $w_1(\beta) \neq w_1(M)$,

(a_2) $\operatorname{coker} \mathrm{st}_2 \simeq 0$ *if also* $w_1(\beta) = w_1(M)$ *and* $w_2(\alpha)\rho(H_2(M;\widetilde{\mathbb{Z}}_\alpha)) = 0$,

(a_3) $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}$ *if also* $w_1(\beta) = w_1(M)$ *and* $w_2(\alpha)\rho(H_2(M;\widetilde{\mathbb{Z}}_\alpha)) \neq 0$ *and*

(b) $\ker \mathrm{st}_2 \simeq H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})/A \oplus \mathbb{Z}_2$, *if also* $w_2(\Phi) = 0$.

Corollary 1.4 *In the special case of tangent plane fields, ie when α is the trivial bundle, β is the tangent bundle of M and $w_1(M) \neq 0$ then st_2 is surjective and $\ker \mathrm{st}_2 \simeq H_1(M;\mathbb{Z}) \oplus \mathbb{Z}_2$.*

2 Preliminaries and notations

Given α^a and β^n vector bundles over M of dimension a and n, denoted by α and β, we will consider the virtual bundles $\Phi = \lambda \otimes p_1^*(\beta-\alpha) - p_1^*\tau_M$, $\Phi_1 = \lambda \otimes p^*(\beta-\alpha) - p^*\tau_M$, which we denote simply by $\lambda \otimes (\beta-\alpha) - \tau_M$, virtual vector bundles over $M \times P^\infty$ and $P(\alpha)$, respectively, where $p_1\colon M \times P^\infty \to M$, $p\colon P(\alpha) \to M$ are the projections. The line bundle λ in the case of Φ is the pullback by p_2 of the canonical line bundle over P^∞ and in the case of Φ_1 denotes the canonical line bundle over $P(\alpha)$.

We recall that for $a=1$, $P(\alpha)=M$ and the virtual vector bundle Φ_1 over $P(\alpha)=M$ becomes $\alpha \otimes \beta - \tau_M$. Throughout this paper we will make use of the auxiliary virtual vector bundle $\eta = \beta - \alpha - \tau_M$ and of its orientation line bundle ξ_η over M. Let

$x = w_1(\lambda)$ and $\Phi_0 = \lambda \otimes p_1^*\beta - p_1^*\tau_M$, which we denote simply by $\lambda \otimes \beta - \tau_M$, over $M \times P^\infty$ which we many times denote simply M^∞.

For $a = 1$, we have

$$w_1(\Phi) = (n+1)x + w_1(\eta)$$

$$w_2(\Phi) = \begin{cases} x(w_1(\beta) + w_1(\alpha)) + w_2(\eta), & n \equiv 1(4) \\ xw_1(M) + w_2(\eta), & n \equiv 2(4) \\ x^2 + x(w_1(\beta) + w_1(\alpha)) + w_2(\eta), & n \equiv 3(4) \\ x^2 + xw_1(M) + w_2(\eta), & n \equiv 0(4) \end{cases}$$

$$w_1(\Phi_1) = nw_1(\alpha) + w_1(\beta) + w_1(M)$$

$$w_2(\Phi_1) = \begin{cases} w_1(\alpha)w_1(\beta) + (w_1(\alpha))^2 + w_2(\eta), & n \equiv 1(4) \\ w_1(\alpha)w_1(M) + (w_1(\alpha))^2 + w_2(\eta), & n \equiv 0(4) \\ w_1(\alpha)w_1(\beta) + w_2(\eta), & n \equiv 3(4) \\ w_1(\alpha)w_1(M) + w_2(\eta), & n \equiv 2(4) \end{cases}$$

$$w_1(\Phi_0) = nx + w_1(\beta - \tau_M).$$

For $a = 2$ and n even, we have

$$w_1(\Phi) = w_1(\eta) = w_1(\beta) + w_1(\alpha) + w_1(M)$$

$$w_2(\Phi) = \begin{cases} x(w_1(M) + w_1(\eta)) + w_2(\eta), & \text{if } n \equiv 2(4) \\ x^2 + x(w_1(M) + w_1(\eta)) + w_2(\eta) & \text{if } n \equiv 0(4) \end{cases}$$

$$w_2(\eta) = w_2(\beta) + w_2(\alpha) + w_1^2(\alpha) + w_1(\beta)w_1(\alpha) + w_2(M) + w_1^2(M) + w_1(\beta)w_1(M) + w_1(\alpha)w_1(M).$$

Given a vector bundle θ^ℓ, of dimension ℓ, we denote $\xi_\theta = \Lambda^\ell\theta$ the corresponding orientation line bundle. If τ_K is the tangent bundle of a smooth manifold K we also write ξ_K for ξ_{τ_K}. If Φ is a virtual vector bundle, ξ_Φ denotes the orientation line bundle determined by $w_1(\xi_\Phi) = w_1(\Phi)$. For more details see Randall–Daccach [8, Section III.6].

Let $\mathbb{Z}_\theta$ (or $\mathbb{Z}_{w_1(\theta)}$) denote the group $\mathbb{Z}$ if $w_1(\theta) = 0$ or $\mathbb{Z}_2$ if $w_1(\theta) \neq 0$, and let $\widetilde{\mathbb{Z}}_\theta$ (or $\widetilde{\mathbb{Z}}_{w_1(\theta)}$) denote the twisted integer coefficient system associated to the orientation line bundle ξ_θ.

