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A parametrized Borsuk–Ulam theorem for a product of spheres with free $\mathbb{Z}_p$–action and free S^1–action

DENISE DE MATTOS
EDIVALDO L DOS SANTOS

In this paper, we prove parametrized Borsuk–Ulam theorems for bundles whose fibre has the same cohomology (mod p) as a product of spheres with any free $\mathbb{Z}_p$–action and for bundles whose fibre has rational cohomology ring isomorphic to the rational cohomology ring of a product of spheres with any free S^1–action. These theorems extend the result proved by Koikara and Mukerjee in [7]. Further, in the particular case where $G = \mathbb{Z}_p$, we estimate the "size" of the $\mathbb{Z}_p$–coincidence set of a fibre-preserving map.

55M20; 55R91, 55R25

1 Introduction

The classical Borsuk–Ulam Theorem gives information about maps $S^m \to \mathbb{R}^k$ where S^m has a free action of either the cyclic group $\mathbb{Z}_2$ or the cyclic group $\mathbb{Z}_p$ when m is odd. In particular, the sphere S^m has a free $\mathbb{Z}_2$–action and, in this case, the classical theorem states that there is at least one orbit which is sent to a single point in $\mathbb{R}^k$ if $m \geq k$. Munkholm [9] and others gave methods to estimate the dimension of the space of orbits in S^m which have the property that each orbit is sent to a single point.

Dold [3] and others extended this problem to a fibrewise setting, by considering maps $f\colon S(E) \to E'$ which preserve fibres; here, $S(E)$ denotes the total space of the sphere bundle associated over B to a vector bundle $E \to B$, and E' is another vector bundle over the B. This problem is the parametrized version of the Borsuk–Ulam theorem, whose general formulation is the following:

Parametrized version of Borsuk–Ulam theorem Consider a bundle $\pi\colon E \to B$ and a vector bundle $\pi'\colon E' \to B$ such that G is fibre-preserving and acts freely on E and $E' - 0$ where 0 stands for the zero section of the bundle $\pi'\colon E' \to B$. For $f\colon E \to E'$ a fibre-preserving G–equivariant map, the parametrized version of the Borsuk–Ulam theorem estimates the cohomological dimension of the set

$$Z_f = \{x \in E; f(x) = 0\}.$$

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Such theorems appeared first in the paper of Dold [3] (for vector bundles with free $\mathbb{Z}_2$–action) and Nakaoka [10] (for vector bundles with $\mathbb{Z}_p$–action, for p prime, and S^1–action).

Characteristic polynomials for vector bundles with free G–actions ($G = Z_p$ or $G = S^1$) were introduced by Dold and Nakaoka, and these are useful tools in studying parametrized Borsuk–Ulam type problems. More specifically, Dold proved in [3] that if $G = \mathbb{Z}_2$ and if m and k are the dimensions of the fibres of E and E', respectively, where $m > k$, then

$$\operatorname{cohom.dim} Z_f \geq \operatorname{cohom.dim}(B) + m - k - 1,$$

where cohom. dim denotes the cohomological dimension.

Other papers closely related to the Dold [3] and Nakaoka [10] articles are the papers of Izydorek and Rybicki [5], Jaworowski [6] (for $G = Z_p$) and Volovikov [13] (for a p–torus action, $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$; in this paper a simple interpretation of characteristic polynomials as equivariant Euler classes of the bundles in question is given).

The technique introduced by Dold to solve the parametrized problem by using characteristic polynomials was also used by Koikara and Mukerjee in [7] to show a parametrized version of the Borsuk–Ulam theorem for bundles whose fibre is a product of spheres, with the free $\mathbb{Z}_2$–action given by the product of the antipodal actions. The goal of this paper is to extend this result of Koikara and Mukerjee to all free Z_p–actions, $p > 2$, and to all free S^1–actions. Specifically, we obtain parametrized Borsuk–Ulam theorems for bundles whose fibre has the same cohomology (mod p) as a product of spheres with any free $\mathbb{Z}_p$–action and for bundles whose fibre has rational cohomology ring isomorphic to the rational cohomology ring of a product of spheres with any free S^1–action. Further, in the particular case where $G = \mathbb{Z}_p$, we estimate the "size" of the $\mathbb{Z}_p$–coincidence set of a fibre-preserving map. When the base B of the involved bundles is a single point, such an estimate coincides with Munkholm's estimate given in the classical Borsuk–Ulam theorem for $\mathbb{Z}_p$–actions.

The paper is organized as follows. In Section 2, we recall definitions, fix notation and state results needed. In Section 3, we state the main theorems of the work. In Section 4, we characterize the cohomology ring of orbit space X/G, where X has the same cohomology as a product of spheres. In Section 5, we define the characteristic polynomials associated to the involved fiber bundles. In Section 6, we prove the main theorems. Finally, in Section 7, we estimate the "size" of the $\mathbb{Z}_p$–coincidence set of a fibre-preserving map.

2 Preliminaries

We start by introducing some basic notions and notation. We assume that all spaces under consideration are paracompact Hausdorff spaces. H^* denotes Čech cohomology, unless otherwise indicated. The symbol $\cong$ denotes an appropriate isomorphism between algebraic objects. The mod p Bockstein cohomology operation associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ will be denoted by β.

