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A sufficient condition for a branched surface to fully carry a lamination

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We give a sufficient condition for a branched surface in a 3 dimensional manifold to fully carry a lamination, giving a piece of answer to a classical question of D Gabai.

57M50; 57N10, 57R30

1 Introduction

Branched surfaces are combinatorial objects which prove to be useful, in particular to study laminations. They are the main tool to construct essential laminations in the works of D Gabai, U Oertel, A. Hatcher, T Li or C Delman and R Roberts for instance. One of the most striking topological results is Theorem 1 of Gabai–Oertel [3].

Theorem 1 (Gabai–Oertel [3]) *If a compact orientable 3 dimensional manifold M admits an essential lamination, then its universal cover is homeomorphic to $\mathbb{R}^3$.*

The characterisation of the branched surfaces fully carrying an essential lamination is now known, after works of D Gabai and U Oertel and of T Li.

Theorem 2 (Gabai–Oertel [3]) *A lamination is essential if and only if it is fully carried by an essential branched surface.*

Theorem 3 (Li [4]) *Let M be a closed and orientable manifold. Then every laminar branched surface in M fully carries an essential lamination, and any essential lamination which is not a lamination by planes is fully carried by a laminar branched surface.*

D Gabai and U Oertel gave a number of necessary and sufficient conditions for a branched surface to fully carry an essential lamination, assuming that this branched surface already fully carries a lamination. The important contribution of T Li was to give a sufficient condition for this kind of branched surface to fully carry a lamination, highlighting the importance of the following problem of D Gabai [3, Problem 3.4] and [2, Problem 2.1]: *when does a branched surface fully carry a lamination?*

This question is complex, as shown by L Mosher's theorem.

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Theorem 4 (L Mosher) *The problem of whether or not a general branched surface abstractly carries a lamination is algorithmically unsolvable.*

Let us give brief explanations of the terms “general branched surface” and “abstractly carries” : the branched surfaces we will use in this text are by definition embedded in a 3–manifold. However, a general branched surface could be defined the same way, but without assuming it is embedded or even immersed in a 3–manifold. In [1] J Christy gives necessary and sufficient conditions for a general branched surface to be immersed or embedded in a 3–manifold, and some examples. “Abstractly carrying” a lamination is the generalization for general branched surfaces of “fully carrying” a lamination. Precise definitions can be found in Mosher–Oertel [5]. A proof of Theorem 4 is given in [2].

The goal of this article is to prove the following result, which is a partial answer to the question of D Gabai.

Theorem 5 *Let M be an oriented manifold of dimension 3, without boundary. Let $\mathcal{B}$ be an orientable branched surface of M without twisted curve. Then $\mathcal{B}$ fully carries a lamination.*

A corollary easily comes from this theorem.

Corollary 6 *Let M be an oriented manifold of dimension 3, without boundary. Let $\mathcal{B}$ be an orientable branched surface of M without twisted curve homotopic to zero in M. Then the lift of $\mathcal{B}$ in the universal cover $\widetilde{M}$ of M fully carries a lamination.*

This result is almost optimal in the following sense : the existence of a twisted curve homotopic to zero implies the existence of a closed curve homotopic to zero and transverse to $\mathcal{B}$. But, according to point (4) of [3, Lemma 2.7], if $\mathcal{B}$ fully carries an essential lamination, such a closed curve cannot exist. The condition “there is no twisted curve homotopic to zero” is then sufficient for the lift of $\mathcal{B}$ in the universal cover of M to fully carry a lamination, but it is also necessary for this lift to fully carry an essential lamination.

This sufficient condition appeared when investigating on the notion of contact structure carried by a branched surface. There is some hope to use this criterion in the study of contact structures via branched surfaces.

The basic definitions about branched surfaces, surfaces of contact and twisted curves are given in Section 2. The principle of the proof of Theorem 5 is the same as the one of the construction of a lamination whose holonomy is strictly negative, in Oertel–Światkowski [7, Section 4]. We will build a resolving sequence of splittings, whose inverse limit induces a null holonomy lamination on the fibred neighbourhood of the neighbourhood of the 1–skeleton of some cell decomposition into disks and half-planes of $\mathcal{B}$. Splittings, resolving sequences and inverse limits are introduced in Section 3. Theorem 5 is then proved in Section 4. The last section contains some remarks about the question of D Gabai.

Finally, I'd like to thank U Oertel and J Światkowski for having found a mistake in an optimistic version of this text, in which I thought I had answered the question of D Gabai.

2 Branched surfaces

Throughout this article, M is a 3 dimensional oriented manifold without boundary. It will be supposed paracompact (because of Remark 2.1.3) and separated. Its universal cover is denoted $\widetilde{M}$.

2.1 First definitions

Definition 2.1.1 A *branched surface* $\mathcal{B}$ in M is a union of smooth surfaces locally modeled on one of the three models of Figure 2.1.2. The *singular locus* $\mathcal{L}$ of $\mathcal{B}$ is the set of points, called *branch points*, none of whose neighbourhoods is a disk. Its *regular part* is $\mathcal{B}\setminus\mathcal{L}$. The closure of a connected component of the regular part is called a *sector* of $\mathcal{B}$.

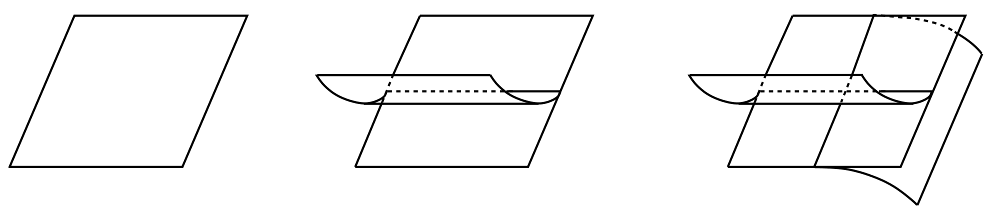

Figure 2.1.2: Local models of a branched surface.

The singular locus may have double points, as it is the case in the third model of figure Figure 2.1.2.

Remark 2.1.3 According to the local models, the double points are isolated, and since M is paracompact, they are countable.

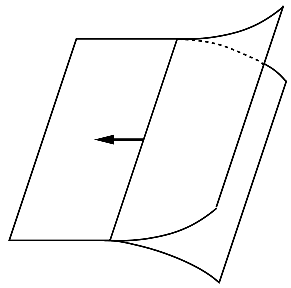

Figure 2.1.4: Branch direction.

At each regular point of $\mathcal{L}$, we can define a *branch direction* as in Figure 2.1.4.

Definition 2.1.5 A *fibred neighbourhood* $N(\mathcal{B})$ of $\mathcal{B}$ is an interval "bundle" over $\mathcal{B}$, as seen on Figure 2.1.6. The boundary of $N(\mathcal{B})$ can be decomposed into an *horizontal boundary* $\partial_h N(\mathcal{B})$ transverse to the fibres and a *vertical boundary* $\partial_v N(\mathcal{B})$, tangent to the fibres (see Figure 2.1.6(a)).

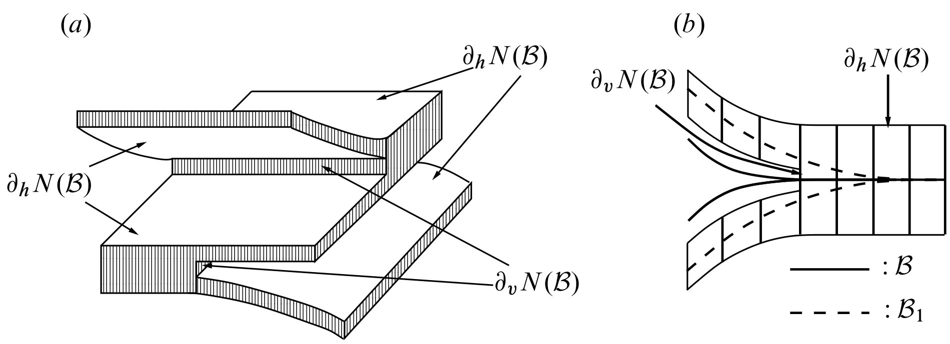

Figure 2.1.6: Fibred neighbourhood of $\mathcal{B}$.

We define the projection map $\pi\colon N(\mathcal{B}) \to \mathcal{B}$ which sends a fibre of $N(\mathcal{B})$ onto its base point. In particular, $\pi(\partial_v N(\mathcal{B})) = \mathcal{L}$. We can also consider $N(\mathcal{B})$, not as an abstract bundle but rather as a part of M, and in this case $\mathcal{B}$ is not included in $N(\mathcal{B})$. However, $N(\mathcal{B})$ contains a branched surface $\mathcal{B}_1$ which is isomorphic to $\mathcal{B}$ (see Figure 2.1.6(b)). The branched surface $\mathcal{B}_1$ is a *splitting* of $\mathcal{B}$ (splittings will be defined in Section 3).

Let's see how we can put a sign on each double point of $\mathcal{L}$. Locally, two smooth parts of $\mathcal{L}$ run through p. They are cooriented by their branch direction, and we call them $\mathcal{L}_1$ and $\mathcal{L}_2$.

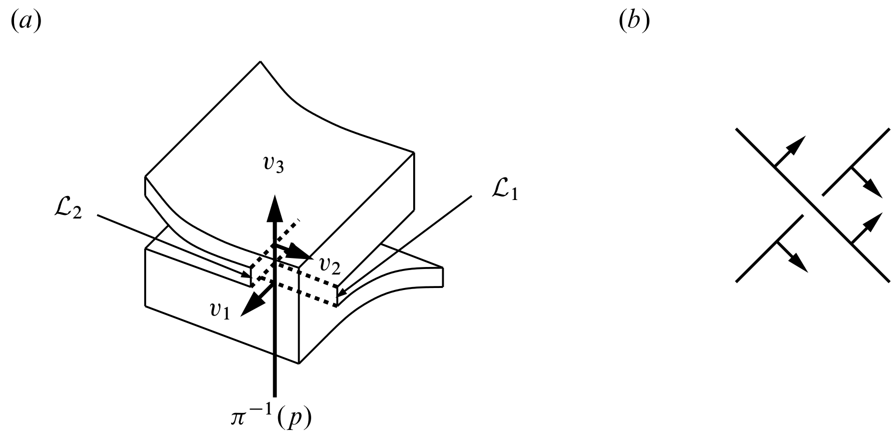

Figure 2.1.7: (a) : $\{v_1, v_2, v_3\}$; (b) positive double point.

Set an orientation of the fibre of $N(\mathcal{B})$ passing through p. Hence, it makes sense to say that one of the branching $\mathcal{L}_1$ or $\mathcal{L}_2$ is over the other at p. Say for example that $\mathcal{L}_1$ is under $\mathcal{L}_2$. Let v_1 be a vector of T_pM defining the branch direction of $\mathcal{L}_1$ at p, and v_2 be a vector of T_pM defining the branch direction of $\mathcal{L}_2$ at p. At last, let v_3 be a vector giving the chosen orientation of the fibre of $N(\mathcal{B})$ passing through p, as seen on Figure 2.1.7(a). We then call p a *positive double point* (resp. *negative double point*) if the base $\{v_1, v_2, v_3\}$ of T_pM is positively (resp. negatively) oriented with respect to the orientation of M. With this convention, the positive double points will be drawn in the plane as on the diagram (b) of Figure 2.1.7.

Remark 2.1.8 The sign of a double point depends on the orientation of M: if this one is reversed, the signs of the double points are reversed as well. Though, this sign is independent of the chosen orientation of the fibre passing through the double point in the preceding definition.

Definition 2.1.9 A codimension 1 *lamination* in a dimension 3 (resp. 2) manifold M is the decomposition of a closed subset λ of M into injectively immersed surfaces (resp. curves) called *leaves*, such that λ is covered by charts of the form $(0,1)^2 \times I$ (resp. $(0,1) \times I$) in which the leaves have the form $(0,1)^2 \times \{\text{point}\}$ (resp. $((0,1) \times \{\text{point}\})$.

Definition 2.1.10 A branched surface $\mathcal{B}$ *carries a lamination* λ of codimension 1 if λ is contained in a fibred neighbourhood of $\mathcal{B}$ and if its leaves are transverse to the fibres. We say that λ is *fully carried* if moreover it meets all the fibres.

