String cohomology groups of complex projective spaces

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Let X be a space and write LX for its free loop space equipped with the action of the circle group $\mathbb{T}$ given by dilation. The equivariant cohomology $H^*(LX_{h\mathbb{T}};\mathbb{Z}/p)$ is a module over $H^*(B\mathbb{T};\mathbb{Z}/p)$. We give a computation of this module when $X=\mathbb{C}\mathrm{P}^r$ for any positive integer r and any prime number p. The computation does not use the fact that $\mathbb{C}\mathrm{P}^r$ is formal, nor does it use the Jones isomorphism and negative cyclic homology.

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1 Introduction

Let LX denote the free loop space on X. The circle acts on itself by rotation, and this action induces an action of the circle group $\mathbb{T}=S^1=SO(2)$ on LX. (The action extends to an $O(2)$–action, but we will not consider the extended action). The homotopy orbit under the circle action is the space $E\mathbb{T}\times_{\mathbb{T}}LX$, which we will also write as $LX_{h\mathbb{T}}$.

The purpose of this paper is to compute the $\mathbb{T}$–Borel cohomology with $\mathbb{F}_p$–coefficients of the free loop space on $\mathbb{C}\mathrm{P}^r$, that is $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}};\mathbb{F}_p)$, as a module over $H^*(B\mathbb{T};\mathbb{F}_p)=\mathbb{F}_p[u]$ using a mixture of Morse theory and homotopy theory.

The computation has already been done via the Jones isomorphism and cyclic homology. The point in doing the same computation by different methods is that these methods applies to different situations than the cyclic homology approach. As examples our methods should apply also for computations in topological K–theory and they could also be useful when working with non-formal globally symmetric spaces.

There are several motivations for studying Borel cohomology of free loop spaces. In differential topology one studies the spectrum $TC(M)$, which is related to the diffeomorphisms of the manifold M. There is a long exact sequence for the spectrum cohomology of $TC(M)$, where the other terms are given by the cohomology and the Borel cohomology of the free loop space on M. So this result gets us closer to understanding $TC(\mathbb{C}\mathrm{P}^r)$.

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The theory of Chas and Sullivan [8] constructs algebraic operations on groups related to the free loop space. Part of this structure has been computed in the case $M = \mathbb{C}\mathrm{P}^r$ (see Cohen, Jones and Yan [9]). It would be interesting to write down as much as possible of the Chas–Sullivan structure for the particularly simple example $M = \mathbb{C}\mathrm{P}^r$. Since one of the groups carrying this structure is the Borel cohomology of the free loops space, a first step is to compute this group.

The usual cohomology of the free loop space has been used to study the existence of closed geodesics (see Gromoll and Meyer [12], Klingenberg [18], and Vigué-Poirrier and Sullivan [31]). A nice survey of the theory can be found in McCleary [24]. The starting point is the following remarkable theorem:

Theorem (Gromoll–Meyer [12]) *Let M be a closed simply connected Riemannian manifold. If the sequence of Betti numbers*

$$(\dim H_n(LM;\mathbb{Q}))_{n\geq 0}$$

is unbounded, then M has infinitely many geometrically distinct closed geodesics in any Riemannian metric.

In [31], Sullivan and Vigué showed that if $H^*(M;\mathbb{Q})$ requires at least two generators as an algebra, then the sequence of Betti numbers of LM is unbounded. One may replace $\mathbb{Q}$ by any field k in the Gromoll–Meyer theorem but whether one may do it in the Sullivan–Vigué theorem is an open problem.

For simply connected rank one globally symmetric spaces, the above gives no information. Here the integral homology of LM is known (see Ziller [34]) and the sequence of Betti numbers is bounded for any coefficient field. It is a far out possibility that the methods presented in this paper could be of some help in settling the geodesics question for $\mathbb{C}\mathrm{P}^r$ in any Riemannian metric.

The method we use is convoluted, so we should try to give an overview of the computation and the main results here.

In Section 2 we study the space of geodesics on $\mathbb{C}\mathrm{P}^r$ equipped with the Fubini–Study metric. Except for the constant "geodesics" $\mathbf{B}_0(\mathbb{C}\mathrm{P}^r) = \mathbb{C}\mathrm{P}^r$ this space consists of an infinite number of homeomorphic components as follows: Let $\mathbf{B}_1(\mathbb{C}\mathrm{P}^r)$ denote the set of primitive parametrized closed geodesics and let $\mathbf{B}_n(\mathbb{C}\mathrm{P}^r)$ for integer $n \geq 1$ denote the set of n fold iterate primitive geodesics. We show that there are diffeomorphisms

$$\mathbf{B}_1(\mathbb{C}\mathrm{P}^r) \cong \mathbf{PV}_2(\mathbb{C}^{r+1}) \cong S(\tau(\mathbb{C}\mathrm{P}^r)),$$

where $\mathbf{PV}_2(\mathbb{C}^{r+1})$ denotes the projective Stiefel manifold $\mathbf{V}_2(\mathbb{C}^{r+1})/\mathrm{diag}_2(U(1))$ and $S(\tau(\mathbb{C}\mathrm{P}^r))$ is the total space of the sphere bundle of the tangent bundle on $\mathbb{C}\mathrm{P}^r$.