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of a space X. The order of $\mathcal{U}$, denoted by $\operatorname{ord}\mathcal{U}$, is the largest number n such that there is a subcollection $\{U_{\lambda_i}\}_{i=1}^n$ of $\mathcal{U}$ satisfying $\bigcap_{i=1}^n U_{\lambda_i} \neq \varnothing$. Equivalently, $\operatorname{ord}\mathcal{U} = n$ if and only if some point of X lies in n elements of $\mathcal{U}$ and no point of X lies in more than n elements of $\mathcal{U}$. We say that a collection $\mathcal{U}$ has *finite order* if $\operatorname{ord}\mathcal{U} = n$, for some natural number n.

A space X is said to be *finitistic*[1] if every open cover of X has an open refinement with finite order. By the definition, all compact spaces and all finite dimensional (in the sense of the covering dimension) paracompact spaces are finitistic spaces, where the covering dimension of space X, denoted by $\dim X$, is defined as follows: for each integer $n \geq 0$, $\dim X \leq n$ if every finite open cover of X can be refined by an open cover of order $\leq n+1$. If $\dim X \leq n$ and the statement $\dim X \leq n-1$ is false, we say $\dim X = n$. If the statement $\dim X \leq n$ is false for all n, then we say $\dim X = \infty$. For the empty set, $\dim \varnothing = -1$.

We denote by $X \sim_p S^m \times S^n$ a finitistic space with mod p cohomology ring isomorphic to a product of spheres $S^m \times S^n$ admitting a free action of the cyclic group $G = \mathbb{Z}_p$ and by $X \sim_{\mathbb{Q}} S^m \times S^n$ a finitistic space with rational cohomology ring isomorphic to a product of spheres $S^m \times S^n$ admitting a free action of the circle group $G = S^1$.

If G is a compact Lie group which acts freely on a paracompact Hausdorff space X, then $X \to X/G$ is a principal G–bundle [1, Chapter II, Theorem 5.8] and one can take

$$h\colon X/G \to BG$$

a classifying map for the G–bundle $X \to X/G$.

The cases of main interest for us in this paper are $G = \mathbb{Z}_p$, p an odd prime, and $G = S^1$. Recall that for $G = \mathbb{Z}_p$, p an odd prime, we have that

$$H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p[s,t]/(s^2) \tag{2–1}$$

with $\deg s = 1$, $\deg t = 2$ and $t = \beta(s)$, where β is the mod p Bockstein cohomology operation.

[1] The concept of finitistic spaces was introduced by Swan [12] for working in fixed point theory.

For $G = S^1$, we have that

$$H^*(BS^1;\mathbb{Q}) \cong \mathbb{Q}[t] \tag{2–2}$$

with $\deg t = 2$.

3 Main theorems

Given a topological space $X \sim_p S^m \times S^n$ (resp. $X \sim_{\mathbb{Q}} S^m \times S^n$), where $0 < m \leq n$ and m is odd, let $\pi\colon X \hookrightarrow E \to B$ be a fibre bundle with the fibrewise free $\mathbb{Z}_p$–action (resp. free S^1–action) such that the quotient bundle $\overline{\pi}\colon \overline{E} \to B$ has the cohomology extension property, in the sense of [11, Chapter 5, Section 7]. Let us consider $\pi'\colon E' \to B$ a k–dimensional vector bundle with fibrewise G–action on E' which is free on $E'/0$, with k even. If $f\colon E \to E'$ is a fibre-preserving G–equivariant map, denote by $Z_f = f^{-1}(0)$ and $\overline{Z}_f$ the quotient by the free G-action induced on Z_f.

Let $H^*(B)[x,y,z]$ be the polynomial ring over $H^*(B)$ in the indeterminates x, y and z. If $G = \mathbb{Z}_p$, in Section 5, we will define the characteristic polynomials $W_1(x,y,z)$ and $W_2(x,y,z)$ in $H^*(B)[x,y,z]$ and we will show that $H^*(\overline{E})$ and $H^*(B)[x,y,z]/(x^2, W_1(x,y,z), W_2(x,y,z))$ are isomorphic as $H^*(B)$-modules. As a result, each polynomial $q(x,y,z)$ in $H^*(B)[x,y,z]$ defines an element of $H^*(\overline{E})$ which we will denote by $q(x,y,z)|_{\overline{E}}$. We will denote by $q(x,y,z)|_{\overline{Z}_f}$ the image of $q(x,y,z)|_{\overline{E}}$ by the $H^*(B)$–homomorphism $i^*\colon H^*(\overline{E}) \to H^*(\overline{Z}_f)$, where i^* denotes the induced by the natural inclusion.

Similarly if $G = S^1$, we will show that $H^*(B)[y,z]/(W_1(y,z), W_2(y,z))$ and $H^*(\overline{E})$ are isomorphic as $H^*(B)$–modules, where $W_1(y,z)$ and $W_2(y,z))$ are characteristic polynomials. Thus, each polynomial $q(y,z)$ in $H^*(B)[y,z]$ defines elements $q(y,z)|_{\overline{E}}$ and $q(y,z)|_{\overline{Z}_f}$ in $H^*(\overline{E})$ and $H^*(\overline{Z}_f)$, respectively.

