The homotopy Lie algebra of the complements of subspace arrangements with geometric lattices

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A subspace arrangement in $\mathbb{C}^l$ is a finite set $\mathcal{A}$ of subspaces of $\mathbb{C}^l$. The complement space $M(\mathcal{A})$ is $\mathbb{C}^l \setminus \cup_{x\in\mathcal{A}} x$. If $M(\mathcal{A})$ is elliptic, then the homotopy Lie algebra $\pi_\star(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$ is finitely generated. In this paper, we prove that if $\mathcal{A}$ is a geometric arrangement such that $M(\mathcal{A})$ is a hyperbolic 1–connected space, then there exists an injective map $\mathbb{L}(u,v) \to \pi_\star(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$ where $\mathbb{L}(u,v)$ denotes a free Lie algebra on two generators.

55P62

1 Introduction

A subspace arrangement in $\mathbb{C}^l$ is a finite set $\mathcal{A} = \{x_1, \dots, x_n\}$ of subspaces of $\mathbb{C}^l$. If every $x_i \in \mathcal{A}$ is an hyperplane, it is called an arrangement of hyperplanes. The complement space is the topological space $M(\mathcal{A}) = \mathbb{C}^l \setminus \cup_{x\in\mathcal{A}} x$. To every subspace arrangement, we can associate the lattice $L(\mathcal{A})$ of intersections, ordered by reverse inclusions. In this paper, we are mainly interested in the (rational) homotopy Lie algebra $\pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}$. In general, the homotopy Lie algebra can be defined for any commutative graded algebra.

Definition 1.1 Let A be a commutative graded algebra. The algebra $\mathrm{Ext}_A(\mathbb{Q},\mathbb{Q})$ is the universal enveloping algebra $U(L_A)$ of a Lie algebra L_A. This Lie algebra is called the *homotopy Lie algebra* of A.

An important tool for the study of arrangements of hyperplanes is the Orlik–Solomon algebra $A(\mathcal{A})$. This algebra, which is constructed using only $L(\mathcal{A})$, is the quotient of an exterior algebra by a homogeneous ideal. Orlik and Solomon [5] showed that there is an isomorphism of graded algebras $H^\star(M(\mathcal{A}),\mathbb{Q}) \simeq A(\mathcal{A})$.

If $\mathcal{A}$ is an arrangement of hyperplanes, the homotopy Lie algebra $L_{A(\mathcal{A})}$ can be complicated. For example, J Roos [6] showed the existence of arrangements such that $L_{A(\mathcal{A})}$ is not finitely generated. In some cases, $L_{A(\mathcal{A})}$ can be described more precisely. Denham and Suciu [2] showed that if $\mathcal{A}$ is an hypersolvable arrangement of

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hyperplanes (with an additional technical condition), then $L_{A(\mathcal{A})}$ splits as a semi-direct product of a Lie algebra and a free Lie algebra.

These results show that the Lie algebra $L_{A(\mathcal{A})}$ can be difficult to grasp. In this paper, we will study the more general case of subspace arrangements. For subspace arrangements, we cannot use the Orlik–Solomon algebra. Instead, we will use a rational model described by Yuzvinsky and Feichtner (see Section 2). It is a differential algebra $(D_{\mathcal{A}}, d)$ generalizing the Orlik–Solomon algebra whose cohomology satisfies $H^\star(D_{\mathcal{A}}, d) \simeq H^\star(M(\mathcal{A}), \mathbb{Q})$.

It is known that if $\mathcal{A}$ is a subspace arrangement with a geometric lattice $L(\mathcal{A})$, then the topological space $M(\mathcal{A})$ is formal (see Yuzvinsky [7]). If $\mathcal{A}$ is also such that $M(\mathcal{A})$ is 1–connected (if $\operatorname{codim}(x) \geq 2$ for all $x \in \mathcal{A}$), then the homotopy Lie algebra of $H^\star(D_{\mathcal{A}}, d)$ has a topological interpretation:

$$L_{H^\star(D_{\mathcal{A}},d)} = \pi_\star(\Omega M(\mathcal{A})) \otimes \mathbb{Q}.$$

Using their results for arrangements of hyperplanes, Denham and Suciu described $L_{A(\mathcal{A})}$ for a very particular class of subspace arrangements (these subspace arrangements have a geometric lattice). This description shows that $L_{A(\mathcal{A})}$ contains a free Lie algebra.

Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a subspace arrangement with a geometric lattice and such that $M(\mathcal{A})$ is 1–connected. The sum $x_1^\perp + \cdots + x_n^\perp$ is a direct sum if and only if $M(\mathcal{A})$ is elliptic (see Debongnie [1]). In that case, $\pi_\star(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$ is finitely generated and abelian. Otherwise, $M(\mathcal{A})$ is hyperbolic and $\pi_\star(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$ is more complicated. The main result of this paper is the following theorem.

Theorem *Let $\mathcal{A}$ be a geometric arrangement such that for every $x \in \mathcal{A}$, we have* $\operatorname{codim}(x) \geq 2$. *If $M(\mathcal{A})$ is rationally hyperbolic, then there exists an injective map* $\mathbb{L}(u, v) \to \pi_\star(\Omega M(\mathcal{A})) \otimes \mathbb{Q}$.

Note that this is a particular case of a conjecture by Avramov–Félix.