The $\mathbb{T}$–action on $L\mathbb{C}\mathrm{P}^r$ restricts to a $\mathbb{T}$–action on $\mathbf{B}_n(\mathbb{C}\mathrm{P}^r)$. Write $\mathbf{PV}_2^{(n)}(\mathbb{C}^{r+1})$ and $S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}$ for the corresponding $\mathbb{T}$–spaces via the above diffeomorphisms. We describe the action on $\mathbf{PV}_2^{(n)}(\mathbb{C}^{r+1})$ explicitly. It is free when $n=1$.

In Section 3 we compute the cohomology of $S(\tau(\mathbb{C}\mathrm{P}^r))$ by comparing to the cohomology of Grassmann spaces. Our results in these first two sections are very closely related to Klingenberg's work in [17]. The main difference is that we are interested in the circle action, while he studies the action of the group $O(2)\supset\mathbb{T}$.

In Section 4 we compute the equivariant (Borel) cohomology of the spaces of geodesics. The action of $\mathbb{T}$ is different on the different homeomorphic components. The equivariant cohomology can distinguish at least some of them.

Theorem 1.1 *For any prime p one has that*

$$H^*(E\mathbb{T}\times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)};\mathbb{F}_p)\cong\begin{cases}\mathbb{F}_p[x_1,x_2]/(Q_r,Q_{r+1}), & p\nmid n,\\ \mathbb{F}_p[u,x,\sigma]/(x^{r+1},\sigma^2), & p\mid n,\ p\mid(r+1),\\ \mathbb{F}_p[u,x,\overline{\sigma}]/(x^r,\overline{\sigma}^2), & p\mid n,\ p\nmid(r+1),\end{cases}$$

where u, x, x_1, x_2 have degree 2 and $\deg(\sigma)=2r-1$, $\deg(\overline{\sigma})=2r+1$. The polynomial Q_k is defined for integer $k\geq 0$ by

$$Q_k(x_1,x_2)=\sum_{i=0}^{k}x_1^i x_2^{k-i}\in\mathbb{F}_p[x_1,x_2].$$

In Section 5 we recall some facts about equivariant cohomology theory, in particular the localization theorem, which shows that the relative equivariant cohomology of $X^{C_p}\subset X$ is annihilated by u.

In Section 6 we study the following abstract construction: Let X be a $\mathbb{T}$–space with action map $\mu\colon\mathbb{T}\times X\to X$. We can twist this action by the power map $\theta_n\colon\mathbb{T}\to\mathbb{T}$; $\theta_n(z)=z^n$ and obtain another $\mathbb{T}$–space $X^{(n)}$. The underlying spaces of X and $X^{(n)}$ are equal, but the action map for $X^{(n)}$ is $\mu_n\colon\mathbb{T}\times X^{(n)}\to X^{(n)}$; $\mu_n(z,x)=\mu(\theta_n(z),x)$. We prove the following result:

Theorem 1.2 *For any prime p and $\mathbb{T}$–space X there is an isomorphism, natural in X:*

$$H^*(X;\mathbb{F}_p)\otimes\mathbb{F}_p[u]\xrightarrow{\ \cong\ }H^*(E\mathbb{T}\times_{\mathbb{T}}X^{(p)};\mathbb{F}_p).$$

In Section 7 we first review Morse theory for the energy integral on the free loop space. Let M be a compact Riemannian manifold. There is a Hilbert manifold model of the free loop space LM. The energy integral

$$E\colon LM \to \mathbb{R}; \quad E(\gamma) = \int_{\mathbb{T}} |\gamma'(z)|^2 dz$$

is smooth function and its critical points are the closed geodesics on M. For a critical value a of E we put $\mathcal{F}(a) = E^{-1}(-\infty, a)$. The sequence of critical values $0 = \lambda_0 < \lambda_1 < \dots$ gives us a $\mathbb{T}$–equivariant filtration $\mathcal{F}(\lambda_0) \subseteq \mathcal{F}(\lambda_1) \subseteq \dots$ of LM. We refer to this as the energy filtration of LM.

If one assumes that the so called Bott non-degeneracy condition holds, then the critical points for E of energy λ_i will be collected on a compact critical submanifold $N(\lambda)$ of LM for all i. (The symmetric spaces satisfy this condition according to Ziller [34, Theorem 2]). Furthermore, the tangent bundle of LM restricted to $N(\lambda_i)$ splits $\mathbb{T}$–equivariantly into a sum of three bundles

$$TLM|_{N(\lambda_i)} \cong \mu^-(\lambda_i) \oplus \mu^0(\lambda_i) \oplus \mu^+(\lambda_i),$$

and there is a $\mathbb{T}$–equivariant homotopy equivalence

$$\mathcal{F}(\lambda_i)/\mathcal{F}(\lambda_{i-1}) \simeq \mathrm{Th}(\mu^-(\lambda_i)).$$

In the last part of Section 7 we prove a localization result for the energy filtration. The idea is that the iteration map, which maps a loop to the same loop run through p times, behaves like the inclusion of the C_p–fixed points, so that it should induce an isomorphism of localized equivariant cohomology and by the results of Section 5 it is easy to compute this equivariant cohomology.