For a topological space X and a virtual bundle $\Phi = \Phi^+ - \Phi^-$ over X, let $\overline{\Omega}_k(X; \xi_\Phi)$ be the group of bordism classes $[K, g, or]$, where K is a smooth closed k dimensional manifold, $g\colon K \to X$ is a continuous map and $or\colon \xi_K \to g^*(\xi_\Phi)$ is an isomorphism.

We also consider the map $f_k\colon \Omega_k(X;\Phi)\to\overline{\Omega}_k(X;\xi_\Phi)$ which forgets the vector bundle isomorphism $\tau_K\oplus g^*(\Phi^+)\xrightarrow{\simeq} g^*(\Phi^-)$ and retains the orientation information.

If X is connected we have that $\Omega_0(X;\theta)\simeq\mathbb{Z}_\theta\simeq\overline{\Omega}_0(X;\xi_\theta)$.

For each n, the homomorphism

$$\Delta_\theta\colon \Omega_n(X;\Phi)\to\Omega_{n-\ell}(X;\Phi+\theta)$$

is defined by considering

$$w=\big[W^n\xrightarrow{g}X,h\colon \tau_W\oplus g^*(\Phi^+)\xrightarrow{\simeq} g^*(\Phi^-)\big]$$

in $\Omega_n(X;\Phi)$, $Z^{n-\ell}\subset W^n$ (the zero set of a generic section of the vector bundle $g^*(\theta)$ over W^n), and the isomorphism $\nu(Z^{n\ell},W^n)\simeq g^*(\theta)|_{Z^{n-\ell}}$ by the formula

$$\Delta_\theta(w)=\big[Z^{n-\ell}\xrightarrow{g|_Z}X,\tau_Z\oplus g|^*(\Phi^++\theta)\xrightarrow{\simeq} g|^*(\Phi^-)\big].$$

The homomorphism Δ_θ and its weak analogue $\overline{\Delta}_\theta\colon \overline{\Omega}_n(X;\xi_\Phi)\to\overline{\Omega}_{n-\ell}(X;\xi_{\Phi+\theta})$ and the homomorphism st_a and its weak analogue $\overline{\mathrm{st}}_a\colon \overline{\Omega}_a(P(\alpha);\xi_{\Phi_1})\to\overline{\Omega}_a(M\times P^\infty;\xi_\Phi)$ fit into the long exact Gysin sequences (1) and (2) below:

$$(1)\quad \cdots\longrightarrow\Omega_j(M^\infty;\Phi)\xrightarrow{\tau_{j+1}}\Omega_{j-a}(M^\infty;\Phi_0)\xrightarrow{\delta'_j}\Omega_{j-1}(P(\alpha);\Phi_1)\xrightarrow{\mathrm{st}_a}\Omega_{j-1}(M^\infty;\Phi)\longrightarrow\cdots$$

$$(2)\quad \cdots\longrightarrow\overline{\Omega}_j(M^\infty;\xi_\Phi)\xrightarrow{\overline{\tau}_{j+1}}\overline{\Omega}_{j-a}(M^\infty;\xi_{\Phi_0})\xrightarrow{\overline{\delta}'_j}\overline{\Omega}_{j-1}(P(\alpha);\xi_{\Phi_1})\xrightarrow{\overline{\mathrm{st}}_a}\overline{\Omega}_{j-1}(M^\infty;\xi_\Phi)\longrightarrow\cdots$$

Here, $\overline{\tau}_2=\overline{\Delta}_{\lambda\otimes\alpha}$, $\overline{\delta}'_2$ and δ'_2 come from Gysin sequences and $\tau_2=\overline{\tau}_2\circ f_2^\infty$, where $f_2^\infty\colon \Omega_2(M\times P^\infty;\Phi)\to\overline{\Omega}_2(M\times P^\infty;\xi_\Phi)$ is the forgetful map and M^∞ denote $M\times P^\infty$ (see [4, Sections 1 and 2] for details).

The sequences above and other two singularity sequences defined in [2, Theorem 9.3], for $a=2$, fit together into the next commutative diagram, where we have exactness in all sequences if the two "pinching conditions", below, hold:

(i) $\Omega_1(M\times P^\infty;\Phi_0)\simeq\overline{\Omega}_1(M\times P^\infty;\xi_{\Phi_0})$

(ii) $\mathrm{st}_1\colon \Omega_1(P(\alpha)\times BO(2);\Phi+\Gamma)\xrightarrow{\simeq}\Omega_1(M\times P^\infty\times BO(2);\Phi+\Gamma)$ is an isomorphism.

In the diagram, we have $\Phi_2 = \Phi + \Gamma$ where Γ is $\gamma_2 \otimes \xi_{\gamma_2} + \xi_{\gamma_2} - \gamma_2$ and γ_2 denotes the canonical plane bundle over $BO(2)$ (see [4, Theorem 3.1]). Also $\mathrm{forg}_2 = \overline{\mathrm{st}}_2 \circ f_2 = f_2^\infty \circ \mathrm{st}_2$, $\mathbb{Z}_2$ is in place of $\Omega_0(M \times P^\infty \times BO(2); \Phi_2)$ (see [2, Section 9.2, Theorem 9.3]), $\mathbb{Z}_{\Phi_0}$ is in place of $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0})$ and M^∞ in the place of $M \times P^\infty$.

