# The Burau estimate for the entropy of a braid

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The topological entropy of a braid is the infimum of the entropies of all homeomorphisms of the disk which have a finite invariant set represented by the braid. When the isotopy class represented by the braid is pseudo-Anosov or is reducible with a pseudo-Anosov component, this entropy is positive. Fried and Kolev proved that the entropy is bounded below by the logarithm of the spectral radius of the braid's Burau matrix, $B(t)$, after substituting a complex number of modulus 1 in place of $t$. In this paper we show that for a pseudo-Anosov braid the estimate is sharp for the substitution of a root of unity if and only if it is sharp for $t=-1$. Further, this happens if and only if the invariant foliations of the pseudo-Anosov map have odd order singularities at the strings of the braid and all interior singularities have even order. An analogous theorem for reducible braids is also proved.

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## 1 Introduction

Artin's braid group and its Burau representation have been extensively studied by many researchers from many points of view. In dynamical applications a braid is often used to describe the motion of a collection of points in the two-dimensional disk. Since the braid depends only on the motion of the points, it is describing an isotopy class of homeomorphisms on the complement of the points. Thus, the interpretation of the braid group on $n$–strings, $B_n$, as a mapping class group of the $n$–punctured disk is frequently used, and so Thurston's classification theorem for surface isotopy classes is an important tool.

The (reduced) Burau matrix, $B(t)$, of a braid $\beta \in B_n$, is an $(n-1)\times(n-1)$ matrix with entries in $\mathbb{Z}[t,t^{-1}]$, ie the entries of the matrix are Laurent polynomials over the integers. In the early 1980s two different but closely related dynamical interpretations of the Burau matrix emerged. Using the construction in Franks' paper [14], the Burau matrix can be interpreted as the signed, linking matrix of a certain Axiom A flow associated with the braid. The signed, linking matrix is an enhanced Markov transition

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matrix which records the linking of the Markov boxes with the strings of the braid as well as the orientations of their images.

The second dynamical interpretation comes from the machinery in Fried's paper [15]. In this case the Burau matrix of a braid arises as the induced action of the associated mapping class on a $\mathbb{Z}$-cover, where the first homology of the cover is given the structure of a module over $\mathbb{Z}[t, t^{-1}]$ (this is a standard description, see, for example, Birman and Brendle [4]). In the more general setting of twisted cohomology, Fried showed that after using the appropriate representation of the fundamental group, the spectral radius of the induced action gives a lower bound on the topological entropy. While these interpretations of the Burau matrix were not explicit in either of these two papers, the two authors were certainly aware of them (P B personal communication, 1984). See Boyland [7] for an introductory exposition of these two interpretations.

Because the topological entropy of a self-map measures a certain kind of exponential growth it is natural to expect that, at least in certain cases, it is detectable from the growth rates of induced maps on various algebraic objects associated with the underlying space. Thus, for example, the growth rate of the induced map on first homology (ie its spectral radius) gives a lower bound for the topological entropy (Manning [21]), as does exponential growth rate of word length in the fundamental group under iteration by $f_*$ (Bowen [6] and Fathi–Laudenbach–Poénaru [12]).

These lower bounds only depend on the homotopy class of the map, and so it is also natural to ask whether the bounds are attained, ie is there a map in the homotopy class that realises the lower bound? For surface homeomorphisms this question was answered by Thurston. One consequence of his classification theorem is that any isotopy class of surface homeomorphisms contains a map with entropy equal to the growth rate on the fundamental group (Thurston [30] and [12]). While this result is invaluable in theory, in practice, the computation of word length growth in non-Abelian groups is very difficult. On the other hand, computations in homology are much more tractable, but frequently give only trivial lower bounds. A fundamental idea in Fried's paper [15] is that there is a middle ground between these two theories provided by the action on twisted homology. In the most concrete case this amounts to providing a systematic way to examine the growth rate of the action on homology in a collection of finite covers.

Thus one sees that the Burau representation provides a lower bound for the topological entropy of the isotopy class represented by a braid. Specifically, if $h$ is a homeomorphism of the $n$–punctured disk which is represented by the braid $\beta \in B_n$ with Burau matrix $B(t)$, then

$$h_{\mathrm{top}}(h) \geq \sup\{\log \mathrm{sr}(B(\eta)) : \eta \in S^1\} \tag{1–1}$$

where $\mathrm{sr}(B(\eta))$ is the spectral radius of the complex matrix $B(\eta)$ obtained by substituting the complex number $\eta$ with $|\eta| = 1$ into the Burau matrix. This estimate was obtained directly with different methods by Kolev [20]. The estimate in (1–1) and its alternative version in (4–1) below will be called the *Burau estimate*. If the inequality in (1–1) is an equality for $\eta = \eta_0$, then the Burau estimate is said to be *sharp at* $\eta_0$.

Since a braid represents an isotopy class, we may use Thurston's classification scheme to classify braids. Thus a braid is said to be pseudo-Anosov, finite order or reducible if its corresponding isotopy class is. If a braid is finite order or reducible with all finite order components, there is a map in the class with zero topological entropy, and so the Burau estimate is already sharp at $\eta = 1$. The main result here for pseudo-Anosov braids is the following Theorem.

**Theorem 1.1** *For a pseudo-Anosov braid $\beta$, the Burau estimate is sharp at the root of unity $\eta_0$ only if $\eta_0 = -1$. Further, sharpness at $-1$ happens if and only if the invariant foliations for a pseudo-Anosov map in the class represented by $\beta$ have odd order singularities at all punctures and all interior singularities are even order.*

A portion of this theorem was obtained in Song, Ko and Los [26] (see Remark 5.2 below). An immediate consequence of the theorem is that the Burau matrix contains nontrivial information about the invariant foliations of a pseudo-Anosov braid $\beta$: if the Burau estimate for $\beta$ attains a maximum at a root of unity other than $-1$, then the invariant foliations for $\beta$ either have an odd-order interior singularity or an even-order puncture singularity (or both). This kind of geometrical information is often difficult to obtain. The substitution of complex numbers on the unit circle which are not roots of unity requires different methods. In a subsequent paper we will show that for a pseudo-Anosov braid, $\mathrm{sr}(B(e^{2\pi i\theta})) < \lambda$ for all $\theta \notin \mathbb{Q}$.

There is an analogous theorem for reducible braids with at least one pseudo-Anosov component. Its full statement is rather complicated (see Theorem 6.2 below), but one useful consequence is the following.

**Theorem 1.2** *For any braid $\beta$ on $n$ strings with at least one pseudo-Anosov component, the Burau estimate is sharp at the root of unity $\eta_0$ only if $\eta_0$ is of the form $e^{2\pi ij/k}$ for some even $k \leq \frac{2}{3}n$ and some odd $j$ with $0 < j < k$.*

There are two main components in the proof of these theorems. The main algebraic tool is Lemma 3.2 which shows that the union of the spectra obtained by substituting all the $k$th roots of unity into the Burau matrix $B(t)$ yields essentially the entire spectrum of the action on first homology of a lift of the corresponding mapping class to the $k$–fold cover. This is coupled with information about the connection of entropy, the Thurston normal form, the action on homology, and the orientability of the invariant foliations of a pseudo-Anosov map.

The investigations of this paper were partly inspired by recent work using the Burau estimate to get lower bounds on the entropy of fluid flows by applying the Burau estimate to the braids generated by large collections of points moving with the fluid (Thiffeault [28], Gouillart, Thiffeault and Finn [17] and Thiffeault and Finn [29]). For these applications a good understanding of "sharpness" of the estimate is necessary and this paper provides a first step. In addition, questions surrounding the Burau representation provide an important, special case of the more general question of dynamics on abelian covers of surfaces which we investigate in subsequent papers.

## 2 Preliminaries

### 2.1 Standing hypotheses and conventions

In this paper surfaces $X$ will always be orientable, perhaps with boundary, and compact except perhaps for a finite number of punctures. We fix a Riemannian metric on $X$ which allows us to speak of the lengths of tangent vectors. Self-homeomorphisms of the surface $f\colon X \to X$ are always orientation-preserving. If no coefficient ring for homology is specified, it is assumed to be the integers $\mathbb{Z}$, and so $H_1(X) := H_1(X;\mathbb{Z})$. The induced map of the homeomorphism $f$ on first homology is denoted $f_*$. If $M$ is a square, complex matrix, then $\mathrm{sr}(M)$ denotes its spectral radius.

The classification theorem for regular connected covering spaces identifies each such cover with a normal subgroup of the fundamental group of the connected base space $X$, or equivalently, with an epimorphism $\pi_1(X) \to G$, where $G$ is the group of deck transformations of the cover (see, for example, Fulton [16]). In this paper $G$ will always be abelian, and so we often designate a cover $\widetilde{X}$ by an epimorphism $\rho\colon H_1(X) \to G$, with the Hurewicz homomorphism $\pi_1(X) \to H_1(X)$ being implicit.

More generally, we shall also need to consider disconnected covers over connected and disconnected base spaces. In these cases it will be convenient to continue to designate the cover by a homomorphism $\rho\colon H_1(X) \to G$, which perhaps is not surjective. As this is less commonly encountered, we describe the cover $\widetilde{X}$ associated to such a homomorphism $\rho$. First suppose $X$ is connected, and let $G' = \mathrm{im}\,\rho \subseteq G$. Then $\rho$ induces an epimorphism $\rho'\colon H_1(X) \to G'$ with the same domain as $\rho$ but with range $G'$, and so as above it determines a connected covering space $\widetilde{X}_0$ over $X$ with deck group $G'$. We define $\widetilde{X}$ to be the disjoint union of copies of $\widetilde{X}_0$, one such copy for each coset of $G'$ in $G$. The action of $G'$ by deck transformations on each copy of $\widetilde{X}_0$ extends to an action of $G$ on $\widetilde{X}$, in such a way that for any $g, g' \in G$, the element $g$ sends the copy of $\widetilde{X}_0$ corresponding to $g' + G$ to that corresponding to $g + g' + G'$. Then $\widetilde{X}/G \cong X$, and so this makes $\widetilde{X}$ a covering space over $X$ with deck group $G$. Finally, if $X$ is disconnected, we define $\widetilde{X}$ to be the disjoint union, over all connected components $Y$ of $X$, of the covering space $\widetilde{Y}$ corresponding to $\rho|_{H_1(Y)}$ as above.

In all cases every homeomorphism of the base $f\colon X \to X$ will satisfy $\rho f_* = \rho$, which implies that $f$ lifts to $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which commutes with all deck transformations $g \in G$.

## 2.2 The braid group and the Burau representation

In this section we briefly survey relevant results from the point of view of Dynamical Systems. The classic references are Artin [2] and Birman [3], and see Birman and Brendle [4] for a survey including recent developments and Boyland [8] for a survey of dynamical applications. Fenn's book [13] and the classic paper [22] of Milnor are good sources of information on homology of cyclic covers.

The braid group on $n$ strings, $B_n$, is defined using generators and relations as

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_k = \sigma_k \sigma_i \text{ if } |i-k| > 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.$$

In this paper we shall be principally concerned with $B_n$ interpreted as a mapping class group, namely, the group of isotopy classes of homeomorphisms of the $n$–punctured disk where all homeomorphisms and all isotopies are required to fix the boundary pointwise. Letting $x_j = j/(n+1)$ for $j = 1, \ldots, n$ and $D_n = \{z \in \mathbb{C} : |z - 1/2| \leq 1/2\} \setminus \{x_1, \ldots, x_n\}$, the generator $\sigma_i$ of $B_n$ corresponds to a homeomorphism that switches $x_i$ and $x_{i+1}$ in a counter-clockwise direction. When we indicate a braid $\beta \in B_n$ we will always be identifying it with its corresponding isotopy class, and so for example, $h \in \beta \in B_n$ means that $h$ is a homeomorphism of $D_n$ that is contained in the isotopy class corresponding to $\beta$.

As with the braid group we shall need an interpretation of the Burau representation with dynamical content, as the action on homology in a particular cover. To construct the $\mathbb{Z}$–cover relevant to the Burau representation, we fix a basepoint $x_0 \in D_n$, and around each puncture $x_i$ we take a small clockwise loop $\Gamma_i$ which we then homotope so it begins and ends at $x_0$. Then $\pi_1(D_n, x_0)$ and $H_1(D_n)$ are freely generated by the set $\{\gamma_i\}$ of homotopy/homology classes of the $\Gamma_i$'s. Let $\tau$ be the epimorphism of $H_1(D_n)$ onto $\mathbb{Z}$ generated by $\tau(\gamma_i) = 1$ for all $i$; the resulting cover is called the *Burau cover* and is denoted $D^{(\infty)}$. By construction the deck group of $D^{(\infty)}$ is isomorphic to $\mathbb{Z}$, and we call its generator $T$. An orientation-preserving homeomorphism $h$ of $D_n$ must permute the punctures of $D_n$ and will therefore act as a permutation on the generators of $H_1(D_n)$, and so $\tau h_* = \tau$. Thus any orientation-preserving homeomorphism $h$ of $D_n$ will lift to a homeomorphism $\widetilde{h}$: $D^{(\infty)} \to D^{(\infty)}$ and further, each lift of $h$ commutes with all deck transformations, or $T^j \widetilde{h} = \widetilde{h} T^j$ for all $j \in \mathbb{Z}$. Note that by definition, any $h \in \beta \in B_n$ is a homeomorphism that fixes the outside boundary of $D_n$ point-wise and this yields a preferred lift of $h$ to $D^{(\infty)}$, namely, the lift which fixes the lift of the outside boundary of $D_n$ point-wise. Unless indicated otherwise any lift to $D^{(\infty)}$ of an $h \in \beta$ will be this preferred lift.