2.2 Surfaces of contact

Let $\mathcal{B}$ be a branched surface.

Definition 2.2.1 A *surface of contact* is the immersion of a surface $\mathcal{S}$ in $\mathcal{B}$, such that the boundary of $\mathcal{S}$ is sent onto smooth circles of the singular locus of $\mathcal{B}$, and such that the branch directions along these boundary components point into $\mathcal{S}$. If we consider a lift of $\mathcal{S}$ into $N(\mathcal{B})$, we see that the existence of such a surface is equivalent to the existence of an immersion $f\colon \mathcal{S} \to N(\mathcal{B})$ satisfying:

(i) $f(Int(\mathcal{S})) \subset \mathrm{Int}(N(\mathcal{B}))$ and is transverse to the fibres ;

(ii) $f(\partial\mathcal{S}) \subset \mathrm{Int}(\partial_v N(\mathcal{B}))$ and is transverse to the fibres.

Hence, the expression *surface of contact* will be used for both definitions.

An example is given in Figure 2.2.4 (a).

Remark 2.2.2 In general, a surface of contact is not a sector, but a union of sectors. The singular locus of the branched surface may meet the interior of the surface of contact. The same is true for the sink surfaces and the twisted surfaces of contact defined further.

Definition 2.2.3 A *sink surface* is the immersion of a surface $\mathcal{S}$ in $\mathcal{B}$, such that the boundary of $\mathcal{S}$ is sent onto piecewise smooth circles of the singular locus of $\mathcal{B}$, at least one of whose is not smooth, and such that the branch directions along these boundary components point into $\mathcal{S}$. A double point in the boundary of $\mathcal{S}$ which is the intersection of two smooth components of the boundary of $\mathcal{S}$ is called a *corner* of $\mathcal{S}$. A sink surface has thus at least one corner. Equivalently, if we consider a non smooth lift of $\mathcal{S}$ into $N(\mathcal{B})$, we can say that a sink surface is an immersion $f\colon \mathcal{S} \to N(\mathcal{B})$ satisfying :

(i) $f(\mathrm{Int}(\mathcal{S})) \subset \mathrm{Int}(N(\mathcal{B}))$ and is transverse to the fibres ;

(ii) $f(\partial\mathcal{S})$ is included in $\mathrm{Int}(\partial_v N(\mathcal{B}))$ except in a finite and non empty number of closed intervals $C_1, \dots, C_k$. Outside these C_i, $f(\partial\mathcal{S})$ is transverse to the fibres of $\partial_v N(\mathcal{B})$. Each C_i is included in a fibre of $N(\mathcal{B})$ corresponding to a double point of $\mathcal{L}$, and must intersect $\mathrm{Int}(N(\mathcal{B}))$. Thus, $\pi(f(\partial\mathcal{S}))$ is not smooth. The C_i s are called the *corners* of $\mathcal{S}$.

An example is given in Figure 2.2.4 (b).

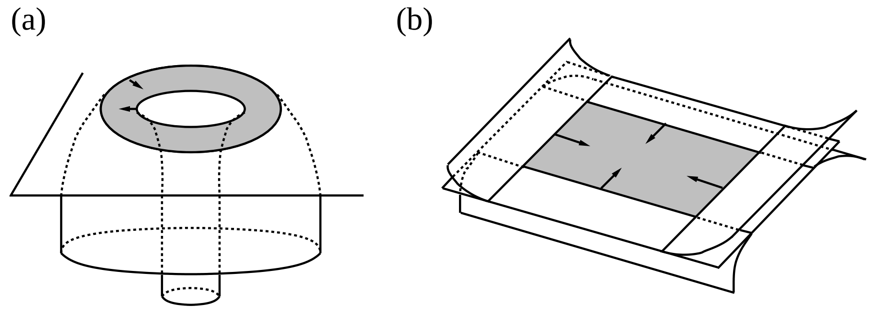

Figure 2.2.4: Annulus of contact and sink disk.

Definition 2.2.5 A *twisted surface of contact* is a sink surface whose corners, which are double points, have the same sign, and which satisfies, for some Riemannian metric for which $\mathcal{L}$ in the neighbourhood of a double point cuts $\mathcal{B}$ into four sectors of angle $\pi/2$, the corners of a twisted surface of contact are all of angle $\pi/2$. The case of a corner of angle $3\pi/2$ is forbidden. A twisted surface of contact is *positive* (resp. *negative*) if all its corners are positive (resp. negative).

An example is given in Figure 2.2.6.

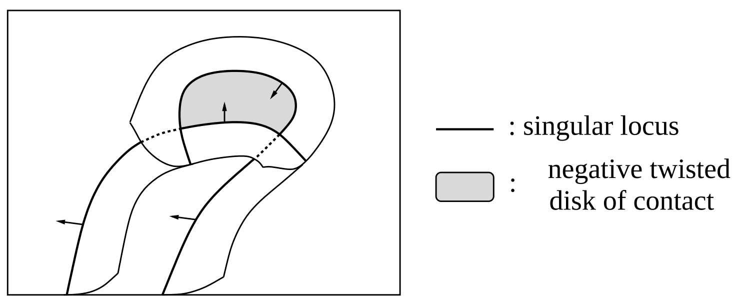

Figure 2.2.6: Negative twisted disk of contact.

Remark 2.2.7 It is well-known that the existence of a twisted disk of contact which is a sector is an obstruction to the existence of a lamination fully carried. This fact is stated in proposition Proposition 4.3.8, and then proved, by using *train tracks*.

2.3 Twisted curves

Definition 2.3.1 A branched surface $\mathcal{B}$ is said *orientable* if there exists a global orientation of the fibres of a fibred neighbourhood $N(\mathcal{B})$ of $\mathcal{B}$.

Remark 2.3.2 An orientable branched surface cannot have any *monogon*, ie a disk $D \subset M \setminus Int(N(\mathcal{B}))$ with $\partial D = D \cap N(\mathcal{B}) = \beta \cup \delta$, where $\beta \subset \partial_v N(\mathcal{B})$ lies in a fibre of $\partial_v N(\mathcal{B})$ and $\delta \subset \partial_h N(\mathcal{B})$ (see Figure 2.3.3).

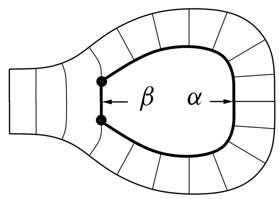

Figure 2.3.3: Monogon.

Throughout this section, $\mathcal{B}$ will now be an orientable branched surface, and $N(\mathcal{B})$ a fibred neighbourhood of $\mathcal{B}$, with a fixed orientation of the fibres.

Definition 2.3.4 A *positive* (resp. *negative*) *twisted curve* γ is an oriented closed curve, immersed in $\mathcal{B}$ and which satisfies:

(i) γ is included in $\mathcal{L}$, and is then a finite union of smooth segments of $\mathcal{L}$. When, along γ, we pass from a smooth segment to another one through a double point of $\mathcal{L}$, this double point is called a *corner* of γ;

(ii) γ has at least one corner;

(iii) at a corner, γ passes from a smooth segment I_1 of $\mathcal{L}$ to a smooth segment I_2 of $\mathcal{L}$. Since $\mathcal{B}$ is oriented, one of these segments is over the other one. If I_2 is over I_1, we then say that the corner is *ascending*, otherwise it is *descending*. The sign of a corner as a double point does not determine whether it is ascending or descending. We then demand all the corners of γ to be ascending (resp. descending).

The existence of a twisted curve γ is equivalent to the existence in $N(\mathcal{B})$ of a curve, still denoted γ, which can be decomposed into a union $\gamma = \gamma_l \cup \gamma_c$, where γ_l and γ_c verify:

(i) γ_l is the *smooth part* of γ : it is a finite union of segments included in $\partial_v N(\mathcal{B})$ and transverse to the fibres of $N(\mathcal{B})$;

(ii) γ_c is a finite and non empty union of segments denoted C_i, $i = 1, \dots, n$, where each C_i is included in a fibre of $N(\mathcal{B})$, in such a way that the orientation of C_i, induced by the one of γ, coincides with (resp. is opposite to) the orientation of this fibre. The C_i 's are at the corners of γ. They are said ascending (resp. descending) if γ is positive (resp. negative).

Remark 2.3.5 If we reverse the orientation of a positive twisted curve, we get a negative twisted curve. The converse is also true.

Remark 2.3.6 The boundary of a twisted surface of contact is a twisted curve.

Definition 2.3.7 A *simple* twisted curve is a twisted curve such that the complement of the corners is embedded in $\mathcal{B}$. Otherwise said, only the corners are of multiplicity 2 or more.

Lemma 2.3.8 *Let γ be a twisted curve. Then there exists a simple twisted curve δ included in γ.*

Proof We also denote $\gamma\colon \mathbb{S}^1 \to \mathcal{B}$ the immersion of the twisted curve γ, $\mathbb{S}^1$ being oriented. We can always suppose that γ is positive. If γ is simple, we obviously have $\delta = \gamma$, otherwise there exist two segments of $\mathbb{S}^1$ denoted $J = [a, b]$ and $K = [c, d]$, whose interiors are disjoint, and whose orientation is the one induced by the orientation of $\mathbb{S}^1$, and such that the images $\gamma(J)$ and $\gamma(K)$ coincide. If $\gamma(J) = \gamma(\mathbb{S}^1)$, we then set $\delta_1 = \gamma(J)$, which is a twisted curve, otherwise the segments J and K are chosen to be maximal, in the sense that for every sufficiently small neighbourhood $\mathcal{V}(J)$ of J in $\mathbb{S}^1$ and every sufficiently small neighbourhood $\mathcal{V}(K)$ of K in $\mathbb{S}^1$, we have $\gamma(\mathcal{V}(J)) \not\subset \gamma(\mathcal{V}(K))$ and $\gamma(\mathcal{V}(K)) \not\subset \gamma(\mathcal{V}(J))$. This means that $\gamma(\mathcal{V}(J))$ and $\gamma(\mathcal{V}(K))$ split at $\gamma(a)$ and at $\gamma(b)$. Thus, the point $A = \gamma(a)$ of $\mathcal{L}$ is a corner of $\gamma(\mathcal{V}(J))$ or of $\gamma(\mathcal{V}(K))$, and it is the same for $B = \gamma(b)$.

We then distinguish two cases.

(i) The orientations of $\gamma(J)$ and $\gamma(K)$ coincide, that is $\gamma(c) = \gamma(a) = A$ and $\gamma(d) = \gamma(b) = B$.

The point B is a corner of $\gamma(\mathcal{V}(J))$ or of $\gamma(\mathcal{V}(K))$. For example, let us suppose it is a corner of $\gamma(\mathcal{V}(J))$. We then set $\delta_1 = \gamma([a, c])$. Since $\gamma(c) = \gamma(a)$, this curve is closed. Since $c \notin \mathrm{Int}(J)$, we have $J = [a, b] \subset [a, c]$, and hence B is a corner of δ_1, because it is a corner of $\gamma(\mathcal{V}(J))$. At last, all the corners of δ_1 are ascending since $[a, c]$ is oriented after $\mathbb{S}^1$, and all the corners of γ are ascending. Therefore, δ_1 is a twisted curve.

(ii) The orientations of $\gamma(J)$ and $\gamma(K)$ are opposite, that is $\gamma(d) = \gamma(a) = A$ and $\gamma(c) = \gamma(b) = B$.

We then set $\delta_1 = \gamma([c,b])$. Since $\gamma(c) = \gamma(b)$, this curve is closed. Since B is a corner of $\gamma(\mathcal{V}(J))$ or of $\gamma(\mathcal{V}(K))$, B is a corner of δ_1. At last, all the corners of δ_1 are ascending since $[c,b]$ is oriented after $\mathbb{S}^1$, and all the corners of γ are ascending. Therefore, δ_1 is a twisted curve.

In every case, γ contains a positive twisted curve δ_1 which has strictly less corners (counted with multiplicity) than γ. If δ_1 is not simple, we iterate the previous construction to δ_1, and we get a positive twisted curve δ_2, having strictly less corners than δ_1. Since γ is closed, it is compact and has a finite number of corners, even with multiplicity. Therefore, in a finite number of steps, we get a positive simple twisted curve included in γ. □

The following corollary follows easily.