Under these conditions, we have the following result:

Theorem 3.1 (Case $G = \mathbb{Z}_p$, p an odd prime) *Suppose that $q(x,y,z)$ in the ring $H^*(B)[x,y,z]$ is a polynomial such that $q(x,y,z)|_{\overline{Z}_f} = 0$. Then there are polynomials $r_1(x,y,z), r_2(x,y,z)$ in $H^*(B)[x,y,z]$ such that*

$$q(x,y,z)W'(x,y) = r_1(x,y,z)W_1(x,y,z) + r_2(x,y,z)W_2(x,y,z)$$

in the ring $H^*(B)[x,y,z]/(x^2)$, where $W'(x,y), W_1(x,y,z)$ and $W_2(x,y,z)$ are characteristic polynomials. [2]

[2] Characteristic polynomials will be defined in Section 5.

As a consequence, we have the following corollary, which is a parametrized version of the Borsuk–Ulam theorem.

Corollary 3.2 *Suppose that the fibre dimension of $E' \to B$ is equal to k. Then $q(x,y,z)|_{\overline{Z}_f} \neq 0$, for all nonzero polynomials in $H^*(B)[x,y,z]$ whose degree in x, y and z is less than $m-k+1$. In other words, the $H^*(B)$–homomorphism*

$$\sum_{i=0}^{(m-k-1)/2} H^*(B).x.y^i \oplus \sum_{i=0}^{(m-k-1)/2} H^*(B).y^i \to H^*(\overline{Z}_f),$$

defined by $x \mapsto x|_{\overline{Z}_f}$ and $y^i \mapsto y^i|_{\overline{Z}_f}$ is a monomorphism. In particular, if $m > k$

$$\operatorname{cohom.dim} \overline{Z}_f \geq \operatorname{cohom.dim}(B) + m - k.$$

Theorem 3.3 (Case $G = S^1$) *Suppose that $q(y,z) \in H^*(B)[y,z]$ is a polynomial such that $q(y,z)|_{\overline{Z}_f} = 0$. Then there are polynomials $r_1(y,z)$ and $r_2(y,z)$ in $H^*(B)[y,z]$ such that*

$$q(y,z)W'(y) = r_1(y,z)W_1(y,z) + r_2(y,z)W_2(y,z),$$

where $W'(y)$, $W_1(y,z)$ and $W_2(y,z)$ are characteristic polynomials in $H^*(B)[y,z]$.

We have the following corollary:

Corollary 3.4 *Suppose that the fibre dimension of $E' \to B$ is equal to k. Then $q(y,z)|_{\overline{Z}_f} \neq 0$, for all nonzero polynomials in $H^*(B)[y,z]$, whose degree in y and z is less than $m-k+1$. In other words, the $H^*(B)$–homomorphism*

$$\sum_{i=0}^{(m-k-1)/2} H^*(B).y^i \to H^*(\overline{Z}_f)$$

defined by $y^i \mapsto y^i|_{\overline{Z}_f}$ is a monomorphism. In particular, if $m > k$

$$\operatorname{cohom.dim} \overline{Z}_f \geq \operatorname{cohom.dim}(B) + m - k - 1.$$

Remark 3.5 Suppose that in Corollary 3.2 (resp. Corollary 3.4) B is a point. Then for any $\mathbb{Z}_p$–equivariant map (resp. S^1–equivariant map) $f\colon X \sim_p S^m \times S^n \to \mathbb{R}^k$, we have that $\operatorname{cohom.dim} \overline{Z}_f \geq m-k$ (resp. $\operatorname{cohom.dim} \overline{Z}_f \geq m-k-1$).

Remark 3.6 Theorem 3.3 and Corollary 3.4 extend the result proved by Nakaoka [10, Theorem 1(ii) and its corollary] for bundles whose fibre has rational cohomology ring isomorphic to rational cohomology ring of a product of spheres, in case of free S^1–action.

4 The cohomology rings of the orbit space X/G

The results of this section are based upon work of Dotzel et al [4]. Using as main tool the Leray–Serre spectral sequence, they determined the possible cohomology algebra of the orbit space X/G, where X and G satisfy the properties required in Section 2. In the same conditions on X and G, we have the following lemmas, which are consequences of [4, Theorems 1 and 3].

Lemma 4.1 (Case $G = \mathbb{Z}_p$, p an odd prime) *Let $X \sim_p S^m \times S^n$ be a topological space, where $0 < m \leq n$ and m is odd. Suppose that $H^*(X;\mathbb{Z})$ is of finite type. Then $H^*(X/G;\mathbb{Z}_p)$ is a free graded module generated by the elements*

$$1, a, b, ab, \dots, b^{(m-1)/2}, ab^{(m-1)/2}, d, ad, abd, \dots, ab^{(m-1)/2}d$$

subject to the relations $a^2 = 0$, $b^{(m+1)/2} = 0$ and $d^2 = 0$, where $a \in H^1(X/G;\mathbb{Z}_p)$, $b = \beta(a) \in H^2(X/G;\mathbb{Z}_p)$ and $d \in H^n(X/G;\mathbb{Z}_p)$.