$$
\begin{tikzcd}
\Omega_3(M^\infty;\Phi) \arrow[rr, bend left, "\tau_3"] \arrow[dr, "f_3^\infty"'] & & \overline{\Omega}_1(M^\infty;\xi_{\Phi_0}) \arrow[rr, bend left, "\overline{\delta}'_2"] \arrow[dr, "\delta'_2"] & & \overline{\Omega}_2(P(\alpha);\xi_{\Phi_1}) \arrow[rr, bend left, "\sigma_{j_2}"] \arrow[dr, "\overline{\mathrm{st}}_2"] & & \mathbb{Z}_2 \\
& \overline{\Omega}_3(M^\infty;\xi_\Phi) \arrow[ur, "\overline{\tau}_3"] \arrow[dr, "\sigma_{j_3}^\infty"] & & \Omega_2(P(\alpha);\Phi_1) \arrow[ur, "f_2"] \arrow[dr, "\mathrm{st}_2"] \arrow[rr, dashed, "\mathrm{forg}_2"] & & \overline{\Omega}_2(M^\infty;\xi_\Phi) \arrow[ur, "\sigma_{j_2}^\infty"] \arrow[dr, "\overline{\tau}_2"] & \\
\overline{\Omega}_3(P(\alpha);\xi_{\Phi_1}) \arrow[ur, "\overline{\mathrm{st}}_3"] \arrow[rr, bend right, "\sigma_{j_3}"'] & & \Omega_1(M^\infty \times BO(2);\Phi_2) \arrow[ur, "\delta_2"] \arrow[rr, bend right, "\delta_2^\infty"'] & & \Omega_2(M^\infty;\Phi) \arrow[ur, "f_2^\infty"] \arrow[rr, bend right, "\tau_2"'] & & \mathbb{Z}_{\Phi_0}
\end{tikzcd}
$$

Figure 1: Diagram for (a=2)

For more details see [2, Section 9] and [4, Section 3].

We recall from [4, Proposition 3.3], that if $a = 2$ and n is even, the two pinching conditions above are equivalent to each one of the conditions:

$$w_2(\beta - \tau_M)[\rho(H_2(M; \widetilde{Z}_{\beta-\tau_M}))] \neq 0$$

or

$$\begin{cases} w_1(\beta) \neq 0 \text{ if } n \equiv 0(4) \\ w_1(M) \neq 0 \text{ if } n \equiv 2(4). \end{cases}$$

For $\Phi = \lambda \otimes (\beta - \alpha) - \tau_M$ and $\eta = \beta - \alpha - \tau_M$, as defined before, we have from [4, Propositions 1.2 and 1.3], the following Proposition.

Proposition 2.1 *For $i \in \mathbb{Z}$*

$$\overline{\Omega}_i(M \times P^\infty; \xi_\Phi) \simeq \begin{cases} \mathfrak{N}_i(M) & \text{if } a \not\equiv n(2) \\ \overline{\Omega}_i(M; \xi_\eta) \oplus \mathfrak{N}_{i-1}(M) & \text{if } a \equiv n(2). \end{cases}$$

3 Proofs of the theorems

We will study the kernel and cokernel of the stabilizing homomorphism

$$\mathrm{st}_a\colon \Omega_a(P(\alpha);\Phi_1) \longrightarrow \Omega_a(M\times P^\infty;\Phi).$$

We recall that

$$\begin{aligned}
\Omega_0(M^\infty;\Phi_0) &\simeq \overline{\Omega}_0(M^\infty;\xi_{\Phi_0})\\
&\simeq \begin{cases}\mathbb{Z} \text{ if } w_1(\Phi_0)=nx+w_1(\beta)+w_1(M)=0\\ \mathbb{Z}_2 \text{ if } w_1(\Phi_0)=nx+w_1(\beta)+w_1(M)\neq 0.\end{cases}\\
\Omega_0(P(\alpha);\Phi_1) &\simeq \overline{\Omega}_0(P(\alpha);\xi_{\Phi_1})\\
&\simeq \begin{cases}\mathbb{Z} \text{ if } w_1(\Phi_1)=nw_1(\alpha)+w_1(\beta)+w_1(M)=0\\ \mathbb{Z}_2 \text{ if } w_1(\Phi_1)=nw_1(\alpha)+w_1(\beta)+w_1(M)\neq 0.\end{cases}
\end{aligned}$$

Let us consider also the following commutative diagram, for $a=1$, where the horizontal sequences are defined in [2, Theorem 9.3] and the vertical sequences come from the Gysin sequences (1) and (2), also for $a=1$. We also recall that in this case $P(\alpha)=M$.

$$\begin{tikzcd}
 & & {} \arrow[d, "\tau_2"'] & {} \arrow[d, "\overline{\tau}_2"] & \\
 & & \Omega_1(M\times P^\infty;\Phi_0) \arrow[d, "\delta_1'"'] & \overline{\Omega}_1(M\times P^\infty;\xi_{\Phi_0}) \arrow[d, "\overline{\delta}_1'"] & \\
\overline{\Omega}_2(P(\alpha);\xi_{\Phi_1}) \arrow[r, "\sigma_{j_2}"] \arrow[d, "\overline{\mathrm{st}}_2"'] & \mathbb{Z}_2 \arrow[r, "\delta_1"] \arrow[d, "\simeq"] & \Omega_1(P(\alpha);\Phi_1) \arrow[r, "f_1"] \arrow[d, "\mathrm{st}_1"'] & \overline{\Omega}_1(M\times P^\infty;\xi_{\Phi_0}) \arrow[r] \arrow[d, "\overline{\mathrm{st}}_1"] & 0 \\
\overline{\Omega}_2(M\times P^\infty;\xi_\Phi) \arrow[r, "\sigma_{j_2}^\infty"'] & \mathbb{Z}_2 \arrow[r, "\delta_1^\infty"'] & \Omega_1(M\times P^\infty;\Phi) \arrow[r, "f_1^\infty"'] \arrow[d, "\tau_1"'] & \Omega_1(M\times P^\infty;\Phi) \arrow[r] \arrow[d, "\overline{\tau}_1"] & 0 \\
 & & \mathbb{Z}_{\Phi_0} \arrow[r, "\overline{f}_0^\infty"'] & \mathbb{Z}_{\Phi_0} &
\end{tikzcd}$$

Figure 2: Diagram for $(a=1)$

Lemma 1 *If σ_{j_2} and $\sigma_{j_2}^\infty$ are surjective or both null maps then* $\ker \mathrm{st}_1 \simeq \ker \overline{\mathrm{st}}_1$.