Corollary 2.3.9 *A branched surface without simple twisted curve does not have any twisted curve at all.*

3 Splittings

3.1 Definitions

Definition 3.1.1 Let $\mathcal{B}$ and $\mathcal{B}'$ be two branched surfaces in M. We say that $\mathcal{B}$' is a *splitting* of $\mathcal{B}$ if there exists a fibred neighbourhood $N(\mathcal{B})$ of $\mathcal{B}$ and an I–bundle J in $N(\mathcal{B})$, over a subsurface of $\mathcal{B}$, such that:

(i) $N(\mathcal{B}) = N(\mathcal{B}') \cup J$;

(ii) $J \cap N(\mathcal{B}') \subset \partial J$;

(iii) $\partial_h J \subset \partial_h N(\mathcal{B}')$;

(iv) $\partial_v J \cap N(\mathcal{B}')$ is included in $\partial_v N(\mathcal{B}')$, has finitely many components, and their fibres are fibres of $\partial_v N(\mathcal{B}')$.

Remark 3.1.2 *When $\mathcal{B}'$ is a splitting of $\mathcal{B}$, the following notation will be used: $\mathcal{B}' \xrightarrow{p} \mathcal{B}$. Actually, $\mathcal{B}'$ is included in a fibred neighbourhood $N(\mathcal{B})$ of $\mathcal{B}$, endowed with a projection π on $\mathcal{B}$, and the restriction p of π to $\mathcal{B}'$ is the projection we wanted.*

Definition 3.1.3 Let $\mathcal{B}$ be a branched surface. Let Σ be a sector of $\mathcal{B}$ whose boundary contains a smooth part α of $\mathcal{L}$ and whose branching direction points into Σ. Let $\gamma\colon I \to \Sigma$ be an embedded arc in Σ such that $\gamma(0) \in \alpha$ and $\gamma(t) \in \mathrm{Int}(\Sigma)$ for $t \neq 0$. A *splitting along* γ is a branched surface $\mathcal{B}'$ defined as in Definition 3.1.1, where J is an I–bundle over a tubular neighbourhood of γ in Σ (see Figure 3.1.4).

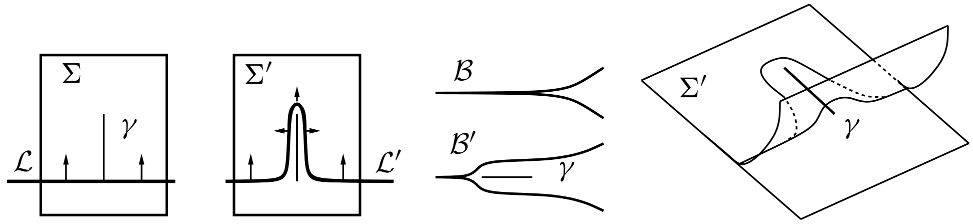

Figure 3.1.4: Splitting along γ.

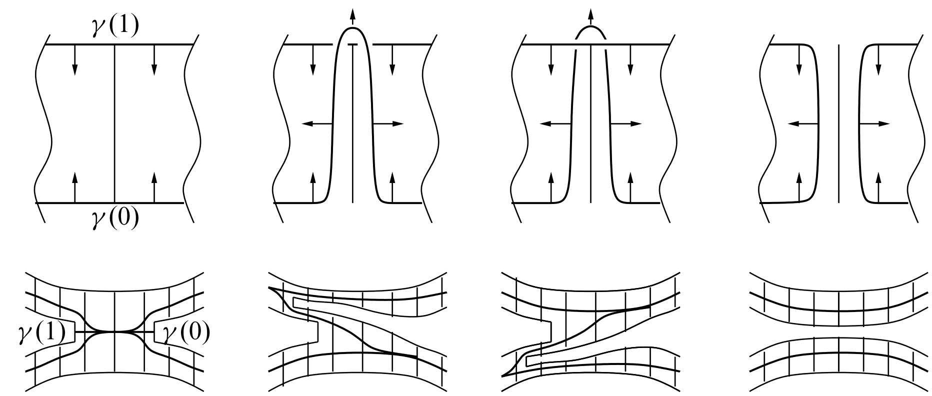

Figure 3.1.5: Over, under and neutral splittings.

Definition 3.1.6 We keep the notations of definition 3.1.3. Suppose now that $\gamma(1)$ is in $\mathcal{L}$, in a point where the branching direction points into Σ as well.

We then say that γ is in *face-to-face* position. If an orientation of the fibres of $N(\mathcal{B})$ along γ is chosen, there are three possible splittings along γ: the *over splitting*, the *under splitting* and the *neutral splitting*, drawn in Figure 3.1.5.

Remark 3.1.7 If Σ is a non compact sector, a splitting can be performed along a non compact arc $\gamma\colon [0,1) \to \Sigma$, verifying the same conditions as in Definition 3.1.3. This splitting can be seen as a neutral splitting “at infinity".

Remark 3.1.8 It is possible to perform a splitting along an arc γ which comes from a sector to another one through the singular locus in the branch direction. In this case, there is only one possible splitting, called a *backward splitting* (see Figure 3.1.9).

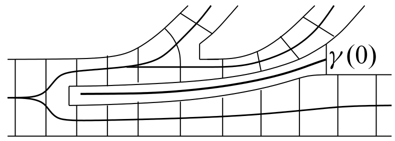

Figure 3.1.9: Backward splitting.

3.2 Inverse limit of a sequence of splittings

Definition 3.2.1 Let $\mathcal{B}$ be a branched surface. A *sequence of splittings* is a sequence $\cdots \mathcal{B}_{k+1} \overset{p_{k+1}}{\to} \mathcal{B}_k \overset{p_k}{\to} \cdots \overset{p_2}{\to} \mathcal{B}_1 \overset{p_1}{\to} \mathcal{B} = \mathcal{B}_0$ of branched surfaces $(\mathcal{B}_i)_{i\in\mathbb{N}}$ such that:

(i) for all i, $\mathcal{B}_{i+1}$ is a splitting of $\mathcal{B}_i$;

(ii) for all i, $\mathcal{B}_i$ is endowed with a fibred neighbourhood $N(\mathcal{B}_i)$, and those fibred neighbourhoods are such that $N(\mathcal{B}_{i+1})$ is fiberwise contained in $N(\mathcal{B}_i)$.

Thus, the fibres of each $N(\mathcal{B}_i)$ are closed subintervals of the fibres of $N(\mathcal{B})$.

For such a sequence, we denote, for all $k \geq 1$:

$$P_k = p_1 \circ p_2 \circ \cdots \circ p_k = \pi|_{B_k} \colon \mathcal{B}_k \to \mathcal{B}$$

the projection from $\mathcal{B}_k$ onto $\mathcal{B}$. We will also denote by $\pi_n\colon N(\mathcal{B}_n) \to \mathcal{B}_n$ the projection along the fibres from $N(\mathcal{B}_n)$ to $\mathcal{B}_n$.

The following definition is inspired by [5].

Definition 3.2.2 A sequence of splittings $\cdots \mathcal{B}_{k+1} \overset{p_{k+1}}{\to} \cdots \overset{p_1}{\to} \mathcal{B} = \mathcal{B}_0$ is called *resolving* if it satisfies:

(i) there exist points of $\mathcal{B}$ denoted $(x_i)_{i\in\mathbb{N}}$, a real number $\rho > 0$ and disks embedded in $\mathcal{B}$ denoted $(d_i)_{i\in\mathbb{N}}$, centred at x_i and of radius ρ for some metric on $\mathcal{B}$, such that the d_i cover $\mathcal{B}$;

(ii) for all integer i, there exists a subsequence $(\mathcal{B}_{\varphi_i(n)})_{n\in\mathbb{N}}$ such that the branch loci of the branched surfaces of this subsequence do not intersect $\pi^{-1}(d_i)$. That is, for all k, $\mathcal{B}_{\varphi_i(k)}$ does not have any branching over d_i: the branch points over d_i have been *resolved*, and $P^{-1}_{\varphi_i(k)}(d_i)$ is thus a union of disjoint disks.

When such a sequence exists, we say that $\mathcal{B}$ *admits a resolving sequence of splittings*.

Remark 3.2.3 In particular, a branched surface admitting a resolving sequence of splittings is *fully splittable* in the sense of Gabai and Oertel [3].

Lemma 3.2.4 (Gabai–Oertel [3], Mosher–Oertel [5]) *Let $\mathcal{B}$ be a branched surface admitting a resolving sequence of splittings. Then $\mathcal{B}$ fully carries a lamination.*

Proof The proof can be found in Mosher and Oertel [5, pages 84–85].

Let us first deal with the case where $\mathcal{B}$ is connected. Let $\cdots\mathcal{B}_{k+1} \overset{p_{k+1}}{\to} \cdots \overset{p_1}{\to} \mathcal{B} = \mathcal{B}_0$ be a resolving sequence. Let us define $\lambda = \bigcap_{n\in\mathbb{N}} N(\mathcal{B}_n)$. As an intersection of closed subsets, λ is closed. We will now find an adapted atlas, whose charts will be the $\pi^{-1}(d_i)$, where the d_i are the disks from point (i) Definition 3.2.2. Let $i \in \mathbb{N}$, and $y \in d_i$. Then $\lambda \cap \pi^{-1}(y)$ is some closed subset T in $[0,1]$. The sequence of splittings being resolving, let us consider the subsequence $(\mathcal{B}_{\varphi_i(n)})_{n\in\mathbb{N}}$ from point (ii) of Definition 3.2.2. Since the $N(\mathcal{B}_n)$ form a decreasing sequence of closed subsets, we get: $\lambda = \bigcap_{n\in\mathbb{N}} N(\mathcal{B}_{\varphi_i(n)})$. But, according to point (ii) of Definition 3.2.2, for all y in d_i and for all integer n, $P^{-1}_{\varphi_i(n)}(x_i) = P^{-1}_{\varphi_i(n)}(y)$, when thought of as closed subsets of some transversal to the chart. Hence, for all i, $\lambda \cap \pi^{-1}(d_i)$ is topologically the product $d_i \times T$, since $\mathcal{B}$ is connected. If the transversal T contains a maximal interval I_T whose interior is non empty, we remove $Int(I_T)$ from T. We then reduce T to a transversal $T' = T \setminus \mathrm{Int}(T) = \partial T$, whose interior is empty, and which is totally discontinuous. Hence $\pi^{-1}(d_i)$ is a laminated chart, the leaves being the $\{t\} \times d_i$, for $t \in T'$. The set $\lambda' = \cup_{i\in\mathbb{N}}(d_i \times T')$ is a lamination. Moreover, λ' meets all the fibres of $N(\mathcal{B})$ transversally. If $\mathcal{B}$ is not connected, we can apply the previous proof to each connected component of $\mathcal{B}$. □

Definition 3.2.5 Let $\cdots\ \mathcal{B}_{k+1} \overset{p_{k+1}}{\to} \cdots \overset{p_1}{\to} \mathcal{B} = \mathcal{B}_0$ be a resolving sequence of splittings. The fully carried lamination $\lambda' = \cup_{i\in\mathbb{N}}(d_i \times T')$ defined in the previous proof is called the *inverse limit* of this sequence of splittings.

4 Proof of Theorem 5

Let $\mathcal{B}$ be a branched surface satisfying the hypotheses of Theorem 5.

4.1 Cell decomposition of $\mathcal{B}$

We will give $\mathcal{B}$ the structure of a 2–complex, following [7, Section 4]. We choose this structure such that the 0–skeleton contains the double points of $\mathcal{L}$, and such that $\mathcal{L}$ is a subset of the 1–skeleton. Hence, each connected component of the complement of the 1–skeleton in $\mathcal{B}$ is contained in a sector. Furthermore, by adding vertices to the 0–skeleton and edges to the 1–skeleton, this 2–complex structure can be made to satisfy two additional properties:

- no edge is a loop (ie its two ends are distinct);
- each connected component of the complement of the 1–skeleton in $\mathcal{B}$ is a disk or a half-plane, whether it is compact or not.

We denote Y the obtained decomposition, which is a cell-decomposition.