Conjecture *If X is finite dimensional, not $\mathbb{Q}$–elliptic, then the homotopy Lie algebra $\pi_\star(\Omega X)) \otimes \mathbb{Q}$ contain a free Lie subalgebra on two generators.*

The rational model of the space $M(\mathcal{A})$ given by Yuzvinsky and Feichtner is described in Section 2. In Section 3, the general situation is set up : a map $\varphi\colon \Lambda(e_1, \dots, e_n) \to H^\star(M(\mathcal{A}), \mathbb{Q}); e_i \mapsto [\{x_i\}]$ is defined and studied. This map and its kernel will play an important role in the proof. Finally, the last two sections contain the proof of the main theorem.

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2 The rational model of subspace arrangements

Let $\mathcal{A}$ be a central arrangement of subspace in $\mathbb{C}^l$. It is known that, with the appropriate choice of the operations $\vee$ and $\wedge$, the set $L(\mathcal{A})$ of non empty intersections of elements of $\mathcal{A}$ is a lattice with a rank function. Yuzvinsky and Feichtner [3] defined the relative atomic differential graded algebra $(D_{\mathcal{A}}, d)$ associated with an arrangement as follows. Choose a linear order on $\mathcal{A}$. The graded vector space $D_{\mathcal{A}}$ has a basis given by all subsets $\sigma \subseteq \mathcal{A}$. For $\sigma = \{x_1, \dots, x_n\}$, we define the differential by the formula

$$d\sigma = \sum_{j : \vee(\sigma \setminus \{x_j\}) = \vee\sigma} (-1)^j (\sigma \setminus \{x_j\})$$

where the indexing of the elements in σ follows the linear order imposed on $\mathcal{A}$. With $\deg(\sigma) = 2\operatorname{codim} \vee\sigma - |\sigma|$, $(D_{\mathcal{A}}, d)$ is a cochain complex. Finally, we need a multiplication on $(D_{\mathcal{A}}, d)$. For $\sigma, \tau \subseteq \mathcal{A}$,

$$\sigma \cdot \tau = \begin{cases} (-1)^{\operatorname{sgn} \epsilon(\sigma,\tau)} \sigma \cup \tau & \text{if } \operatorname{codim} \vee\sigma + \operatorname{codim} \vee\tau = \operatorname{codim} \vee(\sigma \cup \tau) \\ 0 & \text{otherwise,} \end{cases}$$

where $\epsilon(\sigma, \tau)$ is the permutation that, applied to $\sigma \cup \tau$ with the induced linear order, places elements of τ after elements of σ, both in the induced linear order.

A subset $\sigma \subseteq \mathcal{A}$ is said to be independent if $\operatorname{rank}(\vee\sigma) = |\sigma|$. When $\mathcal{A}$ is a a subspace arrangement with a geometric lattice, then $H^\star(M(\mathcal{A}))$ is generated by the classes $[\sigma]$, with σ independent (see [3]).

3 General situation

Let $\mathcal{A} = \{x_1, \dots, x_n\}$ be a subspace arrangement with a geometric lattice such that every $x \in \mathcal{A}$ has $\operatorname{codim}(x) \geq 2$. We will suppose that no element x_i is contained in another one, because otherwise, we can omit it when we consider $M(\mathcal{A})$. We consider the morphism of graded algebras

$$\varphi\colon \Lambda(e_1, \dots, e_n) \to H^\star(M(\mathcal{A}), \mathbb{Q}); e_i \mapsto [\{x_i\}].$$

As we will see, in some sense, the kernel of this map measure the non-ellipticity of the space $M(\mathcal{A})$. The following proposition shows a clear connection between $\ker\varphi$ and ellipticity.

Proposition 3.1 *If the map φ is injective, then the space $M(\mathcal{A})$ is rationally elliptic.*

Proof If this map is injective then, for each sequence $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, we have $\{x_{i_1}\} \cdot \{x_{i_2}\} \cdots \{x_{i_s}\} \neq 0$ because their product is non zero in cohomology. Therefore, for an appropriate choice of sign, we have the following equality $\prod_{j=1}^{s}\{x_{i_j}\} = \pm\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ and $[\{x_{i_1}, \dots, x_{i_s}\}] \neq 0$ (in cohomology). This implies that φ is surjective because, for each independent set $\{x_{i_1}, \dots, x_{i_s}\}$ (which generates $H^\star(M(\mathcal{A}))$), we have $[\{x_{i_1}, \dots, x_{i_s}\}] = \pm\prod_{i=j}^{s}[\{x_{i_j}\}]$, which is in the image of φ. It means that φ is an isomorphism. Therefore, $M(\mathcal{A})$ has the rational homotopy type of a product of odd dimensional spheres and [1, Theorem 5.1] implies that $M(\mathcal{A})$ is rationally elliptic. □

Now, assume that the map φ is not injective. In that case, we can define the natural number $r = \max\{s \mid \ker\varphi \subset \Lambda^{\geq s} e_i\}$. It is clear that $2 \leq r \leq n$. The bigger r is, the smaller $\ker\varphi$ is. Also, we understand quite well $\varphi(\Lambda^{\leq r} e_i) \subset D_{\mathcal{A}}$.

Lemma 3.2 *If $\sigma \in D_{\mathcal{A}}$ with $|\sigma| \leq r$, then $d\sigma = 0$ and $\operatorname{rank} \vee\sigma = |\sigma|$.*

Proof We use induction on s to prove that for $1 \leq s < r$ and for each sequence $1 \leq i_1 < i_2 < \cdots < i_s \leq n$,

(1) $d\{x_{i_1}, \dots, x_{i_s}\} = 0$,

(2) $\varphi(e_{i_1} \cdots e_{i_s}) = [\{x_{i_1}, \dots, x_{i_s}\}] \neq 0$.