Proof It follows from [4, Theorem 1(i)] that the cohomology ring $H^*(X/G;\mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p[x,y,z]/(x^2, y^{(m+1)/2}, z^2)$ as a graded commutative algebra, where m is odd, $\deg x = 1$, $y = \beta(x)$ and $\deg z = n$. Therefore, the homomorphism

$$\mathbb{Z}_p[x,y,z]/(x^2, y^{(m+1)/2}, z^2) \to H^*(X/G;\mathbb{Z}_p) \quad \text{given by} \quad (x,y,z) \mapsto (a,b,d)$$

is an isomorphism of $\mathbb{Z}_p$–algebras. □

Lemma 4.2 (Case $G = S^1$) *Let $X \sim_{\mathbb{Q}} S^m \times S^n$ be a topological space, where $0 < m \leq n$ and m is odd. Then $H^*(X/G;\mathbb{Q})$ is a free graded module generated by the elements*

$$1, b, b^2, \dots, b^{(m-1)/2}, d, db, db^2 \dots, db^{(m-1)/2}$$

subject to the relations $b^{(m+1)/2} = 0$ and $d^2 = 0$, where $b \in H^2(X/G;\mathbb{Q})$ and $d \in H^n(X/G;\mathbb{Q})$.

Proof By [4, Theorem 3(i)], $H^*(X/G;\mathbb{Q})$ is isomorphic to $\mathbb{Q}[y,z]/(y^{(m+1)/2}, z^2)$ as a graded commutative algebra, where m is odd, $\deg y = 2$ and $\deg z = n$. Thus, the homomorphism

$$\mathbb{Q}[y,z]/(y^{(m+1)/2}, z^2) \to H^*(X/G;\mathbb{Q}) \quad \text{given by} \quad (y,z) \mapsto (b,d)$$

is an isomorphism of $\mathbb{Q}$–algebras. □

5 Characteristic polynomials

Using the technique introduced by Dold, in this section we define the characteristic polynomials associated to the fibre bundle (X, E, π, B). As in [7], we need to assume that the quotient bundle $(X/G, \bar{E}, \bar{\pi}, B)$, where G is $\mathbb{Z}_p$ or S^1, has the cohomology extension property and then the Leray–Hirsch theorem can be applied. There are two cases to consider, as follows.

5.1 Case $G = \mathbb{Z}_p$, p an odd prime

Let $(X \sim_p S^m \times S^n, E, \pi, B)$ be a fibre bundle with the same hypotheses of Section 3 and let us consider the quotient bundle $(X/G, \bar{E}, \bar{\pi}, B)$. It follows from the Leray–Hirsch theorem that there exist elements $\mathbf{a} \in H^1(\bar{E})$, $\mathbf{b} \in H^2(\bar{E})$ and $\mathbf{d} \in H^n(\bar{E})$ such that the natural homomorphism $j^*\colon H^*(\bar{E}) \to H^*(X/G)$ maps $\mathbf{a}$ to a, $\mathbf{b}$ to b and $\mathbf{d}$ to d, where a, b and d are as in Lemma 4.1. Furthermore, $H^*(\bar{E})$ is a $H^*(B)$–module, via the induced homomorphism $\bar{\pi}^*$, generated by

$$1, \mathbf{a}, \mathbf{b}, \mathbf{ab}, \dots, \mathbf{b}^{(m-1)/2}, \mathbf{ab}^{(m-1)/2}, \mathbf{d}, \mathbf{ad}, \mathbf{abd}, \dots, \mathbf{ab}^{(m-1)/2}\mathbf{d}. \tag{5–1}$$

Let us first consider natural numbers m and n satisfying $1 < m < n$. We can express the elements

$$\mathbf{b}^{(m+1)/2} \in H^{m+1}(\bar{E}) \quad \text{and} \quad \mathbf{d}^2 \in H^{2n}(\bar{E})$$

in terms of the basis (5–1), that is, there exist unique elements $\omega_i, \nu_i \in H^i(B)$ such that

$$\mathbf{b}^{(m+1)/2} = \omega_{m+1} + \omega_m \mathbf{a} + \omega_{m-1}\mathbf{ab} + \cdots + \omega_2 \mathbf{b}^{(m-1)/2} + \omega_1 \mathbf{ab}^{(m-1)/2} + \alpha \mathbf{d},$$

where $\alpha \in \mathbb{Z}_p$, with $\alpha = 0$ if $n > m+1$ and

$$\mathbf{d}^2 = \nu_{2n} + \nu_{2n-1}\mathbf{a} + \cdots + \nu_{2n-m}\mathbf{ab}^{(m-1)/2} + \nu_n \mathbf{d} + \nu_{n-1}\mathbf{ad} + \cdots + \nu_{n-m}\mathbf{ab}^{(m-1)/2}\mathbf{d}.$$

Definition 5.1 The characteristic polynomials in the indeterminates x, y and z of degrees respectively $1, 2$ and n associated to the fibre bundle $(X \sim_p S^m \times S^n, E, \pi, B)$ are defined as follows:

$$\begin{aligned} W_1(x,y,z) &= \omega_{m+1} + \omega_m x + \cdots + \omega_1 x y^{(m-1)/2} + y^{(m+1)/2} + \alpha z, \\ W_2(x,y,z) &= \nu_{2n} + \nu_{2n-1}x + \cdots + \nu_{2n-m} x y^{(m-1)/2} + \nu_n z + \cdots \\ &\qquad + \nu_{n-m} x y^{(m-1)/2} z + z^2, \end{aligned}$$

where $\omega_i, \nu_i \in H^i(B)$ and $1 < m < n$. If we consider natural numbers m and n such that $1 < m = n$, we can express the elements

