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Representations of surface groups and right-angled Artin groups in higher rank

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We give concrete constructions of discrete and faithful representations of right-angled Artin groups into higher-rank Lie groups. Using the geometry of the associated symmetric spaces and the combinatorics of the groups, we find a general criterion for when discrete and faithful representations exist, and show that the criterion is satisfied in particular cases. There are direct applications towards constructing representations of surface groups into higher-rank Lie groups, and, in particular, into lattices in higher-rank Lie groups.

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1 Introduction

The study of embeddings of surface groups $\Gamma = \pi_1(\Sigma_g)$, where Σ_g is a closed surface of genus g, in Lie groups has a long history, and has been recently the subject of study from a variety of different viewpoints.

The most basic example is when the Lie group $G = PSL(2,\mathbb{R})$, in which case the faithful, discrete representations inside $\mathrm{Hom}(\Gamma, G)/G$ give Teichmüller space. Hitchin [11] expanded the notion of Teichmüller space to representations into $G = SL(n,\mathbb{R})$, showing in that case that $\mathrm{Hom}(\Gamma, G)/G$ has a distinguished connected component, topologically a ball, containing Teichmüller space. He also calculated the number of connected components in the whole representation space. Other examples include real projective structures on surfaces (see Choi and Goldman [3]), Kähler geometry and bounded cohomology (see Burger, Iozzi and Wienhard [1]), and Anosov flows and hyperconvex curves in projective space (see Labourie [13]).

Looking at the representation variety is also a key component in studying vector bundles over surfaces and surface bundles over surfaces. In addition, when the images of the representations lie in a lattice Λ in G, one obtains examples of essential surfaces inside a locally symmetric space, generalizing the study of closed geodesics to higher dimensional submanifolds. However, while much has been proven about general properties of $\mathrm{Hom}(\Gamma, G)/G$, such as the number of connected components, or the

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nature of certain types of embeddings, there has not been very much attention paid towards constructing explicit examples of such objects, save in some special cases.

This paper focuses on the general problem of constructing explicit examples of discrete, faithful representations of surface groups into all sorts of Lie groups, or, even better, into lattices. One can then hope to calculate numerical invariants of those representations, such as Toledo invariants (when the associated symmetric space is Hermitian), to determine which components of the representation variety these representations belong to. We apply techniques and results of geometric group theory, combined with an understanding of the geometry of symmetric spaces, to the problem.

The main part of this work is the construction of numerous new explicit examples of discrete, faithful representations of surface groups into semisimple Lie groups of higher rank. We also construct new representations into lattices in semisimple Lie groups. The method developed can also be adapted to representations of other groups of interest in geometric group theory, such as the fundamental groups of closed hyperbolic 3- and 4-manifolds (see Crisp–Wiest [5] and Januskiewicz–Świątkowski [12]), and graph braid groups [4].

We examine the topic of representations of surface groups using the technique of right-angled Artin groups. Given a finite graph $H=(V,E)$, we define the *right-angled Artin group* $A(H)$ to be the group given by the presentation

$$\langle s_v | v \in V; [s_v, s_w] = 1 \text{ if } vw \in E \rangle.$$

Despite their simple presentations, these groups exhibit a number of interesting properties, and have been the subject of an increasing amount of study in recent years (see, for example, Charney [2]).

The focus on right-angled Artin groups is useful in studying surface groups because of the work of John Crisp and Bert Wiest [4], who have shown that all fundamental groups of closed, oriented surfaces embed in some right-angled Artin group. In particular, the fundamental group of any closed oriented surface of genus at least 2 embeds into the Artin group $A(C_5)$, where C_n denotes the cycle on n vertices.

This is one reason why we primarily focus on Artin groups $A(H)$ where H is a cycle of at least 5 vertices. Another reason is that, since cycles are subgraphs of almost every graph, restrictions on the representations of the corresponding subgroups (or the lack thereof) will translate to information about the full group.

One can show by simple dimension count arguments that representations of $A(C_5)$ exist, but exhibiting faithfulness and, particularly, discreteness requires a stronger argument. Our construction is explicit, and gives a whole class of discrete, faithful representations. In particular, the proof of discreteness exploits the geometry of the symmetric spaces associated to Lie groups.

The paper is in two main parts. In the first, we prove that if there is an arrangement of the maximal flats of the symmetric space $X = G/K$ that mimics the structure of the graph H in a certain sense, then we can find a faithful and discrete representation of $A(H)$ into G. In the second, we primarily focus on the case where H is a cycle, and show that such arrangements of flats exist for certain symmetric spaces.

This process yields the following main results:

Theorem 1 *There are infinitely many conjugacy classes of discrete, faithful representations of* $A(C_5)$ *into* $SL(n,\mathbb{R})$ *for* $n \geq 3$.

This method ought to be generalizable to arbitrary symmetric spaces, but it sometimes requires Artin groups on larger cycles than C_5. For certain symmetric spaces, we will not be able to use our method to embed $A(C_5)$, but only Artin groups on cycles of even length. It is known that these Artin groups contain some, but not all, surface groups (see Droms, Servatius and Servatius [6]).

Theorem 2 *There are infinitely many conjugacy classes of discrete, faithful representations of* $A(C_6)$ *into* $SO(3,2)$.

It is essential that the Lie groups are of $\mathbb{R}$–rank at least two.

Theorem 3 *If* G *is a simple Lie group of* $\mathbb{R}$*–rank 1, and* H *is a non-complete connected graph, then there is no discrete and faithful embedding of* $A(H)$ *into* G.