If σ_{j_2} is the null map and $\sigma_{j_2}^{\infty}$ is a surjective map then $\ker \mathrm{st}_1$ *is an extension of* $\ker \overline{\mathrm{st}}_1$ *by $\mathbb{Z}_2$.*

Proof The proof is obtained by diagram chasing in Figure 2 above. □

Lemma 2 *Let n be odd and $w_2(\Phi_1)=0$ then for $n\equiv 1(4)$, $\sigma_{j_2}^{\infty}=0$, if $w_1(\beta-\alpha)=0$ and $\sigma_{j_2}^{\infty}$ is surjective if $w_1(\beta-\alpha)\neq 0$.*

For $n\equiv 3(4)$, $\sigma_{j_2}^{\infty}=0$, if $w_1(M)=0$ and $(w_1(\alpha)=0$ or $w_1(\beta)=0)$ and $\sigma_{j_2}^{\infty}$ is surjective if $w_1(M)\neq 0$.

Proof Let $w=[K^2,(h_1,h_2),\overline{h}]\in\overline{\Omega}_2(M\times P^{\infty};\xi_{\Phi})\simeq\overline{\Omega}_2(M;\xi_{\eta})\oplus H_1(M;\mathbb{Z}_2)\simeq H_2(M;\widetilde{\mathbb{Z}}_{\eta})\oplus H_1(M;\mathbb{Z}_2)$.

We observe that w corresponds to $([K^2,h_1,h],h_{1_*}([L^1]))$, where $L^1\subset K^2$ is the zero set of a generic section of the pullback vector bundle $h_2^*(\lambda)$ over K^2, and h is equivalent to an isomorphism $h_2^*(\lambda)\simeq\xi_L\oplus h_1^*(\xi_{\eta})$. Then $w_1(\lambda)=w_1(L)+w_1(\eta)$ (see [4, Section 1.8]), for simplicity L^1 and K^2 are denoted by L and K, respectively.

It follows from [4, Section 0.18] that for all $y\in H^1(M;\mathbb{Z}_2)$, $y([L])=yw_1(\lambda)([K])$.

Using [2, Theorem 9.3], we have that

$$\begin{aligned} &\sigma_{j_2}^{\infty}(w)=(w_1(\lambda)w_1(\beta-\alpha)+w_2(\eta))([K]) && \text{if}\quad n\equiv 1(4)\\ \text{and}\quad &\sigma_{j_2}^{\infty}(w)=((w_1(\lambda))^2+w_1(\lambda)(w_1(\beta-\alpha))+w_2(\eta))([K]) && \text{if}\quad n\equiv 3(4).\end{aligned}$$

In the first case we have

$$\sigma_{j_2}^{\infty}(w)=w_1(\beta-\alpha)([L])+w_2(\eta)([K])$$

and in the second case,

$$\begin{aligned}\sigma_{j_2}^{\infty}(w)&=(w_1(\lambda)+w_1(\beta-\alpha))([L])+w_2(\eta)([K])\\ &=(w_1(L)+w_1(\eta)+w_1(\beta-\alpha))([L])+w_2(\eta)([K])\\ &=w_1(M)([L])+w_2(\eta)([K]).\end{aligned}$$

It follows that for $s\in H_2(M;\widetilde{\mathbb{Z}}_{\eta})$ and $t\in H_1(M;\mathbb{Z}_2)$ we have

$$\begin{aligned} &\sigma_{j_2}^{\infty}(w)=w_1(\beta-\alpha)(t)+w_2(\eta)\rho(s) && \text{if}\quad n\equiv 1(4)\\ \text{and}\quad &\sigma_{j_2}^{\infty}(w)=w_1(M)(t)+w_2(\eta)\rho(s) && \text{if}\quad n\equiv 3(4)\end{aligned}$$

The result follows by observing that if $n\equiv 1(4)$, $w_2(\Phi_1)=w_1(\alpha)w_1(\beta-\alpha)+w_2(\eta)$ and if $n\equiv 3(4)$, $w_2(\Phi_1)=w_1(\alpha)w_1(\beta)+w_2(\eta)$. □

Proof of Theorem 1.1 Let $n > 3$ be odd and $a = 1$.

We have that $w_1(\Phi_0) = x + w_1(\beta - \tau_M)$ and $w_1(\Phi_1) = w_1(\eta)$.

Note that $H_1(M \times P^\infty) \simeq H_1(M) \oplus H_1(P^\infty) \simeq H_1(M) \oplus \mathbb{Z}_2$ and, if we call b the generator of the $\mathbb{Z}_2$ factor, we have $x(b) \neq 0$, $w_1(\beta)(b) = 0 = w_1(M)(b)$ and so we have $w_1(\Phi_0) \neq 0$, which gives us $\mathbb{Z}_{\Phi_0} \simeq \mathbb{Z}_2 \simeq \overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0})$ and from [4, Proposition 2.1], $\overline{\mathrm{st}}_1$ and $\overline{\mathrm{st}}_0$ are injective. Therefore, $\overline{\tau}_1$ is onto $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \simeq \mathbb{Z}_2$, see Figure 2.