$$\mathbf{b}^{(m+1)/2} \in H^{m+1}(\bar{E}) \quad \text{and} \quad \mathbf{d}^2 \in H^{2m}(\bar{E})$$

in terms of the basis (5–1), as follows:

$$\mathbf{b}^{(m+1)/2} = \omega_{m+1} + \omega_m \mathbf{a} + \cdots + \omega_1 \mathbf{ab}^{(m-1)/2} + \overline{\omega}_1 \mathbf{d} + \alpha \mathbf{ad}$$

and
$$\mathbf{d}^2 = \nu_{2m} + \nu_{2m-1}\mathbf{a} + \cdots + \nu_m \mathbf{ab}^{(m-1)/2} + \overline{\nu}_m \mathbf{d} + \nu_{m-1}\mathbf{ad} + \cdots + \gamma \mathbf{ab}^{(m-1)/2}\mathbf{d},$$

for unique elements $\omega_i, \overline{\omega}_i, \nu_i, \overline{\nu}_i \in H^i(B)$ and $\alpha, \gamma \in \mathbb{Z}_p$. In this case, the characteristic polynomials are given by

$$W_1(x,y,z) = \omega_{m+1} + \omega_m x + \cdots + \omega_1 x y^{(m-1)/2} + y^{(m+1)/2} + \overline{\omega}_1 z + \alpha x z,$$
$$W_2(x,y,z) = \nu_{2m} + \nu_{2m-1} x + \cdots + \nu_m x y^{(m-1)/2} + \overline{\nu}_m z + \nu_{m-1} x z + \cdots + \gamma x y^{(m-1)/2} z + z^2, \text{ for } 1 < m = n.$$

We can substitute these elements for the indeterminates x, y and z respectively and obtain the homomorphism of $H^*(B)$–algebras

$$\sigma\colon H^*(B)[x,y,z] \to H^*(\bar{E}) \text{ given by } (x,y,z) \mapsto (\mathbf{a},\mathbf{b},\mathbf{d}). \tag{5–2}$$

We have that $\mathrm{Ker}(\sigma)$ is an ideal generated by the characteristic polynomials x^2, $W_1(x,y,z)$ and $W_2(x,y,z)$ and consequently,

$$\frac{H^*(B)[x,y,z]}{(x^2, W_1(x,y,z), W_2(x,y,z))} \cong H^*(\bar{E}). \tag{5–3}$$

Now, given a polynomial $q(x,y,z) \in H^*(B)[x,y,z]$ we will denote by $q(x,y,z)|_{\overline{E}}$ the image of $q(x,y,z)$ by the map σ defined in (5–2) and $q(x,y,z)|_{\overline{Z}_f}$ the image of $q(x,y,z)$ by the composite

$$H^*(B)[x,y,z] \to H^*(\bar{E}) \to H^*(\overline{Z}_f)$$

given by $(x,y,z) \mapsto (\mathbf{a},\mathbf{b},\mathbf{d}) \mapsto (i^*\mathbf{a}, i^*\mathbf{b}, i^*\mathbf{d})$, where i^* denotes the induced by the natural inclusion.

Next, we need to define the characteristic polynomials associated to the k–dimensional vector bundle $\pi'\colon E' \to B$ with fibrewise $\mathbb{Z}_p$–action on E' which is free on $E'/0$, with k even. For this, let us denote by SE' the total space of sphere bundle of $\pi'\colon E' \to B$. Since $\mathbb{Z}_p$ acts freely on SE' we obtain the lens-bundle $(L_p^{k-1}, \overline{SE'}, \overline{\pi'}, B)$ and the principal $\mathbb{Z}_p$–bundle $SE' \to \overline{SE'}$, where L_p^{k-1} denotes the $(k-1)$–dimensional lens space. We have that $H^*(L_p^{k-1};\mathbb{Z}_p) \cong \mathbb{Z}_p[a',b']/((a')^2,(b')^{k/2})$, with $a' = (i')^*(s) \in H^1(L_p^{k-1};\mathbb{Z}_p)$ and $b' = (i')^*(t) \in H^2(L_p^{k-1};\mathbb{Z}_p)$, where $s \in H^1(B\mathbb{Z}_p)$ and $t \in H^2(B\mathbb{Z}_p)$ are defined in (2–1) and $i'\colon L_p^{k-1} \to B\mathbb{Z}_p$ is a classifying map for the principal $\mathbb{Z}_p$–bundle $S^{k-1} \to L_p^{k-1}$. Let us consider the classes $\mathbf{a}' = h^*(s) \in H^1(\overline{SE'})$

and $\mathbf{b}' = h^*(t) \in H^2(\overline{SE'})$, where $h\colon \overline{SE'} \to B\mathbb{Z}_p$ is a classifying map for the principal $\mathbb{Z}_p$–bundle $SE' \to \overline{SE'}$. The $\mathbb{Z}_p$-module homomorphism $\theta\colon H^*(L_p^{k-1};\mathbb{Z}_p) \to H^*(\overline{SE'};\mathbb{Z}_p)$ defined by $a' \mapsto \mathbf{a}'$ and $b' \mapsto \mathbf{b}'$ is a cohomology extension of the fibre. Then, it follows from Leray–Hirsch theorem that $H^*(\overline{SE'})$ is a $H^*(B)$–module, via the induced homomorphism $(\overline{\pi'})^*$, generated by the elements

$$1, \mathbf{a}',\ \mathbf{b}',\ \mathbf{a}'\,\mathbf{b}', \dots, \mathbf{a}'\,(\mathbf{b}')^{(k-2)/2}.$$

We can express $(\mathbf{b}')^{k/2} \in H^k(\overline{SE'})$ as follows:

$$(\mathbf{b}')^{k/2} = \omega'_k + \omega'_{k-1}\mathbf{a}' + \cdots + \omega'_1\mathbf{a}'(\mathbf{b}')^{(k-2)/2},$$

for unique elements $\omega'_i \in H^i(B)$.