Next note that the first homology group $H_1(D^{(\infty)})$ has a natural structure as a module over $R := \mathbb{Z}[t^{\pm 1}]$, the ring of all Laurent polynomials with coefficients in $\mathbb{Z}$, ie the group ring of $\mathbb{Z}$. To describe this structure, lift all the loops $\Gamma_i$ to arcs $\widetilde{\Gamma}_i \in D^{(\infty)}$ with all $\widetilde{\Gamma}_i$ starting at some point $\widetilde{x}_0$ and ending at the point $T\widetilde{x}_0$. Thus for $i = 1, \dots, n-1$, $\zeta_i := [\widetilde{\Gamma}_{i+1} - \widetilde{\Gamma}_i] \in H_1(D^{(\infty)})$. For a Laurent polynomial $p(t) = \sum a_j t^j \in R$, $p(t)\zeta_i$ represents the class $\sum a_j T^j \zeta_i \in H_1(D^{(\infty)})$, and so as an $R$–module, $H_1(D^{(\infty)}) \cong R^{n-1}$. Now since the lift of a homeomorphism, $\widetilde{h}$, commutes with $T$, we see that that $\widetilde{h}$ acts on $H_1(D^{(\infty)})$ by an $R$–module isomorphism. So with respect to the $R$–module basis $\{\zeta_i\}$ of $H_1(D^{(\infty)})$, $\widetilde{h}$ acts by a matrix $B(\widetilde{h}) \in \mathrm{GL}(n-1, R)$.

For a braid $\beta \in B_n$, pick $h \in \beta$ and its preferred lift $\widetilde{h}$ to $D^{(\infty)}$. The matrix $B(\beta) = B(\widetilde{h})$ is called the *reduced Burau matrix* of $\beta$, and the corresponding homomorphism $B_n \to \mathrm{GL}(n-1, R)$ the *reduced Burau representation* of the braid group $B_n$. When the braid $\beta$ is fixed, we often will write its Burau matrix as $B(t)$. The full Burau representation will not be used here, but for completeness we note that it can be defined similarly using the action of $\widetilde{h}$ on the relative homology group $H_1(D^{(\infty)}, F)$, where $F$ is the fiber above the basepoint $x_0$ (which in this case must be a fixed point of $h$).

Since our results work for all $n > 2$, we fix once and for all such an $n$ and suppress the dependence of objects on $n$ when possible. So, for example, we write just $D$ not $D_n$.

## 2.3 The Nielsen–Thurston normal form

Since we have identified the braid group $B_n$ with a surface mapping class group, Thurston's classification theorem will be of central importance here. This theorem identifies "simplest" representatives in any isotopy class. See Fathi–Laudenbach–Poénaru [12] and Thurston [30] for more information. There are minor differences in the literature in how punctures, boundary and the reducible case are handled in stating Thurston's results. The version we give in Theorem 2.1 is adapted to our use with the braid group.

The two main ingredients in Thurston's classification are finite order and pseudo-Anosov maps. A map $\phi$ is *finite order* if $\phi^n = \mathrm{id}$ for some $n \geq 1$. The map $\phi$ is pseudo-Anosov if there are a pair of transverse, $\phi$-invariant measured foliations[1], $\mathcal{F}^u$ and $\mathcal{F}^s$. Under the action of $\phi$ the transverse measures are expanded and contracted by a number $\lambda > 1$, which is called the *expansion factor* of the pseudo-Anosov map. This fact is usually indicated by the notation $\phi_* \mathcal{F}^u = \lambda \mathcal{F}^u$ and $\phi_*^{-1} \mathcal{F}^s = \lambda \mathcal{F}^s$. The supporting surface of a pseudo-Anosov map $\phi$ may be connected or disconnected, but in the latter case we require that $\phi$ cyclically permutes the components.

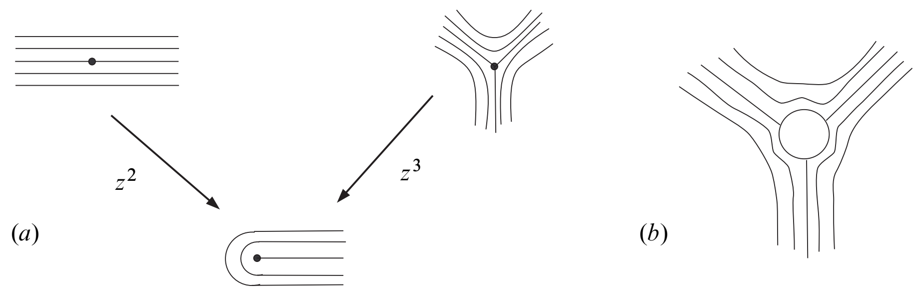



Figure 1: (a) Constructing prongs; (b) a boundary three prong.

Since the structure of the singular points in a measured foliation is central here we describe it in more detail. Near a *regular point* a measured foliation looks locally like the foliation of $\mathbb{R}^2$ by horizontal lines. A measured foliation is also allowed to have a finite number of non-regular or singular points which are required to have a very a specific local structure which is characterized by the order of the singularity, ie by the number of leaves coming directly into the point. The local structure can be succinctly described using covers branched over $\mathbf{0} \in \mathbb{C}$. Starting with the foliation of $\mathbb{C}$ by horizontal lines and projecting under $z \mapsto z^2$, we get the local structure of an *order one or one-prong singularity* at $\mathbf{0}$ (see Figure 1(a)). Lifting the one-prong by the map $z \mapsto z^n$ for $n > 2$ gives an *order $n$ or $n$–prong singularity*. An *order $n$–punctured singularity* is formed by removing the singular point from an $n$–prong, and an *order $n$–boundary singularity* is obtained by replacing this puncture with a boundary circle (see Figure 1(b)).

The invariant foliations associated with a pseudo-Anosov map have a few special additional qualifications. They can have punctured or boundary one-prongs, but interior one-prongs do not occur as they will not persist under isotopy. An interior two-prong is a regular point and is not considered a singularity, but a punctured or boundary two-prong is considered a singularity. Given a measured foliation $\mathcal{F}$ on a surface $X$

[1]The designation “measured foliation” is the standard shortening of the more proper, and much longer name, “foliation with conical singularities with a holonomy invariant transverse measure”.

with genus $g$, the Euler–Poincaré–Hopf formula says that

$$2-2g=\sum(1-\kappa_i/2), \tag{2–1}$$

where the sum is over singularities of all types (interior, boundary and punctured) of $\mathcal{F}$, and $\kappa_i$ is the order of the $i$th singularity.

Now recall that any homeomorphism $h\in\beta\in B_n$ must fix the boundary of $D_n$ pointwise, a property that is not shared by the Thurston representatives under the usual definitions. In addition, the isotopies used in the braid group must fix the boundary pointwise. These two facts require a small alteration in the designation of Thurston representative in an isotopy class. Here is a version of Thurston's classification theorem adapted to our situation. It follows, for example, by altering the constructions in Boyland [9].

**Theorem 2.1** (Thurston) *Let $f$ be a homeomorphism of the possibly disconnected, possibly bordered surface $X$, compact except for a finite number of punctures, and assume that $f$ fixes the boundary of $X$ pointwise. Then there is a homeomorphism $\Phi$ isotopic to $f$ by an isotopy which fixes the boundary pointwise, and a decomposition*

$$X=A\cup\bigcup_{j=1}^{m}X_j$$

*of $X$ into pairwise disjoint $\Phi$-invariant sets, with the following properties.*

(1) *$A$ is a finite disjoint union of embedded open annuli. The waist curve of such an annulus is never null-homotopic; nor are the waist curves of two annuli mutually homotopic. If the waist curve of an annulus $a\in A$ is homotopic to a boundary component $b$ of $X$, then $b$ is a component of $\partial a$ (and by convention we include $b$ in $a$).*
(2) *Each $X_j$ is the union of a collection of connected components of $X\setminus A$ which are permuted cyclically by $\Phi$; and the restriction of $\Phi$ to $X_j$ is either finite order or pseudo-Anosov.*
(3) *The restriction of $\Phi$ to $A$ has zero topological entropy.*

## 2.4 Branched covering spaces and oriented foliations

Given a finite set $B\subset X$, the triple $p\colon Y\to X$ is a called *a covering space branched over $B$* if $p$ is onto and restricts to a covering map (ie a surjective local homeomorphism which is evenly covered over any small neighborhood, cf Fulton [16]) of $Y\setminus p^{-1}B$ over $X\setminus B$. Now let $\mathcal{F}$ be a measured foliation on the surface $X$. The foliation $\mathcal{F}$ is *orientable* provided there is a vector field $\Upsilon$ on $X$, zero at the singularities of $\mathcal{F}$, and everywhere else nonzero and tangent to $\mathcal{F}$. When $\mathcal{F}$ is not orientable, it is often useful to orient it by lifting it to a two-sheeted branched cover constructed as follows. Start by puncturing $X$ at the singularities of the foliation and denote the resulting space by $X'$. Now define $\overrightarrow{X'}$ as the space of unit tangent vectors to $\mathcal{F}$ in $X'$ with the topology induced as a subspace of the unit tangent bundle of $X$. It is evident that $\overrightarrow{X'}$ is a two-sheeted cover over $X'$. By sewing back in the punctures, we obtain a two-fold branched cover $\overrightarrow{p}\colon \overrightarrow{X} \to X$, and pulling back $\mathcal{F}$, we obtain a measured foliation $\overrightarrow{\mathcal{F}}$ on $\overrightarrow{X}$. The space $\overrightarrow{X}$ equipped with the pulled back foliation $\overrightarrow{\mathcal{F}}$ is called the *orientation double-cover* of the non-orientable foliation $\mathcal{F}$.

If $\gamma$ is a smooth loop in $X'$, then since $X$ is an orientable surface, the tangent bundle restricted to $\gamma$ is trivial and so there is a well defined monodromy as we pull along $\gamma$ the unit vectors tangent to $\mathcal{F}$. If this monodromy brings a vector back to itself, we say that *the foliation is oriented along* $\gamma$, and if it brings a vector back to its opposite, *the foliation is disoriented along* $\gamma$. It is evident from the definition of the orientation double cover that in the first case $\gamma$ lifts to a pair of disjoint loops in $\overrightarrow{X}$, while in the latter case, $\gamma$ is covered by a single loop $\widetilde{\gamma} \subset X'$, and the covering map induces a degree-two map $\widetilde{\gamma} \to \gamma$. Thus a foliation is oriented if and only if it is oriented along every loop in the complement of the singularity set, and if $\overrightarrow{\rho}\colon H_1(X') \to \mathbb{Z}_2$ is the epimorphism associated to the cover $\overrightarrow{X'}$, then the foliation is oriented along $\gamma$ if and only if $\overrightarrow{\rho}([\gamma]) = 0$.

The next lemma gives a simple criterion for checking when the foliation is oriented when lifted to a cover. Its proof is standard and we omit it. Implicit in the statement of (ii) is the fact that if $\gamma$ is a small loop surrounding a singularity $P$, then $\overrightarrow{\rho}([\gamma]) = 0$, if $P$ has even order, and $\overrightarrow{\rho}([\gamma]) = 1$, if $P$ has odd order. Thus if all interior singularities of a measured foliation on $X$ are of even order, then we may treat $\overrightarrow{\rho}$ as being defined on $H_1(X)$.

**Lemma 2.2** *Assume that $X$ is a possibly disconnected, possibly bordered surface $X$, compact except for finitely many punctures, and that $\mathcal{F}$ is a measured foliation on $X$. Let $\rho\colon H_1(X) \to G$ be a homomorphism of $H_1(X)$ to a finite abelian group $G$ and let $\widetilde{X}$ be the corresponding covering space of $X$. The following are equivalent:*

(i) *The lift of $\mathcal{F}$ to $\widetilde{X}$ is orientable;*

(ii) *All singularities in the interior of $X$ have even order, and there exists a homomorphism $\delta : \operatorname{im}\rho \to \mathbb{Z}_2$ such that $\delta \circ \rho = \overrightarrow{\rho}$, where $\overrightarrow{\rho}$ is the morphism defining the orientation cover of $\mathcal{F}$.*

# 3 Finite covers and substituting roots of unity

In this section we show that the substitutions of $k$th roots of unity into the Burau representation of a braid give all the essential spectral information about the action of the braid's mapping class on homology in the $k$–fold cover. The reducible case considered in Section 6 requires us to work with more general subsurfaces of $D$ and, in fact, the results of this section apply to fairly general topological spaces.