But $\overline{f}_0^\infty$ is an isomorphism, so τ_1 is also onto $\mathbb{Z}_2 \simeq \Omega_0(M \times P^\infty; \xi_{\Phi_0})$. Then $\operatorname{coker} \mathrm{st}_1 \simeq \mathbb{Z}_2$. It follows also from Diagram IV and the fact that $\overline{\mathrm{st}}_1$ is injective that $\ker \mathrm{st}_1 \subset \operatorname{im} \delta_1$, the image of δ_1, which is zero or isomorphic to $\mathbb{Z}_2$.

If $w_2(\Phi_1)[\rho(H_2(M; \widetilde{\mathbb{Z}}_\eta))] \neq 0$, σ_{j_2} is surjective. Therefore we can conclude that $\Omega_1(P(\alpha); \Phi_1) \simeq \overline{\Omega}_1(P(\alpha); \xi_{\Phi_1}) \simeq H_1(M; \widetilde{\mathbb{Z}}_\eta)$ and recalling that $\overline{\mathrm{st}}_1$ is injective we obtain in Diagram IV that st_1 is also injective.

If $w_2(\Phi_1) = 0$, σ_{j_2} is the null map, and then $\Omega_1(P(\alpha); \Phi_1)$ is an extension of $H_1(M; \widetilde{\mathbb{Z}}_\eta)$ by $\mathbb{Z}_2$. The result follows from Lemmas 1 and 2 above. □

Lemma 3 *Let $n > 3$ be even and $a = 1$.*

If $w_1(\beta) = w_1(M)$, $\overline{\mathrm{st}}_1$ is surjective and $H_1(M; \mathbb{Z})$ is an extension of $H_1(M; \mathbb{Z}_2)$ by $\ker \overline{\mathrm{st}}_1$. We have also that $|H_1(M; \mathbb{Z})| = |\ker \overline{\mathrm{st}}_1|.|H_1(M; \mathbb{Z}_2)|$.

If $w_1(\beta) \neq w_1(M)$, $\operatorname{coker} \overline{\mathrm{st}}_1 \simeq \mathbb{Z}_2$ and $H_1(M; \mathbb{Z}_2)$ is an extension of $\mathbb{Z}_2$ by $H_1(M; \widetilde{\mathbb{Z}}_{\eta_0}) \ker \overline{\mathrm{st}}_1$. We have also $2.|H_1(M; \widetilde{\mathbb{Z}}_{\eta_0})| = |\ker \overline{\mathrm{st}}_1|.|H_1(M; \mathbb{Z}_2)|$.

Proof Let us consider the long exact Gysin sequence (2).

If $w_1(\beta) = w_1(M), w_1(\Phi_0) = 0$ and so $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \simeq \mathbb{Z}$ is a free abelian group. Since n is even, $\overline{\Omega}_1(M \times P^\infty; \xi_\Phi) \simeq H_1(M; \mathbb{Z}_2)$ is a torsion group so $\overline{\tau}_1$ is zero and then $\overline{\mathrm{st}}_1$ is surjective. It follows that $H_1(M; \mathbb{Z}) / \ker \overline{\mathrm{st}}_1 \simeq H_1(M; \mathbb{Z}_2)$.

If $w_1(M) \neq w_1(\beta)$, $\overline{\mathrm{st}}_0$ is injective from [4, Proposition 2.1] and $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \simeq \mathbb{Z}_2$. Therefore, $\mathbb{Z}_2 \simeq \operatorname{coker} \overline{\mathrm{st}}_1 \simeq H_1(M; \mathbb{Z}_2) / \ker \overline{\tau}_1$, where $\ker \overline{\tau}_1 \simeq H_1(M; \widetilde{\mathbb{Z}}_{\eta_0}) / \ker \overline{\mathrm{st}}_1$. □

Proof of Theorem 1.2 Let $n > 3$ be even and $a = 1$. In this case we recall that $w_1(\Phi_0) = w_1(\eta_0) = w_1(\Phi_1)$. If $w_1(\beta) = w_1(M)$ it follows from Lemma 3 that $\overline{\mathrm{st}}_1$ is surjective. Therefore, by Figure 2, st_1 is also surjective. If $w_1(\beta) \neq w_1(M), w_1(\Phi_0) \neq 0$ and so $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \simeq \mathbb{Z}_2$. We have again that $\overline{\mathrm{st}}_0$ is injective and so $\overline{\tau}_1$ is onto $\mathbb{Z}_2 \simeq \overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0})$.

Since f_0^∞ is an isomorphism, τ_1 is onto $\mathbb{Z}_2 \simeq \Omega_0(M \times P^\infty; \Phi_0)$ and $\operatorname{coker} \mathrm{st}_1 \simeq \mathbb{Z}_2$. If $w_2(\Phi_1)[\rho(H_2(M;\widetilde{\mathbb{Z}}_\eta))] \neq 0$, σ_{j_2} and $\sigma_{j_2}^\infty$ are surjective maps, then, $\ker \mathrm{st}_1 \simeq \ker \overline{\mathrm{st}}_1$. We recall that if $w_2(\Phi_1) = 0$, the map $\sigma_{j_2} = 0$, and so $\Omega_1(M;\Phi_1)$ is an extension of $H_1(M;\widetilde{\mathbb{Z}}_{\eta_0})$ by $\mathbb{Z}_2$.