Definition 5.2 The characteristic polynomial in the indeterminates x and y of degrees respectively 1 and 2, associated to the vector bundle $E' \to B$ is defined as follows:

$$W'(x,y) = \omega'_k + \omega'_{k-1}x + \cdots + \omega'_1 x y^{(k-2)/2} + y^{k/2},$$

where $\omega'_i \in H^i(B)$.

From similar arguments to those used above, we have the following isomorphism of $H^*(B)$–algebras:

$$\frac{H^*(B)[x,y]}{(x^2, W'(x,y))} \cong H^*(\overline{SE'}) \quad \text{defined by} x \mapsto \mathbf{a}' \text{ and } y \mapsto \mathbf{b}'.$$

5.2 Case $G = S^1$

Let $(X \sim_{\mathbb{Q}} S^m \times S^n, E, \pi, B)$ be a fibre bundle as in Section 3 and consider the quotient bundle $(X/G, \overline{E}, \overline{\pi}, B)$. It follows from the Leray–Hirsch theorem and Lemma 4.2 that

$$\frac{H^*(B)[y,z]}{(W_1(y,z), W_2(y,z))} \cong H^*(\overline{E}) \tag{5–4}$$

where, for $1 < m < n$,

$$W_1(y,z) = \omega_{m+1}1 + \omega_{m-1}y + \cdots + \omega_2 y^{(m-1)/2} + y^{(m+1)/2} + \alpha z$$

and
$$W_2(y,z) = \nu_{2n} + \nu_{2n-2}y + \cdots + \nu_{2n-(m-1)}y^{(m-1)/2} + \nu_n z + \nu_{n-2}yz + \cdots + \nu_{n-(m-1)}y^{(m-1)/2}z + z^2$$

are the characteristic polynomials associated to $(X \sim_{\mathbb{Q}} S^m \times S^n, E, \pi, B)$, where $\omega_i, \nu_i \in H^i(B)$, $\alpha \in \mathbb{Q}$, and $\alpha = 0$ if $n > m+1$.

In case that $1 < m = n$, we have the characteristic polynomials

$$W_1(y,z) = \omega_{m+1} + \omega_{m-1} y + \cdots + \omega_2 y^{(m-1)/2} + y^{(m+1)/2} + \bar{\omega_1} z$$

and $$W_2(y,z) = \nu_{2m} + \nu_{2m-2} y + \cdots + \nu_{2m-(m-1)} y^{(m-1)/2} + \nu_m z + \nu_{m-2} yz + \cdots + \nu_1 y^{(m-1)/2} z + \gamma z,$$

where $\omega_i, \bar{\omega_i} \in H^i(B)$ and $\gamma \in \mathbb{Q}$.

Now, let us consider the k–dimensional vector bundle $\pi'\colon E' \to B$ with fibrewise S^1–action on E' which is free on $E'/0$, with k even. Let us denote by SE' the total space of sphere bundle of $\pi'\colon E' \to B$. Since S^1 acts freely on SE' we obtain the complex projective-bundle $(P_{(k-2)/2}(\mathbb{C}), \overline{SE'}, \overline{\pi'}, B)$ and the principal S^1–bundle $SE' \to \overline{SE'}$, where $P_{(k-2)/2}(\mathbb{C}) = S^{k-1}/S^1$ denotes the $(k-2)$–dimensional complex projective space . We have that $H^*(P_{(k-2)/2}(\mathbb{C});\mathbb{Q}) \cong \mathbb{Q}[b']/((b')^{k/2})$, with $b' = i^*(t) \in H^2(P_{(k-2)/2}(\mathbb{C});\mathbb{Q})$, where $t \in H^2(BS^1;\mathbb{Q})$ is defined in (2–2) and $i\colon P_{(k-2)/2}(\mathbb{C}) \to BS^1$ is a classifying map for the principal S^1–bundle $S^{k-1} \to P_{(k-2)/2}(\mathbb{C})$.

Following the same argument of the previous case, we have that

$\dfrac{H^*(B)[y]}{(W'(y))} \cong H^*(\overline{SE'})$, where $W'(y) = \omega'_{m+1} 1 + \omega'_{m-1} y + \cdots + \omega'_2 y^{(k-2)/2} + y^{k/2}$ is the characteristic polynomial associated to vector bundle $E' \to B$.