Let $h : X \to X$ be a homeomorphism of the perhaps disconnected space $X$. Suppose that $\rho$: $H_1(X) \to \mathbb{Z}$ is some homomorphism which satisfies $\rho h_* = \rho$, and denote by $p^{(\infty)}$: $X^{(\infty)} \to X$ the covering space associated to $\rho$ as in Section 2.1. The covering $p^{(\infty)}$ is thus generated by a deck group isomorphic to $\mathbb{Z}$, and we denote the generator of this deck group by $T$. Let $h^{(\infty)}$ be a lift of $h$ to $X^{(\infty)}$. The condition $\rho h_* = \rho$ implies that $h^{(\infty)}$ commutes with $T$ and hence also with every other element of the deck group.

The main example for our purposes is where $X = D$ is the $n$–punctured disk, $h$: $D \to D$ is a representative of the braid $\beta \in B_n$, and $\rho = \tau$ is the homomorphism defining the Burau cover $D^{(\infty)}$ of $D$ (see Section 2.2). To deal with reducible braids we will also have to consider certain $h$-invariant subsurfaces of $D$ (which may be disconnected). In general, what we shall assume about the space $X$ is that (in addition to having a universal cover over each component) it has the homotopy type of a compact 1–dimensional cell complex.

To construct cyclic covers, for each integer $k > 0$, let $\xi_k$: $\mathbb{Z} \to \mathbb{Z}_k$ be the quotient homomorphism and define $\rho_k = \xi_k \circ \rho$. Let $p^{(k)}$: $X^{(k)} \to X$ be the covering space associated to $\rho_k$, so $X^{(k)} = X^{(\infty)}/T^k$. Let $q^{(k)}$: $X^{(\infty)} \to X^{(k)}$ be the covering projection. The image under $q_*^{(k)}$ of $H_1(X^{(\infty)})$ is a subgroup of $H_1(X^{(k)})$ which we denote by $S^{(k)}$.

The map $h^{(\infty)}$ pushes down to a well-defined lift $h^{(k)}$ of $h$ on $X^{(k)}$. Also, the deck group generator $T$ of $X^{(\infty)}$ pushes down to a generator of the deck group for $p^{(k)}$, which we also denote by $T$. Both $h_*^{(k)}$ and $T_*$ leave $S^{(k)}$ invariant by definition.

As in Section 2.2, we write $R = \mathbb{Z}[t^{\pm 1}]$ for the ring of all Laurent polynomials with coefficients in $\mathbb{Z}$ (ie the group ring of $\mathbb{Z}$) and we treat the integral homology group $H_1(X^{(\infty)})$ as a module over $R$. The next lemma generalizes Section 2.2 to show that, just as for the case of the Burau cover $D^{(\infty)}$, the first homology of $X^{(\infty)}$ is a *free* module of finite rank over $R$.

**Lemma 3.1** *Suppose $H_1(X)$ is a free Abelian group of rank $r_1$, and let $r_0$ be the number of connected components $X_0 \subseteq X$ such that $\rho|_{H_1(X_0)}$ is not identically $0$. Then $H_1(X^{(\infty)})$ is a free $R$–module of rank $r = r_1 - r_0$.*

**Proof** First assume $X$ has one connected component, and let $\kappa\colon K \to X$ be a homotopy equivalence from a one-dimensional cell complex $K$, having only one vertex, to $X$. Let $x \in X$ be the $\kappa$–image of the vertex of $K$ and for $1 \le i \le r_1$ let $\gamma_i$ be the $\kappa$–image of the $i$th edge of $K$, regarded as a path in $X$. Let $\tilde{x}$ be a lift of $x$ to $X^{(\infty)}$ and for each $i$ let $\tilde{\gamma}_i$ be a lift of $\gamma_i$ with initial point $\tilde{x}$. Also set $Z = (p^{(\infty)})^{-1}(x)$ so that $\tilde{x} \in Z$. Because $\kappa$ is a homotopy equivalence, the relative homology group $H_1\big(X^{(\infty)}, Z\big)$ may be identified with the chain group of the appropriate covering space over $K$, and it follows that $H_1(X^{(\infty)}, Z)$ is freely generated as an $R$–module by the homology classes $[\tilde{\gamma}_1], \dots, [\tilde{\gamma}_{r_1}]$.

Since $Z$ is discrete $H_1\big(X^{(\infty)}\big)$ may be regarded as the kernel of the boundary operator $\partial\colon H_1\big(X^{(\infty)}, Z\big) \to H_0(Z)$. Because $X$ is connected $H_0(Z)$ is freely generated as an $R$–module by $\tilde{x}$. We claim that $\operatorname{im}\partial \subseteq H_0(Z)$ is a free submodule, ie that it is either 0 or generated by a single element. To see this, note that $\partial(\tilde{\gamma}_i) = (t^{\rho(\gamma_i)} - 1)\cdot\tilde{x}$ for each $i$. If $\rho$ is identically zero on $H_1(X)$ then this implies $\operatorname{im}\partial = 0$. Otherwise, let $g$ denote the greatest common divisor of the nonzero elements in the list $|\rho(\gamma_1)|, \dots, |\rho(\gamma_{r_1})|$. Then for each $i$ with $\rho(\gamma_i) > 0$ we have

$$t^{\rho(\gamma_i)} - 1 = \big(t^g - 1\big)\Bigg(\sum_{j=0}^{\rho(\gamma_i)/g-1} t^{jg}\Bigg) \tag{3–1}$$

so that $t^g - 1$ divides $t^{\rho(\gamma_i)} - 1$. A similar expression holds when $\rho(\gamma_i) < 0$. Moreover using (3–1) and an expression for $g$ as a linear combination of the $\rho(\gamma_i)$'s, it is not hard to write $t^g - 1$ as an $R$–linear combination of the $(t^{\rho(\gamma_i)} - 1)$. This shows that $\big(t^g - 1\big)\cdot\tilde{x}$ is a generator of $\operatorname{im}\partial$, and hence $\operatorname{im}\partial$ is free and of rank 1.

Since $\operatorname{im}\partial$ is free, the exact sequence

$$0 \longrightarrow H_1\big(X^{(\infty)}\big) \longrightarrow H_1\big(X^{(\infty)}, Z\big) \xrightarrow{\partial} \operatorname{im}\partial \longrightarrow 0$$

splits. Therefore $H_1\big(X^{(\infty)}\big)$ is a direct summand of the free module $H_1\big(X^{(\infty)}, Z\big)$, hence free[2] by Swan [27]. The rank of $H_1\big(X^{(\infty)}\big)$ is $r_1$ minus the rank of $\operatorname{im}\partial$, ie

$$r = \begin{cases} r_1 - 1 & \text{if } \rho \not\equiv 0 \\ r_1 & \text{Otherwise.} \end{cases}$$

This completes the proof when $X$ is connected.

[2] One can also produce a free basis for $H_1\big(X^{(\infty)}\big)$ explicitly by applying Gaussian elimination to the columns of the matrix of $\partial$.

Finally, if $X$ has more than one connected component, the lemma follows by writing $H_1\big(X^{(\infty)}\big)$ as the direct sum of $H_1\big((p^{(\infty)})^{-1}(Y)\big)$ over all connected components $Y$ of $X$, and applying the above calculation to each such component. $\square$

Let $\zeta_1,\dots,\zeta_r$ be a basis for $H_1(X^{(\infty)})$. With respect to this basis, the module isomorphism $h_*^{(\infty)}$ is given by a square matrix $M = M(t) \in \mathrm{GL}(r,R)$ with entries in $R$. Note that $M$ depends on the morphism $\rho$ used to define $X^{(\infty)}$, on the homotopy class of $h$, and also on the choice of lift of $h$ to $X^{(\infty)}$. If $X = D$ is the $n$–punctured disk, $h$ is a representative of the braid $\beta$ on $n$ strings, $\rho$ is the homomorphism $\tau$ of Section 2.2 and $h^{(\infty)}$ is the preferred lift of $h$ to $D^{(\infty)}$, then we have $M = B(\beta)$, the reduced Burau matrix of $\beta$.

For a complex number $\eta \in \mathbb{C}$, denote by $M(\eta)$ the complex matrix obtained from $M$ by substituting $\eta$ in place of $t$. If $\eta \neq 0$ then this matrix is invertible, just as $M$ is, and so it acts as a linear isomorphism of $\mathbb{C}^r$ to itself. We denote by $S_{\mathbb{C}}^{(k)}$ the complexification of $S^{(k)}$ and treat $S_{\mathbb{C}}^{(k)}$ as a subspace of $H_1(X^{(k)},\mathbb{C})$.

The next lemma connects the action on the $k$–fold cover to substitutions of complex $k$th roots of unity into the matrix $M(t)$. It is based on well-known, elementary facts. Depending on the chosen perspective, it follows, for example, from the splitting of a representation of $Z_k$ into the sum of irreducibles, or from the invertibility of the order–$k$ discrete Fourier transform. Rather than abstract the necessary algebra and then apply it to the situation at hand, it is simpler to maintain an algebraic topology perspective and give a direct proof using the invariance of an eigen-decomposition.

**Lemma 3.2** *Let $T$ be the generator of the deck group for the covering $p^{(k)}\colon X^{(k)} \to X$, and let $h^{(k)}$ and $h^{(\infty)}$ be the lifts of $h$ to $X^{(k)}$ and $X^{(\infty)}$. The eigenvalues of $T_*$ restricted to $S_{\mathbb{C}}^{(k)}$ are $1, \eta_k, \eta_k^2, \dots, \eta_k^{k-1}$ where $\eta_k = e^{2\pi i/k}$. Denote by $E_0,\dots,E_{k-1}$ the corresponding eigenspaces in $S_{\mathbb{C}}^{(k)}$. Then each subspace $E_m$ is $h_*^{(k)}$–invariant, and the action of $h_*^{(k)}$ on $E_m$ is given by the matrix $M(\eta_k^m)$, obtained by substituting $\eta_k^m$ into the matrix $M(t)$ of $h_*^{(\infty)}$.*

**Proof** Let $R_{\mathbb{C}}^{(k)}$ be the ring of all complex Laurent polynomials in a variable $s$ which satisfies $s^k = 1$, so $R_{\mathbb{C}}^{(k)}$ is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}_k]$. As with $H_1(X^{(\infty)})$, we can treat $H_1(X^{(k)},\mathbb{C})$ as a module over $R_{\mathbb{C}}^{(k)}$. Note that $T_*$ and $h_*^{(k)}$ act by module isomorphisms, and $S_{\mathbb{C}}^{(k)}$ is an $R_{\mathbb{C}}^{(k)}$-submodule that is invariant under both $T_*$ and $h_*^{(k)}$.

Since $T_*$ has order $k$ its eigenvalues are as given. Letting $\zeta_j^{(k)} = q_*^{(k)}(\zeta_j)$, a general element of $S_{\mathbb{C}}^{(k)}$ has the form $\sum_{i\in\mathbb{Z}_k} s^i \big(\sum_{j=1}^r a_{i,j}\zeta_j^{(k)}\big)$, for complex numbers $a_{i,j} \in \mathbb{C}$.

For $m = 0, \dots, k-1$, let $E_m$ be the set of elements which when written in this form satisfy $a_{i+1,j} = \eta_k^{-m} a_{i,j}$ for each $i \in \mathbb{Z}_k$ and all $j$. Since $T_*$ acts on $S_{\mathbb{C}}^{(k)}$ by multiplication by $s$, the set $E_m$ consists of eigenvectors of $T_*$ with eigenvalue $\eta_k^m$. Further, the dimension of $E_m$ as a complex vector space is $r$ and so $E_0 \oplus \cdots \oplus E_{k-1} = S_{\mathbb{C}}^{(k)}$.

Since $h_*^{(k)}$ commutes with $T_*$, each $E_m$ is $h_*^{(k)}$-invariant. By definition, the matrix $M \in \mathrm{GL}(r, R)$ is the matrix of $h_*^{(\infty)} : H_1(X^{(\infty)}) \to H_1(X^{(\infty)})$ relative to the chosen basis $\{\zeta_i\}$ of $H_1(X^{(\infty)})$. We decompose this matrix as

$$M = \sum_{i \in \mathbb{Z}} t^i M_i,$$

with each $M_i \in \mathrm{Mat}(r, \mathbb{Z})$. Projecting this action to $S_{\mathbb{C}}^{(k)}$ we have that $h_*^{(k)}$ acts as an $R_{\mathbb{C}}^{(k)}$–module homomorphism on $S_{\mathbb{C}}^{(k)}$ by the matrix

$$M(s) := \sum_{i \in \mathbb{Z}_k} s^i \left( \sum_{l \in \mathbb{Z}} M_{lk+i} \right) \in \mathrm{GL}(r, R_{\mathbb{C}}^{(k)}). \tag{3–2}$$

On the other hand, since $\eta_k^k = 1$, the matrix $M(\eta_k^m)$ is given by

$$M(\eta_k^m) = \sum_{i \in \mathbb{Z}} \eta_k^{mi} M_i = \sum_{i=0}^{k-1} \eta_k^{mi} \left( \sum_{l \in \mathbb{Z}} M_{lk+i} \right). \tag{3–3}$$

If $v \in E_m$ then by (3–2) and (3–3) we have

$$\begin{aligned} h_*^{(k)}(v) &= \sum_{i \in \mathbb{Z}_k} s^i \left( \sum_{l \in \mathbb{Z}} M_{lk+i} \right) \cdot v \\ &= \sum_{i \in \mathbb{Z}_k} \left( \sum_{l \in \mathbb{Z}} M_{lk+i} \right) \cdot \eta_k^{mi} v \\ &= M(\eta_k^m) \cdot v \end{aligned}$$

as claimed. □

Although $H_1(X^{(k)})$ is larger than $S^{(k)}$, the next lemma indicates that all of the *growth* of $h_*^{(k)}$ on $H_1(X^{(k)})$ occurs in $S^{(k)}$.