It is interesting that this method can also be used to construct representations of surface groups into lattices in semi-simple Lie groups. These constructions yield examples of essential surfaces inside locally symmetric spaces.

Theorem 4 *There are infinitely many conjugacy classes of representations of* $A(C_5)$ *into* $SL(5,\mathbb{Z})$.

As a result of Theorem 1, we obtain explicit representations of surface groups.

Theorem 5 *For any closed surface group* $\Gamma = \pi_1(\Sigma_g)$ *with* $g \geq 2$ *and* $n \geq 3$*, there are infinitely many conjugacy classes of discrete, faithful representations of* Γ *into* $SL(n,\mathbb{R})$.

There is a little bit of work required to keep the infinitely many conjugacy classes, but otherwise this is an immediate corollary. Certain discrete and faithful representations of surface groups into $PSL(3,\mathbb{R})$ correspond to convex projective structures on the surface, and have been extensively studied (see Choi–Goldman [3] and Goldman [10]). For the most part, however, the techniques used in previous work have not yielded algebraically explicit examples. Our construction gives a way of constructing these representations explicitly.

All examples constructed via this method lie in the connected component of the trivial representation, so these representations do not lie in Hitchin's Teichmüller component or have non-zero Toledo invariant (when the target symmetric space is Hermitian). These are, however, the first class of explicit non-trivial examples of representations, and it is hoped that the geometric nature of the construction (and, in particular, the ease in which it yields discreteness) will be helpful in future study of the representation variety.

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2 Preliminaries

2.1 Symmetric spaces

Throughout this paper, G will denote a connected, semisimple Lie group with no compact or Euclidean factors, K a maximal compact subgroup of G, and $X = G/K$ its associated symmetric space of non-compact type.

The *rank* of X is the largest integer r such that there exists a totally geodesic subspace of X isometric to $\mathbb{R}^r$. A symmetric space of non-compact type has rank 1 if and only if its sectional curvatures are negative. A *maximal flat* in a symmetric space X is a totally geodesic submanifold F which is isometric to $\mathbb{R}^r$, where r is the rank of X. Note that, under the identification of T_pX with $\mathfrak{p}$, T_pF is a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$.

Every geodesic of X is contained in at least one maximal flat. A geodesic γ is called *regular* if there is exactly one maximal flat containing γ; if there is more than one maximal flats containing γ, it is *singular*. Likewise, a point $z \in X(\infty)$ is called regular (resp. singular) if a geodesic γ with $\gamma(\infty) = z$ is regular (resp. singular).

An isometry ϕ of X having the property that there is some geodesic γ such that $\phi(\gamma(t)) = \gamma(t + t_0)$ for all t is called *axial*. In this case, we call γ an axis for ϕ and t_0 the translation distance.

For these and other facts on symmetric spaces, see (for instance) Eberlein [9].

2.2 Artin groups

Given a finite graph $H = (V, E)$, we define the *right-angled Artin group* $A(H)$ to be the group given by $\langle s_v | v \in V; [s_v, s_w] = 1 \text{ if } vw \in E \rangle$.

Crisp and Wiest have shown that all but three surface groups embed in some right-angled Artin group. In particular, for closed orientable surfaces, we have the following:

Theorem 6 (Crisp–Wiest [4]) *There is a faithful homomorphism from the fundamental group of any closed oriented surface* Σ_g *into the Artin group* $A(C_5)$*, where* C_n *denotes the cycle on* n *vertices.*

Other Artin groups also contain certain surface groups as subgroups:

Theorem 7 (Droms–Servatius–Servatius [6]) *If* $n \geq 5$*, then* $A(C_n)$ *contains the surface group* $\pi_1(\Sigma_g)$*, where* $g = 1 + (n-4)2^{n-3}$.

We will therefore concentrate on the question of finding discrete and faithful representations of right-angled Artin groups, particularly $A(C_5)$, into semi-simple Lie groups.

It will be important that the Lie group be of rank greater than 1. There are examples of faithful embeddings of $A(C_5)$ into Lie groups of rank 1, but they are far from being discrete.

Theorem 8 *If* G *is a simple Lie group of* $\mathbb{R}$*–rank 1, and* H *is a non-complete connected graph, then there is no discrete and faithful embedding of* $A(H)$ *into* G.

Proof Let v and w be two adjacent vertices of H, and s_v and s_w the corresponding generators of the right-angled Artin group. Given a representation $\sigma\colon A(H) \to G$, we examine the action of $\sigma(s_v)$ on the symmetric space $X = G/K$, which will be of negative curvature.

If $\sigma(s_v)$ fixes some point $p \in X$, then $\sigma(s_v)$ lies in a subgroup of G which is compact. Since s_v has infinite order in $A(H)$, this means that σ is either non-faithful or has non-discrete image.

Thus $\sigma(s_v)$ fixes exactly one or two points in $X(\infty)$. If it fixes two, then there is some geodesic $\gamma\colon \mathbb{R} \to X$ and some $t_0 \in \mathbb{R}$ such that $\sigma(s_v)(\gamma(t)) = \gamma(t+t_0)$ for all t. This will be the unique geodesic that is fixed by $\sigma(s_v)$, and thus γ will be fixed by $\sigma(s_w)$ as well. There will be some $t_1 \in \mathbb{R}$ such that $\sigma(s_w)(\gamma(t)) = \gamma(t+t_1)$ for all t.

But this means that, for any $\epsilon > 0$, we can find $k, l \in \mathbb{Z} - \{0\}$ such that $\sigma(s_v^k s_w^l)$ translates $\gamma(0)$ a distance less than ϵ (possibly 0), so either σ is not faithful or it does not have a discrete image.