6 Proof of the main theorems

Proof of Theorem 3.1 Let $q(x,y,z)$ be a polynomial in $H^*(B)[x,y,z]$ such that $q(x,y,z)|_{\overline{Z}_f} = 0$. It follows from the continuity of the cohomology theory, that there is an open subset V in $\bar{E}$, with $V \supset \overline{Z}_f$ such that $q(x,y,z)|_V = 0$. One has that from the exact sequence

$$\cdots \longrightarrow H^*(\bar{E}, V) \xrightarrow{j_1^*} H^*(\bar{E}) \longrightarrow H^*(V) \longrightarrow \cdots$$

there exists $\mu \in H^*(\bar{E}, V)$ such that $j_1^*(\mu) = q(x,y,z)|_{\bar{E}}$, where the map $j_1\colon \bar{E} \to (\bar{E}, V)$ denotes the natural inclusion. One can take the map of the orbit spaces $\bar{f}\colon \bar{E} - \overline{Z}_f \to \overline{E'} - \{0\}$ induced by the equivariant map $f\colon E \to E'$. Since the induced cohomology homomorphism $\bar{f}^*$ is a $H^*(B)$–homomorphism and $W'(\mathbf{a}', \mathbf{b}') = 0$, we have that

$$W'(x,y)|_{\bar{E}-\overline{Z}_f} = W'(\mathbf{a}, \mathbf{b}) = W'(\bar{f}^*(\mathbf{a}'), \bar{f}^*(\mathbf{b}')) = \bar{f}^*(W'(\mathbf{a}', \mathbf{b}')) = 0.$$

On the other hand, from the exact sequence

$$\cdots \longrightarrow H^*(\bar{E}, \bar{E} - \overline{Z}_f) \xrightarrow{j_2^*} H^*(\bar{E}) \longrightarrow H^*(\bar{E} - \overline{Z}_f) \longrightarrow \cdots$$

there exists $\theta \in H^*(\bar{E}, \bar{E} - \overline{Z}_f)$ satisfying the condition $j_2^*(\theta) = W'(x, y)|_{\bar{E}}$, where $j_2\colon \bar{E} \to (\bar{E}, \bar{E} - \overline{Z}_f)$ is the natural inclusion. Hence,

$$q(x, y, z)W'(x, y)|_{\bar{E}} = j_1^*(\mu) j_2^*(\theta) = j^*(\mu \smile \theta)$$

by naturality of cup product. Let us observe that

$$\mu \smile \theta \in H^*(\bar{E}, V \cup (\bar{E} - \overline{Z}_f)) = H^*(\bar{E}, \bar{E})$$

which implies $\mu \smile \theta = 0$. Thus, $q(x, y, z)W'(x, y)|_{\bar{E}} = 0$ and by (5–3) we conclude that there exist polynomials $r_1(x, y, z)$ and $r_2(x, y, z)$ in $H^*(B)[x, y, z]$ such that

$$q(x, y, z)W'(x, y) = r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z)$$

in the ring $H^*(B)[x, y, z]/(x^2)$. This completes the proof. □

Proof of Corollary 3.2 Let $q(x, y, z) \in H^*(B)[x, y, z]$ be a nonzero polynomial satisfying the condition $\deg q(x, y, z) < m - k + 1$ and suppose by contradiction that $q(x, y, z)|_{\overline{Z_f}} = 0$. One then has, by Theorem 3.1 that

$$q(x, y, z)W'(x, y) = r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z)$$

in $H^*(B)[x, y, z]/(x^2)$. Note that $\deg W'(x, y) = k$, $\deg W_1(x, y, z) = m + 1$ and $\deg W_2(x, y, z) = 2n$, implies $\deg q(x, y, z) \geq m + 1 - k$, which is impossible. □

Proof of Theorem 3.3 Let $q(y, z)$ be a polynomial in the ring $H^*(B)[y, z]$ such that $q(y, z)|_{\overline{Z}_f} = 0$. By similar arguments used in the proof of Theorem 3.1, we conclude that $q(y, z)W'(y)|_{\bar{E}} = 0$. Therefore, by (5–4) we have that there are polynomials $r_1(y, z)$ and $r_2(y, z)$ in $H^*(B)[y, z]$ such that

$$q(y, z)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z).$$ □

Proof of Corollary 3.4 Let $q(y, z) \in H^*(B)[y, z]$ be a nonzero polynomial such that $\deg q(y, z) < m - k + 1$. If $q(y, z)|_{\overline{Z}_f} = 0$, by Theorem 3.3

$$q(y, z)W'(y) = r_1(y, z)W_1(y, z) + r_2(y, z)W_2(y, z),$$

where $\deg W'(y) = k$, $\deg W_1(y, z) = m + 1$ and $\deg W_2(y, z) = 2n$. Thus, we conclude that $\deg q(y, z) \geq m + 1 - k$, which is a contradiction. □

7 Estimating the size of the $\mathbb{Z}_p$–coincidence set

Let $(X \sim_p S^m \times S^n, E, \pi, B)$ be a fibre bundle as in Section 3. Now consider $E'' \to B$ a vector bundle of dimension l and let $f\colon E \to E''$ be a fibre-preserving map (here, we do not assume that E'' has a $\mathbb{Z}_p$–action). Suppose that $T\colon E \to E$ is a generator of the free $\mathbb{Z}_p$–action in E. The $\mathbb{Z}_p$–coincidence set $A(f)$ is the set of points x in E such that f maps the entire $\mathbb{Z}_p$–orbit of x to a single point, that is,

$$A(f) = \{x \in E; f(T^i(x)) = f(x),\ \forall i = 1, \dots, p-1\}.$$

In the above conditions, one has the following:

Theorem 7.1 cohom. dim $A(f) \geq$ cohom. dim$(B) + m - (p-1)l$.