The localization theorem needs a finite dimensionality condition. This condition is not just technical, but essential. Since the free loop space is infinite dimensional, we have to reduce the situation to a finite dimensional situation. We do this step using Morse theory on the Hilbert manifold version of the free loop space.

Assume that $N(a) \subseteq LM$ is a non-degenerate critical submanifold in the sense of Bott of energy a. Assume also that its p–fold iterate $P_p(N(a)) \subseteq N(p^2 a)$ is a non-degenerate critical submanifold, and that there are no critical values in the open intervals $(a-\epsilon, a)$ and $(p^2(a-\epsilon), p^2 a)$ where $\epsilon > 0$. Then we prove the following result:

Theorem 1.3 *The p–fold iteration map induces an isomorphism in cohomology localized away from u as follows:*

$$\widetilde{H}^*((\mathcal{F}(a)/\mathcal{F}(a-\epsilon))_{h\mathbb{T}}; \mathbb{F}_p)[1/u] \cong \widetilde{H}^*((\mathcal{F}(p^2 a)/\mathcal{F}(p^2(a-\epsilon)))^{(p)}_{h\mathbb{T}}; \mathbb{F}_p)[1/u].$$

In Section 8 we set up the Morse spectral sequences. The energy filtration gives us a non-equivariant spectral sequence

$$\{E_r^{n,m}(\mathcal{M})(LM)\} \Rightarrow H^{n+m}(LM), \quad E_1^{n,m}(\mathcal{M})(LM) \cong \widetilde{H}^m(\mathrm{Th}(\mu^-(\lambda_n)))$$

and an equivariant spectral sequence

$$\{E_r^{n,m}(\mathcal{M})(LM_{h\mathbb{T}})\} \Rightarrow H^{n+m}(LM_{h\mathbb{T}}),\ E_1^{n,m}(\mathcal{M})(LM_{h\mathbb{T}}) \cong \widetilde{H}^m(\mathrm{Th}(\mu^-(\lambda_n)_{h\mathbb{T}})).$$

By Theorem 1.3 we get the following result:

Theorem 1.4 *Consider the Morse spectral sequences with $\mathbb{F}_p$–coefficients. Up to a re-indexing of the columns the p–fold iteration map induces a natural isomorphisms of spectral sequences*

$$E_*(\mathcal{M})(LM_{h\mathbb{T}})[1/u] \cong E_*(\mathcal{M})(LM) \otimes \mathbb{F}_p[u, u^{-1}].$$

In Section 9 we specialize to $M = \mathbb{C}\mathrm{P}^r$. We use the results from Sections 3 and 4 and the Thom isomorphism to write down explicitly the E_1 pages of the two Morse spectral sequences.

For this space we do know that the non-equivariant spectral sequence collapses, and it follows from Section 7 that the localization of the Morse spectral sequence collapses. This does not imply that the un-localized Morse spectral sequence for the equivariant cohomology collapses from the E_1 page, but it is sufficient to prove collapsing from the E_p page.

In Section 10 there is a change of scene towards the the Serre spectral sequence for the fibration sequence $L\mathbb{C}\mathrm{P}^r \to L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}} \to B\mathbb{T}$. Since the cohomology of $L\mathbb{C}\mathrm{P}^r$ is known, we know the E_2 page of this spectral sequence. The d_2 differential is given if you can compute the action map $\mathbb{T} \times L\mathbb{C}\mathrm{P}^r \to L\mathbb{C}\mathrm{P}^r$ in the ordinary cohomology of the free loop space. We compute the cohomology of the free loop space together with the action differential in the slightly more general case of spaces whose cohomology are truncated polynomial algebras on one generator. The computation can be found in Appendix A. We use methods from [5]. The central part is a computation in non-abelian homological algebra. The reader may choose to use Hochschild homology to bypass the proof in the appendix since $\mathbb{C}\mathrm{P}^r$ is a formal space. One can use Connes' exact sequence to get the action differential.

In Section 11 we write down the E_3 page of the Serre spectral sequence as an algebra, and we compute its Poincaré series. Furthermore we use the $\mathbb{T}$–transfer map to show that some classes must survive to E_∞.

Finally, in Section 12 we show how the transfer map forces the existence of non-trivial differentials in the Morse spectral sequence. It seems to be hard to pinpoint these differentials exactly, but we obtain sufficient information to compute the E_∞ term as a module over $\mathbb{F}_p[u]$. We obtain the following main result:

Theorem 1.5 *As a graded $\mathbb{F}_p[u]$–module, $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}};\mathbb{F}_p)$ is isomorphic to the direct sum*

$$\mathbb{F}_p[u] \oplus \bigoplus_{2k\in\mathcal{I}F(r,p)} \mathbb{F}_p[u]f_{2k} \oplus \bigoplus_{2k\in\mathcal{I}F'(r,p)} \mathbb{F}_p[u]f_{2k-1} \oplus \bigoplus_{2k\in\mathcal{I}T(r,p)} (\mathbb{F}_p[u]/(u))t_{2k-1}.$$

Here the lower index on a generator denotes its degree. The index sets are defined as follows:

$$\mathcal{I}F(r,p) = \{2(ri+j) \mid \chi_p(r+1) \le j \le r, 0 \le i \text{ and } p \mid ((r+1)i+j)\} \setminus \{0\},$$
$$\mathcal{I}T(r,p) = \{2(ri+j) \mid \chi_p(r+1) \le j \le r, 0 \le i \text{ and } p \nmid ((r+1)i+j)\},$$
$$\mathcal{I}F'(r,p) = \mathcal{I}F(r,p) \text{ if } p \nmid (r+1),$$
$$\mathcal{I}F'(r,p) = (\mathcal{I}F(r,p) \setminus 2r\mathbb{N}) \cup (2+2r\mathbb{N}) \text{ if } p \mid (r+1),$$

where $\chi_p(s)=0$ if $p \mid s$ and $\chi_p(s)=1$ if $p \nmid s$.