**Lemma 3.3** *With notation as above, the eigenvalues of $h_*^{(k)}$ acting on $H_1(X^{(k)})$ are those of its restriction to $S^{(k)}$ together with some roots of unity. In particular, the spectral radius of $h_*^{(k)}$ equals that of its restriction to $S^{(k)}$.*

**Proof** We return to treating $H_1(X^{(k)})$ and $S^{(k)}$ as Abelian groups. For each connected component $Y$ of $X^{(k)}$, any element $c \in H_1(Y)$ can be represented by a closed loop in $Y$. This loop lifts to a loop in $X^{(\infty)}$ if and only if $\rho \circ p_*^{(k)}(c) = 0$. Moreover, suppose $\widetilde{c} \in H_1(X^{(\infty)})$ is a general element of $(q_*^{(k)})^{-1}(c)$ and represent $\widetilde{c}$ as a sum $\sum \sigma_i$ of weighted simplices. By applying a deck transformation of $(q^{(k)})^{-1}(Y)$ over $Y$ to each $\sigma_i$, we may modify $\widetilde{c}$ within $(q_*^{(k)})^{-1}(c)$ so that all its simplices lie in one and the same component, say $\widetilde{Y}$, of $X^{(\infty)}$ covering $Y$. Then $\widetilde{c}$ may be represented by a loop in $\widetilde{Y}$ which, as we have shown, is possible if and only if $\rho \circ p_*^{(k)}(c) = 0$. This shows that $S^{(k)} \cap H_1(Y)$ is the kernel of the map $\rho \circ p_*^{(k)}|_{H_1(Y)}$. Since $\operatorname{im} \rho \subseteq \mathbb{Z}$, we get that $H_1(Y)/(S^{(k)} \cap H_1(Y))$ is either trivial or isomorphic to $\mathbb{Z}$.

Writing $H_1(X^{(k)})$ as the direct sum of the first homology groups of the components of $X^{(k)}$, we obtain $H_1(X^{(k)})/S^{(k)} \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ where each factor corresponds to a connected component of $X^{(k)}$ on which $\rho \circ p_*^{(k)}$ is not identically zero. Since $h$ permutes the components of $X^{(k)}$, it acts on this splitting by permuting the factors. Thus $h_*^{(k)}$ may be represented by a matrix of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

in which $A$ represents the action of $h_*^{(k)}$ on $S^{(k)}$, and $C$ is a permutation matrix representing the action of $h_*^{(k)}$ on $H_1(X^{(k)})/S^{(k)}$. The eigenvalues of $C$ are all roots of unity, so this proves the lemma. $\square$

Lemma 3.3 together with Lemma 3.2 yields the following.

**Theorem 3.4** *Let $h\colon X \to X$ be a homeomorphism of the locally path-connected, semi-locally simply connected space $X$ having the homotopy type of a compact $1$–dimensional cell complex. Suppose $\rho\colon H_1(X) \to \mathbb{Z}$ is a homomorphism which satisfies $\rho h_* = \rho$, and let $X^{(\infty)}$ and $X^{(k)} = X^{(\infty)}/T^k$ denote the covering spaces over $X$ corresponding to $\rho$ and $\xi_k \circ \rho$, with covering projection $q^{(k)}\colon X^{(\infty)} \to X^{(k)}$. Let $h^{(\infty)}$ and $h^{(k)}$ denote lifts of $h$ to these covering spaces. If $M = M(t) \in GL(r, R)$ denotes the matrix of $h_*^{(\infty)} : H_1(X^{(\infty)}) \to H_1(X^{(\infty)})$ as an $R$–module isomorphism, then the action of $h_*^{(k)}$ on the invariant subspace $S_{\mathbb{C}}^{(k)} = q_*^{(k)}(H_1(X^{(\infty)}, \mathbb{C}))$ is given by the direct sum*

$$h_*^{(k)} = M(1) \oplus M(\eta_k) \oplus \cdots \oplus M(\eta_k^{k-1})$$

*where $M(\eta_k^j)$ denotes the complex matrix obtained by substituting $\eta_k^j = e^{2\pi i j/k}$ into $M$. Furthermore, any eigenvector of $h_*^{(k)}$ not lying in $S^{(k)}$ has eigenvalue which is a root of unity.*

# 4 Entropy and first homology

The topological entropy, $h_{\mathrm{top}}(f)$, is a well-known measure of the complexity of the dynamics of a self-map $f$ of a compact metric space. See Adler, Konheim and McAndrew [1], Denker [11] or Katok [18] for more information. The next well-known lemma contains part of the main idea in Fried's paper [15]: passing to a finite cover often allows one to detect more growth on homology.

**Lemma 4.1** *If $f\colon Y \to Y$ is a continuous map of the compact manifold $Y$, then*

$$h_{\mathrm{top}}(f) \geq \sup\bigl\{\log\bigl(\mathrm{sr}\bigl(\widetilde{f}_*\bigr)\bigr)\bigr\}$$

*where the supremum is over all lifts, $\widetilde{f}$, of $f$ to a finite cover $\widetilde{Y}$ and $\widetilde{f}_*$ is the action of $\widetilde{f}$ on first homology of the cover $H_1(\widetilde{Y};\mathbb{R})$.*

**Proof** The lemma follows directly from two classic results. Manning proved in [21] that $h_{\mathrm{top}}(f) \geq \log(\mathrm{sr}(f_*))$ for any continuous $f\colon Y \to Y$ and Bowen proved in [5] that entropy is preserved under finite to one factors, in particular, $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(\widetilde{f})$ for any lift $\widetilde{f}$ of $f$ to a finite cover. □

For a braid $\beta$ we define $h_{\mathrm{top}}(\beta) = \inf\{h_{\mathrm{top}}(h) : h \in \beta\}$. Now Theorem 3.4 says that the maximal spectral radius of $B(e^{2\pi i j/k})$ for $0 \leq j < k$ gives the spectral radius on the first homology of the lift to the $k$–fold Burau cover $D^{(k)}$ of an $h \in \beta$. Thus by Lemma 4.1 we have what was referred to in the Introduction as the *Burau estimate*.

$$h_{\mathrm{top}}(\beta) \geq \sup\bigl\{\log(\mathrm{sr}(B(e^{2\pi i j/k}))) : j/k \in \mathbb{Q}\bigr\}.$$

Since the entries in $B(t)$ are polynomials, $\mathrm{sr}(B(t))$ is continuous in $t$. Thus we have the result of Fried [15] and Kolev [20].

**Lemma 4.2** *If $\beta \in B_n$ with Burau representation $B(t)$, then*

$$h_{\mathrm{top}}(\beta) \geq \sup\{\log(\mathrm{sr}(B(\eta))) : \eta \in S^1\}. \tag{4–1}$$

A fundamental result in Nielsen–Thurston theory says that $h_{\mathrm{top}}(\beta) = h_{\mathrm{top}}(\Phi)$ where $\Phi$ as given in Theorem 2.1 is the Thurston representative in the isotopy class $\beta$. A natural question is whether this value is detected by the action on homology in some finite cover, or in the current context, whether the estimate in (4–1) is ever sharp.

To investigate this question we must first understand a simpler question, namely, for a pseudo-Anosov map $\phi$ of a surface $X$, when is $h_{\mathrm{top}}(\phi) = \mathrm{sr}(\phi_*)$? In other words, when is Manning's estimate in [21] sharp for a pseudo-Anosov map? The answer turns out to depend exactly on the orientability of the $\phi$–invariant foliations. It is well-known that the oriented, measured foliation $\mathcal{F}^u$ gives rise to a homology class $v \in H_1(X;\mathbb{R})$, for example, as an asymptotic direction as in Schwartzman [25] or a geometric current as in Ruelle and Sullivan [23]. Briefly, $v$ is the average direction in homology obtained from flowing along the one-dimensional leaves of $\mathcal{F}^u$. This average exists and is unique because the unstable foliation of a pseudo-Anosov map is uniquely ergodic [12].

Now if the unstable foliation $\mathcal{F}^u$ gives rise to $v^u \in H_1(X;\mathbb{R})$, then since $\phi_*\mathcal{F}^u = \lambda\mathcal{F}^u$, on first homology we have $\phi_* v^u = \lambda v^u$. Thus $v^u$ is an eigenvector of $\phi_*$ with eigenvalue $\lambda$, and so $\mathrm{sr}(\phi_*) \geq \lambda$. On the other hand, since pseudo-Anosov maps have entropy equal to the logarithm of their expansion constant, Lemma 4.1 yields that $\lambda \geq \mathrm{sr}(\phi_*)$. Thus when the invariant foliations are orientable we see that the spectral radius on first homology gives the entropy of a pseudo-Anosov map. The converse of this fact doesn't seem as well-known so we include a proof below for completeness.

**Lemma 4.3** *Suppose $\phi$: $X \to X$ is a pseudo-Anosov homeomorphism of the orientable surface $X$ having $\ell$ connected components (which, according to the definition of pseudo-Anosov map, must be permuted cyclically by $\phi$), and let $\lambda$ be the expansion constant of $\phi$.*

(a) *The pseudo-Anosov map $\phi$ has orientable invariant foliations in some (hence each) connected component if and only if* $\mathrm{sr}(\phi_*) = \lambda$.

(b) *Suppose $\phi$ has oriented invariant foliations, and let $\epsilon = \pm 1$ according to whether $\phi$ preserves or reverses the orientation of the unstable foliation. Then each complex number of the form $\epsilon e^{2\pi i j/\ell}\lambda$ is a simple eigenvalue of $\phi_*$, and every other eigenvalue $\mu$ satisfies $|\mu| < \lambda$.*

**Proof** We first suppose that $X$ is connected, ie that $\ell = 1$, and prove 4.3 and then 4.3. The proof when $\ell > 1$ is an easy modification.

When $\ell = 1$, 4.3 states that if the invariant foliations of $\phi$ are oriented, then $\epsilon\lambda$ is a simple eigenvalue of $\phi_*$, and every other eigenvalue is smaller in modulus. Because $X$ is orientable, orientability of the unstable foliation of $\phi$ is equivalent to orientability of the stable foliation of $\phi$ (see Camacho and Lins Neto [10]), so we need only talk about orientability of the former.

In [24] Rykken shows that if the pseudo-Anosov map $\phi$ has an orientable unstable foliation, then except for zeros and roots of unity, the eigenvalues of its action on first homology are the same as those of its Markov transition matrix, $A_{\mathcal{P}}$, with respect to any Markov partition $\mathcal{P}$. Using this result 4.3 then follows from the Perron–Frobenius Theorem applied to $A_{\mathcal{P}}$ (see, for example, Kitchens [19]).

We noted above the theorem how one implication in 4.3 follows from treating the foliation as an asymptotic cycle. We prove the contrapositive of the converse and so assume that the unstable foliation $\mathcal{F}^u$ of $\phi$ is not orientable. Let $\overrightarrow{X} \xrightarrow{\overrightarrow{p}} X$ be the orientation double-cover of $\mathcal{F}^u$ constructed in Section 2.4; $\phi$ lifts to a pseudo-Anosov $\overrightarrow{\phi}$ of $\overrightarrow{X}$ with unstable foliation denoted $\overrightarrow{\mathcal{F}}^u$. The leaves of $\overrightarrow{\mathcal{F}}^u$ of necessity project under $\overrightarrow{p}$ to those of $\mathcal{F}^u$. Since $\overrightarrow{\phi}$ also has expansion constant $\lambda$, by part 4.3 $\overrightarrow{\phi}_*\colon H_1(\overrightarrow{X}) \to H_1(\overrightarrow{X})$ has a simple eigenvalue equal to $\epsilon\lambda$ and all its other eigenvalues have smaller modulus.

If we let $\theta$ denote the generator of the deck group of $\overrightarrow{X}$, then $\theta$ will take leaves of $\overrightarrow{\mathcal{F}^u}$ to leaves reversing the orientation while preserving the transverse measure, and so $\theta_* v^u = -v^u$. Now since $\theta^2 = \mathrm{id}$, $H_1(\overrightarrow{X}, \mathbb{R}) = V_+ \oplus V_-$ with $V_\pm$ the eigenspace of $\theta_*$ with eigenvalue $\pm 1$. Moreover, we now show that $V_-$ is precisely the kernel of the map induced on homology by the covering projection $\overrightarrow{p_*}\colon H_1(\overrightarrow{X}, \mathbb{R}) \to H_1(X, \mathbb{R})$. Suppose $\tau$ is a 1–cycle in $\overrightarrow{X}$ avoiding the singular points of $\overrightarrow{\mathcal{F}}$ such that $\overrightarrow{p_*}[\tau] = 0 \in H_1(X)$. Then we may write $\overrightarrow{p_*}\tau = \partial\Delta$ where $\Delta = \sum_i a_i \Delta_i$ is some 2–chain. Lift each $\Delta_i$ to a 2–simplex $\Delta'_i$ in $\overrightarrow{X}$ and let $\overrightarrow{\Delta} = \sum_i a_i\big(\Delta'_i + \theta\Delta'_i\big)$. Clearly, $\partial\overrightarrow{\Delta} = \tau + \theta\tau$ proving that $[\tau] + [\theta\tau] = 0 \in H_1(\overrightarrow{X})$. In other words $[\tau] \in V_-$. So $\ker \overrightarrow{p_*} \subset V_-$. Conversely if $v \in V_-$, then $2\overrightarrow{p_*}(v) = \overrightarrow{p_*}(v) + \overrightarrow{p_*}(\theta(v)) = \overrightarrow{p_*}(v) - \overrightarrow{p_*}(v) = 0$, and hence $V_- = \ker \overrightarrow{p_*}$ as claimed.