We are left with the case where $\sigma(s_v)$ fixes exactly one point x at infinity. The commutation relations then force every element of the image of σ to fix x, so σ is a representation into a horospherical subgroup N of G. But N is a nilpotent group (see Eberlein [8]) and is therefore amenable; since $A(H)$ contains a non-abelian free subgroup, it cannot embed faithfully into N. □

3 Discrete and faithful representations

Given a connected graph H with no triangle subgraphs, we will show that the right-angled Artin group $A(H)$ embeds into a higher-rank Lie group G if one can find a configuration of geodesics and flats in the symmetric space $X = G/K$ that mimics the graph H. That is to say, if we view the singular geodesics throuigh some point $p_0 \in X$ as the vertices of a graph H', and the maximal flats containing p_0 as the edges of the graph, we wish to find an induced subgraph of H' which is isomorphic to H. We will use the terminology of graph theory and say that two geodesics through p_0 are *adjacent* if they lie in a common maximal flat.

There will be a few technical conditions, so the exact statement is as follows:

Theorem 9 *Let $X = G/K$ be a symmetric space of non-compact type of rank at least 2, and let $p_0 \in X$. Suppose that for each vertex v of H, we can find a geodesic $\gamma_v\colon \mathbb{R} \to X$ with $\gamma_v(0) = p_0$ with the following properties:*

- *If vw is an edge of H, then γ_v and γ_w are adjacent, there is exactly one maximal flat F_{vw} containing both γ_v and γ_w, and the set of singular geodesics S_{vw} contained in the copy of $\mathbb{R}^2 \subset F_{vw}$ determined by γ_v and γ_w is finite.*
- *If vw is not an edge of H, γ_v and γ_w are not adjacent.*

- *If* $v_1 v_2$ *and* $w_1 w_2$ *are disjoint edges in* H*, then no elements of* $S_{v_1 v_2}$ *and* $S_{w_1 w_2}$ *are adjacent, with the exception of* γ_{v_i} *and* γ_{w_j} *if* $v_i w_j$ *is an edge of* H*.*
- *If* vw *and* wx *are two edges of* H *sharing an endpoint, then no element of* $S_{vw} - \{\gamma_w\}$ *is adjacent to any element of* $S_{wx} - \{\gamma_w\}$*.*

Then there are infinitely many conjugacy classes of discrete, faithful representations $\sigma\colon A(H) \to G$*.*

The proof of the theorem is a modification of the "ping-pong" technique. The classic "ping-pong" argument was first used to prove that two elements α, β in $SL(2,\mathbb{R})$ generated a free subgroup. It employed four open sets $U_\alpha, U_{\alpha^{-1}}, U_\beta, U_{\beta^{-1}} \subset \mathbb{H}^2(\infty)$ and showed that a word in α and β sent a base point $z \in \mathbb{H}^2(\infty)$ to the open set corresponding to the first letter in the word, utilizing the fact that α and β had disjoint repelling and attracting fixed sets.

Our proof follows the same spirit, but there are complicating factors due to the fact that $A(H)$ is not free and the more intricate geometry of higher rank symmetric spaces. For instance, our open sets will necessarily intersect one another, and we will be unable to use separate open sets for a generator and its inverse.

We begin the proof with a lemma.

Lemma 10 *Let* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *be the Cartan decomposition given by a point* $p_0 \in X$*, and let* $A \in \mathfrak{p}$ *be a non-zero vector. Let* F *denote the union of all maximal flats containing the geodesic* $\gamma(t) = \exp(tA)p$*. Let* $z_0 \in X(\infty)$*; then* $\lim_{t\to\infty} \exp(tA)z_0$ *will exist, and lie in* $F(\infty)$*.*

Proof Let z be any accumulation point of the sequence $\{\exp(tA)z_0\}$, and let $y = \gamma(-\infty)$. Since X is non-positively curved, the function $f(t) = \angle_{\gamma(t)}(y, z_0)$ will be non-increasing, and therefore there is some β such that $\lim_{t\to-\infty} f(t) = \beta$.

We claim that $\angle_{\gamma(t_0)}(y, z) = \beta$ for all t_0. Indeed, for any $\epsilon > 0$,

$$\beta - \epsilon \le \angle_{\gamma(-t)}(y, z_0) \le \beta$$

for $t \gg 0$ so, applying the isometry $\exp((t + t_0)A)$, we see that

$$\beta - \epsilon \le \angle_{\gamma(t_0)}(y, \exp((t + t_0)A)z_0) \le \beta$$

for $t \gg 0$. Hence $\angle_{\gamma(t_0)}(y, z)$ must lie in the interval $[\beta - \epsilon, \beta]$. Since ϵ was arbitrary, $\angle_{\gamma(t_0)}(y, z) = \beta$, and the claim is proven.

That the claim implies the lemma is precisely Claim E5 in Eberlein [7]. □

Let $\{e_i\}$ denote the set of edges of H, and S_i the corresponding sets of singular geodesics defined in the statement of the theorem, and denote the unique maximal flat containing the geodesics of S_i be denoted by F_i. Let S be the union of all S_i, and let $F(\gamma)$ denote the union of all flats containing γ.

Two geodesics γ, η are adjacent if and only if $F(\gamma)(\infty)$ and $F(\eta)(\infty)$ are disjoint. Therefore, for each geodesic $\gamma \in S$, we can find an open set $U_\gamma \subset X(\infty)$ containing $F(\gamma)(\infty)$ such that $\overline{U}_\gamma \cap \overline{U}_\eta \neq \varnothing$ iff there is some i such that $\gamma, \eta \in S_i$. Let $U_i = \bigcap_{\gamma \in S_i} U_\gamma$; this will be an open set containing $F_i(\infty)$.