Remark 4.1.1 The “boundary of a 2–cell" is not the topological boundary, but the combinatorial one. An edge can be found twice, with different orientations, in the boundary of the same 2–cell.

4.2 First splitting

The first step is to perform a first splitting of $\mathcal{B}$, denoted $\mathcal{B}_1$, which is fully carried by $N(\mathcal{B})$, as in definition Definition 2.1.5. Let us describe it more precisely.

Let us fix some metric on M. It induces a metric on $\mathcal{B}$. Let ε be a non negative real number, such that the edges of Y are all strictly longer than 5ε (we shall see why later), and such that the ε–tubular neighbourhood of $\mathcal{L}$ in M is regular. The intersection of $\mathcal{B}$ with this ε–neighbourhood is the union of $\mathcal{L}$ and of two other parts, which meet together at the double points: one part lies behind $\mathcal{L}$, for the coorientation of $\mathcal{L}$ given by the branch directions, and the other part, denoted $T_{\mathcal{L}}$, lies in front of $\mathcal{L}$. The boundary of $T_{\mathcal{L}}$ is included in the union of $\mathcal{L}$ with a parallel copy of $\mathcal{L}$, called $\mathcal{L}_1$. It is just “included in” and not “equal to” this union, because of what happens at the double points. The first splitting is a splitting over $T_{\mathcal{L}}$, which means that we remove from $N(\mathcal{B})$ an I–bundle over $T_{\mathcal{L}}$. The branched surface $\mathcal{B}_1$ we get is isomorphic to $\mathcal{B}$, and its singular locus is $\mathcal{L}_1$ (see Figure 4.2.1).

The trace of $\pi^{-1}(\mathcal{L})$ on $\mathcal{B}_1$ (ie the intersection of $\pi^{-1}(\mathcal{L})$ and $\mathcal{B}_1$) is made of two copies of $\mathcal{L}$, drawn on two different sectors (at least locally), as seen on Figure 4.2.1. The trace of $\pi^{-1}(Y)$ on $\mathcal{B}_1$, denoted Y_1, is then more complicated than a cell decomposition into disks and half-planes, since some of the cells and some of the edges are branched. But all these branchings lie in a closed ε–neighbourhood of $\mathcal{L}_1$, and $\mathcal{B}_1$ minus a closed ε–neighbourhood of the 1–skeleton of Y_1 is the same union of disks and half-planes as $\mathcal{B}$ minus the 1–skeleton of Y.

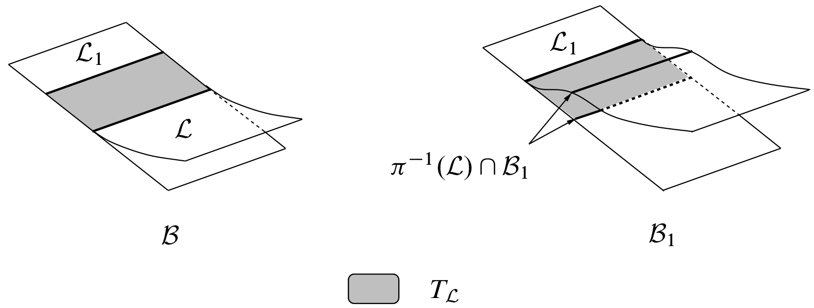

Figure 4.2.1: First splitting.

4.3 Train tracks

Each 2–cell Σ of Y inherits from $N(\mathcal{B})$ an interval bundle $N(\Sigma)$ built in the following way: we denote by $N(\mathrm{Int}(\Sigma))$ the set of all the fibres of $N(\mathcal{B})$ whose base point lies in $Int(\Sigma)$ and we set $N(\Sigma) = \overline{N(Int(\Sigma))}$.

The boundary of this $N(\Sigma)$ can be decomposed into an horizontal boundary (included in $\partial_h N(\mathcal{B})$) and a vertical boundary (not included in $\partial_v N(\mathcal{B})$). Since all the compact 2–cells of Y are disks and are orientable, the vertical boundary of $N(\Sigma)$, denoted $\partial_v N(\Sigma)$, is in fact of the form $\mathbb{S}^1 \times I$. For the non compact 2–cells, the vertical boundary is of the form $\mathbb{R} \times [0,1]$. For each 2–cell Σ, let us look at the trace of $\mathcal{B}_1$ on $\partial_v N(\Sigma)$, which is also the boundary of $\Sigma_1 = P_1^{-1}(\Sigma)$. It is a *train track*, ie a branched curve fully carried by $\partial_v N(\Sigma)$. This train track does not have a boundary and avoids the trace of $\partial_v N(\mathcal{B})$ on $\partial_v N(\Sigma)$. It is compact if and only if Σ is compact. Figure 4.3.1 shows two examples of compact train tracks.

An orientation of a 2–cell induces an orientation of its boundary. The corresponding train track is then oriented as well. For each 2–cell Σ, we set an orientation of the fibres of $\partial_v N(\Sigma)$. We introduce the following definitions.

Definition 4.3.2 A branching of a train track is called *direct* when a track followed in the direct way divides itself into two tracks at this branching, and it is called *backward* when two tracks followed in the positive direction meet at this branching.

We can go a bit further in the classification of the branchings of a train track .

Definition 4.3.3 Let $\mathcal{V}$ be an oriented compact train track without boundary, fully carried by a trivial bundle $\mathbb{S}^1 \times [0,1]$. We set an orientation of the fibres. Let $\mathcal{C}$ be a

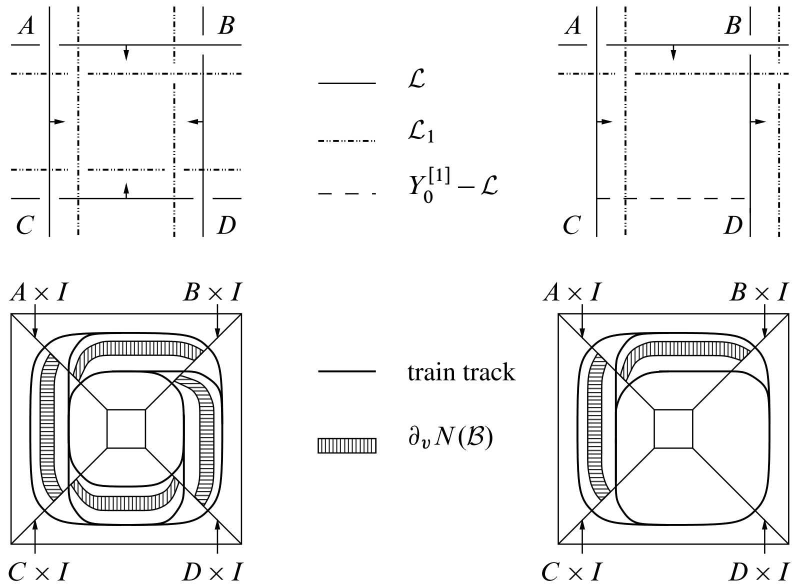

Figure 4.3.1: Train tracks.

smooth closed curve of $\mathcal{V}$. It cuts $\mathbb{S}^1 \times [0,1]$ into two parts: $(\mathbb{S}^1 \times [0,1])^+$, containing the points which lie over $\mathcal{C}$ for the orientation of the fibres and $(\mathbb{S}^1 \times [0,1])^-$ containing the points which lie under. A branching along a smooth closed curve $\mathcal{C}$ is called an *over branching* (resp. *under branching*) if the branch which leaves or meets $\mathcal{C}$ there lies in $(\mathbb{S}^1 \times [0,1])^+$ (resp. $(\mathbb{S}^1 \times [0,1])^-$).

We can thus state the following lemma.

Lemma 4.3.4 *Let Σ be a compact 2–cell of Y (Σ is a disk), and Σ_1 be its trace on $\mathcal{B}_1$. Let $\mathcal{V}$ be the train track associated to the boundary of Σ_1. It is an oriented compact train track without boundary fully carried by a bundle $\mathbb{S}^1 \times [0,1]$. We set an orientation of the fibres. The three following assertions are equivalent:*

(i) *when we follow a smooth closed curve of $\mathcal{V}$, either no under branching is met or at least one direct under branching and one backward under branching are met;*

(ii) *$\mathcal{V}$ can be split into a union of smooth circles;*

(iii) *Σ is not a twisted disk of contact.*

Proof (i)$\Rightarrow$(ii) If this is true for each connected component of $\mathcal{V}$, then it is true for $\mathcal{V}$. So we assume $\mathcal{V}$ connected and different from a smooth curve. For $\theta \in \mathbb{S}^1$

we define $\max(\theta) = \max\{t \in [0,1] \mid (\theta, t) \in \mathcal{V}\}$, which is in [0,1], and then we define $\max(\mathcal{V}) = \{(\theta, \max(\theta)), \theta \in \mathbb{S}^1\}$. This $max(\mathcal{V})$ is a smooth circle of $\mathcal{V}$, along which we meet at least one direct under branching and one backward under branching, and no over branching. In particular, there exists an oriented arc $\mathcal{A}$ of $max(\mathcal{V})$, going (for the orientation of $\mathcal{V}$), from a direct branching to a backward branching, with no branching between the two previous ones. Then $\mathcal{V} \backslash \mathcal{A}$ is an oriented compact train track without boundary denoted $\mathcal{V}_1$, fully carried by $\mathbb{S}^1 \times [0,1]$, and $\mathcal{V} = \mathcal{V}_1 \cup max(\mathcal{V})$. Each smooth closed curve of $\mathcal{V}_1$ is a smooth closed curve of $\mathcal{V}$ as well, and its under branchings remain unchanged by the previous splitting. Hence, $\mathcal{V}_1$ satisfies point (i) of the lemma. If $\mathcal{V}_1$ is not a circle, we perform the same operation again using $max(\mathcal{V}_1)$, and after a finite number of steps, we have decomposed $\mathcal{V}$ into a union of smooth circles. An example is shown in Figure 4.3.5.

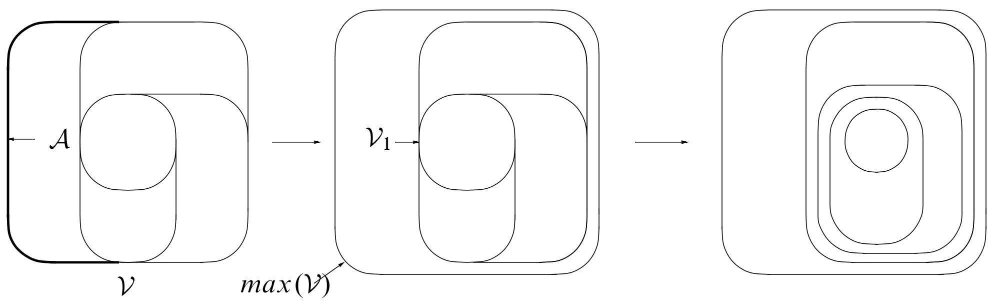

Figure 4.3.5: Splitting of a train track into a union of smooth circles.

$\neg$(i)$\Rightarrow\neg$(ii) Let $\mathcal{C}$ be a smooth closed curve of $\mathcal{V}$ having, for example, only direct under branchings. If we follow $\mathcal{C}$ in the direct way, and if we take a direct under branching, then, whatever the smooth path we follow on $\mathcal{V}$, we will never be able to go on $\mathcal{C}$ again, for it would imply the existence of a backward under branching along $\mathcal{C}$. Thus, no branch leaving $\mathcal{C}$ by a direct under branching is included in smooth closed circle of $\mathcal{V}$, and $\mathcal{V}$ is not a union of smooth circles.

$\neg$(iii)$\Rightarrow\neg$(i) The trace of $\mathcal{B}_1$ when Σ is a twisted disk of contact is always as on Figure 4.3.6, ie it is the union of two smooth circles and of segments joining them at branch points.

The top circle has only under branchings, and it has at least one under branching because a twisted disk of contact has at least one corner. Moreover, these branchings are of one type because the corners of a twisted disk of contact all have the same sign.