This main result can be reformulated as follows:

Theorem 1.6 *Let $\{E_*\}$ be the mod p Serre spectral sequence for the fibration sequence $L\mathbb{C}\mathrm{P}^r \to (L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}} \to B\mathbb{T}$. That is*

$$E_2^{*,*} = H^*(B\mathbb{T};\mathbb{F}_p) \otimes H^*(L\mathbb{C}\mathrm{P}^r;\mathbb{F}_p) \Rightarrow H^*((L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}};\mathbb{F}_p).$$

For any positive integer r and any prime p one has that $E_3 = E_\infty$. Furthermore, the Poincaré series $P_{r,p}(t)$ for $H^*((L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}};\mathbb{F}_p)$ is given by

$$P_{r,p}(t) = \frac{1}{1-t}\Bigl(1 + \sum_{k\in\mathcal{I}F(r,p)} t^k\Bigr).$$

If p divides $r+1$ we can rewrite this as

$$P_{r,p}(t) = \frac{1-t^{2(r+1)}}{(1-t)(1-t^{2r})(1-t^{2p})}.$$

Our analysis of the Morse spectral sequence has some geometrical consequences, which we describe at the end of Section 12. Recall that the Morse spectral sequence will eventually collapse, not from the E_1 page, but only from the E_p page. There are definitely non-trivial differentials. Not all of these differentials start at the $\mathbb{T}$–fixed points, that is at the space of constant curves. The non-triviality of the differentials can be interpreted as a geometrical statement about n–fold iterated geodesics. It is a consequence of our calculation that on $\mathbb{C}\mathrm{P}^r$ with the standard metric for any n there is always a curve close to an n–fold geodesic, such that if the curve moves according to the dynamics given by the gradient of the energy integral E, then it will eventually approximate an $(n-1)$–fold iterated geodesic.

Finally, let us outline what is known by computations in cyclic homology. The complex projective space $\mathbb{C}\mathrm{P}^r$ as well as the quaternionic projective space $\mathbb{H}\mathrm{P}^r$ are known to be p–formal for any prime p. One reference for this is the proof of Menichi [25, Proposition 36] combined with that of El Haouari [10, Theorem 1.4.3]. In Ndombol and Thomas [27, Example 4.2] it is even shown that $\mathbb{C}\mathrm{P}^r$ is shc–formal for every p. By the Jones isomorphism [14] it then suffices to compute negative cyclic homology of a truncated polynomial ring. This computation is a corollary of a result of Loday [19, Proposition 5.4.16] where one has to keep track of the x–weight. Thus the following result is well known:

Theorem 1.7 *Let k be a commutative ring with unit and consider the graded truncated polynomial ring $k[x]/(x^{r+1})$ where $r \geq 1$ and the degree of x is an even number α. Put $\rho = (r+1)\alpha - 2$ and write ${}_nk$ and k/nk for the kernel and cokernel of the map $n\cdot\colon k \to k$ respectively. Write $k[u_2]$ for $HC_*^-(k)$.*

(1) There is an isomorphism of $k[u_2]$–modules

$$\begin{aligned} HC_*^-(k[x]/x^{r+1}) \cong \\ & k[u_2]\otimes_k\Big(ke_0 \oplus \bigoplus_{0\leq i}({}_{r+1}ke_{\rho(i+1)} \oplus k/(r+1)ke_{\rho(i+1)+1}) \\ & \qquad \oplus \bigoplus_{0\leq i, 1\leq j\leq r} \big({}_{(i(r+1)+j)}kf_{\rho i+\alpha j} \oplus k/(i(r+1)+j)ku_2t_{\rho i+\alpha j-1}\big)\Big) \\ & \oplus \bigoplus_{0\leq i, 1\leq j\leq r} kt_{\rho i+\alpha j-1}, \end{aligned}$$

where the lower indices on the generators refers to the grading in which a $z \in HC_n^-(k[x]/x^{r+1})$ has the degree $\alpha\cdot(x\text{–weight of } z) - n$.

(2) If we take $k = \mathbb{F}_p$ for any prime p, $k = \mathbb{Q}$ or $k = \mathbb{Z}$, then the above is isomorphic to $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}; k)$ for $\alpha = 2$ and to $H^*(L\mathbb{H}\mathrm{P}^r_{h\mathbb{T}}; k)$ for $\alpha = 4$ as modules over $H^*(B\mathbb{T}; k)$.

Notation We fix a prime p. Cohomology groups will always have coefficients in the field $\mathbb{F}_p$, unless we explicitly state otherwise. We state otherwise mainly in Section 3.