It is true for $s = 1$. Now suppose that it is true for $s-1$. If $d\{x_{i_1}, \dots, x_{i_s}\} \neq 0$, then $d\{x_{i_1}, \dots, x_{i_s}\}$ is a non zero linear combination $\sum \rho_j \{x_{j_1}, \dots, x_{j_{s-1}}\}$ and

$$0 = \left[\sum \rho_j \{x_{j_1}, \dots, x_{j_{s-1}}\}\right] = \varphi\left(\sum \rho_j\, e_{j_1} \cdots e_{j_{s-1}}\right)$$

which is impossible because φ restricted to $\Lambda^{<r}(e_1, \dots, e_n)$ is injective. This shows that $d\{x_{i_1}, \dots, x_{i_s}\} = 0$.

The map φ is extended in a multiplicative way, therefore, by the induction hypothesis, we have:

$$\varphi(e_{i_1} \cdots e_{i_s}) = \varphi(e_{i_1})\varphi(e_{i_2} \cdots e_{i_s}) = [\{x_{i_1}\}][\{x_{i_2} \cdots x_{i_s}\}].$$

But $s < r$, so $\varphi(e_{i_1} \cdots e_{i_s}) \neq 0$ and we have $\varphi(e_{i_1} \cdots e_{i_s}) = [\{x_{i_1}, \dots, x_{i_s}\}]$. This proves the assertion (2). This proof by induction showed that $d\sigma = 0$ if $|\sigma| < r$. But the exact same reasoning can be done for $|\sigma| = r$. So, $d\sigma = 0$ if $|\sigma| \leq r$.

In order to prove that $\operatorname{rank} \vee\sigma = |\sigma|$, let's prove by induction that if $1 \leq s \leq r$, then for each sequence $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, $\operatorname{rank} \vee\{x_{i_1}, \dots, x_{i_s}\} = s$. It is obviously true for $s = 1$. Assume that it is true until $s - 1 < r$. By the induction hypothesis, $\vee\{x_{i_1},\dots,x_{i_{s-2}}\} < \vee\{x_{i_1},\dots,x_{i_{s-2}},x_{i_s}\}$ is a maximal chain (if $s=2$, then $\vee\{x_{i_1},\dots,x_{i_{s-2}}\} = \vee\varnothing = \mathbb{C}^l$). But the lattice $L(\mathcal{A})$ is geometric. So,

$$\vee\{x_{i_1},\dots,x_{i_{s-2}}\}\vee x_{i_{s-1}} \leq \vee\{x_{i_1},\dots,x_{i_{s-2}},x_{i_s}\}\vee x_{i_{s-1}}$$

is also a maximal chain. The first part of this lemma shows that $d\{x_{i_1},\dots,x_{i_s}\}=0$, which implies that

$$\vee\{x_{i_1},\dots,x_{i_{s-1}}\} \neq \vee\{x_{i_1},\dots,x_{i_{s-2}},x_{i_{s-1}},x_{i_s}\}.$$

Hence, $\operatorname{rank}\vee\{x_{i_1},\dots,x_{i_s}\} = \operatorname{rank}\vee\{x_{i_1},\dots,x_{i_{s-1}}\}+1 = s$. $\square$

To make the next sections easier to read, we will use the following notations. For a commutative differential graded algebra (A,d), let us denote by $L_{(A,d)}$ the homotopy Lie algebra associated to its Sullivan minimal model. And for every $1\leq i_1<i_2<\dots<i_{r+1}\leq n$, let us denote by $[e_{i_1},\dots,e_{i_{r+1}}]$ the element

$$\sum_{j=1}^{r+1}(-1)^j e_{i_1}\cdots\widehat{e_{i_j}}\cdots e_{i_{r+1}}.$$

4 Main result

We will study the situation described in Section 3 with $\ker\varphi\neq 0$ (if $\ker\varphi = 0$, Proposition 3.1 shows that we are in the elliptic case, which is studied in [1]). There are two slightly different cases that can arise : either $\ker\varphi$ contains a monomial $e_{i_1}\cdots e_{i_r}$ or $\ker\varphi$ does not contain such a monomial. The next two propositions shows the existence of an injective map $\mathbb{L}(u,v)\to\pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}$ in these two cases. Then, the main Theorem 4.3 is proved.

Proposition 4.1 *If* $\ker\varphi$ *contains a monomial* $e_{i_1}\cdots e_{i_r}$ *with* $1\leq i_1<\dots<i_r\leq n$, *then there exists an injective map*

$$\mathbb{L}(u,v)\to\pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}.$$

Proof We define $(A_4,0)=\left(\frac{\Lambda(e_{i_1},\dots,e_{i_r})}{e_{i_1}\cdots e_{i_r}},0\right)$ and we construct the map $\psi\colon(D_{\mathcal{A}},d)\to(A_4,0)$ in the following way: if $\{k_1,\dots,k_t\}\subseteq\{i_1,\dots,i_r\}$ and $k_1<\dots<k_t$, then $\psi(\{k_1,\dots,k_t\}) = [e_{k_1}\cdots e_{k_t}]$. Otherwise, $\psi(\{k_1,\dots,k_t\})=0$. Since $\ker\varphi\cap\Lambda^{<r}(e_1,\dots,e_n)=0$, a simple check shows that ψ is multiplicative. Lemma 3.2 shows that $\psi(d\sigma)=\psi(0)=0=d\psi(\sigma)$. Hence, ψ is a morphism of differential graded algebras.