$\neg$(i)$\Rightarrow\neg$(iii) Suppose that there exists a closed smooth curve C of $\mathcal{V}$ whose under branchings are all of a single type, for example direct, and which has at least one under

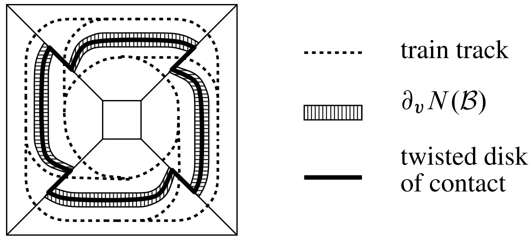

Figure 4.3.6: $\neg(i) \Leftrightarrow \neg(ii)$.

branching. Look at the trace of $\partial_v N(\mathcal{B})$ on $\partial_v(\Sigma)$. Each of its connected components has a vertical boundary with two connected components and a horizontal boundary, also with two connected components. Each component of the vertical boundary is included in a fibre of $\partial_v N(\Sigma)$ whose base point is a branch point of $\mathcal{B}$. Stand at a point p on C, and follow C in the direct way. When we meet the first branch point p_1, C divides into two branches: the top branch passes over a component b_1 of $\partial_v N(\mathcal{B})$, and the bottom branch passes under b_1. We go on until we meet the fibre where b_1 ends, whose base point is some branch point p_2. If p_2 is not a double point, then the branch of $\mathcal{V}$ which is over b_1 joins the branch which is under b_1. But these two branches are the two previous branches, and that would imply that there is a backward branching on C. Hence p_2 is a double point. At p_2, there are thus two branchings, one is direct and the other is backwards. One of them is on C, so this is the direct one. Again, C divides into two branches which surrounds another component b_2 of the trace of $\partial_v N(\mathcal{B})$. Since this branching is direct, b_2 lies over b_1 at p_2. We carry on following C until we return at p_1. We have then met k components $b_1, \ldots, b_k$ of $\partial_v N(\mathcal{B})$ and k double points $p_1, \ldots, p_k$. Each b_i goes from p_i to p_{i+1} for $i = 1, \ldots, k$ modulo k. At p_i, b_{i-1} lies under b_i, for all i. As a result, all the double points have the same sign, and Σ is a twisted disk of contact with k corners. □

Remark 4.3.7 In the proof of point (i) $\Rightarrow$ (ii), we could also define $\min(\mathcal{V})$ in the same way as $\max(\mathcal{V})$, and show that points (ii) and (iii) are equivalent to a point (i'): when we follow a smooth closed curve of $\mathcal{V}$, either we meet no over branching, or we meet at least one direct over branching and at least one backward over branching. Points (i) and (i') are thus equivalent.

With the same ideas, we can also prove the following well known proposition, whose result has already been mentioned in Remark 2.2.7.

Proposition 4.3.8 *Let $\mathcal{B}$ be a branched surface having a disk sector $\mathcal{D}$ which is a twisted disk of contact as well. Then $\mathcal{B}$ cannot fully carry a lamination.*

Proof Suppose that $\mathcal{B}$ fully carries a lamination λ. We consider $N(\mathcal{D})$, the fibred neighbourhood over $\mathcal{D}$ in $N(\mathcal{B})$. Let $\mathcal{V}$ be the train track which is the trace of $\mathcal{B}_1$ in $\partial_v(N(\mathcal{D}))$. The intersection of the leaves of λ passing over $\mathcal{D}$ with $\partial_v N(\mathcal{D})$ is a union of disks. Their boundaries are circles which form a 1-dimensional lamination fully carried by some fibred neighbourhood of $\mathcal{V}$. However, as seen in the example on Figure 4.3.9 (b), $\mathcal{V}$ is the union of two smooth circles, and of segments which join these two circles at branch points. And since $\mathcal{V}$ cannot be decomposed into an union of circles, there is no circle carried by a fibred neighbourhood of $\mathcal{V}$ which passes over one of these segments (c), which is a contradiction. $\square$

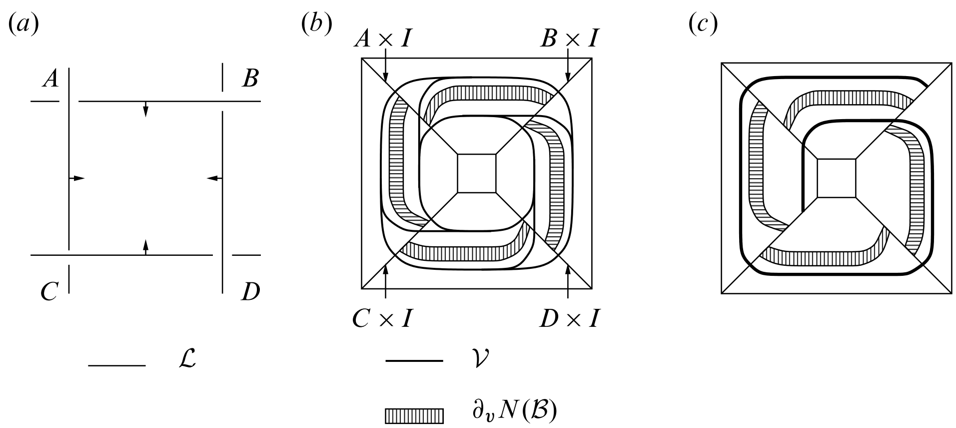

Figure 4.3.9: Twisted disk of contact.

That is why the existence of a twisted disk of contact prevents the proof to work. That is also why we have refined the first cell decomposition of $\mathcal{B}$ in Section 4.1.

Remark 4.3.10 For the non compact cells, we have a simple equivalent statement: a train track fully carried by a fibred neighbourhood $\mathbb{R} \times [0,1]$ can always be decomposed into a union of smooth lines.

4.4 Resolving sequence of splittings

We keep the real number $\varepsilon > 0$ defined in Section 4.2 for the splitting from $\mathcal{B}$ to $\mathcal{B}_1$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers such that for all n, $\frac{\varepsilon}{2} < \varepsilon_n < \varepsilon$. Let Y_ε be a $\frac{\varepsilon}{2}$–neighbourood of $Y^{[1]}$ in $\mathcal{B}$. Denote by $\mathcal{B}_1'$ the trace of $\mathcal{B}_1$ on $\pi^{-1}(Y_\varepsilon)$, which is a branched surface with boundary.

The purpose of this subsection is to explain how to build a sequence of splittings of $\mathcal{B}$, which induces a resolving sequence of splittings of $\mathcal{B}'_1$.

We denote by $(y_i)_{i\in J}$ the set of vertices of Y, where J is a subset of $\mathbb{N}$, finite or not. To each vertex y_i there correspond several vertices of Y_1, at least 2 and at most 3, depending on whether y_i is a double point or a regular point of the singular locus. We denote these vertices $y_i(j)$ for $j = 1$, 2 or 3. We then call $d_i(j)$ the projection by p_1 of the disk of $\mathcal{B}_1$ centred at $y_i(j)$ and of radius 2ε, such that $d_i = \cup_{j\in\{1,2,3\}} d_i(j)$ is a branched disk neighbourhood of y_i in $\mathcal{B}$ (see Figure 4.4.1).

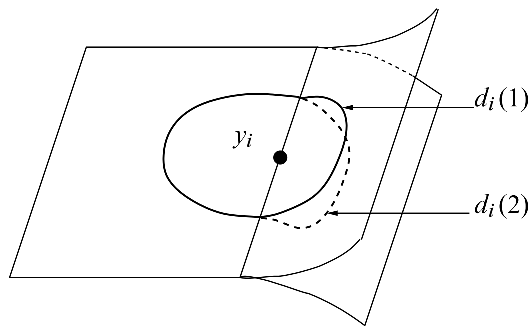

Figure 4.4.1

The singular locus of $\mathcal{B}'_1$ is included in the union of the $p_1^{-1}(d_i(j)))$, for all the i and j. Notice that the singular locus of $\mathcal{B}'_1$ has no double point. Moreover, since we supposed that the edges of Y are strictly longer than 5ε, if y_{i_1} is different from y_{i_2}, then d_{i_1} and d_{i_2} are disjoint.

At last, we define a sequence of vertices of Y, $(y_{\psi(n)})_{n\in\mathbb{N}}$, for ψ a map from $\mathbb{N}$ to J, such that each vertex appears infinitely many times. This is possible since there is only countably many vertices in Y.

Define now the splittings from $\mathcal{B}_1$ to $\mathcal{B}_2$. We take all the edges of Y which have $y_{\psi(1)}$ as a vertex. We orient them from $y_{\psi(1)}$ to their second vertex. Let $\mathfrak{a}$ be one of these edges. Since $\mathcal{B}$ has no twisted curve, its second vertex, y_k, is different from $y_{\psi(1)}$. We call $\mathcal{V}_{\mathfrak{a}}$ the trace of $\mathcal{B}_1$ on $\pi^{-1}(\mathfrak{a})$. Since $\mathfrak{a}$ is oriented, it makes sense to talk of direct and backward branchings along $\mathcal{V}_{\mathfrak{a}}$.

Because of the definition of $\mathcal{B}_1$, the backward branchings all lie in $\pi^{-1}(d_{\psi(1)})$, and the direct branchings all lie in $\pi^{-1}(d_k)$. Moreover, each branching lies at a distance ε from the ends of $\mathcal{V}_{\mathfrak{a}}$. Actually, at this step of the sequence of splittings, there is at most one direct branching and one backward branching along $\mathcal{V}_{\mathfrak{a}}$. If there is no backward branching, no splitting will be made along $\mathcal{V}_{\mathfrak{a}}$, otherise we will perform a splitting along a path inscribed on $\mathcal{V}_{\mathfrak{a}}$, going from the backward branching to the direct one if it exists, or to the end of $\mathcal{V}_{\mathfrak{a}}$, in an ε_1–neighbourhood of this path. If a direct branching is met, this splitting can be an over, under or neutral splitting. Section 4.5 will describe which one must be chosen. If it is the neutral splitting, the splitting stops at this branching point, otherwise we can split on along a path in $\mathcal{V}_{\mathfrak{a}}$ which goes to the end of $\mathcal{V}_{\mathfrak{a}}$. Since $\varepsilon_1 < \varepsilon$, along this path, no other backward branching is met, and hence, there is no backward splitting. The same process is applied to the other edges having $y_{\psi(1)}$ as a vertex. The second splitting takes place in an ε_2–neighbourhood of the corresponding path, the third splitting takes place in an ε_3–neighborhhod of the corresponding path, and so on.

The fact that the ε_i are decreasing allows us to avoid backward splittings. The order of the edges does not matter.

After these splittings, we get a branched surface $\mathcal{B}_2$. We take all the edges of Y which have $y_{\psi(2)}$ as a vertex. We orient them from $y_{\psi(2)}$ to their second vertex. Let $\mathfrak{a}$ be one of these edges. Its second vertex is y_k, different from $y_{\psi(2)}$. We call $\mathcal{V}_{\mathfrak{a}}$ the trace of $\mathcal{B}_2$ on $\pi^{-1}(\mathfrak{a})$. The situation is as previously, except for one detail: there can now be more than one direct branching and one backward branching along $\mathcal{V}_{\mathfrak{a}}$. However, all the backward branchings lie in $\pi^{-1}(d_{\psi(2)})$, and all the direct branchings lie in $\pi^{-1}(d_k)$. All these branchings lie at a distance at least ε_i from the ends of $\mathcal{V}_{\mathfrak{a}}$, where i is the number of splittings performed on $\mathcal{B}_1$. Look at the backward branchings of $\mathcal{V}_{\mathfrak{a}}$: there are j such branchings. Since the successive splittings have been performed in smaller and smaller neighbourhoods, we can order these branchings from the furthest from $\pi^{-1}(y_{\psi(2)})$ to the nearest. We note them $b_1, \dots, b_j$, b_i being strictly further than b_{i+1}. We will make splittings along paths going from the b_i, in smaller and smaller neighbourhoods, whose size is set by the (ε_n) sequence. To avoid any backward splitting, we begin by the splitting along a path starting from b_1. The second splitting will start from b_2, and so on until the last splitting, which will start from b_j. When a direct branching is met, one of the over, under and neutral splittings must be chosen: this is done in Section 4.5. As previously, if the neutral splitting is chosen, the splitting stops here, otherwise we can split on until another direct branching is met, or until the end of $\mathcal{V}_{\mathfrak{a}}$. Again, thanks to the choice of the ε_n, backward splittings are avoided. The same process is applied to all the edges having $y_{\psi(2)}$ as a vertex. The order of the edges does not matter.