Thus $\phi_*$ acting on $H_1(X, \mathbb{R})$ is conjugate to $\overrightarrow{\phi}_*$ acting on $V_+$. Since $v^u \in V_-$ and the eigenvalue $\epsilon\lambda$ is geometrically simple, it follows that every eigenvalue of $\overrightarrow{\phi}_*$ having an eigenvector lying in $V_+$ has modulus strictly smaller than $\lambda$, so that $\mathrm{sr}(\phi_*) < \lambda$ as required. □

# 5 The Burau estimate for pseudo-Anosov braids

A braid $\beta \in B_n$ is said to be *pseudo-Anosov* if in the Thurston normal form $\Phi \in \beta$ of Theorem 2.1 the set $A$ consists of just one annulus $a$ which is the collar of $\partial D$, and the restriction of $\Phi$ to the complement of $a$ is a pseudo-Anosov map $\phi$. In this case we will call $\Phi$ a *collared pseudo-Anosov* map and consider its invariant foliations to be those of $\phi$.

For simplicity of notation we let $r(\theta) = \mathrm{sr}(B(e^{2\pi i\theta}))$ and just consider $\theta \in [0, 2\pi)$. If $\beta$ is pseudo-Anosov with expansion constant $\lambda = e^{h_{\mathrm{top}}(\Phi)}$, Lemma 4.1 shows that

$r(\theta) \leq \lambda$ for all $\theta \in [0, 2\pi)$. In this section we investigate when the Burau estimate (4–1) is sharp for a pseudo-Anosov braid and the substitution of a root of unity, that is, when do we have $r(j/k) = \lambda$ for some $j/k \in \mathbb{Q}$. The first observation is that according to Theorem 3.4 when $r(j/k) = \lambda$ occurs, then the action of $h_*^{(k)}$ on the first homology of the $k$–fold Burau cover $H_1(D^{(k)})$ has some eigenvalue of modulus $\lambda$. By Lemma 4.3, this eigenvalue must be either $\lambda$ or $-\lambda$, it must be simple, and all other eigenvalues of $h_*^{(k)}$ must be of smaller modulus. So, in particular, $r(\ell/k) < \lambda$ for all $\ell \neq j$.

The second observation is that the function $r(\theta)$ is periodic with period one and in addition, since the coefficients of the Laurent polynomial entries of $B(t)$ are real, $\mathrm{sr}(B(\eta)) = \mathrm{sr}(B(\overline{\eta}))$, where $\overline{\eta}$ denotes the complex conjugate of $\eta$. Thus $r$ is an even, one-periodic function and so it is even about $1/2$.

Putting the two observations together we have that for a given $k$, $r(j/k) = \lambda$ for at most one $j$ and since $r(1/2 + x) = r(1/2 - x)$, that can only happen if $j/k = 1/2$. Thus if $k$ is even, the only possibility for the Burau estimate to be sharp is that $r((k/2)/k) = \lambda$ and so $r(j/k) < \lambda$ for other $j$. On the other hand, if $k$ is odd, $r(j/k) < \lambda$ for all $j$, so the Burau estimate is never sharp for substitutions with $k$ odd.

In the next Proposition we connect these observations with the structure of the invariant foliations from Section 2.4 and Lemma 4.3.

**Theorem 5.1** *Suppose $\beta \in B_n$ is a braid represented by the collared pseudo-Anosov map $\Phi$: $D \to D$ having expansion factor $\lambda > 1$. The following are equivalent:*

(a) $\mathrm{sr}\big(B(e^{2\pi i j/k})\big) = \lambda$ *for some $k > 0$ and some $0 \leq j < k$;*

(b) $\mathrm{sr}\big(B(-1)\big) = \lambda$ *and $-1$ is the only root of unity for which this occurs.*

(c) *The invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ have an odd-order singularity at each puncture of $D$, and all singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$ in the interior of $D$ have even order.*

(d) $D^{(2)}$ *is the orientation double-cover of $\mathcal{F}^u$ and $\mathcal{F}^s$ (after removing the collaring).*

**Proof** The observations above the Theorem prove the equivalence of 5.1 and 5.1.

Let $\Phi^{(2)}$ be the preferred lift of $\Phi$ to $D^{(2)}$ and $\Phi_*^{(2)}$ its action on $H_1(D^{(2)})$. By Theorem 3.4, the eigenvalues of $\Phi_*^{(2)}$ are those of $B(1) \oplus B(-1)$, together with some roots of unity. Since $B(1)$ is a permutation, $\lambda$ (or $-\lambda$) is an eigenvalue of $\Phi_*^{(2)}$ if and only if it is an eigenvalue of $B(-1)$. In addition, by Lemma 4.3, since $\Phi^{(2)}$ is collared pseudo-Anosov, one of $\lambda$ or $-\lambda$ is an eigenvalue of $\Phi_*^{(2)}$ if and only the invariant foliations of $\Phi^{(2)}$ are oriented. Thus 5.1 holds exactly when the lift of $\mathcal{F}^u$ to $D^{(2)}$ is oriented, and so in particular 5.1 implies 5.1.

As remarked above Lemma 2.2, if $\gamma$ is a small loop surrounding a singularity $P$, then recalling that $\overrightarrow{\rho}$ denotes the epimorphism defining the orientation double cover, we have $\overrightarrow{\rho}([\gamma]) = 1$ iff $P$ has odd order. On the other hand, the epimorphism $\tau_2\colon H_1(D) \to \mathbb{Z}_2$ which yields the two-fold Burau cover $D^{(2)}$ is defined by $\tau_2([\gamma']) = 1$ for $\gamma'$ a small loop around a puncture. This shows the equivalence of 5.1 and 5.1.

Now we will show 5.1 implies 5.1 by proving its contrapositive. If $\mathcal{F}^u$ has an interior singularity of odd order, by Lemma 2.2 the lifted foliations to $D^{(2)}$ are not oriented. Now note that by Euler–Poincaré–Hopf formula (2–1), $\mathcal{F}^u$ always has a one-pronged singularity at some puncture $x_i$ of $D$. If $\mathcal{F}^u$ has an even order singularity at some other puncture $x_j$, we consider the homotopy class $\alpha = \gamma_j^{-1}\gamma_i$. We then have $\overrightarrow{\rho}(\alpha) = 0 + 1 = 1$ and $\tau_2(\alpha) = -1 + 1 = 0$. Thus there can be no homomorphism $\delta$ with $\delta \circ \tau_2 = \overrightarrow{\rho}$, and so again by Lemma 2.2, the lifted foliations to $D^{(2)}$ are not oriented. Since we have just seen that 5.1 holds exactly when the lift of $\mathcal{F}^u$ to $D^{(2)}$ is oriented, we have that 5.1 implies 5.1. □

**Remark 5.2** Song, Ko and Los [26, Lemma 5 and Theorem 7] contain a portion of the above Theorem in slightly different language, namely, the implications 5.1 $\implies$ 5.1 and 5.1 $\implies$ 5.1.

# 6 The Burau estimate for reducible braids

## 6.1 Pseudo-Anosov components and Burau orientability

The Thurston normal form $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ of a general braid $\beta \in B_n$ can be quite complicated. Using the notation of Theorem 2.1, there may be several reducing annuli forming the set $A$, each component $\Phi_i = \Phi|_{X_i}$ of $\Phi$ may be either periodic or pseudo-Anosov, and $X_i$ itself may have one or more connected components. In this section we will consider this more complicated decomposition and obtain general results on the sharpness of the entropy bound provided by the Burau representation.

The simplest case is when each $\Phi_i$ is periodic, and so $h_{\mathrm{top}}(\Phi) = 0$. Since all the eigenvalues of $\Phi_*$ are roots of unity, we have $\mathrm{sr}(B(1)) = 1$, giving a sharp bound on the entropy. Thus from now on we suppose that some component of $\Phi$ is pseudo-Anosov, and we let $\lambda = e^{h_{\mathrm{top}}(\Phi)}$. The main theorem of this section gives necessary and sufficient conditions for a root of unity $\omega$ to make the Burau estimate (4–1) sharp, ie to satisfy $\mathrm{sr}(B(\omega)) = \lambda$.

To state these conditions, we first define for each component $\Phi_i$ a positive integer $a_i$ expressing the manner in which the surface $X_i$ on which $\Phi_i$ is supported surrounds the punctures of $D$. Begin by choosing a component $\Phi_i$ of $\Phi$. The supporting surface $X_i$ of $\Phi_i$ may have several connected components; by definition, these are permuted cyclically by $\Phi_i$. Let $X_{i0}$ be one of these components. Thus $X_{i0}$ is a finitely punctured subdisk of $D$ from which a finite collection of open punctured subdisks have been deleted (see Figure 2(a)). Let $x_1, \ldots, x_{r'}$ denote the punctures in $X_{i0}$, where $0 \le r' \le n$, and let $O_{r'+1}, \ldots, O_r$ be the deleted subdisks. For $r' < j \le r$ we write $m_j$ for the number of punctures of $D$ which lie in $O_j$; and for $1 \le j \le r'$ we define $m_j = 1$. Finally, we define $a_i$ by

$$a_i = \gcd(m_1, m_2, \ldots, m_r). \tag{6–1}$$

Because $\Phi$ permutes the connected components of $X_i$ cyclically, and sends punctures to punctures, this definition is independent of the choice of $X_{i0}$.

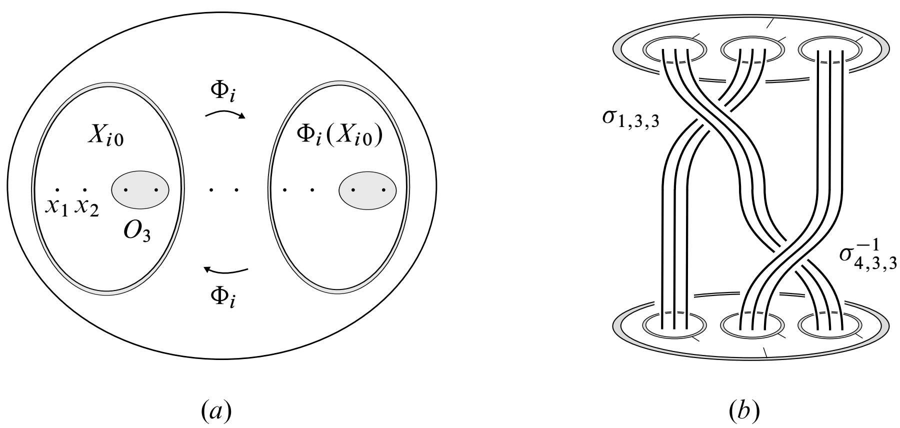



Figure 2: (a) A typical supporting surface $X_i$. Here $r' = 2$, $r = 3$, $m_3 = 2$ and $a_i = 1$. (b) the braid $\beta_3$ and its Thurston decomposition.

The next lemma gives another interpretation of the number $a_i$ which follows from the definition of the morphism $\tau\colon H_1(D) \to \mathbb{Z}$ associated to the covering space $D^{(\infty)}$.

**Lemma 6.1** *Let $\ell_i$ be the number of connected components of the supporting surface $X_i$ of $\Phi_i$, and let $X_{i0}$ be the chosen connected component of $X_i$ as above. Write $X_i^{(\infty)} = (p^{(\infty)})^{-1}(X_i)$ and $X_{i0}^{(\infty)} = (p^{(\infty)})^{-1}(X_{i0})$. The number $a_i$ just defined* *is equal to the number of connected components of* $X_{i0}^{(\infty)}$*. Hence the number of connected components of* $X_i^{(\infty)}$ *is* $\ell_i a_i$*.*

**Proof** According to Thurston's Theorem 2.1, the waist curve of any reducing annulus $a \in A$ is neither null-homotopic nor homotopic to a puncture. It follows that the inclusion $X_i \subset D$ induces an injection $H_1(X_i) \to H_1(D)$, and so we can consider $H_1(X_i)$ as a subgroup of $H_1(D)$. Then $X_i^{(\infty)} \to X_i$ is (isomorphic to) the covering space defined by the homomorphism $\tau|_{H_1(X_i)}$. By definition $\tau$ sends a small clockwise loop around the puncture $x_j$ (respectively, the hole $O_j$) of $X_{i0}$ to $m_j \in \mathbb{Z}$, and therefore $a_i$ is the generator of $\operatorname{im} \tau|_{H_1(X_{i0})}$. This proves the first statement of the lemma. Since $h$ permutes the connected components of $X_i$ cyclically and since $\tau\Phi_* = \tau$, we have that $X_i^{(\infty)}$ has the same number of connected components above each component of $X_i$, proving the second statement. □

Let us say that a pseudo-Anosov component $\Phi_i$ of $\Phi$ is *Burau orientable* provided the lifts of its invariant foliations to some $D^{(k)}$ are orientable. The main theorem of this section is the following.