Let $z \in X(\infty)$ be a point not contained in the closure of any of the U_γ.

Lemma 11 *It is possible to find real numbers t_v such that the isometries $\phi_v = \exp(t_v A_v)$ have the following properties:*

- *If $\gamma \in S$ is not adjacent to γ_v, then $\phi_v^k(U_\gamma) \subset U_{\gamma_v}$ for all $k \in \mathbb{Z}^\times$.*
- *For any $k \in \mathbb{Z}^\times$, $\phi_v^k(z) \in U_{\gamma_v}$.*
- *If $e_i = vw$ is an edge in H, and $\gamma \in S$ is adjacent to neither γ_v nor γ_w, then for any $k, l \in \mathbb{Z}^\times$, $\phi_v^k \circ \phi_w^l(U_\gamma) \subset \bigcup_{\eta \in S_i} U_\eta$.*
- *If $e_i = vw$ is an edge in H, then for any $k, l \in \mathbb{Z}^\times$, $\phi_v^k \circ \phi_w^l(z) \in \bigcup_{\eta \in S_i} U_\eta$.*

Proof The first two conditions are satisfied as long as the t_v are large enough, by the lemma.

To show that the second two are satisfiable, for a given edge $e_i = vw$, let $\mathfrak{h} \subset \mathfrak{p}$ be the subspace generated by A_v and A_w, and let $\tau\colon \mathfrak{h} \to X(\infty)$ be given by $\tau(A) = \exp(A)(z')$, for some $z' \notin \bigcup_{\eta \in S_i} U_\eta$. We will show that $\tau^{-1}(\bigcup_{\eta \in S_i} U_\eta)$ has a bounded open set in $\mathfrak{h}$ as its complement. This will suffice, for then we can simply pick t_v and t_w such that $\tau^{-1}(\bigcup_{\eta \in S_i} U_\eta) \subset \{aA_v + bA_w \mid |a| > 1 \text{ or } |b| > 1\}$.

To do so, we simply apply the lemma to each geodesic $\exp(tA)(p_0)$, where $A \in \mathfrak{h}$ has length 1. All but finitely many will be regular geodesics, and for these there is some minimum T_A such that $\exp(tA)(z') \subset U_i$ whenever $|t| > T_A$. If $\exp(tA)(p_0) = \eta \in S_i$, then we can still find some T_A such that $\exp(tA)(z') \subset U_\eta$ when $|t| > T_A$. Thus along any one-dimensional subspace of $\mathfrak{h}$, the image of τ will eventually land in $\bigcup_{\eta \in S_i} U_\eta$. The function $A \to T_A$ will be continuous, and thus there is some T such that $\tau(A) \subset \bigcup_{\eta \in S_i} U_\eta$ whenever $|A| > T$. □

Once we have chosen our ϕ_v, we then define the representation $\sigma\colon A(H) \to G$ by $\sigma(s_v) = \phi_v$. If vw is an edge of H, we know that γ_v and γ_w are adjacent, and thus ϕ_v and ϕ_w commute. Therefore the relations of $A(H)$ are satisfied, and this is a group homomorphism. To show that the image of σ is faithful and discrete, we will show that if h is a non-identity element of $A(H)$, then $\sigma(h)(z) \in \bigcup_{\eta \in S_i} U_\eta$ for some i.

For each element $h \in A(H)$, let $\ell(h)$ denote the minimum integer n such that h has a representation $h = s_{v_1}^{k_1} s_{v_2}^{k_2} \dots s_{v_n}^{k_n}$ as a product of generators of $A(H)$ (where the k_i can be either positive or negative integers).

By induction on n, we will show that any element h has a representation as a word in the generators $h = s_{v_1}^{k_1} s_{v_2}^{k_2} \dots s_{v_n}^{k_n}$ where $n = \ell(h)$ and either $\sigma(h)(z) \in U_{v_1}$ if $v_1 v_2$ is not an edge of H, or $\sigma(h)(z) \in \bigcup_{\eta \in S_i} U_\eta$ if $v_1 v_2 = e_i$ is an edge of H.

When $\ell(h) = 1$, this is true by our choice of the ϕ_v. Now assume that this is true for all $n < m$, and suppose that $\ell(h) = m$. Pick a representation $h = s_{v_1}^{k_1} s_{v_2}^{k_2} \dots s_{v_m}^{k_m}$. By transposing commuting generators, it might be possible to switch $s_{v_1}^{k_1}$ to a position farther right; suppose that the farthest right it can go is position j, in the word $h = s_{w_1}^{l_1} s_{w_2}^{l_2} \dots s_{w_m}^{l_m}$. Thus we know that s_{w_i} commutes with s_{w_j} for all $i < j$, and since there are no triangles in H, s_{w_i} does not commute with $s_{w_{i+1}}$ if $i < j-1$.

Also, by our induction hypothesis, we can transpose commuting generators so that the word $h_j = s_{w_{j+1}}^{l_{j+1}} \dots s_{w_m}^{l_m}$ has the property that $\sigma(h_j)(z)$ is in $U_{w_{j+1}}$ if $w_{j+1} w_{j+2}$ is not an edge of H, or $\bigcup_{\eta \in S_i} U_\eta$ if $w_{j+1} w_{j+2}$ is the edge e_i of H. We know that $w_j w_{j+1}$ cannot be an edge of H (else we could do another transposition to get the original left-most generator s_{v_1} farther right). Therefore, if $j = 1$, we must have $\sigma(h)(z) \in U_{w_1}$, and we are done.