With the condition $w_2(\Phi_1) = 0$ we have that if $n \equiv 2(4)$ and $w_1(\alpha) = 0 = w_1(\beta)w_1(M)$, $\sigma_{j_2}^\infty(s,t) = (w_1(M)w_1(\beta-\alpha) + w_2(\eta))(s) = 0$ and so we have $\ker \mathrm{st}_1 \simeq \ker \overline{\mathrm{st}}_1$. If $n \equiv 0(4)$ or $n \equiv 2(4)$ and $(w_1(\alpha))^2 \neq w_1(M)w_1(\beta)$, $\sigma_{j_2}^\infty$ is surjective and $\ker \mathrm{st}_1$ is an extension of $\ker \overline{\mathrm{st}}_1$ by $\mathbb{Z}_2$.

Therefore, the results follow by remarking that if $\sigma_{j_2}^\infty$ is surjective, then

$$\Omega_1(M \times P^\infty; \Phi) \simeq H_1(M;\mathbb{Z}_2)$$

and if $\sigma_{j_2}^\infty = 0$, then $\Omega_1(M \times P^\infty; \Phi)$ is an extension of $H_1(M;\mathbb{Z}_2)$ by $\mathbb{Z}_2$. □

Lemma 4 *Let $n > 5$ be even, $a = 2$ and $w_1(\beta) = w_1(M)$. Then $\overline{\tau}_2 = 0$ if and only if $\tau_2 = 0$.*

Proof Since $\tau_2 = \overline{\tau}_2 \circ f_2^\infty$, $\overline{\tau}_2 = 0$ implies $\tau_2 = 0$.

Let $\tau_2 = 0$. In Figure 1, $\Omega_0(M \times P^\infty \times BO(2); \Phi_2) \simeq \mathbb{Z}_2$ (see Koschorke [2, Section 9.2, Theorem 9.3]), $\overline{\Omega}_0(M \times P^\infty; \xi_{\Phi_0}) \simeq \mathbb{Z}$ and $\overline{\Omega}_2(M \times P^\infty; \xi_\Phi)/f_2^\infty(\Omega_2(M \times P^\infty; \Phi))$ is isomorphic to $\sigma_{j_2}^\infty(\overline{\Omega}_2(M \times P^\infty; \xi_\Phi))$.

Then, in case $\sigma_{j_2}^\infty = 0$, f_2^∞ is surjective and since $\tau_2 = 0$ we have $\overline{\mathrm{st}}_2$ also surjective and so $\overline{\tau}_2 = 0$.

If $\sigma_{j_2}^\infty(\overline{\Omega}_2(M \times P^\infty; \xi_\phi)) \simeq \mathbb{Z}_2$, the image of f_2^∞ has index 2 and the image of $\overline{\mathrm{st}}_2$ has index at most 2, and the result follows. □

Proof of Theorem 1.3 Let $n > 5$ be even and $a = 2$.

We have $w_2(\beta - \tau_M)\rho(H_2(M;\widetilde{Z}_{\beta-\tau_M})) \neq 0$ or $w_1(\beta) \neq 0$ if $n \equiv 0(4)$ and $w_1(M) \neq 0$ if $n \equiv 2(4)$ so the pinching conditions are valid and we can use Figure 1.

Let $z = \left[K^2 \xrightarrow{(g_1,g_2)} M \times P^\infty, \overline{g}\right] \in \overline{\Omega}_2(M \times P^\infty; \xi_\Phi) \simeq H_2(M;\widetilde{Z}_\eta) \oplus H_1(M;\mathbb{Z}_2)$. We denote this isomorphism by φ and $\varphi(z) = (s,t)$. Then

$$\mu(\overline{\tau}_2(z)) = w_2(\alpha)\rho(s) + w_1(\beta - \tau_M)(t).$$

Let us consider $w_1(\beta) = w_1(M)$. Then if $w_2(\alpha)\rho(H_2(M;\widetilde{Z}_\eta)) = 0$, $\overline{\tau}_2 = 0$, and it follows by Lemma 4 that $\tau_2 = 0$ and then st_2 is surjective. If $w_2(\alpha)\rho(H_2(M;\widetilde{Z}_\eta)) \neq 0$, $\overline{\tau}_2 \neq 0$ and so $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}$.

If $w_1(\beta) \neq w_1(M)$, $\overline{\tau}_2$ is onto $\mathbb{Z}_2$ by [4, Proposition 2.1] and so is τ_2. If $w_2(\Phi) = 0$ then f_2^∞ is surjective. Therefore $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}_2$. If $w_2(\Phi) \neq 0$, we have that $\sigma_{j_2}^\infty$ is onto $\mathbb{Z}_2$. Since $\mathrm{forg}_2 = f_2^\infty \circ \mathrm{st}_2$, we have that $\operatorname{coker} \mathrm{forg}_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If st_2 is surjective, forg_2 and f_2^∞ have the same image and then the index of this image in $\overline{\Omega}_2(P^\infty \times M; \xi_\Phi)$ is 2, which is a contradiction. It follows that $\tau_2 \neq 0$ and again $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}_2$.

Let us now analyze $\ker \mathrm{st}_2$ when $w_2(\Phi) = 0$. We have that δ_2 and δ_2^∞ are injective (see footnote of [2, Theorem 9.3]). In the Figure 1 we see that $\sigma_{j_3}^\infty = 0$, and then f_3^∞ is surjective. As a consequence, $\operatorname{im} \overline{\tau}_3$ is equal to $\operatorname{im} \tau_3$, and then we have that $\ker \mathrm{st}_2 \simeq H_1(M; \widetilde{\mathbb{Z}}_{\beta - \tau_M}) \oplus \mathbb{Z}_2 / \overline{\tau}_3(\ker \sigma_{j_3}^\infty)$, where $\ker \sigma_{j_3}^\infty = \overline{\Omega}_3(M \times P^\infty; \xi_\Phi)$.