We iterate these operations at each step: the splittings from $\mathcal{B}_n$ to $\mathcal{B}_{n+1}$ are performed along arcs whose image by π is included in an edge having $y_{\psi(n)}$ as a vertex. The backward branchings are always over $d_{\psi(n)}$: they are more and more numerous, but they are always strictly ordered, from the furthest to the nearest. Moreover, the singular

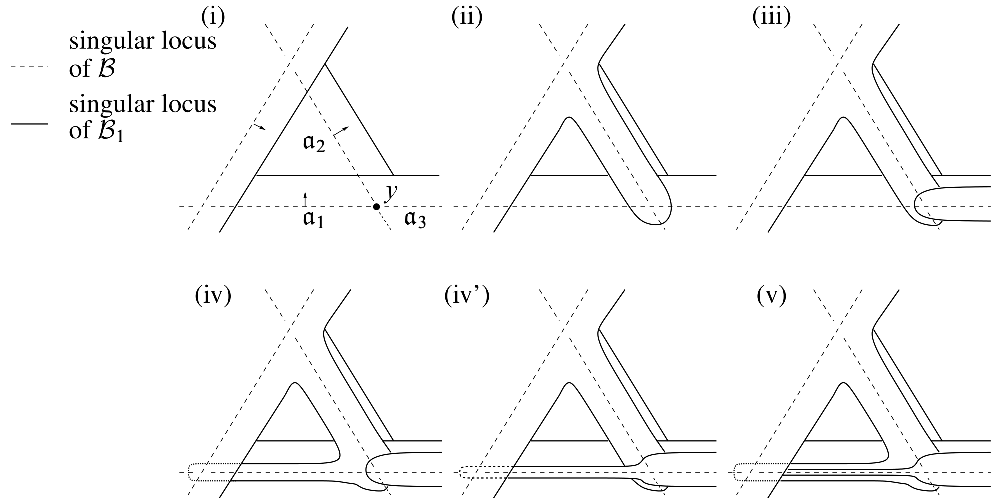

Figure 4.4.2: Example of a sequence of splittings.

locus of $\mathcal{B}'_{n+1}$ does not intersect $P_{n+1}^{-1}(d_{\psi(n)})$ any more: the singularities over $d_{\psi(n)}$ have then been *resolved*. Since the vertex $y_{\psi(n)}$ will reappear infinitely many times in the sequence $(y_{\psi(n)})_{n\in\mathbb{N}}$, the sequence of splittings is resolving. Figure 4.4.2 shows an example of such a sequence of splittings. On this figure, the branch loci are seen "from above", and only the top parts are drawn. The three first points show a sequence of splittings at the end of which there are several direct branchings along some edges having y as a vertex. The first splitting to be performed along $\mathfrak{a}_1$ is the one drawn in (iv), but not the one drawn in (iv'), where a backward splitting occurs. The second splitting is the one drawn in (v). It then remains to split along $\mathfrak{a}_2$ and $\mathfrak{a}_3$.

4.5 Adapted splittings

We will now see how it is possible to perform the splittings along the edges previously defined, in such a way that none of the $\mathcal{B}_n$ has a twisted curve, and then a twisted disk of contact.

If an arc of splitting from $\mathcal{B}_n$ to $\mathcal{B}_{n+1}$ is not in a face-to-face position, then the singular locus of $\mathcal{B}_{n+1}$ remains the same as the singular locus of $\mathcal{B}_n$: it is deformed, but there is no new double point.

When the arc of splitting is in a face-to-face position, then we have the following fact: an over splitting introduces two new double points in the singular locus, a positive and a negative one, and an under splitting introduces two double points at the same place but of opposite signs. Figure 3.1.5 shows this.

Being given an arc to split along, we now have to find a splitting which will not create a twisted curve. Such a splitting will be said ti be *adapted*.

The following proposition is fundamental (we keep the previous notations).

Proposition 4.5.1 *Let $\mathcal{B}_n$ be a branched surface obtained from $\mathcal{B}_1$ by a sequence of splittings, and which does not have any twisted curve. We denote $\mathcal{L}_n$ its singular locus. Then, for every arc of splitting in a face-to-face position in $\mathcal{B}'_n$ (the branched surface induced by $\mathcal{B}_n$ on $\pi^{-1}(Y_\varepsilon)$), at least one of the three splittings, over, under or neutral, is adapted.*

Proof According to corollary Corollary 2.3.9, it is possible to consider only simple twisted curves. This will be done throughout this proof. Let us denote $\mathfrak{a}$ the splitting path used to go from $\mathcal{B}_n$ to $\mathcal{B}_{n+1}$. We suppose that $\mathfrak{a}$ lies in a face-to-face situation. Consider all the possible splittings along $\mathfrak{a}$. When we perform one of these splittings, the singular locus is only modified in some neighbourhood $\mathcal{V}(\mathfrak{a})$ of $\mathfrak{a}$.

We begin by performing the neutral splitting. We note $\mathcal{B}_{\mathrm{nul}}$ the obtained branched surface. We use the notations of Figure 4.5.2 (b), where $\mathfrak{a}_g = [p_g, q_g]$, $\mathfrak{a}_d = [q_d, p_d]$, l_g and l_d are smooth oriented segments of the singular locus. We will say that an oriented curve passes *positively* through one of these segments if it passes through this segment with the same orientation as this segment. We will say that it passes *negatively* if it passes through this segment with the opposite orientation.

Suppose that the neutral splitting is not adapted. Suppose, for instance, that γ is a positive twisted curve passing positively through $\mathfrak{a}_g$ in $\mathcal{B}_{\mathrm{nul}}$. The other cases can be dealt with in the same way. We distinguish the case (A), where γ does not pass through $\mathfrak{a}_d$, from the case (B), where it passes through $\mathfrak{a}_d$.

Lemma 4.5.3 *If γ is in case (B), then it passes negatively through $\mathfrak{a}_d$.*

Proof of Lemma 4.5.3 Suppose that it passes positively through $\mathfrak{a}_d$. The immersion of γ in $\mathcal{B}_{\mathrm{nul}}$ gives an immersion γ' of a closed curve in $\mathcal{B}_n$, which coincides with γ outside $\mathcal{V}(\mathfrak{a})$. It remains to define γ' inside $\mathcal{V}(\mathfrak{a})$. Let us define two oriented edges in $\mathcal{B}_n$, denoted a_g and a_d, which go from p_g to q_g and from q_d to p_d. Those edges correspond to $\mathfrak{a}_g$ and $\mathfrak{a}_d$, but they are not in the singular locus of $\mathcal{B}_n$. We define γ' inside $\mathcal{V}(\mathfrak{a})$ in the same way as γ, replacing $\mathfrak{a}_g$ and $\mathfrak{a}_d$ by a_g and a_d. Each corner of γ' different from p_g, p_d, q_g and q_d is ascending. If we consider γ' as a loop based

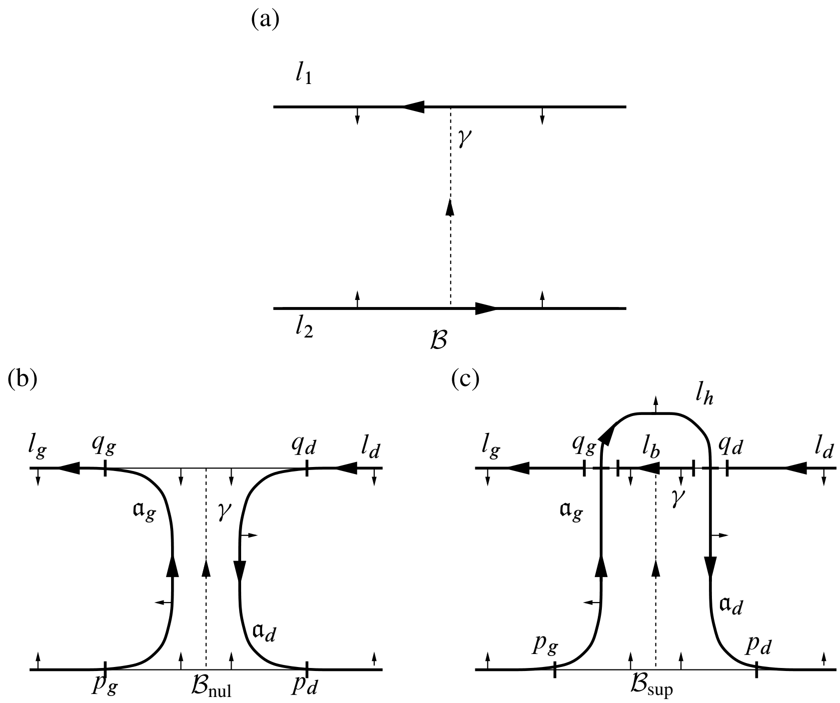

Figure 4.5.2

in q_g, we can write $\gamma' = \gamma'_1 * a_d * \gamma'_2 * a_g$, where γ'_1 is the part ofγ' going from q_g to q_d, and γ'_2 is the part of γ' going from p_d to p_g. We then set $\beta_1 = \gamma'_1 * [q_d, q_g]$ and $\beta_2 = \gamma'_2 * [p_g, p_d]$. These two curves lie in the singular locus of $\mathcal{B}_n$. Moreover, a corner of γ' is necessarily a corner of β_1 or of β_2, and *vice-versa*. Therefore, one of the curves β_1 or β_2 has at least one corner, and all of its corners are ascending. It is a positive twisted curve of $\mathcal{B}_n$, existing before splitting, which is a contradiction. □

In case (B), we can write $\gamma = \gamma_1 * \mathfrak{a}_d^{-1} * \gamma_2 * \mathfrak{a}_g$, where γ_1 is the part of γ going from q_g to p_d, and γ_2 the one from q_d to p_g. We have a second lemma.

Lemma 4.5.4 *There is at least one corner on γ_1 and at least one corner on γ_2.*

Proof of Lemma 4.5.4 Suppose for instance that there is no corner on γ_2. As in the proof of lemma Lemma 4.5.3, there is a closed curve γ' in $\mathcal{B}_n$ corresponding to γ, and which can be written $\gamma' = \gamma'_1 * a_d^{-1} * \gamma'_2 * a_g$. We set in $\mathcal{B}_n$, $\beta = \gamma'_1 * [p_d, p_g] * \gamma_2'^{-1} * [q_d, q_g]$. This is a closed curve, immersed in $\mathcal{B}_n$, and included in the singular locus of $\mathcal{B}_n$. Each corner of γ is a corner of γ_1', and thus of β, and each corner of β is a corner of γ. Hence, β is a positive twisted curve in $\mathcal{B}_n$, which is a contradiction. □

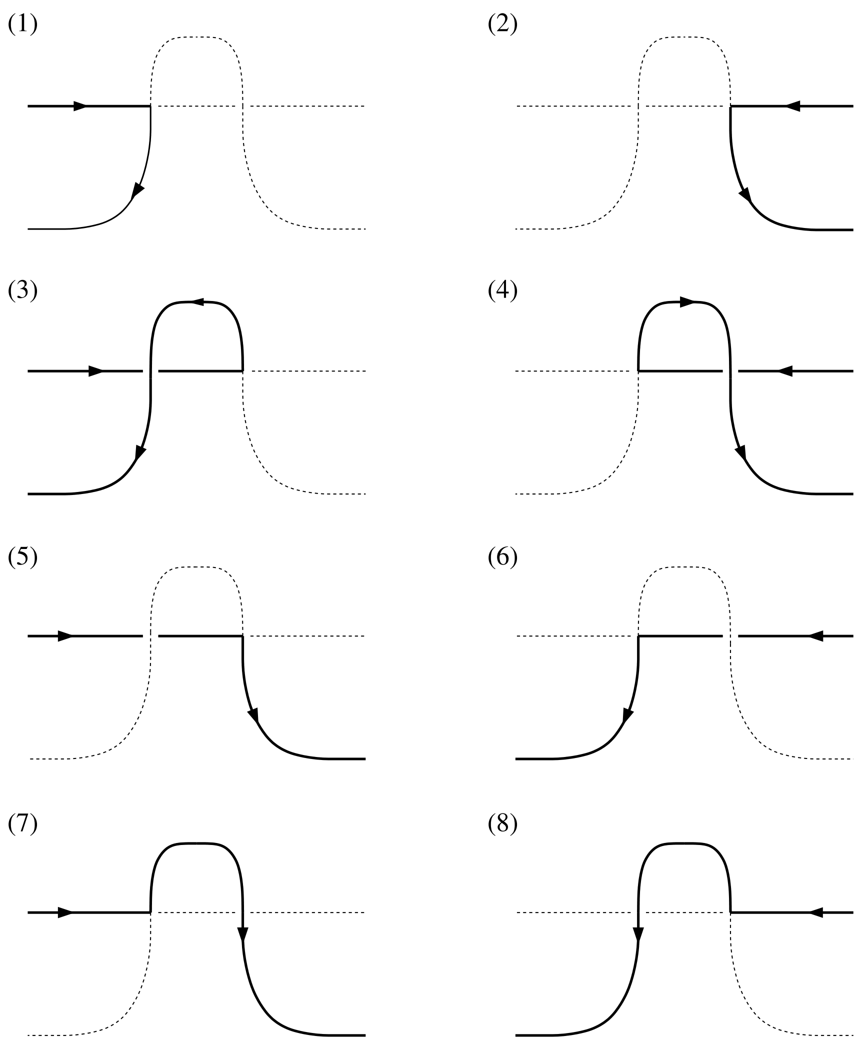

Figure 4.5.5: Positive twisted curves passing through $\mathcal{V}(\mathfrak{a})$.