If $j \geq 2$, we see that $\phi_{w_{j-1}}^{l_{j-1}} \circ \phi_{w_j}^{l_j}$ must take $\sigma(h_j)(z)$ into $\bigcup_{\eta \in S_i} U_\eta$, where $w_{j-1} w_j$ is the edge e_i of H. If $j = 2$, we are then done. If $j > 2$, since $w_i w_{i+1}$ is not an edge of H for $i < j-1$, we can therefore conclude that $\sigma(h)(z) \in U_{w_1}$. This concludes the induction argument.

It remains to be shown that we can get infinitely many conjugacy classes. In the proof of Lemma 11, we were free to choose the t_v, the translation length of the axial isometries ϕ_v, to be any numbers, as long as they were sufficiently large. Given finitely many discrete and faithful representations σ_i, there are only countably many translation lengths of axial isometries in their images, so as long as we choose a t_v distinct from all of them, we are guaranteed a new conjugacy class.

4 Finding geodesic configurations

Since the result of Crisp and Wiest holds for a right-angled Artin group on a 5-cycle, we are interested in finding arrangements of 5 singular geodesics that have the properties required by the theorem in the previous chapter.

4.1 The model group

Theorem 12 *Let* $X = SL(3,\mathbb{R})/SO(3)$, *and* $p_0 \in X$. *Then there are infinitely many conjugacy classes of sets of 5 geodesics through* p_0 *satisfying the conditions of Theorem 9 for the graph* $H = C_5$.

Our goal is to find a different flat F_{12} which intersects the original flat F_{01} in one of the singular geodesics γ_1, choose a different singular geodesic γ_2 in F_{12}, find another flat F_{23} which intersects F_{12} in γ_2, and so on, until we are able to find a flat F_{40} which intersects F_{01} in a geodesic γ_0, completing the cycle. To find successive flats, we will apply "rotations" around the singular geodesics - that is, isometries fixing the geodesic pointwise - to move one flat to the next (see diagram). We will see that the condition that this process cycle back around to the original flat essentially is a condition on the product of the chosen rotations.

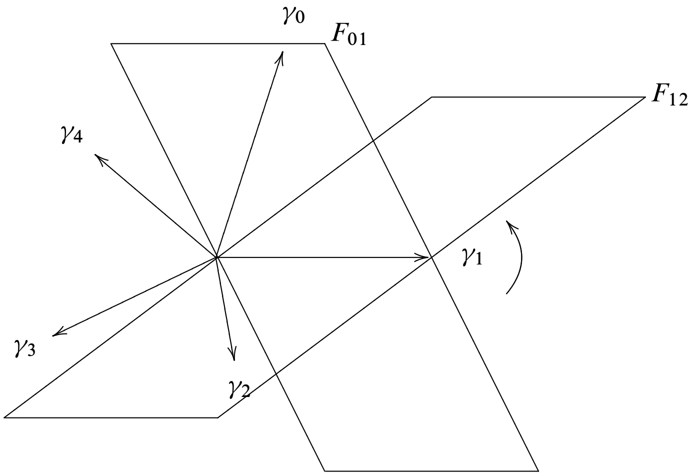

Figure 1: Flats and geodesics in a five-cycle configuration

Without loss of generality, we may take p_0 to be the identity coset $K = SO(3)$.

Consider the geodesics $\lambda_1(t) = \mathrm{diag}(e^{-2t}, e^t, e^t)p_0$, $\lambda_2(t) = \mathrm{diag}(e^t, e^{-2t}, e^t)p_0$, and $\lambda_3(t) = \mathrm{diag}(e^t, e^t, e^{-2t})p_0$. These are all contained in a unique common flat F_{01}. Let $A_i = \{g \in K | g\lambda_i = \lambda_i\}$ for each i. Note that A_1 is the set of rotations in $SO(3)$ that fix the x–axis, A_2 fixes the y–axis, and A_3 the z–axis.

Pick any non-identity elements $R_1 \in A_1$ and $R_2 \in A_2$.

Let Z be the vector $(0,0,1)$; consider $Y = R_2 R_1 Z$. There is then an $R_3 \in A_3$ such that $R_3 Y$ has second coordinate equal to 0, and thus a unique $R_4 \in A_1$ such that $R_4 R_3 R_2 R_1 Z = Z$.

Let $\gamma_0 = \lambda_2$, $\gamma_1 = \lambda_1$, $\gamma_2 = R_4 \lambda_3$, $\gamma_3 = R_4 R_3 \lambda_2$ and $\gamma_4 = R_4 R_3 R_2 \lambda_1$.

Note that γ_1 and γ_2 both lie in the flat $F_{12} = R_4 F_{01}$ since R_4 leaves γ_1 invariant. Similarly, γ_2 and γ_3 both lie in the flat $F_{23} = R_4 R_3 F_{01}$, and γ_3 and γ_4 both lie in $F_{34} = R_4 R_3 R_2 F_{01}$. Since $R_4 R_3 R_2 R_1 \in A_2$, both γ_4 and γ_0 are in $F_{40} = R_4 R_3 R_2 R_1 F_{01}$.

Thus the five geodesics γ_i satisfy the first condition of Theorem 9. There are finitely many non-adjacency requirements; to verify that geodesics η_1 and η_2 are non-adjacent, we simply need to confirm that axial isometries with η_1 and η_2 as axes do not commute.