Proof Consider a vector bundle $M \to B$, which is the Whitney sum of p copies of the l–dimensional vector bundle $E'' \to B$. One then has that $M = E'' \oplus \ldots \oplus E''$ admits an action of the cyclic group $\mathbb{Z}_p$, generated by a periodic homeomorphism $t_M\colon M \to M$ of period p given by

$$t_M(m_1, \dots, m_{p-1}, m_p) = (m_p, m_1, \dots, m_{p-1}),$$

for each $(m_1, \dots, m_p) \in M$.

Denote by Δ the subspace of M consisting of the all points $(m_1, \dots, m_p)$ in M such that $m_1 = \cdots = m_p$. Therefore $\Delta \to B$ is a subbundle of $M \to B$, which is called diagonal bundle. Each fibre M_b of M can be represented as a direct sum $\Delta_b \oplus \Delta_b^{\perp}$, where $\Delta_b^{\perp}$ is the orthogonal complement of Δ_b. The bundle $M \to B$ is the Whitney sum of the bundles $\Delta \to B$ and $\Delta^{\perp} \to B$. Observe that $\Delta^{\perp}$ is a $\mathbb{Z}_p$–subspace of M and $\mathbb{Z}_p$ acts freely on the sphere bundle $S\Delta^{\perp} \subset \Delta^{\perp}$. Since $\Delta \to B$ is a l–dimensional bundle, the fibre dimension of $\Delta^{\perp} \to B$ is equal to $k = (p-1)l$, which is even. Consider the fibre-preserving $\mathbb{Z}_p$–equivariant map $F\colon E \to M$ defined by

$$F(x) = (f(x), f(Tx), \dots, f(T^{p-1}x)).$$

The linear projection along of the diagonal defines an equivariant fibre-preserving map $r\colon (M, M - \Delta) \to (\Delta^{\perp}, \Delta^{\perp} - 0)$, where 0 is the zero section of $\Delta^{\perp}$. Let $h = F \circ r$ be the composition given by

$$(E, E - A(f)) \to (M, M - \Delta) \to (\Delta^{\perp}, \Delta^{\perp} - 0)$$

with $Z_h = h^{-1}(0) = (F \circ r)^{-1}(0) = F^{-1}(\Delta) = A(f)$. Since $h\colon E \to \Delta^{\perp}$ is an equivariant fibre-preserving map, it follows from Corollary 3.2 that

$$\text{cohom. dim}\,\overline{Z}_h = \text{cohom. dim}\,\overline{A(f)} \geq \text{cohom. dim}(B) + m - (p-1)l, \qquad \square$$

Remark 7.2 In the particular case where B is a single point and $f\colon X \sim_p S^m \times S^n \to \mathbb{R}^l$ is a continuous map, with $m > (p-1)l$, one has that $A(f) \neq \varnothing$. Moreover,

$$\text{cohom. dim}\, A(f) \geq m - (p-1)l.$$

This result extends to spaces which have the same cohomology (mod p) as a product of spheres the classical version of the Borsuk–Ulam theorem for $\mathbb{Z}_p$–actions proved by Munkholm in [9].

Remark 7.3 Koikara and Mukerjee [8] obtained an estimate of the "size" of the $\mathbb{Z}_p$–coincidence set $A(f)$ for maps of fibre bundles, with closed orientable differentiable manifolds as fibres, under certain conditions. Observe that the estimate determined by Theorem 7.1 and Remark 7.2 cannot be obtained from [8], since in Theorem 7.1 the fibres of the bundles are, respectively, $X \sim_p S^m \times S^n$ (which in general is not a differentiable manifold) and $\mathbb{R}^l$ (which is an open manifold).

Remark 7.4 Consider a bundle $\pi\colon E \to B$ whose fibre is either a sphere or a product of spheres, and a vector bundle $\pi''\colon E'' \to B$ such that $\mathbb{Z}_p$ is fibre-preserving and acts freely on E (here, we do not assume that E'' has a $\mathbb{Z}_p$–action). Let $f\colon E \to E''$ be a fibre-preserving map. Suppose that $T\colon E \to E$ is a generator of the free $\mathbb{Z}_p$–action on E and define the following set:

$$\begin{aligned} A(f,q) = \{x \in E \mid \text{there exist } i_1, i_2, \dots, i_q \text{ with } 0 < i_1 < i_2 < \cdots < i_q \leq p \\ \text{and } f(T^{i_1}x) = f(T^{i_2}x) = \cdots = f(T^{i_q}x)\} \end{aligned}$$

An interesting question, raised by the referee, is to estimate the cohomological dimension of the set $A(f,q)$. Note that if $q = p$, we have that $A(f,q) = A(f)$ and the question is answered by Theorem 7.1. Estimates of the above type are given by Cohen and Lusk [2] in the nonparametrized case.

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Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

Departamento de Matemática, Universidade Federal de São Carlos
Caixa Postal 676, São Carlos, SP 13565-905, Brazil

deniseml@icmc.usp.br, edivaldo@dm.ufscar.br

http://www.icmc.usp.br/~topologia/, http://www2.dm.ufscar.br/~edivaldo/

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