We compute now the image of $\overline{\Omega}_3(M \times P^\infty; \xi_\Phi)$ by $\overline{\tau}_3$. Let

$$\begin{aligned} w = \big[K^3 \xrightarrow{(g_1,g_2)} M \times P^\infty; \overline{g}\big] \in \overline{\Omega}_3(M \times P^\infty; \xi_\Phi) &\simeq H_3(M; \widetilde{\mathbb{Z}}_\eta) \oplus \mathfrak{N}_2(M) \\ &\simeq H_3(M; \widetilde{\mathbb{Z}}_\eta) \oplus H_2(M; \mathbb{Z}_2) \oplus \mathbb{Z}_2. \end{aligned}$$

We remark that w corresponds to $\big(\big[K^3 \xrightarrow{g_1} M, \overline{g}\big], \big[L^2 \xrightarrow{g_1|_{L^2}} M\big]\big)$, where L^2 is the zero set of a generic section of the vector bundle λ over K, and

$$[L^2 \xrightarrow{g_1|_{L^2}} M]$$

corresponds to $((g_1|_{L^2})_*(\mu(L^2)), [P^2 \xrightarrow{c} M])$, where c is a constant map. Notice that for all $y \in H^1(M; \mathbb{Z}_2)$, $y([P^2]) = 0$ and $y([\mu(\overline{\tau}_3(w))]) = (y w_2(\lambda \otimes \alpha^2))([M])$ and the image of $\overline{\tau}_3 \subset H_1(M; \widetilde{\mathbb{Z}}_{\beta - \tau_M})$. Therefore, $\ker \mathrm{st}_2 \simeq H_1(M; \widetilde{\mathbb{Z}}_{\beta - \tau_M})/A \oplus \mathbb{Z}_2$, where

$$\begin{aligned} A = \big\{z \mid y(\rho(z)) &= (y w_2(\alpha))\rho(c_3) + y w_1(\beta - \tau_M)(c_2), \\ &c_3 \in H_3(M; \widetilde{\mathbb{Z}}_\eta), c_2 \in H_2(M; \mathbb{Z}_2), \forall y \in H^1(M; \mathbb{Z}_2)\big\}. \end{aligned}$$

This completes the proof. □

Proof of Corollary 1.4 Since $w_1(\beta) = w_1(M) \neq 0$ and α is the trivial bundle, it follows from Theorem 1.3 that st_2 is surjective.

Using Figure 1 we obtain that $\ker \mathrm{st}_2 \simeq H_1(M; \mathbb{Z}) \oplus \mathbb{Z}_2 / \overline{\tau}_3(f_3^\infty(\Omega_3(M \times P^\infty; \Phi)))$. But under our hypothesis $\overline{\tau}_3 = 0$, then $\ker \mathrm{st}_2 \simeq H_1(M; \mathbb{Z}) \oplus \mathbb{Z}_2$. □

4 Examples

Example 1 Let p_1, p_2, p_3 and q_1, q_2, q_3 be arbitrary integers and consider the vector bundles $\alpha^2 = \gamma_1' \otimes \gamma_2' \otimes \gamma_3'$ and $\beta^6 = \gamma_1 \times \gamma_2 \times \gamma_3$ over $M^6 = S_1^2 \times S_2^2 \times S_3^2$. Each factor S_i^2 is identified with $CP(1), \gamma_i'$ and γ_i are complex line bundles characterized respectively by $c_1(\gamma_i') = p_i \cdot g^i$ and by $c_1(\gamma_i) = q_i \cdot g^i$, where g^i denotes the generator of $H^2(S_i^2; \mathbb{Z})$ and c_1 is the first Chern class. Therefore, we have that

$$\begin{aligned} w_2(\alpha) &= p_1\rho(g^1) + p_2\rho(g^2) + p_3\rho(g^3), \\ w_2(\beta) &= q_1\rho(g^1) + q_2\rho(g^2) + q_3\rho(g^3), \\ w_2(M) &= 0. \end{aligned}$$

From [5, Proposition 4.3] we know that there is a non-stable monomorphism $u_0 : \alpha \hookrightarrow \beta$ in the following cases

(i) $p_1 + p_2 + p_3 = 0$ and at least one of the q_i's is zero and

(ii) $p = p_1 + p_2 + p_3 \neq 0$ and $(q_1 - p).(q_2 - p).(q_3 - p)$ is divisible by $4p$.

Since $w_1(M), w_1(\beta)$ and $w_2(M)$ are all zero, to satisfy the pinching conditions we need $w_2(\beta) \neq 0$, that is, at least one of the q_i's is $\neq 0$.

Different choices of the parameters give rise to different examples.

(1) If $p = 0$, $p_i \equiv 0(2)$, for some $k, q_k \equiv 1(2)$ and for some $j, q_j = 0$, we have $w_2(\alpha) = 0$ and $w_2(\beta) \neq 0$ and so st_2 is surjective.

(2) If for some k, $p_k \equiv 1(2)$, $p = 0$, for some $l, q_l \equiv 1(2)$ and for some $j, q_j = 0$, then $w_2(\alpha) \neq 0$ and $w_2(\beta) \neq 0$ and so $\operatorname{coker} \mathrm{st}_2 \simeq \mathbb{Z}$.
Note that

$$\begin{aligned} w_2(\Phi) = w_2(\eta) = w_2(\alpha) + w_2(\beta) \\ = (p_1+q_1)\rho(g^1) + (p_2+q_2)\rho(g^2) + (p_3+q_3)\rho(g^3). \end{aligned}$$

(3) If $p_i \equiv q_i (2) i = 1, 2$ and 3, $p = 0, q_k = 0$ for some k and for some $l, q_l \equiv 1(2)$, then $w_2(\alpha) = w_2(\beta)$ and so $w_2(\Phi) = 0$ and $\ker \mathrm{st}_2 \simeq \mathbb{Z}_2$.