There are ten singular geodesics through p_0 in the five flats, namely the five γ_i, plus $\eta_0 = \lambda_3$ in F_{01}, $\eta_1 = R_4 \lambda_2$ in F_{12}, $\eta_2 = R_4 R_3 \lambda_1$ in F_{23}, $\eta_3 = R_4 R_3 R_2 \lambda_3$ in F_{34}, and $\eta_4 = R_4 R_3 R_2 R_1 \lambda_3$ in F_{40}. Let $T_1 = \mathrm{diag}(\frac{1}{4}, 2, 2)$, $T_2 = \mathrm{diag}(2, \frac{1}{4}, 2)$, and $T_3 = \mathrm{diag}(2, 2, \frac{1}{4})$. Table 1 gives axial isometries for each of the geodesics.

Geodesic	Isometry
γ_0	T_2
γ_1	T_1
γ_2	$R_4 T_3 R_4^{-1}$
γ_3	$R_4 R_3 T_2 R_3^{-1} R_4^{-1}$
γ_4	$R_4 R_3 R_2 T_1 R_2^{-1} R_3^{-1} R_4^{-1}$
η_0	T_3
η_1	$R_4 T_2 R_4^{-1}$
η_2	$R_4 R_3 T_1 R_3^{-1} R_4^{-1}$
η_3	$R_4 R_3 R_2 T_3 R_2^{-1} R_3^{-1} R_4^{-1}$
η_4	$R_4 R_3 R_2 R_1 T_3 R_1^{-1} R_2^{-1} R_3^{-1} R_4^{-1}$

Table 1: Singular geodesics and axial isometries

Pairs of geodesics that we need to be non-adjacent are γ_i and γ_j if $j - 1 \neq \pm 1$ (mod 5), η_i and η_j for any $i \neq j$, and η_i and γ_j if $j - i \neq 0, 1$ (mod 5).

Recall that we have complete freedom to choose R_1 and R_2 in A_1 and A_2 respectively, but once they are chosen, R_3 and R_4 are determined. Since commuting is an analytic condition, if we see that these pairs of isometries do not commute for one particular choice of R_1 and R_2, then they will not commute for almost any choice of R_1 and R_2.

We make the choices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, R_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

This means that R_3 and R_4 are the following:

$$R_3 = \begin{pmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} & 0 \\ -\sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{3}{8}} \\ 0 & \sqrt{\frac{3}{8}} & \sqrt{\frac{5}{8}} \end{pmatrix}.$$

Verification that the appropate pairs of matrices in Theorem 12 do not commute is done by explicitly calculating their commutators. These are relegated to a separate appendix which is published on the author's web site.

Theorem 13 *For any closed hyperbolic surface group* Γ *and* $n \geq 3$, *there are infinitely many conjugacy classes of discrete, faithful representations of* Γ *into* $SL(n, \mathbb{R})$.

The only thing remaining to prove in this theorem is that we can obtain infinitely many conjugacy classes. This is not immediate, since representations of $A(C_5)$ which are not conjugate might still yield representations of a $\pi_1(\Sigma)$ which are conjugate. However, the flexibility of our construction proves that you can in fact guarantee different conjugacy classes for the surface groups.

Take any discrete, faithful representation $\sigma\colon A(C_5) \to G$ given by our construction. Pick $x \in X = G/K$. Since σ is discrete, we know that for any $r > 0$, the set $B_r = \{a \in A(C_5) | d(\sigma(a)x, x) < r\}$ is finite.

Let $a_1, a_2, \dots a_5$ be the generators of $A(C_5)$. Then the representation σ_n given by $\sigma_n(a_i) = \sigma(a_i)^n$ will be discrete and faithful. The image of σ_n will be the image ofσ restricted to the subgroup A_n of $A(C_5)$ generated by the a_i^n; we can choose n large enough so that $A_n \cap B_r = \{e\}$, since B_r is finite.

Thus for every r, we can find a discrete, faithful representation $\tau\colon A(C_5) \to G$ such that $\min\{d(x, \tau(a)x) | a \in A(C_5) - \{e\}\} > r$.

Now suppose that we have a finite set of representations $\tau_1 \dots \tau_n\colon \pi_1(\Sigma) \to G$. Take some $a \in \pi_1(\Sigma) \subset A(C_5)$ and let $r = \max d(x, \tau_i(a)x)$. If $\tau\colon A(C_5) \to G$ is a discrete, faithful representation such that $\min\{d(x, \tau(a)x) | a \in A(C_5) - \{e\}\} > r$, then τ restricted to $\pi_1(\Sigma)$ cannot be conjugate to any of the σ_i.

4.2 Other groups

The example of $SL(3,\mathbb{R})$ serves as a model for embeddings of Artin groups on cycles into other Lie groups of rank 2.

Crisp and Wiest's result about embedding surface groups holds only for $A(C_5)$, so if the cycle is larger than C_5 we would not be able to embed all surface groups into G via this Artin group technique. However, we would still be able to embed certain surface groups, thanks to the result of Droms, Servatius and Servatius.

For instance, we have the following:

Theorem 14 *There are infinitely many conjugacy classes of discrete, faithful representations of $A(C_6)$ and $\pi_1(\Sigma_{17})$ into $SO(3,2)$.*

Here the symmetric space is $X = SO(3,2)/SO(3)\times SO(2)$, again of rank 2.

Let p_0 be the identity coset $K = SO(3)\times SO(2)$. We wish to find a collection of six geodesics through p_0 satisfying the conditions of Theorem 9.