**Theorem 6.2** *Let* $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ *be the Thurston normal form of the braid* $\beta$*, and for each* $i$ *such that* $\Phi_i$ *is pseudo-Anosov, let* $a_i$ *be as in* (6–1).

(1) *The pseudo-Anosov component* $\Phi_i$ *is Burau orientable if and only if (with notation as in the definition of* $a_i$*) the invariant foliations of* $\Phi_i$ *have a singularity of odd order at each puncture* $x_j$ *of* $X_{i0}$*, a singularity of odd (respectively, even) order on the boundary of each deleted disk* $O_j$ *such that* $m_j/a_i$ *is odd (even), and all singularities in the interior of* $X_{i0}$ *are even order.*

(2) *Let* $I$ *be the set of* $1 \le i \le m$ *such that* $\Phi_i$ *is pseudo-Anosov, Burau orientable, and satisfies* $h_{\text{top}}(\Phi_i) = h_{\text{top}}(\Phi)$*. Then the set of roots of unity* $\omega$ *for which* $\operatorname{sr}(B(\omega)) = \lambda$*, is equal to the union over* $i \in I$ *of the set of all* $a_i$*th roots of* $-1$*. In particular, if* $I$ *is empty, then* $\operatorname{sr}(B(\omega)) < \lambda$ *for every root of unity* $\omega$*.*

Before embarking on the proof of Theorem 6.2, we will illustrate the theorem with some examples.

## 6.2 Examples

It is convenient to use the following notation. Let $n > 0$. If $i, n_1, n_2$ are positive integers with $i + n_1 + n_2 - 1 \le n$, we let $\sigma_{i,n_1,n_2} \in B_n$ denote the braid which moves the group of $n_1$ consecutive strings starting at string $i$ over the group of $n_2$ consecutive strings starting at string $i+n_1$:

$$\sigma_{i,n_1,n_2} = (\sigma_{i+n_1-1}\cdots\sigma_{i+n_1+n_2-2})(\sigma_{i+n_1-2}\cdots\sigma_{i+n_1+n_2-3})\ldots(\sigma_i\cdots\sigma_{i+n_2-1}).$$

In particular $\sigma_{i,1,1}=\sigma_i$ for all $1\le i\le n-1$.

**Example 6.3** Let $n'\ge 1$, and define a braid $\beta_{n'}$ on $3n'$ strings by setting

$$\beta_{n'} = \sigma_{1,n',n'}\sigma^{-1}_{n'+1,n',n'}.$$

Figure 2(b) shows $\beta_{n'}$ when $n'=3$.

The Thurston normal form $\Phi$ of $\beta_{n'}$ reduces along 4 annuli $A=A_0\cup A_1\cup A_2\cup A_3$: there is one such annulus collaring the boundary of $D$, and one surrounding each of the three groups of $n'$ punctures. The component of $\Phi$ in the outer connected component of $D\setminus A$ is pseudo-Anosov. Call this component $\Phi_1$ and its supporting surface $X_1$. The other component of $\Phi$ is periodic, and it cyclically permutes the three inner connected components of $D\setminus A$. The Euler–Poincaré–Hopf formula (2–1) shows that the invariant foliations of $\Phi_1$ have four 1-pronged singularities, one at each of the four boundary components of $X_1$, and no other singularities. According to (6–1) we have $a_1=m_1=m_2=m_3=n'$, and hence $\Phi_1$ is Burau orientable by the first part of Theorem 6.2.

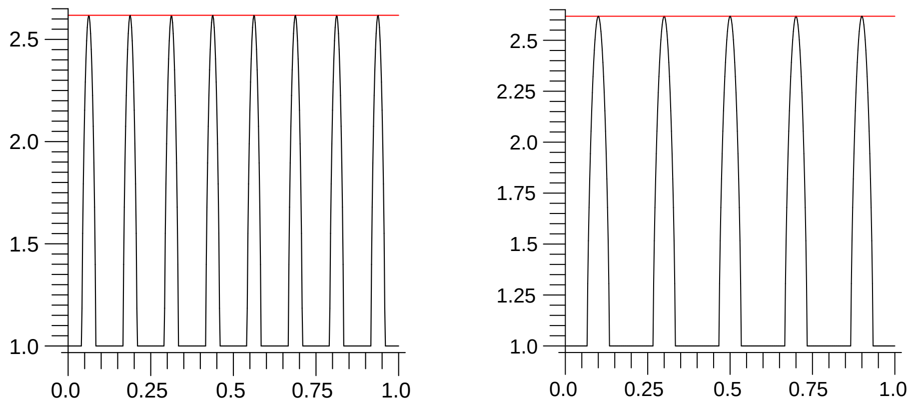
The growth rate of $\beta_{n'}$ is $\lambda = e^{h_{\mathrm{top}}(\Phi_1)} \sim 2.618$. Figure 3 shows the graph of the map sending $\theta\in[0,1]$ to the spectral radius of the substituted Burau matrix $B(e^{2\pi i\theta})$ of $\beta_{n'}$, in the two cases $n'=8=2^3$ and $n'=5$.

Here is one part of the justification of Theorem 6.2 for this example. According to Lemma 6.6 below, if $k$ is a multiple of $2n'$, the invariant foliations of $\Phi_1$ lift to orientable foliations in the $k$–fold covering space $X_1^{(k)}$ of $X_1$. Moreover this covering space has exactly $n'$ connected components, each fixed by the lift of $\Phi$. Therefore by Theorem 3.4 and Lemma 4.3 we expect the matrices $B(\omega)$, for $\omega$ a $k$th root of unity, to contribute exactly $n'$ eigenvalues equal to $\lambda$ or $-\lambda$ (counted with multiplicity). Theorem 6.2 states, in addition, that it is precisely the $n'$th roots of $-1$ which contribute these eigenvalues, a fact clearly reflected in Figure 3.

**Example 6.4** Let $\beta_1'\in B_9$ and $\beta_2'\in B_8$ be the braids

$$\beta_1' = \sigma_{1,3,3}\cdot\sigma^2_{4,3,3}\cdot\sigma^3_{1,3,3} \qquad\text{and}\qquad \beta_2' = \sigma_{1,3,3}\cdot\sigma_{4,3,2}\cdot\sigma_{4,2,3}\cdot\sigma^3_{1,3,3}.$$

Note that $\beta_1'$ respects the grouping of the punctures into consecutive groups of three, while $\beta_2'$ respects the grouping of the punctures into two consecutive groups of three




$(a)$ $(b)$

Figure 3: The Burau estimate (a) for $\beta_8$ and (b) for $\beta_5$. The horizontal lines represent the growth rate $\lambda \sim 2.618$.

and one of two. As for the previous example, the Thurston normal form $\Phi$ of $\beta'_i$ has one reducing annulus around each of the three groups of punctures. Each such annulus encloses a periodic (in fact fixed) component of $\Phi$. If $X_1$ denotes the outer component of $X \setminus A$, then $\Phi_1 = \Phi|_{X_1}$ is pseudo-Anosov with growth rate $\lambda \sim 5.828$. Again the invariant foliations of $\Phi_1$ have a one-pronged singularity on each boundary component of $X_1$.

Now for $\beta'_1$ we have $a_1 = m_1 = m_2 = m_3 = 3$, and so Theorem 6.2 shows that $\Phi_1$ is Burau orientable. Therefore the Burau estimate is sharp at all of the cubic roots of $-1$ (see Figure 4(a)). For $\beta'_2$, however, we have $m_1 = m_2 = 3$ and $m_3 = 2$, and so $a_1 = \gcd(2,3) = 1$. Thus the conditions of the first part of Theorem 6.2 fail to hold and $\Phi_1$ is not Burau orientable, so the Burau estimate is never sharp (see Figure 4(b)).

**Example 6.5** Let $\beta'' \in B_{10}$ be the braid

$$\beta'' = \beta_3 \cdot \sigma_{1,9,1} \cdot \sigma_{1,1,9}$$

where $\beta_3 \in B_9 \subset B_{10}$ is as in Example 6.3 above[3]. See Figure 5(a). The Thurston normal form $\Phi$ of $\beta''$ has one pseudo-Anosov component $\Phi_1 = \Phi|_{X_1}$, topologically conjugate to the pseudo-Anosov component of the Thurston normal form of $\beta_3$. By Theorem 6.2 this component is Burau orientable and the Burau estimate is sharp at

[3] As is usual we regard $B_9$ as the subgroup of $B_{10}$ generated by the first eight generators $\sigma_1, \dots, \sigma_8$.

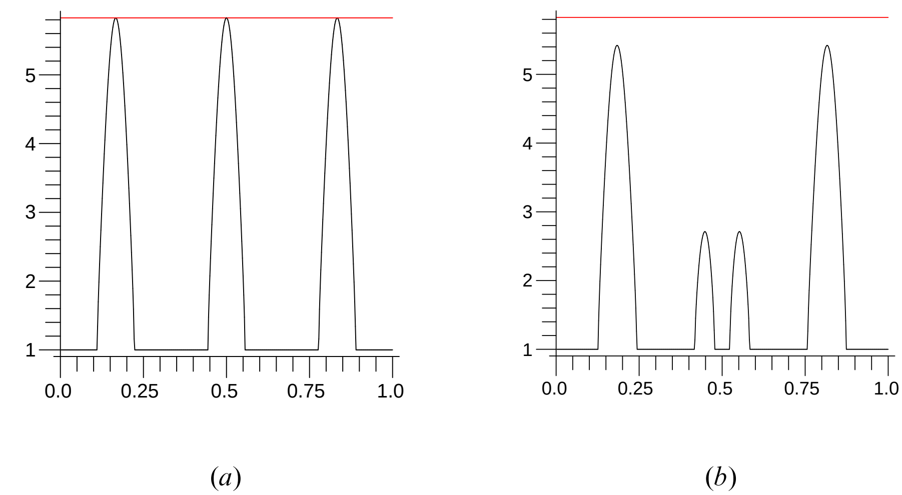
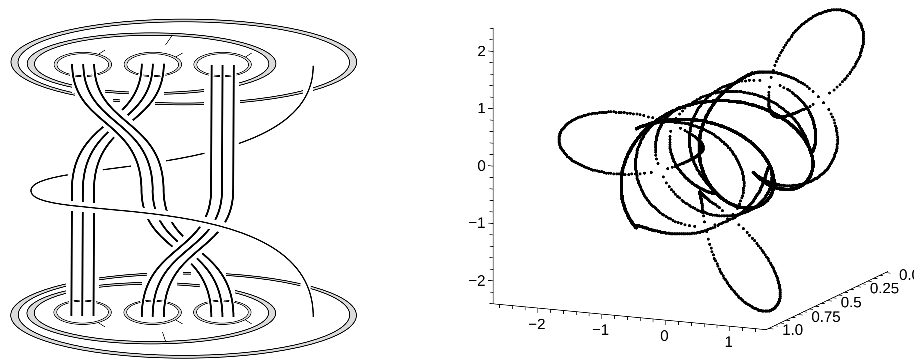



Figure 4: The Burau estimate (a) for $\beta_1'$ and (b) for $\beta_2'$. The horizontal lines represent the growth rate $\lambda \sim 5.828$.

each of the cubic roots of $-1$. However, $\beta''$ is distinguished from $\beta_3$ by the fact that the eigenvalue of $B(-e^{2\pi ij/3})$ of modulus $\lambda$ is not always real (see Figure 5(b)). This can be seen as follows. Lemma 6.6 below shows that for $k$ a multiple of 6 the lift of $\Phi_1$ to the $k$–fold cover $X_1^{(k)}$ has orientable foliations. Because $\Phi$ acts as a full twist in the outer component of $D$, the three connected components of $X_1^{(k)}$ are permuted cyclically by $\Phi^{(k)}$. So by Theorem 3.4 and Lemma 4.3, the matrices $B(\omega)$, for $\omega$ a $k$th root of unity, contribute to $\Phi_*^{(k)}$ exactly three eigenvalues of modulus $\lambda$, differing from each other by the cubic roots of unity. These eigenvalues are the extremal points of the curve in Figure 5(b).

## 6.3 Proof of Theorem 6.2

We will prove Theorem 6.2 by a series of lemmas. The main idea of the proof, already suggested by the above examples, is the following. As we lift a pseudo-Anosov component $\Phi_i$ of the Thurston normal form $\Phi$ of the chosen braid $\beta$ into successive covering spaces $D^{(k)}$, the supporting surface of the lift of $\Phi_i$ may become disconnected. If the lifted invariant foliations also become orientable, then Lemma 4.3 implies the existence of several eigenvalues of modulus $\lambda$ (counted with geometric multiplicity). By Theorem 3.4 these eigenvalues are distributed among those of the matrices $B(\omega)$, where $\omega$ is an $k$th root of unity. An analysis of the action of $\Phi^{(\infty)}$ on the first homology




(*a*) (*b*)

Figure 5: (a) The braid $\beta''$; (b) the eigenvalues of $B(e^{2\pi i\theta})$ for $\theta \in [0,1]$. Here the rightmost axis represents the unit interval $[0,1]$ and the other two axes represent the complex plane.

of $X_i^{(\infty)}$ then shows that it is precisely the $a_i$th roots of $-1$ which contribute such an eigenvalue.