The group $\mathbb{T}$ is the circle group. Spaces are supposed to be homotopy equivalent to C-W-complexes, and $\mathbb{T}$–spaces are supposed homotopy equivalent to $\mathbb{T}$–CW–complexes.

2 Geodesics on $\mathbb{C}\mathrm{P}^r$

In this section we study the spaces of closed parametrized geodesics on the complex projective space $\mathbb{C}\mathrm{P}^r$ equipped with the Fubini–Study metric. Apart from the 'constant' geodesics it consists of an infinite number of homeomorphic components. We identify each of these with a complex projective Stiefel manifold, and also with the sphere bundle of the tangent bundle of $\mathbb{C}\mathrm{P}^r$. The $\mathbb{T}$–action on each component is described, and we give an alternative description of the associated quotient space.

Let us fix some notation and recall some basic facts. Consider $\mathbb{C}^{r+1}$ with the usual Hermitian inner product $\langle\,,\,\rangle_{\mathbb{C}}$, and write $\langle\,,\,\rangle$ for its real part. Thus $\langle\,,\,\rangle$ is the standard inner product on $\mathbb{R}^{2r+2} \cong \mathbb{C}^{r+1}$.

Let S^{2r+1} be the unit sphere in $\mathbb{C}^{r+1}$. Since the left action of $\mathbb{T}$ on S^{2r+1} given by $(z,v) \mapsto zv$ is a free proper action, the Hopf map

$$Q\colon S^{2r+1} \to S^{2r+1}/\mathbb{T} = \mathbb{C}\mathrm{P}^r$$

is a smooth submersion. We equip $\mathbb{C}\mathrm{P}^r$ with the Fubini–Study Hermitian metric. This means that for any $x \in S^{2r+1}$ the map

$$a_x\colon (\mathbb{C}x)^{\perp} \subseteq T_x(S^{2r+1}) \xrightarrow{T_x Q} T_{Q(x)}(\mathbb{C}\mathrm{P}^r) \tag{1}$$

is a $\mathbb{C}$–linear isometry, where $(\mathbb{C}x)^{\perp} = \{v \in \mathbb{C}^{r+1} | \langle v, x\rangle_{\mathbb{C}} = 0\}$. Note that the following identity holds

$$a_{zx}(zv) = a_x(v) \text{ for } z \in S^1. \tag{2}$$

See for example Madsen and Tornehave [22, Lemma 14.4] for these facts.

We equip S^{2r+1} with the Riemannian metric coming from $\langle\,,\,\rangle$ and use the real part of the Fubini–Study metric as Riemannian metric on $\mathbb{C}\mathrm{P}^r$. Let H_x be the horizontal subspace ie. the orthogonal complement of $\ker(T_x Q) = \mathbb{R}ix$ in $T_x(S^{2r+1})$. Since

$$H_x = \{v \in \mathbb{C}^{r+1} | \langle v, x\rangle = \langle v, ix\rangle = 0\} = (\mathbb{C}x)^{\perp}$$

we see that $a_x\colon H_x \to T_{Q(x)}(\mathbb{C}\mathrm{P}^r)$ is an $\mathbb{R}$–linear isometry. So the Hopf map Q is a Riemannian submersion (see Gallot, Hulin and Lafontaine [11, Chapter 2]).

We use the Levi–Civita connection corresponding to the standard metrics on S^{2r+1} and $\mathbb{C}\mathrm{P}^r$. Let $\mathbf{B}_1(\mathbb{C}\mathrm{P}^r)$ denote the set of primitive closed geodesics $f\colon [0,1] \to \mathbb{C}\mathrm{P}^r$

and let $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$ for integer $q \geq 1$ be the set of geodesics of the form $f(qt)$ for $f \in \mathbf{B}_1(\mathbb{C}\mathrm{P}^r)$. Note that a geodesic in $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$ has length $q\pi$ and energy $q^2\pi^2$. The free loop space $L\mathbb{C}\mathrm{P}^r$ defined as the set of closed curves $g\colon [0,1] \to \mathbb{C}\mathrm{P}^r$ of class H^1 is a Hilbert manifold modelled on $L(\mathbb{R}^{2r})$ (see Klingenberg [16, Section 1]) and by Klingenberg [17, Section 1.4] we have that $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$ is a (critical) submanifold of $L\mathbb{C}\mathrm{P}^r$.

The left $\mathbb{T}$ action on $L\mathbb{C}\mathrm{P}^r$ restricts to an action on $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$. We view an element in $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$ as a periodic geodesic and then

$$\mathbb{T} \times \mathbf{B}_q(\mathbb{C}\mathrm{P}^r) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^r); \quad (e^{2\pi i\theta} * f)(t) = f(t-\theta), \quad \theta \in \mathbb{R}.$$

We will now give an alternative description of $\mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$. The description can be found in [17, Section 1], but we will write down some explicit maps and add information on the $\mathbb{T}$–action. Write $\mathbf{V}_2(\mathbb{C}^{r+1})$ for the Stiefel manifold of complex *orthonormal* 2–frames in $\mathbb{C}^{r+1}$. We can rotate a 2–frame (x,v) by an angle $\omega \in \mathbb{R}$ as follows:

$$R(\omega)(x,v) = (\cos(\omega)x + \sin(\omega)v, -\sin(\omega)x + \cos(\omega)v).$$

Write $\mathbf{PV}_2(\mathbb{C}^{r+1})$ for the complex projective Stiefel manifold $\mathbf{V}_2(\mathbb{C}^{r+1})/\mathrm{diag}_2(U(1))$.