Let $\mathfrak{a}$ be the abelian subalgebra of $\mathfrak{g}$ generated by the two elements

$$Y_0 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix}.$$

Then $\exp(\mathfrak{a})p_0$ is a flat F_{01} in X, and $\gamma_0(t) = \exp(tY_0)p_0$ and $\gamma_1(t) = \exp(tY_1)p_0$ are singular geodesics in F_{01} and are perpendicular. The other singular geodesics in F_{01} through p_0 are the angle bisectors λ_0, λ_1 of the right angles formed by γ_0 and γ_1. Thus if $T_0 = \exp(aY_0)$ and $T_1 = \exp(aY_1)$, T_0T_1 and $T_0T_1^{-1}$ are axial isometries of X whose axes are these singular geodesics λ_0 and λ_1. We choose $a = \log(2+\sqrt{3})$, so

$$T_0 = \begin{pmatrix} 2&0&0&\sqrt{3}&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ \sqrt{3}&0&0&2&0\\ 0&0&0&0&1 \end{pmatrix}; T_1 = \begin{pmatrix} 1&0&0&0&0\\ 0&2&0&0&\sqrt{3}\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&\sqrt{3}&0&0&2 \end{pmatrix}.$$

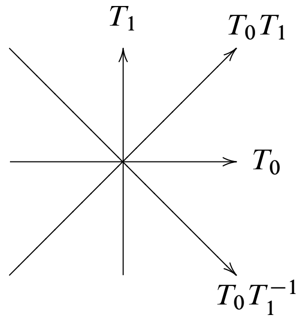

Figure 2: Singular geodesics in a flat in $SO(3,2)/SO(3)\times SO(2)$

Let A_0 and A_1 be the subgroups of K fixing γ_0 and γ_1, respectively. There are isomorphisms τ_i between $SO(2)$ and each of the A_i; they are defined by

$$\tau_0\colon \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in SO(2) \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & -b & a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in A_0$$

$$\tau_1\colon \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in SO(2) \longmapsto \begin{pmatrix} a & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -b & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in A_1$$

Copying our proof of Theorem 12, if we can find $R_1, R_3, R_5 \in A_1$ and $R_2, R_4 \in A_0$ such that $R_5R_4R_3R_2R_1 \in A_0$, then γ_0, γ_1, $\gamma_2 = R_5\gamma_0$, $\gamma_3 = R_5R_4\gamma_1$, $\gamma_4 = R_5R_4R_3\gamma_0$ and $\gamma_5 = R_5R_4R_3R_2\gamma_1$ are six geodesics which are adjacent in a 6-cycle pattern.

Since τ_0 and τ_1 have image only in the first factor of $K = SO(3)\times SO(2)$, the condition that $R_5R_4R_3R_2R_1 \in A_0$ is essentially just a requirement that a product of rotations in $\mathbb{R}^3$ be a rotation around the x–axis. Just like in our discussion in the case of $SL(3,\mathbb{R})$, we see that given any choice of R_1, R_2, R_3, we know that we can find a rotation R_4 around the x–axis and a rotation R_5 around the y–axis such that the product $R_5R_4R_3R_2R_1$ fixes the x–axis.

As was the case in Theorem 12, we only need to check non-adjacency for one particular set of choices of R_i. We choose

$$R_1 = \tau_1 \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}, \qquad R_2 = \tau_0 \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}, \qquad R_3 = \tau_1 \begin{pmatrix} -\frac{31}{481} & -\frac{480}{481} \\ \frac{480}{481} & -\frac{31}{481} \end{pmatrix}.$$

This forces R_4 and R_5 to be the following:

$$R_4 = \tau_0 \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}, \qquad R_5 = \tau_1 \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

Axial isometries with the γ_i as axes are T_0, T_1, $R_5T_0R_5^{-1}$, $R_5R_4T_1R_4^{-1}R_5^{-1}$, $R_5R_4R_3T_0R_3^{-1}R_4^{-1}R_5^{-1}$, $R_5R_4R_3R_2T_1R_2^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$. In addition, the extra singular geodesics in the six flats will have axial isometries listed in the following table, where F_{ij} denotes the flat containing γ_i and γ_j.

Flat	Isometries
F_{01}	T_0T_1 $T_0T_1^{-1}$
F_{12}	$R_5T_0T_1R_5^{-1}$ $R_5T_0T_1^{-1}R_5^{-1}$
F_{23}	$R_5R_4T_0T_1R_4^{-1}R_5^{-1}$ $R_5R_4T_0T_1^{-1}R_4^{-1}R_5^{-1}$
F_{34}	$R_5R_4R_3T_0T_1R_3^{-1}R_4^{-1}R_5^{-1}$ $R_5R_4R_3T_0T_1^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$
F_{45}	$R_5R_4R_3R_2T_0T_1R_2^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$ $R_5R_4R_3R_2T_0T_1^{-1}R_2^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$
F_{50}	$R_5R_4R_3R_2R_1T_0T_1R_1^{-1}R_2^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$ $R_5R_4R_3R_2R_1T_0T_1^{-1}R_1^{-1}R_2^{-1}R_3^{-1}R_4^{-1}R_5^{-1}$

Table 2: Axial isometries by flat

Again, checking the non-adjacency requirements (of which there are 93) is an explicit computation of commutators of these matrices, and is published in a separate appendix on the author's web site.

The proof that there are infinitely many conjugacy classes proceeds in exactly the same way as in Theorem 12.

We have been unable to find a configuration of five maximal flats in X that satisfy all of the requirements of Theorem 9. It is possible to find flats satisfying the first two conditions, but in every configuration analyzed, there are always other geodesics in the flats which are adjacent. Changing the Artin group to $A(C_6)$ allows for more freedom in choosing the rotations around each singular geodesic and allows us to avoid this problem.