In any case, we will prove that the over splitting is adapted. We perform this splitting. We call $\mathcal{B}_{\text{sup}}$ the obtained branched surface. We use the notations of Figure 4.5.2 (c), where q_g and q_d are double points of the singular locus of $\mathcal{B}_{\text{sup}}$. Suppose that $\mathcal{B}_{\text{sup}}$ contains a positive simple twisted curve δ passing inside $\mathcal{V}(\mathfrak{a})$, ie the over splitting makes δ appear. We will actually prove that this cannot happen. Figure 4.5.5 shows all the possible local configurations of positive simple twisted curves passing in $\mathcal{V}(\mathfrak{a})$. If we reverse the orientations of these curves, we get all the possible local configurations of negative simple twisted curves passing through $\mathcal{V}(\mathfrak{a})$.

Diagrams (1) and (3) of this figure are equivalent, in the sense that there is a positive simple twisted curve as in (1) if and only if there is a positive simple twisted curve as in (3). In the same way, diagrams (5) and (7) are equivalent, as are diagrams (2) and (4) and diagrams (6) and (8). There are thus only four cases to study for δ. Notice that the immersion of γ into $\mathcal{B}_{\mathrm{nul}}$ is also an immersion of γ into $\mathcal{B}_{\mathrm{sup}}$. To avoid confusion, the image of this last immersion is called γ_{sup}. However, q_g and q_d are smooth points of γ, whereas they are descending corners of γ_{sup}. The curve γ_{sup} is not twisted, but all its corners different from q_g and q_d are ascending. We now study all the possible cases, starting with those where γ is in case (A).

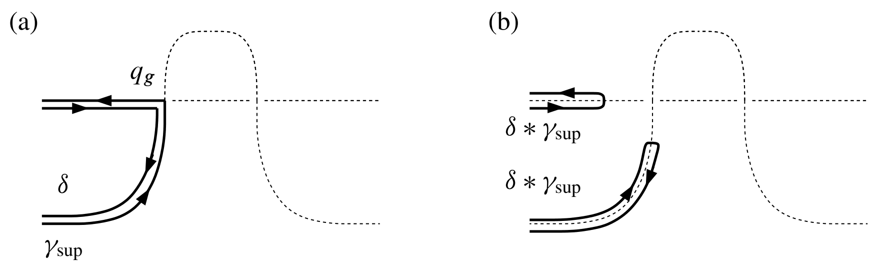

Figure 4.5.6: (A.1).

(A.1) Figure 4.5.6 shows what happens. We consider γ_{sup} and δ as two loops based in q_g, and we set $\beta = \delta * \gamma_{\mathrm{sup}}$. This loop is freely homotopic in $\mathcal{B}_{\mathrm{sup}}$ to an immersed loop which does not pass through $\mathcal{V}(\mathfrak{a})$ anymore, and whose corners are all ascending. This last loop contains, according to lemma Corollary 2.3.9, a positive simple twisted curve which does not pass through $\mathcal{V}(\mathfrak{a})$ either. This curve is a positive simple twisted curve in $\mathcal{B}_n$, which is a contradiction.

(A.4) Figure 4.5.7 shows what happens. We consider γ_{sup} and δ as two loops based in q_g, and we set $\beta = \delta * \gamma_{\mathrm{sup}}$. This loop is immersed, and q_g is no longer a corner of β. All the corners of β are ascending. This β contains a positive simple twisted curve. If this curve passes through $\mathcal{V}(\mathfrak{a})$, it passes in a row either through $\mathfrak{a}_g$, l_h and $\mathfrak{a}_d$, positively, or through l_d, l_b and l_g positively. This curve is a positive simple twisted curve in $\mathcal{B}_n$, which is a contradiction.

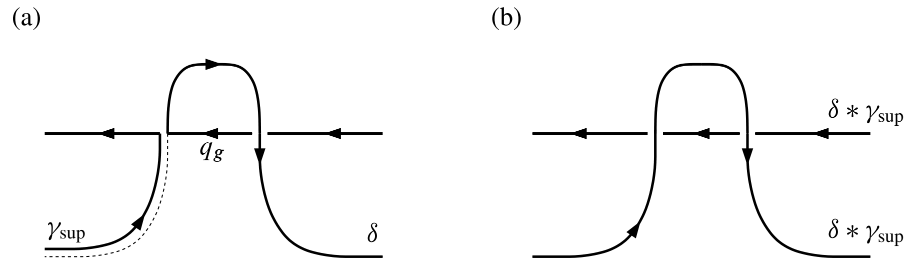

Figure 4.5.7: (A.4).

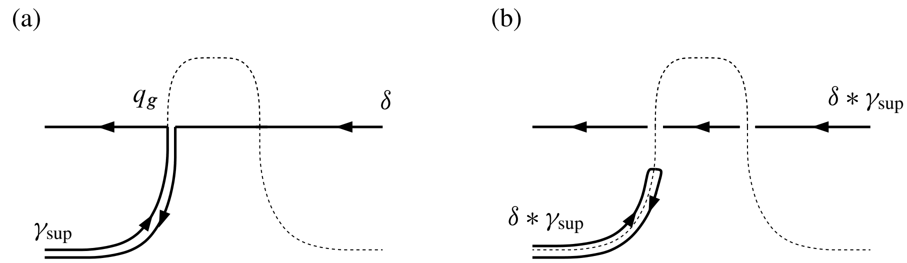

Figure 4.5.8: (A.6).

(A.6) Figure 4.5.8 shows what happens. We consider γ_{sup} and δ as two loops based in q_g, and we set $\beta = \delta * \gamma_{\text{sup}}$. This β is homotopic in $\mathcal{B}_{\text{sup}}$ to an immersed loop for which q_g is not a corner, and which contains a positive simple twisted curve. If this curve passes through $\mathcal{V}(\mathfrak{a})$, it passes in a row either through l_d, l_b and l_g positively, and nowhere else in $\mathcal{V}(\mathfrak{a})$. This curve is a positive simple twisted curve in $\mathcal{B}_n$, which is a contradiction.

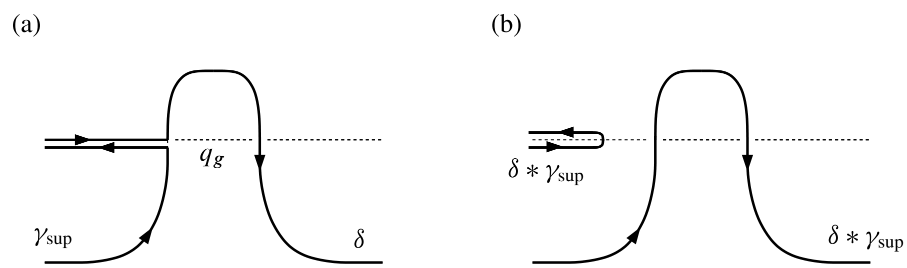

Figure 4.5.9: (A.7).

(A.7) Figure 4.5.9 shows what happens. We consider γ_{sup} and δ as two loops based in q_g, and we set $\beta = \delta * \gamma_{\text{sup}}$. This β is homotopic in $\mathcal{B}_{\text{sup}}$ to an immersed loop for which q_g is not a corner, and which contains a positive simple twisted curve. If this curve passes in $\mathcal{V}(\mathfrak{a})$, it passes in a row either through $\mathfrak{a}_g$, l_h and $\mathfrak{a}_d$ positively, and nowhere else in $\mathcal{V}(\mathfrak{a})$. This curve is a positive simple twisted curve in $\mathcal{B}_n$, which is a contradiction.

We then deal with the cases where γ is in case (B).

(B.1) In the same way as we can build γ_{sup} from γ, we can build a closed curve δ_{nul} immersed in $\mathcal{B}_{\text{nul}}$, from δ. This curve has the same corners as δ, minus q_g. If q_g is the only corner of δ, then δ_{nul} has no corner, otherwise it is a positive twisted curve. We consider δ_{nul} and γ as two loops in $\mathcal{B}_{\text{nul}}$ based in q_g, and we set $\beta = \delta_{\text{nul}} * \gamma$. This loop is freely homotopic in $\mathcal{B}_{\text{sup}}$ to an immersed loop which does not pass through $\mathfrak{a}_g$, and whose corners are all ascending. This last loop contains, according to Corollary 2.3.9, a positive simple twisted curve which does not pass through $\mathfrak{a}_g$ either. We still call this curve β. If this curve does not pass in $\mathcal{V}(\mathfrak{a})$, it is a positive simple twisted curve in $\mathcal{B}_n$, which is a contradiction. If this curve passes in $\mathcal{V}(\mathfrak{a})$, it passes in a row either through l_d and $\mathfrak{a}_d$, positively, and nowhere else in $\mathcal{V}(\mathfrak{a})$. The immersion of β in $\mathcal{B}_{\text{nul}}$ implies the existence of the immersion of a closed curve β_{sup} in $\mathcal{B}_{\text{sup}}$, and which passes only through $\mathfrak{a}_d$ and l_d in $\mathcal{V}(\mathfrak{a})$. We then modify δ by adding to it a loop $l_b * l_h$, so that we get a curve δ' modeled on diagram (4) of Figure 4.5.5. We get a contradiction in the same manner as in point (A.4), by using β_{sup} and δ'.

(B.2) By symmetry, this point can be dealt in the same way as the previous one, (B.1).

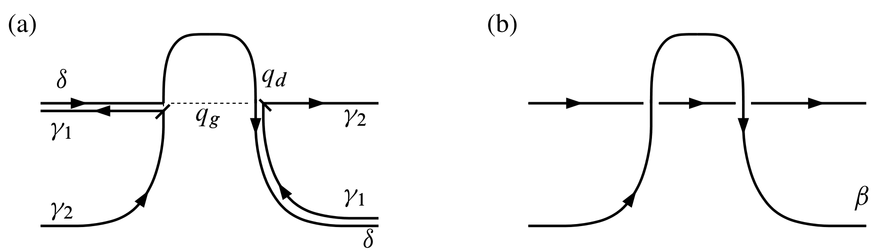

Figure 4.5.10: (B.7).

(B.7) This case is shown on Figure 4.5.10. As previously, we write $\gamma_{\text{sup}} = \gamma_1 \cup \gamma_2$, where γ_1 has q_g as first end and q_d as last end. In $\mathcal{B}_{\text{sup}}$, we have the following loop, based at q_g, $\beta = \delta * l_b * \gamma_2$. Neither q_g nor q_d are corners of this loop. The corners of this loop are the ones of γ_2 and the ones of δ, except q_d. They are thus all ascending, and according to Lemma 4.5.4, there is at least one. Thus, β is a positive simple twisted curve, which was there before splitting, which is a contradiction.

(B.8) By symmetry, this point can be dealt in the same way as the previous one, (B.7).