Definition 2.1 For integer $q \geq 1$ we let $\mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1})$ denote $\mathbf{PV}_2(\mathbb{C}^{r+1})$ equipped with the well-defined $\mathbb{T}$–action

$$\mathbb{T} \times \mathbf{PV}_2(\mathbb{C}^{r+1}) \to \mathbf{PV}_2(\mathbb{C}^{r+1}); \quad e^{2\pi i\theta} * [x,v] = [R(-q\pi\theta)(x,v)], \quad \theta \in \mathbb{R}.$$

Proposition 2.2 *There is a diffeomorphism $\phi_q\colon \mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^r)$ defined by $\phi_q([x,v]) = Q \circ c(q,x,v)$ where*

$$c(q,x,v)(t) = \cos(q\pi t)x + \sin(q\pi t)v, \quad 0 \leq t \leq 1.$$

Furthermore, ϕ_q is a $\mathbb{T}$–equivariant map.

Proof By Gallot, Hulin and Lafontaine [11, 2.109 and 2.110] the map ϕ_q is a bijection. A direct computation using the trigonometric addition formulas shows that ϕ_q is $\mathbb{T}$–equivariant. □

Note that the geodesics $Q \circ c(q,x,v)$ can easily be extended from $[0,1]$ to the open interval $(-\epsilon, 1+\epsilon)$ for $\epsilon > 0$.

Lemma 2.3 *There is a $\mathbb{T}$–equivariant diffeomorphism*

$$\sigma_q\colon \mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1}) \to \widetilde{\mathbf{PV}}_2^{(q)}(\mathbb{C}^{r+1}); \quad [x,v] \mapsto \Big[\frac{x+iv}{\sqrt{2}}, \frac{x-iv}{\sqrt{2}}\Big],$$

where $\widetilde{\mathbf{PV}}_2^{(q)}(\mathbb{C}^{r+1}) = \mathbf{PV}_2(\mathbb{C}^{r+1})$ *equipped with the well-defined* $\mathbb{T}$*–action*

$$e^{2\pi i\theta} \star [a,b] = [e^{-\pi q i\theta}a, e^{\pi q i\theta}b].$$

Proof We have an automorphism σ of $\mathbf{V}_2(\mathbb{C}^{r+1})$ as follows:

$$\sigma(x,v) = \tfrac{1}{\sqrt{2}}(x+iv, x-iv) \quad , \quad \sigma^{-1}(a,b) = \tfrac{1}{\sqrt{2}}(a+b, -i(a-b)).$$

This automorphism respects the diagonal $U(1)$ action. Furthermore,

$$\sigma(R(\omega)(x,v)) = \tfrac{1}{\sqrt{2}}(e^{-i\omega}(x+iv), e^{i\omega}(x-iv)).$$

The result follows. □

Note that the $\mathbb{T}$–action on $\widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^{r+1})$ is free so by the lemma the $\mathbb{T}$–action on $\mathbf{PV}_2^{(1)}(\mathbb{C}^{r+1})$ is also free.

We have another interpretation of the space of primitive geodesics. Let $\eta\colon S(\tau(\mathbb{C}\mathrm{P}^r)) \to \mathbb{C}\mathrm{P}^r$ denote the sphere bundle of the tangent bundle.

Proposition 2.4 *There is a diffeomorphism* $\psi\colon \mathbf{PV}_2(\mathbb{C}^{r+1}) \to S(\tau(\mathbb{C}\mathrm{P}^r))$ *defined by*

$$\psi([x,v]) = (a_x(v))_{Q(x)}.$$

The composition of $\eta \circ \psi$ *is projection on the first factor* $[x,v] \mapsto [x]$.

Proof The map ψ is well-defined by (2) and has the inverse $\psi^{-1}(v_{Q(x)}) = [x, a_x^{-1}(v)]$. Both ψ and ψ^{-1} are smooth maps. □

We equip $S(\tau(\mathbb{C}\mathrm{P}^r))$ with the $\mathbb{T}$–action which makes ψ an equivariant map. There is a fiber bundle sequence $\mathbf{PV}_2^{(1)}(\mathbb{C}^2) \to \mathbf{PV}_2^{(1)}(\mathbb{C}^{r+1}) \to \mathbf{G}_2(\mathbb{C}^{r+1})$ since the Grassmannian is the quotient of the Stiefel manifold by the group $U(2)$. Equivalently, we have a fiber bundle

$$S(\tau(\mathbb{C}\mathrm{P}^1)) \to S(\tau(\mathbb{C}\mathrm{P}^r)) \to \mathbf{G}_2(\mathbb{C}^{r+1}).$$

The action of $\mathbb{T}$ on the total space is free and preserves the fibers. After dividing by it, we get another fiber bundle

$$\Delta_o(\mathbb{C}\mathrm{P}^1) \to \Delta_o(\mathbb{C}\mathrm{P}^r) \to \mathbf{G}_2(\mathbb{C}^{r+1}). \tag{3}$$

Note that we may view a point in $\Delta_o(\mathbb{C}\mathrm{P}^r)$ as the trace of a simple geodesics together with an orientation. Klingenberg [17] does not consider $\Delta_o(\mathbb{C}\mathrm{P}^r)$. He divides out by the entire $O(2)$ action instead and denote the resulting quotient space by $\Delta(\mathbb{C}\mathrm{P}^r)$.