Since $e_{i_1}\cdots e_{i_r}\in\ker\varphi$, we can define another map $\rho\colon (A_4,0)\to H^\star((D_{\mathcal{A}},d),\mathbb{Q})$ by letting $\rho([e_{i_s}])=[\{x_{i_s}\}]$. This is a morphism of graded algebras. Now, we have the following maps :

$$A_4 \xrightarrow{\rho} H^\star((D_{\mathcal{A}},d),\mathbb{Q}) \xrightarrow{H^\star\psi} A_4.$$

Those maps verify the following property: $(H^\star\psi)\circ\rho=\mathrm{id}$, which means that $H^\star\psi$ is a retraction of ρ. Since $M(\mathcal{A})$ is a formal space (proved in [3]), the Lemma 5.6 implies then the existence of an injective map $h\colon L_{(A_4,0)}\to\pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}$. By Lemma 5.4, there is an injective map $\mathbb{L}(u,v)\to L_{(A_4,0)}$. The composition of these two maps gives us the needed map. $\square$

Proposition 4.2 *If* $\ker\varphi$ *does not contain a monomial* $e_{i_1}\cdots e_{i_r}$*, then there exists an injective map*

$$\mathbb{L}(u,v)\to\pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}.$$

Proof Since $\ker\varphi\cap\Lambda^r(e_1,\dots,e_n)\neq\varnothing$, there exists a non zero linear combination $\sum\lambda_{i_1,\dots,i_r}e_{i_1}\dots e_{i_r}$ such that $\varphi(\sum\lambda_{i_1,\dots,i_r}e_{i_1}\cdots e_{i_r})=0$. So,

$$\Big[\sum\lambda_{i_1,\dots,i_r}\{x_{i_1},\dots,x_{i_r}\}\Big]=0$$

in $H^\star(D_{\mathcal{A}},d)$ and there exists a $\sigma\in D_{\mathcal{A}}$ such that $d\sigma=\sum\lambda_{i_1,\dots,i_r}\{x_{i_1},\dots,x_{i_r}\}\neq 0$. From this, we deduce that there exists $1\le i_1<\cdots<i_{r+1}\le n$ such that $d\{x_{i_1},\dots,x_{i_{r+1}}\}\neq 0$.

Let $X=x_{i_1}\vee x_{i_2}\vee\cdots\vee x_{i_{r+1}}$ and $B=\{x\in\mathcal{A}\mid x<X\}=\{x_{j_1},\dots,x_{j_m}\}$. Using Lemma 3.2 and the fact that $d\{x_{i_1},\dots,x_{i_{r+1}}\}\neq 0$, we observe that $\operatorname{rank}X=r$. Also, Lemma 3.2 shows that for any subset $\sigma\subset B$ with r elements, $\operatorname{rank}\vee\sigma=r=\operatorname{rank}X$, so $\vee\sigma=X$. It implies that any $r+1$ product $\prod_{i=1}^{r+1}\{x_{k_i}\}=0$ for x_{k_i} in B. It allows us to define the following map :

$$\rho\colon \frac{\Lambda(e_{j_1},\dots,e_{j_m})}{\Lambda^{\ge r+1}(e_{j_1},\dots,e_{j_m})}\to H^\star(D_{\mathcal{A}},d); e_j\mapsto[\{x_j\}].$$

Let us prove that $\ker\rho\subset\Lambda^r(e_{j_1},\dots,e_{j_m})$ is generated by the $[e_{i_1},\dots,e_{i_{r+1}}]$ with $\{i_1,\dots,i_{r+1}\}\subseteq\{j_1,\dots,j_m\}$:

- It is clear that $\rho[e_{i_1},\dots,e_{i_{r+1}}]=d\{x_{i_1},\dots,x_{i_{r+1}}\}$ (because $\operatorname{rank}X=r$, $\ker\varphi$ does not contain any monomial of degree r and by Lemma 3.2, $\operatorname{rank}\vee\{x_{i_1},\dots,\widehat{x}_{i_j},\dots,x_{i_{r+1}}\}=r$).
- If $\{i_1,\dots,i_r\}\subseteq\{j_1,\dots,j_m\}$ and $y\in\mathcal{A}\setminus B$, then $d\{x_{i_1},\dots,x_{i_r},y\}$ is a sum with no term equal to $\{x_{i_1},\dots,x_{i_r}\}$. Therefore, if $u\in\ker\rho$, then $\rho u=d\sigma$ where σ is a linear combination of $\{x_{i_1},\dots,x_{i_{r+1}}\}$ with $\{i_1,\dots,i_{r+1}\}\subseteq\{j_1,\dots,j_m\}$.

In other words, since $\ker\varphi$ does not contain any monomial of degree r, u is a linear combination of $[e_{i_1},\dots,e_{i_{r+1}}]$, as required.

Let $A_5 = \frac{\Lambda(e_{j_1},\dots,e_{j_m})}{\Lambda^{\geq r+1}(e_{j_1},\dots,e_{j_m})\oplus\ker\rho}$. The map ρ induces an injective map $\overline{\rho}$, and we define a map ψ in the opposite direction

$$A_5 \xrightarrow{\overline{\rho}} H^\star(D_{\mathcal{A}},d) \xrightarrow{\psi} A_5$$

by sending $\{x_i\}$ to $[e_i]$ if $i\in\{j_1,\dots,j_m\}$ and zero if $i\notin\{j_1,\dots,j_m\}$. These two maps are morphisms of graded algebras and verify the following property: $\psi\circ\overline{\rho}=\mathrm{id}$. Finally Lemma 5.5 and Lemma 5.6 give us two injective maps $\mathbb{L}(u,v)\to L_{(A_5,0)}\to \pi_\star\Omega M(\mathcal{A})\otimes\mathbb{Q}$. □

With the two previous propositions, the next theorem is almost completely proved. We just need to put everything in place.