We begin with a lemma which provides a stronger version of the first part of Theorem 6.2. Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ be the Thurston normal form of the braid $\beta \in B_n$, let $\Phi_i$ be a pseudo-Anosov component of $\Phi$, and let $\mathcal{F}^u$ and $\mathcal{F}^s$ denote the invariant foliations of $\Phi_i$. For $k \geq 1$ we consider the $k$–fold Burau covering space $p^{(k)}\colon D^{(k)} \to D$ and write $X_i^{(k)} = (p^{(k)})^{-1}(X_i)$ and $\Phi_i^{(k)} = \Phi^{(k)}|_{X_i^{(k)}}$. As in the definition of $a_i$ in (6–1), let $X_{i0}$ be a connected component of $X_i$, and let $x_1, \ldots, x_{r'}$ be the punctures in $X_{i0}$ and $O_{r'+1}, \ldots, O_r$ the deleted disks. Further, for each $1 \leq j \leq r$ let $\kappa_j$ be the order of the singularity which $\mathcal{F}^u$ and $\mathcal{F}^s$ exhibit at $x_j$ (if $j \leq r'$) or on $\partial O_j$ (if $r' < j \leq r$). We have the following Lemma.

**Lemma 6.6** *Let $\Phi_i = \Phi|_{X_i}$ be a pseudo-Anosov component of the Thurston normal form $\Phi$ of $\beta$, and let $a_i$ be defined as in (6–1). Write $a = a_i$ and suppose that $a = 2^u a'$ where $a'$ is odd. Then the following are equivalent:*

(1) *$\Phi_i$ is Burau orientable, ie there exists $k \geq 1$ such that the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(k)}$ are orientable;*

(2) *The set of $k$ for which the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(k)}$ are orientable is the set of multiples of $2^{u+1}$;*

(3) *All singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$ in the interior of $X_{i0}$ have even order, and for all $1 \le j \le r$, $m_j/a \equiv \kappa_j \mod 2$. In other words $m_j$ is an odd (respectively, even) multiple of $a$ whenever $\kappa_j$ is odd (even).*

**Proof** The implication 6.6 $\implies$ 6.6 is trivial; we will prove 6.6 $\implies$ 6.6 and 6.6 $\implies$ 6.6. Except for an application of Lemma 2.2, the proof is essentially algebraic – it involves only the relevant morphisms of $H_1(X_i)$ into $\mathbb{Z}$ and $\mathbb{Z}_k$. For $1 \le j \le r$ let $\alpha_j$ be the homology class of a small clockwise loop around the puncture $x_j$ (if $j \le r'$) or around the disk $O_j$ (if $j > r'$). Then $H_1(X_{i0})$ can be identified with the subgroup of $H_1(D)$ generated by the $\alpha_j$'s. By definition $\tau(\alpha_j) = m_j$ for each $j$, and $\tau_k(\alpha_j) = \xi_k \tau(\alpha_j) = m_j \mod k$, where $\xi_k\colon \mathbb{Z} \to \mathbb{Z}_k$ is the quotient homomorphism. When all singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$ in the interior of $X_i$ have even order, we let $\overrightarrow{\rho}\colon H_1(X_i) \to \mathbb{Z}_2$ be the morphism associated to the orientation cover of $\mathcal{F}^u$ and $\mathcal{F}^s$. Because $\Phi_i$ preserves the foliations and permutes the components cyclically, $\overrightarrow{\rho}$ is determined by its values on $H_1(X_{i0})$: namely $\overrightarrow{\rho}(\alpha_j) = \kappa_j \mod 2$ for all $j$.

According to Lemma 2.2 we know that for $k \ge 1$ the following two statements are equivalent:

(i) The lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(k)}$ are orientable;

(ii) All singularities in the interior of $X_i$ have even order, and there exists a homomorphism $\delta_k\colon \operatorname{im} \tau_k|_{H_1(X_i)} \to \mathbb{Z}_2$ such that $\delta_k \circ \tau_k = \overrightarrow{\rho}$ on $H_1(X_i)$.

In addition, since $\tau_k \Phi_* = \tau_k$, the formula $\delta_k \circ \tau_k = \overrightarrow{\rho}$ holds on $H_1(X_i)$ whenever it holds on $H_1(X_{i0})$.

We note some elementary facts about cyclic groups. For $q \ge 1$ odd, there is no nontrivial homomorphism $\mathbb{Z}_q \mapsto \mathbb{Z}_2$, while for $q$ even there is only one such homomorphism: namely, the homomorphism which sends odd multiples of the generator to 1 and even multiples to 0 (a property which is independent of the choice of generator). And if $q$ is odd then every subgroup of $\mathbb{Z}_q$ has odd order.

Suppose that 6.6 holds, that is, that $\Phi_i$ is pseudo-Anosov and for some $k > 0$ the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X^{(k)}$ are orientable. Since $p^{(k)}\colon X_i^{(k)} \to X_i$ is ramified only around the punctures, this implies that all singularities in the interior of $X_i$ have even order. Since $\operatorname{im} \tau|_{H_1(X_i)} = a\mathbb{Z}$, we have $\operatorname{im} \tau_k|_{H_1(X_i)} = \xi_k(a\mathbb{Z}) = \xi_k(a)\mathbb{Z}_k$. The Euler–Poincaré–Hopf formula (2–1) shows that $\mathcal{F}^u$ and $\mathcal{F}^s$ have at least one 1–pronged singularity in $X_{i0}$, which must lie at a puncture or on the boundary of a deleted disk. Thus $\overrightarrow{\rho}$ is onto, so from the previous paragraph we see that (ii) holds precisely when $k$ is even and for all $1 \le j \le r$ we have: $\tau_k(\alpha_j)$ is an odd (even) multiple of $\xi_k(a)$ iff $\kappa_j$ is odd (even). If $k$ is even then $\xi_k$ sends odd (even) multiples of $a$ to odd (even) multiplies of $\xi_k(a)$ and so this implies the same statement in $\mathbb{Z}$: that is, that 6.6 holds. Thus 6.6 $\implies$ 6.6.

On the other hand suppose 6.6 holds and set $k = 2^{u+1}$. Since $a = 2^u a'$ where $a'$ is odd, we see that $\operatorname{im} \tau_k|_{H_1(X_i)}$ is the subgroup $\{0, \xi_k(a)\}$ of $\mathbb{Z}_k$ having order two. Let $\delta_k$ be the isomorphism of this subgroup onto $\mathbb{Z}_2$. Again, since $k$ is even, to say that $\xi_k(m_j) = \xi_k(a)$ is to say that $m_j$ is an odd multiple of $a$. By (3) this happens precisely when $\kappa_j$ is odd; hence $\delta_k \tau_k = \overrightarrow{\rho}$ on $H_1(X_i)$. Therefore since (i) $\implies$ (ii) the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(k)}$ are orientable. Therefore the set of $k$ for which the lifted foliations in $X^{(k)}$ are orientable contains $2^{u+1}$, and, it follows, also all multiples of $2^{u+1}$.

To complete the proof that 6.6 $\implies$ 6.6, suppose for a contradiction that 6.6 holds and that for some $k$ which is not a multiple of $2^{u+1}$, the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(k)}$ are orientable. The last condition implies that $k$ is even, so we must have $u > 0$. Let $\delta_k$ be the homomorphism supplied by (ii). Since $\overrightarrow{\rho}$ is onto, $\delta_k$ must send the generator $\xi_k(a)$ of $\operatorname{im} \tau_k|_{H_1(X_i)}$ to $1 \in \mathbb{Z}_2$. Now write $k = 2^{u'} k'$ where $k'$ is odd and $u' \le u$. The order of $\xi_k(a)$ in $\mathbb{Z}_k$ is given as $\zeta = \operatorname{lcm}(a,k)/a = \operatorname{lcm}(a',k')/a'$, which is odd. But then $0 = \delta_k(0) = \delta_k(\zeta \xi_k(a)) = \zeta \delta_k \xi_k(a) = \delta_k \xi_k(a) = 1$, a contradiction. Therefore no such $k$ can exist. This completes the proof. $\square$

To prove the second part of Theorem 6.2, we will require two more lemmas. The first of these expresses $H_1(D^{(\infty)})$ in terms of the first homology groups of the $X_i^{(\infty)}$'s. Since each $X_i^{(\infty)}$ is $\Phi^{(\infty)}$–invariant, we will be able to use this lemma to factorize the characteristic polynomial of the reduced Burau matrix $B(\beta)$.

**Lemma 6.7** *Let $\Phi = \Phi_1 \cup \cdots \cup \Phi_m \in \beta$ be as above and let $\Phi^{(\infty)}$ be the lift of $\Phi$ to $D^{(\infty)}$. Let $X_i^{(\infty)} = (p^{(\infty)})^{-1}(X_i)$ be the pull-back to $D^{(\infty)}$ of the supporting surface $X_i$ of $\Phi_i$. Then, as an Abelian group, $H_1(D^{(\infty)})$ decomposes as a direct sum of subgroups*

$$H_1(D^{(\infty)}) = \Big(\bigoplus_{i=1}^{m} H_1(X_i^{(\infty)})\Big) \oplus V \tag{6–2}$$

*in which each $H_1(X_i^{(\infty)})$ is $\Phi_*^{(\infty)}$–invariant, and $V$ is a free abelian group of finite rank.*

**Proof** We use the Mayer–Vietoris sequence inductively. If $D$ has a collar (ie an annulus $a \in A$ whose boundary contains $\partial D$) we first remove it from $D$ and from $A$

and remove its pull-back from $D^{(\infty)}$. This does not affect the homology calculation. Let $A^{(\infty)} = (p^{(\infty)})^{-1}(A)$ be the set of points of $D^{(\infty)}$ covering the reducing annuli in $D$. For $1 \le i \le m$ define $A_i$ to be the set of those connected components of $A^{(\infty)}$ whose boundary intersects $X_i^{(\infty)}$, but does not intersect $X_j^{(\infty)}$ for any $j < i$. Note that $A_m = \varnothing$ because (since we removed the collar) the boundary of each reducing annulus intersects two of the $X_i$. Write $Z_i = X_i^{(\infty)} \cup A_i$, and for $0 \le i \le m$ let $Y_i' = \bigcup_{j>i} Z_j$ and when $i > 0$ let $Y_i = Y_i' \cup A_i$. We thus have a sequence of inclusions

$$\varnothing = Y_m' = Y_m \subset Y_{m-1}' \subseteq Y_{m-1} \subset \cdots \subset Y_1' \subseteq Y_1 \subset Y_0' = D^{(\infty)}.$$

Moreover, these spaces satisfy

$$Z_i \cap Y_i = A_i, \quad Z_i \cup Y_i = Y_{i-1}'$$

for each $1 \le i \le m$. Since $X_i^{(\infty)}$ is $\Phi^{(\infty)}$-invariant for each $i$, so too are $A_i, Z_i, Y_i$ and $Y_i'$.

For any reducing annulus $a \in A$, each boundary component of $a$ is also the boundary of some (possibly punctured) subdisk $D'$ of $D$. In fact, since no component of $\partial a$ is allowed to be null-homotopic, $D'$ must have at least one puncture and so we will have $\tau([\partial D']) \neq 0$. It follows that any connected component of $A^{(\infty)}$ covering $a$ is an infinite strip, isomorphic as a covering space to the universal cover of $a$. In particular $H_1(A_i) = 0$ for each $i$.

Therefore, regarding $Z_i$ and $Y_i$ as subspaces of their union $Y_{i-1}'$, the Mayer-Vietoris sequence for the pair $(Z_i, Y_i)$ contains the segment

$$0 \longrightarrow H_1(Z_i) \oplus H_1(Y_i) \longrightarrow H_1(Y_{i-1}') \xrightarrow{\partial} H_0(A_i) \longrightarrow \cdots \tag{6–3}$$

All groups in the above sequence are free abelian. Noting that $X_i^{(\infty)}$ is a deformation retract of $Z_i$ and that $Y_{i-1}'$ is a deformation retract of $Y_{i-1}$, we have

$$H_1(Y_{i-1}) \cong H_1(X_i^{(\infty)}) \oplus H_1(Y_i) \oplus V_i$$

where $V_i$ is canonically isomorphic to $\operatorname{im} \partial \subset H_0(A_i)$. Since $A_i$ has finitely many connected components, $H_0(A_i)$ and $V_i$ each have finite rank. The lemma is proved by applying this formula inductively, starting with $H_1(Y_{m-1}) = H_1(X_m^{(\infty)})$, and setting $V = \bigoplus_{i=1}^m V_i$. □

**Corollary 6.8** *For each $1 \le i \le m$ let $\Phi_i^{(\infty)}$ be the restriction of $\Phi^{(\infty)}$ to $X_i^{(\infty)}$, and denote by $g_i$ the action of $\Phi_i^{(\infty)}$ on $H_1(X_i^{(\infty)})$. As in Section 3, regard $H_1(D^{(\infty)})$ and $H_1(X_i^{(\infty)})$ as modules over the ring $R = \mathbb{Z}[t^{\pm 1}]$. Then $\bigoplus_{i=1}^m H_1(X_i^{(\infty)})$ is a* *submodule of $H_1(D^{(\infty)})$ of full rank. Moreover, the characteristic polynomial of the module isomorphism $h_*^{(\infty)}\colon H_1(D^{(\infty)}) \to H_1(D^{(\infty)})$ is given by*

$$\chi(h_*^{(\infty)}) = \prod_{i=1}^{m} \chi(g_i).$$

**Proof** Let $G = \bigoplus_{i=1} H_1(X_i^{(\infty)})$. Because the $R$–module structure on $H_1(D^{(\infty)})$ and on $H_1(X_i^{(\infty)})$ are each defined using the same group of deck transformations acting respectively on $D^{(\infty)}$ and on $X_i^{(\infty)}$, they coincide, making $G$ a submodule of $H_1(D^{(\infty)})$.