Example 2.5 In the case $r=1$ we have that $\mathbb{C}\mathrm{P}^1$ is the standard round sphere S^2 with radius $\frac{1}{2}$. A tangent vector $w \in T_v(S^2)$ is part of a unique ordered basis $(v, w, v \times w)$. There is a unique $A \in SO(3)$ so that $v = Ae_1$, $w = Ae_2$, $v \times w = Ae_3$. This establishes a diffeomorphism between $SO(3)$ and $S(\tau(S^2))$. The induced action of $e^{i\theta} \in S^1$ on $SO(3)$ is given by right multiplication $A \mapsto A\rho(\theta)$ where $\rho(\theta)$ is the rotation matrix by angle θ around the axis e_3. The map $A \mapsto Ae_3$ is invariant under the action. It defines a map $\Delta_o(\mathbb{C}\mathrm{P}^1) \to S^2$ which is a diffeomorphism.

The example combines with the fibrations to show that we have a diagram of fibration sequences

(4)
$$\begin{tikzcd}
\mathbb{T} \arrow[r, equal] \arrow[d] & \mathbb{T} \arrow[r] \arrow[d] & * \arrow[d] \\
SO(3) \arrow[r] \arrow[d] & S(\tau(\mathbb{C}\mathrm{P}^r)) \arrow[r, "q"] \arrow[d, "q'"] & \mathbf{G}_2(\mathbb{C}^{r+1}) \arrow[d, equal] \\
S^2 \arrow[r] & \Delta_o(\mathbb{C}\mathrm{P}^r) \arrow[r, "\bar{q}"] & \mathbf{G}_2(\mathbb{C}^{r+1})
\end{tikzcd}$$

Let γ_2 be the canonical complex 2–dimensional vector bundle over $\mathbf{G}_2(\mathbb{C}^{r+1})$ and let $\mathbb{P}(\gamma_2)$ be the associated projective bundle. A point in the total space of $\mathbb{P}(\gamma_2)$ is a flag $V_1 \subset V_2 \subset \mathbb{C}^{r+1}$ where V_i has complex dimension i.

Lemma 2.6 *The fiber bundle $S^2 \to \Delta_o(\mathbb{C}\mathrm{P}^r) \to \mathbf{G}_2(\mathbb{C}^{r+1})$ is isomorphic to the fiber bundle $\mathbb{C}\mathrm{P}^1 \to \mathbb{P}(\gamma_2) \to \mathbf{G}_2(\mathbb{C}^{r+1})$.*

Proof It is enough to show that there is a bundle map $\widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T} \to \mathbb{P}(\gamma_2)$ which is a fiber wise diffeomorphism. We have a smooth map

$$f\colon \mathbf{V}_2(\mathbb{C}^{r+1}) \to \mathbb{P}(\gamma_2) \quad ; \quad (a,b) \mapsto (\mathbb{C}\{a\} \subset \mathbb{C}\{a,b\} \subset \mathbb{C}^{r+1}),$$

which factors through $\widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T}$ since multiplying the generators by units does not change a linear span. It suffices to see that $\bar{f}\colon \widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T} \to \mathbb{P}(\gamma_2)$ is a diffeomorphism when restricted to the fibers over $\mathbb{C}^2 \subset \mathbb{C}^{r+1}$ since we can then get the result for a general fiber by changing basis. The map of fibers is $\widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^2) \to \mathbb{C}\mathrm{P}^1$ which under the standard identification $U(2)/(U(1)\times U(1)) \cong \mathbb{C}\mathrm{P}^1$ corresponds to the quotient map

$$\Big(\frac{U(2)}{\mathrm{diag}_2(U(1))}\Big)/\mathbb{T} \to \frac{U(2)}{U(1)\times U(1)}.$$

This map is a diffeomorphism because of the following identity of 2×2 diagonal matrices:

$$D(e^{2\pi i\alpha}, e^{2\pi i\beta}) = D(e^{\pi i(\alpha+\beta)}, e^{\pi i(\alpha+\beta)}) D(e^{\pi i(\alpha-\beta)}, e^{\pi i(\beta-\alpha)}); \quad \alpha,\beta\in\mathbb{R}. \quad \square$$

3 The cohomology of spaces of geodesics

The purpose of this section is to compute the cohomology of $\Delta_o(\mathbb{C}\mathrm{P}^r)$ and $S(\tau(\mathbb{C}\mathrm{P}^r))$. It turns out to be convenient to do this computations for cohomology with coefficients in the integers. We first determine the cohomology of the base Grassmann manifold.