Theorem 4.3 *Let $\mathcal{A}$ be a geometric arrangement such that every $x\in\mathcal{A}$ has $\mathrm{codim}(x)\geq 2$. Then $M(\mathcal{A})$ is rationally hyperbolic if and only if there is an injective map $\mathbb{L}(u,v)\to\pi_\star(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$.*

Proof Suppose that $M(\mathcal{A})$ is rationally hyperbolic. As shown at the beginning of this section, the map $\varphi\colon \Lambda(e_1,\dots,e_n)\to H^\star(M(\mathcal{A}),\mathbb{Q})$ can not be injective, otherwise $M(\mathcal{A})$ would be elliptic. Therefore $\ker\varphi\neq 0$ and Proposition 4.1 and Proposition 4.2 show that there exists an injective map $\mathbb{L}(u,v)\to\pi_\star(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$.

Now, assume that such a map exists. In that case, the dimension of $\pi_\star(\Omega M(\mathcal{A}))\otimes\mathbb{Q}$, as a graded rational vector space, is not finite. Hence, the same is true for $\pi_\star M(\mathcal{A})\otimes\mathbb{Q}$ and $M(\mathcal{A})$ is rationally hyperbolic. □

5 Technical results

This section contains the technical lemmas concerning A_4 and A_5 used in Section 4. The aim is to prove the Lemma 5.4, Lemma 5.5 and Lemma 5.6. With that in mind, we consider the following differential graded algebras:

$$(A_1,0)=\left(\Lambda(e_{i_2},\ldots,e_{i_r})\oplus(\oplus_{s\geq 1}\mathbb{Q}u_s),0\right),|u_s|=\sum_{i=1}^{r}|e_{i_r}|+(s-1)|e_{i_1}|-s,$$
$$(A_2,d)=\left(\frac{\Lambda(e_{i_1},\ldots,e_{i_r},t,a)}{te_{i_1},\ldots,te_{i_r},t^2},d\right)\text{ with }de_{i_j}=0,dt=e_{i_1}\cdots e_{i_r},da=e_{i_1},$$
$$(A_3,d)=\left(\frac{\Lambda(e_{i_1},\ldots,e_{i_r},t)}{te_{i_1},\ldots,te_{i_r},t^2},d\right)\text{ with }de_{i_j}=0,dt=e_{i_1}\cdots e_{i_r},$$
$$(A_4,0)=\left(\frac{\Lambda(e_{i_1},\ldots,e_{i_r})}{e_{i_1}e_{i_2}\ldots e_{i_r}},0\right),$$
$$(A_5,0)=\left(\frac{\Lambda(e_{j_1},\ldots,e_{j_m})}{I},0\right).$$

where I is the ideal of $\Lambda(e_{j_1},\ldots,e_{j_m})$ generated by the elements $e_{i_1}\cdots e_{i_{r+1}}$ and $[e_{i_1},\ldots,e_{i_{r+1}}]$. In $(A_1,0)$, the products $u_se_{i_j}=0$ and $u_su_{s'}=0$ for all s,s' and j. Remark: (A_2,d) is equal to $(A_3\otimes\Lambda a,d)$ with $da=e_{i_1}$.

In order to reach our goal, we will need to understand a few properties of these algebras. The proofs make heavy use of rational homotopy theory (especially Sullivan minimal models). The theory and notations are explained in Felix–Halperin–Thomas [4].

Lemma 5.1 *There exists two quasi-isomorphisms* $(A_1,0)\xrightarrow{\simeq}(A_2,d)$ *and* $(A_4,0)\xrightarrow{\simeq}(A_3,d)$.

Proof It is clear that the inclusion $(A_4,0)\to(A_3,d)$ is a quasi-isomorphism, because, as a vector space, $A_3=A_4\oplus V$ where V admits $e_{i_1}\cdots e_{i_r}$ and t as basis elements. Let us prove that there exists a quasi-isomorphism $\theta\colon(A_1,0)\to(A_2,d)$. Consider the subalgebra $(B,d)=(\Lambda(e_{i_1},\ldots,e_{i_r},a),d)$ of (A_2,d). Since $d(A_2)\subset B$, the differential in A_2/B is zero. Therefore, we have a short exact sequence of complexes:

$$0\to(B,d)\to(A_2,d)\to(A_2/B,0)\to 0,$$

and a long exact sequence in cohomology with $A_2/B=\oplus_{s\geq 0}\mathbb{Q}ta^s$. By the connecting map, an element ta^s of $H^\star(A_2/B,0)$ is sent on the cohomology class of $d(ta^s)=e_{i_1}\cdots e_{i_r}a^s$ in B. But $d\left(\frac{1}{s+1}e_{i_2}\cdots e_{i_r}a^{s+1}\right)=e_{i_1}\cdots e_{i_r}a^s$. Therefore, the connecting map is zero. It means that we have a short exact sequence of the cohomology algebras:

$$0\to H^\star(B,d)\to H^\star(A_2,d)\to H^\star(A_2/B,0)\to 0.$$

The cohomology of (B,d) is obviously $\Lambda(e_{i_2},\ldots,e_{i_r})$ and the cohomology of $(A_2/B,0)$ is A_2/B. Consider the map $\theta\colon(A_1,0)\to(A_2,0)$ defined by $\theta(e_{i_j})=e_{i_j}$, $j=2,\ldots,n$

and $\theta(u_s)=\frac{a^s}{s}e_{i_2}\dots e_{i_r}-a^{s-1}t$. It is a morphism of differential graded algebras. This gives us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda(e_{i_2},\dots,e_{i_r}) \arrow[r] \arrow[d, "\simeq"] & \Lambda(e_{i_2},\dots,e_{i_r})\oplus\oplus_{s\geq 1}\mathbb{Q}u_s \arrow[r] \arrow[d, "H^\star\theta"] & \oplus_{s\geq 1}\mathbb{Q}u_s \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^\star(B,d) \arrow[r] & H^\star(A_2,d) \arrow[r] & H^\star(A_2/B,0) \arrow[r] & 0
\end{tikzcd}$$