As an $R$–module, $H_1(D^{(\infty)})$ is free and of rank $n-1$. We claim that $G \subseteq H_1(D^{(\infty)})$ also has rank $n-1$. For suppose $\mathrm{rank}(G) < n-1$. We may choose $w \in H_1(D^{(\infty)})$ so that $G \cup \{w\}$ spans a free submodule of $H_1(D^{(\infty)})$ of rank strictly larger than $\mathrm{rank}(G)$. Since $H_1(D^{(\infty)})$ is free, it follows that $w+G$ generates a free $R$–submodule of $H_1(D^{(\infty)})/G$. But in the notation of Lemma 6.7, we have $H_1(D^{(\infty)})/G \cong V$ as Abelian groups. Therefore $H_1(D^{(\infty)})/G$ has finite rank as an Abelian group, a contradiction.

By definition $h_*^{(\infty)}|_G = \bigoplus_{i=1}^{m} g_i$. Because $G$ is a submodule of $H_1(D^{(\infty)})$, the characteristic polynomial of $h_*^{(\infty)}|_G$ divides the characteristic polynomial of $h_*^{(\infty)}$ on $H_1(D^{(\infty)})$. Since $G$ has full rank both polynomials have the same leading coefficient $x^{n-1}$ and so they are equal. This completes the proof. $\square$

The next lemma shows how the characteristic polynomial $\chi(g_i)$ of $g_i$ reflects the permutation induced by $\Phi_i^{(\infty)}$ on the set of connected components of $X_i^{(\infty)}$.

**Lemma 6.9** *Let $\Phi_i = \Phi|_{X_i}$ be a component of the Thurston normal form $h$ of $\beta$, and let ${\Phi_i}^{(\infty)}$ be the lift of $\Phi_i$ to $X_i^{(\infty)}$. Let $g_i$ denote the action of ${\Phi_i}^{(\infty)}$ on $H_1(X_i^{(\infty)})$. Suppose $X_i$ has $\ell_i$ connected components, and let $d \in \mathbb{Z}$ be chosen so that $(\Phi^{(\infty)})^{\ell_i}(Y) = T^d(Y)$ for any connected component $Y$ of $X_i^{(\infty)}$. Then there exists an integer $e > 0$ such that the characteristic polynomial of $g_i$ is of the form*

$$\chi(g_i) = x^{\ell_i e} + t^d P_1(t) x^{\ell_i(e-1)} + t^{2d} P_2(t) x^{\ell_i(e-2)} + \cdots + t^{ed} P_e(t) \tag{6–4}$$

*where each $P_j \in \mathbb{Z}[t^{\pm a_i}]$.*

**Proof** As before, we let $X_{i0}$ be a connected component of $X_i$ and $X_{i0}^{(\infty)}$ its pull-back to $D^{(\infty)}$. Let $e$ be the dimension of $H_1(X_{i0}^{(\infty)})$ as a module over $\mathbb{Z}[t^{\pm 1}]$. Let $\{\upsilon_j : 1 \le j \le e\}$ be a basis for $H_1(X_{i0}^{(\infty)})$. We choose this basis so that all of the $\upsilon_j$'s are represented by loops in one and the same connected component, $Y$ say, of $X_{i0}^{(\infty)}$. Pushing forward this basis under each of the first $\ell_i$ iterates of $g_i$, we obtain a basis for $H_1(X_i^{(\infty)})$. With respect to this basis, the matrix of $g_i$ has block form

$$g_i = \begin{pmatrix} 0 & & & \dots & \Omega \\ \mathrm{id} & 0 & & & \\ & \mathrm{id} & 0 & & \vdots \\ & & \ddots & \ddots & \\ & & & \mathrm{id} & 0 \end{pmatrix},$$

having $\ell_i \times \ell_i$ blocks each of dimension $e \times e$, where $\Omega$ is some matrix in $\mathrm{GL}(e, \mathbb{Z}[t^{\pm 1}])$.

The connected components of $X_{i0}^{(\infty)}$ are in one to one correspondence with the cosets of $a_i\mathbb{Z}$ in $\mathbb{Z}$. Now $(\Phi^{(\infty)})^{\ell_i}$ sends the chosen connected component $Y \subset X_{i0}^{(\infty)}$ to the possibly different connected component $T^d Y$ of $X_{i0}^{(\infty)}$, where $d \in \mathbb{Z}$ is well-defined up to adding a multiple of $a_i$. It follows that the entries of $\Omega$ lie in the coset $t^d\mathbb{Z}[t^{\pm a_i}]$ of $\mathbb{Z}[t^{\pm a_i}]$ in $\mathbb{Z}[t^{\pm 1}]$.

Any product of $j$ elements of $\Omega$ therefore lies in the coset $t^{jd}\mathbb{Z}[t^{\pm a_i}]$ of $\mathbb{Z}[t^{\pm a_i}]$. The lemma is proved when we note that the characteristic polynomial of the matrix of $g_i$ is of the form

$$x^{\ell_i e} + \omega_1 x^{\ell_i(e-1)} + \cdots + \omega_e$$

where for each $1 \le j \le e$, $\omega_j$ is a sum of products of $j$ elements of $\Omega$, so that $\omega_j \in t^{jd}\mathbb{Z}[t^{\pm a_i}]$. □

**Corollary 6.10** *If $(x,t)$ is a root of $\chi(g_i)$, so too is $(\mu x, \nu t)$ where $\nu$ is any $a_i$ th root of unity and $\mu$ is any $\ell_i$ th root of $\nu^d$. Hence the roots of $\chi(g_i)$ are naturally divided into sets each of $\ell_i a_i$ roots.*

**Proof** Let $(x,t) \in \mathbb{C} \times \mathbb{C}$ and let $\mu$ and $\nu$ be as in the statement. A quick calculation using (6–4) shows that the value of $\chi(g_i)$ at $(\mu x, \nu t)$ is exactly $\nu^{ed}$ times its value at $(x,t)$. In particular if $(x,t)$ is a root so too is $(\mu x, \nu t)$. □

**Proof of Theorem 6.2** Lemma 6.6 establishes the first part of Theorem 6.2. To prove the second part, recall that $I$ is the set of $1 \le i \le m$ such that $\Phi_i$ is pseudo-Anosov, Burau orientable, and satisfies $h_{\mathrm{top}}(\Phi_i) = h_{\mathrm{top}}(\Phi)$. For $k > 0$ we write $\eta_k = e^{2\pi i/k}$. Also, for each $i$ we write $\Phi_i^{(k)} = \Phi^{(k)}|_{X_i^{(k)}}$ and $g_i^{(k)} = \Phi_*^{(k)}|_{H_1(X_i^{(k)})}$ as above.

Let $i \in I$. With notation as above, the number of connected components of $X_i^{(\infty)}$ is $\ell_i a_i$. Since $\tau_k = \xi_k \tau$, the same statement is true if we replace $X_i^{(\infty)}$ by ${X_i}^{(k)}$, for any $k$ which is a multiple of $a_i$.

Suppose in fact that $k$ is a multiple of $2a_i$. By Lemma 6.6, the invariant foliations of $\Phi_i^{(k)}$ in ${X_i}^{(k)}$ are orientable. The $\ell_i a_i$ connected components of ${X_i}^{(k)}$ are permuted by $\Phi_i^{(k)}$ in cycles all having the same number of components. Suppose the number of such cycles is $L$, so that each cycle contains $\ell_i a_i / L$ components. Lemma 4.3 applies to each such cycle, and we conclude that $g_i^{(k)}$ has exactly $\ell_i a_i / L$ eigenvalues of modulus $\lambda$, each having geometric multiplicity $L$. Indeed $\epsilon\lambda$ is one such eigenvalue, where $\epsilon \in \{1, -1\}$ is chosen according to whether $\Phi_i^{(k)}$ preserves or reverses the orientation of its unstable foliation. Counted with geometric multiplicity, there are exactly $\ell_i a_i$ eigenvalues of modulus $\lambda$.

Theorem 3.4 now implies that these $\ell_i a_i$ eigenvalues are distributed among the eigenvalues of the matrices $M(\eta_k^j)$, where $M$ denotes the matrix of $g_i\colon H_1(X_i^{(\infty)}) \to H_1(X_i^{(\infty)})$ as in Section 3. Because $M(\eta_k^j)$ is obtained by substituting $\eta_k^j$ into $M$, an eigenvalue $x$ of $M(\eta_k^j)$ corresponds to a root of $\chi(g_i)$ of the form $(x, \eta_k^j)$. In particular, setting $k = 2a_i$, we conclude that there is some $0 \le j_0 < 2a_i$ such that $(\epsilon\lambda, \eta_{2a_i}^{j_0})$ is a root of $\chi(g_i)$.

Meanwhile, since by Lemma 6.6 the lifts of $\mathcal{F}^u$ and $\mathcal{F}^s$ to $X_i^{(a_i)}$ are not orientable, a similar argument shows that no $a_i$th root of unity can occur in such a root of $\chi(g_i)$. Therefore $j_0$ must be odd. By Corollary 6.10 we now see that every element of the set

$$J_i = \{(\mu\epsilon\lambda, \eta_{2a_i}^{j_0+j}) \mid j \text{ is even and and } \mu \text{ is an } \ell_i\text{th root of } \eta_{2a_i}^{jd}\}$$

is a root of $\chi(g_i)$, where $d$ is as in Lemma 6.9. Since $j_0$ is odd, the roots of unity occurring in elements of $J_i$ are precisely the $a_i$th roots of $-1$.

For $k$ a multiple of $2a_i$, we have thus accounted for all of the $\ell_i a_i$ eigenvalues of $g_i^{(k)}$ of modulus $\lambda$. Furthermore, every root of unity can be written as a $k$th root of unity for some $k$ which is a multiple of $2a_i$. It follows that $J_i$ is precisely the set of roots $(x, \nu)$ of $\chi(g_i)$ such that $\nu$ is a root of unity and $|x| = \lambda$.

By Corollary 6.8, the characteristic polynomial of $B(\beta)$ is the product of those of the $g_i$. Therefore the set of roots of $\chi(B(\beta))$ is the union of those of the $g_i$. For $i \in I$, we have just accounted for all of the roots $(x, \nu)$ of $\chi(g_i)$ with $|x| = \lambda$ and $\nu$ a root of unity. For $i \notin I$ a similar argument shows that $\chi(g_i)$ can have no such roots. This completes the proof. □

**Proof of Theorem 1.2** Fix $\beta \in B_n$ and let $\Phi = \Phi_1 \cup \cdots \cup \Phi_m$ be its Thurston normal form, and as above let $I$ be the set of indices for which $\Phi_i$ is pseudo-Anosov, Burau orientable, and satisfies $h_{\mathrm{top}}(\Phi_i) = h_{\mathrm{top}}(\Phi)$. For $i \in I$ let $X_i$ be the supporting surface of $\Phi_i$ and $X_{i0} \subset X_i$ a connected component. Adopt the notation of Lemma 6.1 and the definition of $a_i$. In particular $r$ denotes the number of punctures and disks deleted from $X_{i0}$ and $\ell_i$ the number of connected components of $X_i$. Because $\Phi_i$ is pseudo-Anosov, we must have $r \geq 3$ by the Euler–Poincaré–Hopf formula. Since $a_i \leq \inf_j m_j$ and $\sum_j m_j \leq n/\ell_i$ by definition, we therefore have $a_i \leq n/r\ell_i \leq n/3$.

By Theorem 6.2 the Burau estimate is sharp at the root of unity $\eta_0$ if and only if $\eta_0$ is an $a_i$th root of $-1$, for some $i \in I$. Any such root is of the form $\eta_0 = e^{2\pi i j/k}$ where $k = 2a_i$ and $0 < j < k$ is odd, and $k \leq \frac{2}{3}n$ as we have just shown. □

**Remark 6.11** We remark that if $n = 3n'$ where $n'$ is a power of two, then for the braid $\beta_{n'} \in B_n$ constructed after Theorem 6.2, the Burau estimate is sharp at each root of $-1$ of order $\frac{1}{3}n$ but not at any root of unity of order less than $\frac{2}{3}n$. Thus for some braids the bound on $k$ in Theorem 1.2 is attained.

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