Example 2 Let $M = P^2 \times S^4$, and let α and β be trivial bundles over M. Since $w_1(M) \neq 0$, $w_1(M) \neq w_1(\beta)$. In this case

$$\begin{aligned} w_2(\Phi) &= 0, \\ \ker \mathrm{st}_2 &\simeq H_1(P^2 \times S^4; \mathbb{Z}_{\beta - \tau_M}) \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2, \\ \operatorname{coker} \mathrm{st}_2 &\simeq \mathbb{Z}_2. \end{aligned}$$

Example 3 We consider some examples of the special case of tangent vector fields, that is, when α is the trivial line bundle and β is the tangent bundle τ_M over a n–dimensional manifold M with $n > 3$.

(1) $M = S^1 \times S^{2k}$. We have $\operatorname{coker} \mathrm{st}_1 \simeq \mathbb{Z}_2$, that is, half of the stable vector fields over $S^1 \times S^{2k}$ come from nonstable vector fields. Also, $\ker \mathrm{st}_1 = 0$, so we have at most one nonstable vector field stabilizing to any stable one.

(2) $M = S^1 \times S^{4k+3}$. In this case, we have infinitely many nonstable vector fields and two stable ones. The homomorphism st_1 is surjective, so, infinitely many nonstable vector fields are associated to each of the two stable ones.

(3) $M = P^{2k+1} \times S^{2k+1}$. In this case st_1 is an isomorphism, so to any stable vector field there is a unique nonstable one stabilizing to it.

(4) $M = P^{2k+1}$. In this case, $w_1(M) = 0$ and $\operatorname{coker} \mathrm{st}_1 \simeq \mathbb{Z}_2$. We have 4 stable and 4 nonstable vector fields. So only 2 stable ones come from nonstable ones and there are two of them for each stable one.

Example 4 We give now an example where α is a non trivial orientation line bundle ξ_{M^n} over a manifold M^n $(n \geq 3)$ and β is τ_{M^n}. We observe that if M^n is orientable ξ_{M^n} is trivial, a case already considered above. So, we are taking M^n to be a nonorientable manifold.

$M = S^1 \times P^{4k}$. Then $H_1(M;\mathbb{Z}) \simeq \mathbb{Z}_2$, (see Borsari–Gonçalves [1, Theorem 2.5]) and st_1 and $\overline{\mathrm{st}}_1$ are injective. In the Diagram IV we get $|\Omega_1(M;\Phi_1)| = |\Omega_1(P(\alpha);\Phi_1)| = 4$ and $|\Omega_1(M \times P^\infty;\Phi)| = 8$; so $\operatorname{coker} \mathrm{st}_1 \simeq \mathbb{Z}_2$, that is, only half of the homotopy classes of stable monomorphism contains unstable ones, and if so, each class contains exactly one element.

Acknowledgement

We would like to express our thanks to Ulrich Koshorke for his careful reading and helpful suggestions, which has substantially improved the early versions of this manuscript.

This work was partially supported by FAPESP, Projecto Temático Topologia Algébrica, Geométrica e Diferencial-2004/10229-6.

References

[1] **L D Borsari**, **D L Gonçalves**, *The first (co)homology group with local coefficients* (2005) preprint

[2] **U Koschorke**, *Vector fields and other vector bundle morphisms – a singularity approach*, Lecture Notes in Mathematics 847, Springer, Berlin (1981) MR611333

[3] **U Koschorke**, *The singularity method and immersions of m–manifolds into manifolds of dimensions $2m-2$, $2m-3$ and $2m-4$*, from: "Differential topology (Siegen, 1987)", Lecture Notes in Mathematics 1350, Springer, Berlin (1988) 188–212 MR979341

[4] **U Koschorke**, *Nonstable and stable monomorphisms of vector bundles*, Topology Appl. 75 (1997) 261–286 MR1430089

[5] **U Koschorke**, *Complex and real vector bundle monomorphisms*, Topology Appl. 91 (1999) 259–271 MR1666651

[6] **A K M Libardi**, **V M do Nascimento**, **I C Rossini**, *The cardinality of some normal bordism groups and its applications*, Rev. Mat. Estatíst. 21 (2003) 107–114 MR2058498

[7] **A K M Libardi**, **I C Rossini**, *Enumeration of nonstable monomorphisms: the even case*, from: "XI Brazilian Topology Meeting (Rio Claro, 1998)", World Sci. Publ., River Edge, NJ (2000) 80–84 MR1835691

[8] **D Randall**, **J Daccach**, *Cobordismo Normal e Aplicacoes*, Notas do Instituto de Ciências Matemáticas de São Carlos, ICMC-USP (1988)

Departamento de Matemática, IME – Universidade de São Paulo
Caixa Postal 66281–Agê ncia Cidade de São Paulo, 05311–970, São Paulo–SP, Brazil

Departamento de Matemática, IGCE – Universidade Estadual Paulista Júlio de Mesquita Filho
Av. 24A, 1515, 13506–700, Rio Claro – SP, Brazil

Departamento de Matemática, ICMC – Universidade de São Paulo
Av. Trabalhador Sãocarlense, 400 – Caixa Postal 668, 13560–970, São Carlos – SP, Brazil

dlgoncal@ime.usp.br, alicekml@rc.unesp.br, ozimneto@icmc.usp.br

Received: 13 November 2006 Revised: 23 March 2007