The 5–lemma proves that $H^\star\theta$ is an isomorphism, or, in other words, that θ is a quasi-isomorphism. □

Lemma 5.2 *Let $m\colon(\Lambda V,d)\to(A_3,d)$ and $m'\colon(\Lambda W,d)\to(A_2,d)$ be the Sullivan minimal models of (A_3,d) and (A_2,d), and $f\colon(\Lambda V,d)\to(\Lambda W,d)$ a minimal model of the injection $(A_3,d)\to(A_2,d)$. Then $Qf\colon V\to W$ is surjective.*

Proof Let $(v_1,v_2,\dots)$ be a basis of V. Since $de_{i_1}=0$, rational homotopy theory shows that we can construct the map m with the property that $m(v_1)=e_{i_1}$. We form then the relative Sullivan model: $(\Lambda V\otimes\Lambda a,d)$ with $da=v_1$. The map $m\otimes\mathrm{id}\colon(\Lambda V\otimes\Lambda a,d)\to(A_3\otimes\Lambda a,d)$ extends the map m and makes commutative the following diagram.

$$\begin{tikzcd}
(\Lambda V,d) \arrow[r, hook, "i"] \arrow[d, "m"'] & (\Lambda V\otimes\Lambda a,d) \arrow[d, "m\otimes\mathrm{id}"] \\
(A_3,d) \arrow[r, hook, "j"] & (A_3\otimes\Lambda a,d)=(A_2,d)
\end{tikzcd}$$

Since m is a quasi-isomorphism, $m\otimes\mathrm{id}$ is also a quasi- isomorphism (see [4, Lemma 14.2]). This shows that $m\otimes\mathrm{id}$ is a Sullivan model of the map $j\circ m$.

The relative Sullivan algebra $(\Lambda V\otimes\Lambda a,d)$ is a Sullivan algebra, and almost minimal: to make it minimal, we only need to divide by the ideal generated by a and v_1. The projection map $p\colon(\Lambda V\otimes\Lambda a,d)\to(\Lambda(v_2,v_3,\dots),d)$ is such a quasi-isomorphism. So, $(\Lambda(v_2,v_3,\dots),d)$ is a minimal model of (A_2,d). We conclude by letting $f=p\circ i$. The map f is such that the linear map Qf is simply the projection $V\to V/v_1$, which is surjective. □

Lemma 5.3 *Let $(\Lambda V,d)$ be a minimal algebra and $f\colon(\Lambda V,d)\to(E,d)$ be a quasi-isomorphism of differential graded algebras. If there exists $x,y\in V$ such that x and y are linearly independent, $dx=dy=0$ and $f(xy)=f(x^2)=f(y^2)=0$, then there exists two morphisms of Lie algebras $\mathbb{L}(u,v)\xrightarrow{i}L_{(\Lambda V,d)}\xrightarrow{p}\mathbb{L}(u,v)$ such that $p\circ i=\mathrm{id}$. In particular, i is injective.*

Proof Let us consider the differential graded algebra $(B,0) = (\mathbb{Q} \oplus \mathbb{Q}x' \oplus \mathbb{Q}y', 0)$ with all products equal to zero and $|x'| = |x|, |y'| = |y|$. We can define a morphism of differential graded algebras $\theta\colon (B,0) \to (E,d)$ with $\theta(x') = f(x)$ and $\theta(y') = f(y)$.

Notice that $(B,0)$ is a model of a wedge of two spheres. Its minimal Sullivan model $\rho\colon (\Lambda W,d) \to (B,0)$ is such that $L_{(\Lambda W,d)} = \mathbb{L}(u,v)$ with $|u| = |x'| - 1$ and $|v| = |y'| - 1$. Without loss of generality, we can assume that $|x'| \le |y'|$.

The existence of the Sullivan minimal model is proved by an inductive process. Looking closely at this construction, we can easily (in low degree) construct a basis for W.

- If $|x'|$ is odd or if $|x'| = |y'|$, then $\Lambda W = \Lambda(x', y', t, \dots)$ with $dt = x'y'$. In degree less than $|y'|$, W has only two generators : x', y'.
- If $|x'|$ is even and if $|x'| < |y'|$, then $\Lambda W = \Lambda(x', y', t_1, t_2, \dots)$ with $dt_1 = x'^2$ and $dt_2 = x'y'$.

Let us construct a map $\psi\colon (\Lambda W,d) \to (\Lambda V,d)$. By the lifting lemma, such a map can be obtained by lifting $\theta \circ \rho$ along f. But we can have more: the lift ψ can be constructed inductively along a basis of W, so we can choose $\psi(x') = x$ and $\psi(y') = y$.

$$\begin{tikzcd} & (\Lambda V,d) \arrow[d, "f"] \\ (\Lambda W,d) \arrow[ur, dashed, "\psi"] \arrow[r, "\theta\circ\rho"'] & (E,d) \end{tikzcd}$$

Now, let's see what happens for the induced map $L_{(\Lambda V,d)} \to \mathbb{L}(u,v)$.