After these eight points, the over splitting is adapted. The other cases where the neutral splitting is not adapted are dealt with in the same way, and are the following:

- γ is positive and passes negatively through $\mathfrak{a}_g$: the under splitting is adapted;
- γ positive and passes positively through $\mathfrak{a}_d$: the under splitting is adapted;
- γ is positive and passes negatively through $\mathfrak{a}_d$: the over splitting is adapted.

This completes the proof of Proposition 4.5.1. □

4.6 Conclusion

After the two previous subsections, we have built a sequence of splittings of $\mathcal{B}$, none of which has a twisted disk of contact. This sequence induces a resolving sequence of splittings of $\mathcal{B}'_1$, whose inverse limit is a lamination λ fully carried by $\mathcal{B}'_1$. We next prove that λ has null holonomy. Here is the meaning that be can given to "null holonomy": Let Σ be a 2–cell of Y, and $\partial\Sigma \times [0,1]$ be the subbundle of $N(\mathcal{B})$ over $\partial\Sigma$. Then $\lambda \cap (\partial\Sigma \times I)$ is an oriented dimension 1 lamination denoted l_Σ, fully carried by $\partial\Sigma \times I$, and obtained as the inverse limit (in the sense of Definition 3.2.5) of the oriented train tracks $v_n = \mathcal{B}_n \cap (\partial\Sigma \times I)$. We will say that λ has null holonomy if, when Σ is compact, the lamination l_Σ is a lamination by circles. If Σ is not compact, there are no holonomy problems since there is no first-return map on a fibre. Then, suppose that Σ is compact.

Definition 4.6.1 Let λ be an oriented lamination carried by a trivial bundle $\mathbb{S}^1 \times [0,1]$. An *increasing leaf* (resp. a *decreasing leaf*) of λ is a leaf which goes, in the direct way, from a point $p_1 = (\theta, t_1)$ to a point $p_2 = (\theta, t_2)$, with $t_1 < t_2$ (resp. $t_1 > t_2$).

Lemma 4.6.2 *The lamination l_Σ is a lamination by circles.*

Proof We denote $N(v_n) = N(\mathcal{B}_n) \cap (\partial\Sigma \times I)$, which is a fibred neighbourhood of v_n. We call $L_\Sigma = \cap_{n\in\mathbb{N}} N(v_n)$, and we then have $l_\Sigma = \partial L_\Sigma$, according to Definition 3.2.5.

Let L be an increasing leaf of l_Σ. This leaf is a spiral with two limit circles C^+ and C^-, where C^+ is the limit of L followed in the positive direction, and C^- is the limit of L followed in the negative direction. We call A the annulus between C^+ and C^-. Look at $L_\Sigma \cap A$. By construction, this intersection is not equal to A. This means that $A \backslash L_\Sigma$ contains some subset of the form $\gamma \times [0,1]$, where γ is a compact oriented path fully carried by $\partial\Sigma \times I$, and which is increasing (see Figure 4.6.3).

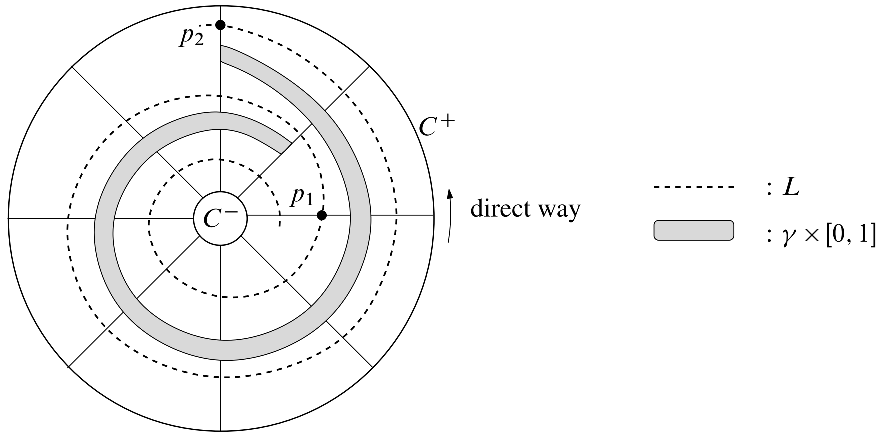

Figure 4.6.3

Hence, there exists an integer N such that for all integer n greater than N, we have $(\gamma \times [0,1]) \cap N(\mathcal{B}_n) = \varnothing$. If not, there would exist a sequence of points (q_n) such that $q_n \in (\gamma \times [0,1]) \cap N(\mathcal{B}_n)$. Since $(\gamma \times [0,1]) \cap N(\mathcal{B}_n)$ is compact, there would be a subsequence of (q_n) converging towards a point q contained in $\cap_{n\in\mathbb{N}}((\gamma \times [0,1]) \cap N(\mathcal{B}_n))$. But this last set is equal to $\gamma \cap L_\Sigma$, which is empty. It is then impossible to find a path in v_N going in the positive direction from p_2 to p_1, where p_1 and p_2 are two points of L placed as in Figure 4.6.3. However, because Σ is not a twisted disk of contact and according to Lemma 4.3.4, the existence of a path of v_N going in the positive direction from p_1 to p_2 implies the existence of a path of v_N going in the negative direction from p_1 to p_2. This is a contradiction, and L must be a circle. In the same way, l_Σ does not have any decreasing leaf. □

Hence, λ has null holonomy.

To get a lamination fully carried by $\mathcal{B}$, it only remains to "fill the holes" of leaves of λ, these holes being in fact diffeomorphic to the 2–cells of Y, which are disks and half planes. This is possible because λ is null holonomic. This ends the proof of Theorem 5.

5 Some remarks about the problem of determining whether a branched surface fully carries a lamination or not

Proposition 4.3.8 states that a necessary condition for a branched surface to fully carry a lamination is that it does not have any twisted disk of contact which is a sector. The obstruction to the existence of a lamination fully carried seems to be essentially linked to the phenomenon of the twisted disks of contact. However, we will show in two examples that the non-existence of twisted disks of contact is not a necessary condition nor a sufficient one for a branched surface to fully carry a lamination.

5.1 No twisted disk of contact is not necessary

Let $\mathcal{B}$ fully carry a lamination λ and have a twisted disk of contact $\mathcal{D}$. According to Proposition 4.3.8, $\mathcal{D}$ cannot be a sector. Then using the arguments of the proof of Proposition 4.3.8, over the boundary of $\mathcal{D}$, the trace of some leaf l of λ must be a spiral. Hence, l cannot pass all over $\mathcal{D}$. That means that l passes over $\mathcal{D}$ along some annulus around $\partial\mathcal{D}$, and then quits $\mathcal{D}$. This separation of l and $\mathcal{D}$ can be done only along a piecewise smooth circle $\mathcal{C}$ of the branch locus such that $\mathcal{C}$ and $\partial\mathcal{D}$ bound a sink annulus in $\mathcal{D}$. Moreover, if we look at all the connected surfaces immersed in $\mathcal{B}$ and bounded by $\mathcal{C}$ so that the branch directions along the boundary component $\mathcal{C}$ point outwards, then one of these surfaces is not a disk. If they were all disks, l could not exist, again with the arguments of the proof of Proposition 4.3.8. This surface which is not a disk allows the existence of the holonomy of the spiral traced by l over $\partial\mathcal{D}$, because its π_1 is not zero. The branched surface of Figure 5.1.1 is an example of such a branched surface.

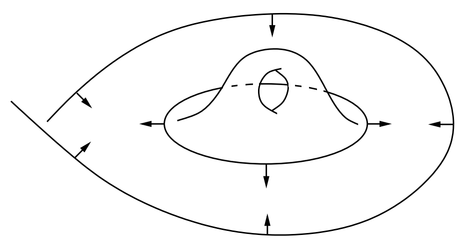

Figure 5.1.1: A branched surface with a twisted disk of contact fully carrying a lamination.

5.2 No twisted disk of contact is not sufficient

Indeed, a twisted disk of contact may be "hidden" in the branched surface, as shown in the example of Figure 5.2.1.

This figure shows the singular locus of a branched surface. This branched surface cannot fully carry a lamination, although it has no twisted disk of contact. Indeed, were it the case, a lamination λ fully carried would have to boundary leaves (ie there is

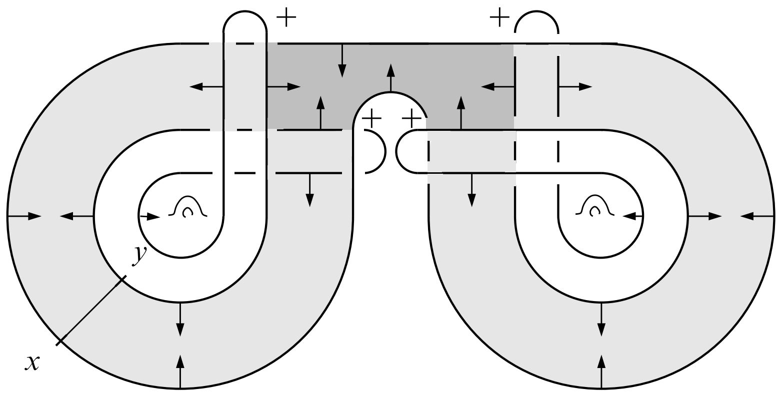

Figure 5.2.1

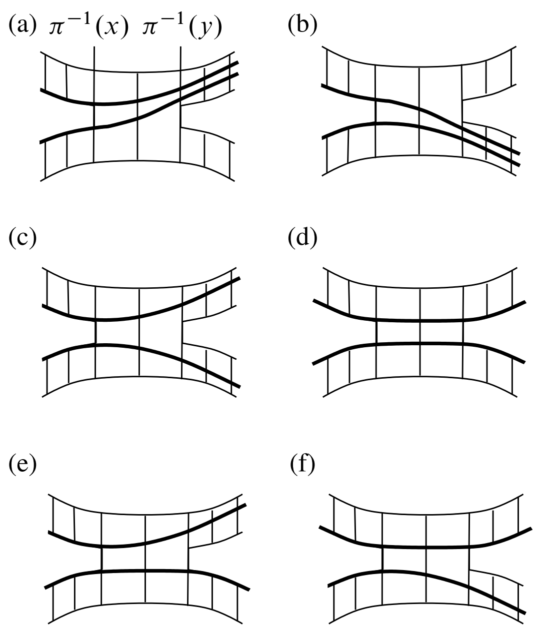

Figure 5.2.2: Trace of two boundary leaves of λ.

no other leaf between them and some component of the horizontal boundary) passing through $\pi^{-1}([x,y])$, as in one of the situations of figure Figure 5.2.2.

In case (a) of this figure, the branched surface obtained after an over splitting along $[x, y]$ still fully carries λ. In case (b) only the under splitting has this property, in case (c) the three splittings fit, in case (d) only the neutral splitting fits, in case (e) the neutral and the over splittings fit, and in case (f) the neutral and the under splittings fit. However, if any of these splittings is performed along $[x, y]$, a twisted disk of contact appears and the new branched surface cannot fully carry a lamination.

Actually, it is possible to give a definition of this kind of twisted surface of contact, which always give birth to a twisted disk of contact, whatever is the splitting performed along some arcs. The non existence of this kind of twisted surface of contact could thus be a sufficient condition for a lamination fully carried to exist. More generally, it is natural to ask whether a branched surface without any twisted surface of contact fully carries a lamination or not. We could then try to adapt the proof of theorem Theorem 5, that is to perform an infinite sequence of splittings called adapted, which does not create any twisted disk of contact, or even any twisted surface of contact. It is possible to show that a splitting along an arc $\mathfrak{a}$ cannot create a twisted surface of contact on one side of $\mathfrak{a}$ and another one of the same sign of the other side of $\mathfrak{a}$. But I cannot find an argument to show that there exists a splitting which does not create a twisted surface of contact on each side, but also which does not create two twisted surfaces of contact of the same sign and on the same side of $\mathfrak{a}$. The issue is that the boundary of a twisted surface of contact can be too "wild", and can pass several times through $\mathfrak{a}$. A way to avoid this problem is to consider twisted curves, bounding or not a surface, and to use Lemma 2.3.8 and Corollary 2.3.9.

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