It seems likely that Artin groups on cycles of 6 or more should be flexible enough to allow representations into most semisimple Lie groups, barring a few obstructions.

For instance, suppose there are only two singular geodesics in a flat through a given point. This is the case when $X = SU(2,n)/S(U(2) \times U(n))$, for instance. In this situation, it will be impossible to embed $A(C_n)$ into the associated Lie group via this method if n is odd, since there will have to be two adjacent geodesics which are conjugates of the same geodesic in the original flat.

Another obstruction occurs for non-irreducible symmetric spaces. If X is a product of rank one symmetric spaces $X_1 \times X_2$, no $A(C_n)$ (for $n > 4$) will embed faithfully by this method. In this case, if $p_0 = (p_1, p_2)$, singular geodesics through p_0 are either $\{\eta_1\} \times \{p_2\}$ or $\{p_1\} \times \{\eta_2\}$, where η_i is a geodesic through p_i in X_i, and any geodesic of the first type will be adjacent to any geodesic of the second, so there is no hope of satisfying the conditions of Theorem 9 for graphs which are not complete and bipartite.

The general conjecture looks like this:

Conjecture 15 *If $X = G/K$ is an irreducible symmetric space of rank at least two, and given any maximal flat F and point $p_0 \in F$, there are at least three singular geodesics in F passing through p_0, then there are infinitely many conjugacy classes of discrete, faithful representations of $A(C_n)$ into G for any $n \geq 6$.*

If $X = G/K$ is an irreducible symmetric space of rank two, and given any maximal flat F and point $p_0 \in F$, there are exactly two singular geodesics in F passing through p_0, then there are infinitely many conjugacy classes of discrete, faithful representations of $A(C_{2n})$ into G, for any $n \geq 3$.

It would be helpful and interesting to find general methods that would allow for attacking this problem in ways other than a case-by-case analysis.

5 Lattices

We can also embed Artin groups inside some lattices in higher rank Lie groups. Recall that a *lattice* in G is a discrete subgroup $\Gamma \subset G$ such that $\Gamma \backslash G$ has finite volume. Embeddings of surface groups into lattices give examples of essential surfaces inside locally symmetric spaces.

Theorem 16 *There are infinitely many conjugacy classes of representations of* $A(C_5)$ *into* $\Gamma = SL(5,\mathbb{Z})$.

We will find five geodesics to apply the main theorem to.

Let p_0 be the identity coset in the symmetric space $X = SL(5,\mathbb{R})/SO(5,\mathbb{R})$. Fix an integer $n \geq 2$, and let A_i be the following elements of Γ:

$$A_1 = \begin{pmatrix} n & n-1 & 0 & 0 & 0 \\ n+1 & n & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & n & n-1 & 0 \\ 0 & 0 & n+1 & n & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} n & 0 & 0 & 0 & n+1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ n-1 & 0 & 0 & 0 & n \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & n & n-1 & 0 & 0 \\ 0 & n+1 & n & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & n & n-1 \\ 0 & 0 & 0 & n+1 & n \end{pmatrix}.$$

As isometries of X, these are all axial hyperbolic isometries. We see easily that A_i and A_j commute if and only if $i - j \equiv \pm 1 \pmod 5$, so the geodesics $\gamma_i(t) = \exp(t \log A_i)SO(5,\mathbb{R})$ mimic the graph C_5 in the appropriate way.

The symmetric space is of rank 4, so the requirement that the set of singular directions in the span of $\log A_i$ and $\log A_j$ be finite is non-trivial. It is relatively easy to check, however. The maximal flat containing both γ_1 and γ_2 can be taken to the maximal flat F of diagonal matrices via some element k_{12} of K which conjugates A_1 and A_2 to the matrices

$$B_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where λ and λ^{-1} are the eigenvalues of the matrix $\left(\begin{smallmatrix} n & n-1 \\ n+1 & n \end{smallmatrix}\right)$. In the Lie algebra $\mathfrak{a}$ of diagonal traceless matrices, the span corresponds to the subalgebra $\mathfrak{b}$ of all elements of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the only singular directions in $\mathfrak{a}$ are those with equal entries on the diagonal, we see that there are only 4 singular directions in $\mathfrak{b}$, namely those where $a=0, b=0, a=b$, or $a=-b$.

It remains to check the non-adjacency requirements. Let F_{ij} denote the two-plane containing γ_i and γ_j. Then the singular geodesics through p_0 in F_{ij} are precisely the axes of the isometries A_i, A_j, A_iA_j, and $A_iA_j^{-1}$. There are 125 pairs of geodesics that are supposed to be non-adjacent, but by symmetry arguments we can focus on those involving one geodesic from F_{12} and one geodesic from either F_{23} or F_{34}.

Since A_1 commutes with A_2 but not with A_3, it will not commute with either A_2A_3 or $A_2A_3^{-1}$. Neither A_1A_2 nor $A_1A_2^{-1}$ commute with A_3, so they also will not commute with A_2A_3 and $A_2A_3^{-1}$. Thus all of the pairs of geodesics in F_{12} and F_{23} behave the way they ought.

Checking that no geodesic in F_{12} is adjacent to one in F_{34} (save γ_2 being adjacent γ_3) requires a few more explicit matrix calculations, which we relegate to a separate appendix on the author's web site.

Thus the geodesics satisfy the requirements of Theorem 9, and so high enough powers of the A_i will give a discrete and faithful subgroup of $SL(5,\mathbb{Z})$ isomorphic to $A(C_5)$.

Note that we could get the image of the representation to lie in any congruence subgroup that we desire, since for any prime p, arbitrarily high powers of the A_i will be congruent to the identity modulo p.

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