- If $|x'|$ is odd or if $|x'| = |y'|$, then the linear map $Q\psi\colon W \to V$ is injective in degree $\le |y'|$ (it is completely described by $Q\psi(x') = x$ and $Q\psi(y') = y$). So, the dual map is surjective. It implies that $L\psi\colon L_{(\Lambda V,d)} \to \mathbb{L}(u,v)$ is surjective in degree $\le |v|$, which means that u and v are in the image of $L\psi$.
- If $|x'|$ is even and if $|x'| < |y'|$, then we can do exactly the same reasoning if $|t_1| > |y'|$. If $|t_1| \le |y'|$, then there is a slight difference. In that case, $x^2 = \psi(x'^2) = \psi(dt_1) = d\psi(t_1)$. So, x^2 is a boundary. There is a $z \in V$ such that $x^2 = dz$. The map $Q\psi$ in degree $\le |y'|$ is completely described by $Q\psi(x') = x$, $Q\psi(y') = y$ and $Q\psi(t_1) = z$. It is injective in degree $\le |y'|$. So, the dual map is surjective in degree $\le |v|$, which also means that u and v are in the image of $L\psi$.

In both cases, the map $L\psi\colon L_{(\Lambda V,d)} \to \mathbb{L}(u,v)$ has u and v in its image. Therefore, we can choose $a, b \in L_{(\Lambda V,d)}$ such that $L\psi(a) = u$ and $L\psi(b) = v$. Let $p = L\psi$

and consider the map $i\colon \mathbb{L}(u,v) \to L_{(\Lambda V,d)}$ defined by $i(u)=a$ and $i(v)=b$. These two maps verify $p\circ i=\mathrm{id}$. □

Now, the preliminary work is done. The main lemmas of this section can be proved.

Lemma 5.4 *There exists an injective map $\mathbb{L}(u,v) \to L_{(A_4,0)}$.*

Proof Let $L_1 = L_{(A_1,0)}$ and $L_2 = L_{(A_4,0)}$. The proof will be done by showing the existence of two injective maps

$$\mathbb{L}(u,v) \xrightarrow{g_1} L_1 \xrightarrow{g_2} L_2.$$

Step 1: constructing the map g_2 By Lemma 5.1, $(A_1,0) \xrightarrow{\simeq} (A_2,d)$ and $(A_4,0) \xrightarrow{\simeq} (A_3,d)$, so $L_1 = L_{(A_2,0)}$ and $L_2 = L_{(A_3,d)}$. The Lemma 5.2 gives us a map $f\colon (\Lambda V,d) \to (\Lambda W,d)$ between the Sullivan minimal models of (A_3,d) and (A_2,d). Applying the homotopy Lie algebra functor to the map gives a map $Lf\colon L_1 \to L_2$. The surjectivity of Qf implies that Lf is injective (see [4, Chapter 21]). Now, $g_2 = Lf$ is the required map.

Step 2: constructing the map g_1 By Lemma 5.3, we only need to show that if $m\colon (\Lambda V,d) \to (A_1,0)$ is a Sullivan minimal model, then there exists $x,y\in V$ such that x,y are linearly independent, $dx=dy=0$ and $m(xy)=m(x^2)=m(y^2)=0$.

Since m is a quasi-isomorphism, $H^\star m\colon H^\star(\Lambda V,d) \to (A_1,0)$ is surjective. So, there exists $[x]$ and $[y]$ in $H^\star(\Lambda V,d)$ such that $H^\star m([x]) = e_{i_2}$ and $H^\star m([y]) = u_1$. It gives us x and y in $(\Lambda V,d)$ such that $dx=dy=0$, $m(x)=e_{i_2}$ and $m(y)=u_1$. But x and y can not be in $\Lambda^{\geq 2}V$ because, otherwise, $e_{i_2}=m(x)$ would be in $\Lambda^{\geq 2}(e_{i_2},\dots,e_{i_r})$ and $u_1=m(y)$ would be in $\Lambda^{\geq 2}(u_1)/u_1^2$. Therefore, x and y are in V. Finally, the Lemma 5.3 gives us the map g_1. □

Lemma 5.5 *There exists an injective map $\mathbb{L}(u,v) \to L_{(A_5,0)}$.*

Proof Recall that A_5 is the quotient of $\Lambda(e_{j_1},\dots,e_{j_m})$ by the ideal I generated by the elements $e_{i_1}\cdots e_{i_{r+1}}$ and $[e_{i_1},\dots,e_{i_{r+1}}]$. It is clear that $A_5^{>r}=0$. Let us prove that a basis of A_5^r is given by the classes of the elements $e_1e_{j_2}\cdots e_{j_r}$ with $1<j_2<\cdots<j_r\leq m$.

- Let $e_{i_1}\cdots e_{i_r}\in A_5^r$, with $i_1<\cdots<i_r$. If $i_1=1$, then it is trivially a linear combination of elements $e_1e_{j_2}\cdots e_{j_r}$. If $i_1>1$, then we know that $[e_1,e_{i_1},\dots,e_{i_r}]=0$. So, it is also a linear combination of such elements. It shows that these elements generate A_5^r.

- If $1 < i_1 < \cdots < i_{r+1} \leq m$, then

$$\sum_{j=1}^{r+1}(-1)^{j+1}[e_1, e_{i_1}, \dots, \widehat{e}_{i_j}, \dots, e_{i_{r+1}}] =$$

$$\sum_{j=1}^{r+1}(-1)^{j+1}\Big(-e_{i_1}\cdots\widehat{e}_{i_j}\cdots e_{i_{r+1}} + \sum_{k=1}^{j-1}(-1)^{k+1}e_1e_{i_1}\cdots\widehat{e}_{i_k}\cdots\widehat{e}_{i_j}\cdots e_{i_{r+1}} + \sum_{k=j+1}^{r+1}(-1)^k e_1e_{i_1}\cdots\widehat{e}_{i_j}\cdots\widehat{e}_{i_k}\cdots e_{i_{r+1}}\Big)$$

$$= \sum_{j=1}^{r+1}(-1)^j e_{i_1}\cdots\widehat{e}_{i_j}\cdots e_{i_{r+1}} = [e_{i_1}, \dots, e_{i_{r+1}}].$$

  It shows that the vector space generated by every $[e_{i_1}, \dots, e_{i_{r+1}}]$ is equal to the vector space generated by the elements $[e_1, e_{i_2}, \dots, e_{i_{r+1}}]$ with $1 < i_2 < \cdots < i_{r+1}$.

- Let us consider the following short exact sequence

$$0 \to \langle[e_{i_1}, \dots, e_{i_{r+1}}]\rangle \to \Lambda^r(e_{j_1}, \dots, e_{j_m}) \to A_5^r \to 0.$$

  Let d_1 be the dimension of the vector space generated by the elements $e_1e_{i_2}\cdots e_{i_r}$, with $1 < i_2 < \cdots < i_r$, in $\Lambda^r(e_{j_1}, \dots, e_{j_m})$ and d_2 be the dimension of the vector space generated by the elements $e_{i_1}e_{i_2}\cdots e_{i_r}$ with $1 < i_1 < \cdots < i_r$. We have: $\dim \Lambda^r(e_{j_1}, \dots, e_{j_m}) = d_1 + d_2$, $\dim A_5^r \leq d_1$, $\dim\langle[e_{i_1}, \dots, e_{i_{r+1}}]\rangle \leq d_2$. So, $\dim A_5^r = d_1$, and the elements $e_1e_{j_2}\cdots e_{j_r}$ form a basis of A_5^r.

Let I be the set of every sequence $1 < i_1 < \cdots < i_{r+1}$ and (B, d) the differential graded algebra defined by

$$B = \frac{\Lambda(e_{j_1}, \dots, e_{j_m})}{\Lambda^{>r}(e_{j_1}, \dots, e_{j_m})} \oplus (\oplus_{i\in I} a_i)$$

and $d(a_i) = [e_1, e_{i_2}, \dots, e_{i_{r+1}}]$. The product in B is defined by $a_i \cdot a_j = a_i \cdot e_j = 0$. The ideal generated by the a_i et the da_i is acyclic, and the quotient map is a quasi-isomorphism: $\varphi\colon (B, d) \to (A_5, 0)$.

Therefore, the differential graded algebras $(A_5, 0)$ and (B, d) have the same minimal model. Let us consider the minimal model of (B, D) given by $\theta\colon (\Lambda W, d) \to (B, d)$. The vector space W is generated in low degree by $e_1, \dots, e_m$ and $(a_i)_{i\in I}$ with $\theta(e_i) = e_i$, $\theta(a_i) = a_i$. Because $\theta(e_i^2) = \theta(e_ia_j) = \theta(a_j^2) = 0$, Lemma 5.3 shows that $L_{(\Lambda W, d)} = L_{(A_5, 0)}$ contains a Lie subalgebra $\mathbb{L}(u, v)$. □

Lemma 5.6 *If $(A,0)$ is a 1–connected differential graded algebra, X is a formal space and if there exists two maps $f\colon A \to H^\star(X,\mathbb{Q})$ and $g\colon H^\star(X,\mathbb{Q}) \to A$ such that $g \circ f = \mathrm{id}_A$, then there exists two morphisms of Lie algebras $\widetilde{f}\colon L_X \to L_{(A,0)}$ and $\widetilde{g}\colon L_{(A,0)} \to L_X$ such that $\widetilde{f} \circ \widetilde{g}$ is an isomorphism. In particular, $\widetilde{g}$ is an injective map.*

Proof Let $(\Lambda V, d) \xrightarrow{m} (A,0)$ and $(\Lambda V', d') \xrightarrow{m'} (H^\star(X,\mathbb{Q}), 0)$ be the minimal Sullivan models of $(A,0)$ and X respectively (the map m' exists because X is a formal space). Since these maps are quasi-isomorphisms, they are surjective. The lifting lemma shows that there exists maps $\overline{f}$ and $\overline{g}$ such that $m' \circ \overline{f} = f \circ m$ and $m \circ \overline{g} = g \circ m'$.

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\bar{f}"] \arrow[d, "m"'] & (\Lambda V', d') \arrow[r, "\bar{g}"] \arrow[d, "m'"'] & (\Lambda V, d) \arrow[d, "m"'] \\
(A,0) \arrow[r, "f"'] & (H^\star(X,Q),0) \arrow[r, "g"'] & (A,0)
\end{tikzcd}$$

The maps $\overline{f}$ and $\overline{g}$ verify $m \circ (\overline{g} \circ \overline{f}) = (g \circ f) \circ m = m$. Since $g \circ f$ is an isomorphism, $\overline{g} \circ \overline{f}$ is a quasi-isomorphism between 1–connected minimal Sullivan algebras. It implies that it is an isomorphism.

Applying the homotopy Lie algebra functor to $(\Lambda V, d) \xrightarrow{\bar{f}} (\Lambda V', d') \xrightarrow{\bar{g}} (\Lambda V, d)$ gives us the maps $\widetilde{f} = L\overline{f}$ and $\widetilde{g} = L\overline{g}$.

$$L_{(A,0)} \xleftarrow{\widetilde{f}} L_X \xleftarrow{\widetilde{g}} L_{(A,0)}.$$

Since L is a functor, $\widetilde{f} \circ \widetilde{g} = (L\overline{f}) \circ (L\overline{g}) = L(\overline{g} \circ \overline{f})$ is an isomorphism. □

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