Theorem 3.1 *Let c_1, c_2 be the two first Chern classes of the canonical bundle γ_2 over $\mathbf{G}_2(\mathbb{C}^{r+1})$. Then one has*

$$H^*(\mathbf{G}_2(\mathbb{C}^{r+1});\mathbb{Z}) \cong \mathbb{Z}[c_1,c_2]/(\phi_r,\phi_{r+1}),$$

where $\phi_i\in\mathbb{Z}[c_1,c_2]$ is the polynomial defined inductively by

$$\phi_0 = 1; \quad \phi_1 = -c_1; \quad \phi_i = -c_1\phi_{i-1} - c_2\phi_{i-2}.$$

Proof According to Borel [7, Proposition 31.1] we have an isomorphism

$$H^*(\mathbf{G}_n(\mathbb{C}^{n+m});\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,\dots,x_n]^{\Sigma_n}\otimes\mathbb{Z}[x_{n+1},\dots,x_{n+m}]^{\Sigma_m}}{(\mathbb{Z}[x_1,\dots,x_{n+m}]^{\Sigma_{n+m}})^+} \tag{5}$$

Here the degree of x_i is 2 for all i and Σ_k denotes the symmetric group. The plus sign means forming the ideal generated by elements in positive degrees. By [7, 30.2] one sees that this isomorphism comes from the fibration

$$\frac{U(n+m)}{U(n)\times U(m)} \to \frac{EU(n+m)}{U(n)\times U(m)} \to \frac{EU(n+m)}{U(n+m)},$$

or equivalently the fibration

$$\mathbf{G}_n(\mathbb{C}^{n+m}) \xrightarrow{j_{n,m}} BU(n)\times BU(m) \xrightarrow{B\rho_{n,m}} BU(n+m),$$

where $\rho_{n,m}\colon U(n)\times U(m)\to U(n+m)$ is the usual block matrix inclusion. The isomorphism above appears as the factorization of $j^*_{n,m}$ through the positive degree part of the image of $(B\rho_{n,m})^*$.

One can describe $j_{n,m}$ as the composite map

$$j_{n,m}\colon \mathbf{G}_n(\mathbb{C}^{n+m}) \to \mathbf{G}_n(\mathbb{C}^{n+m})\times\mathbf{G}_m(\mathbb{C}^{n+m}) \to \mathbf{G}_n(\mathbb{C}^\infty)\times\mathbf{G}_m(\mathbb{C}^\infty),$$

where the first map is given by $V \mapsto (V, V^\perp)$. Using the fact that the pullback of the canonical bundle γ_m along the map $\mathbf{G}_n(\mathbb{C}^{n+m}) \to \mathbf{G}_m(\mathbb{C}^{n+m})$, $V \mapsto V^\perp$ is the orthogonal complement $\gamma_n^\perp$, one sees that the Chern classes maps as follows:

$$j_{n,m}^*(c_i(\gamma_n(\mathbb{C}^\infty))) = c_i(\gamma_n) \quad , \quad j_{n,m}^*(c_i(\gamma_m(\mathbb{C}^\infty))) = c_i(\gamma_n^\perp).$$

Put $c_i = c_i(\gamma_n)$ and $\overline{c}_j = c_j(\gamma_n^\perp)$. By the dimension of the bundles we have that $c_i = 0$ for $i > n$ and $\overline{c}_j = 0$ for $j > m$. Since $\gamma_n \oplus \gamma_n^\perp \cong \epsilon^{n+m}$ we also have that $\sum_{i+j=k} c_i\overline{c}_j = 0$ for $k > 0$. This gives us a quotient map into the cohomology of the Grassmann manifold, which by (5) is an isomorphism

$$H^*(\mathbf{G}_n(\mathbb{C}^{n+m}); \mathbb{Z}) \cong \mathbb{Z}[c_i, \overline{c}_j \,|\, i, j > 0]/(c_i \,|\, i > n) + (\overline{c}_j \,|\, j > m) + \left(\sum_{i+j=k} c_i\overline{c}_j \,|\, k > 0\right).$$

In the special case $n = 2$ and $m = r - 1$ we have that

$$\mathbb{Z}[c_i, \overline{c}_j \,|\, i, j > 0]/(c_i \,|\, i > 2) + \left(\sum_{i+j=k} c_i\overline{c}_j \,|\, k > 0\right) \cong$$
$$\mathbb{Z}[c_1, c_2, \overline{c}_1, \overline{c}_2, \dots]/(\overline{c}_k - \phi_k(c_1, c_2) \,|\, k > 0).$$

Dividing by $(\overline{c}_j \,|\, j > r - 1)$ we see that

$$H^*(\mathbf{G}_2(\mathbb{C}^{r+1}); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2]/(\phi_i \,|\, i \geq r).$$

However, it follows from the inductive definition that the ϕ_i is contained in the ideal generated by ϕ_r, ϕ_{r+1} for $i \geq r$ and this finishes the proof. □

We now turn to the projective bundle over the Grassmann space.

Theorem 3.2 *Let $\pi\colon \mathbb{P}(\gamma_2) \to \mathbf{G}_2(\mathbb{C}^{r+1})$ be the projective bundle of the canonical bundle γ_2. There is an isomorphism*

$$H^*(\mathbb{P}(\gamma_2); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/(Q_r, Q_{r+1}),$$

where x_1 and x_2 have degree 2 and

$$Q_k(x_1, x_2) = \sum_{i=0}^{k} x_1^i x_2^{k-i} \in \mathbb{Z}[x_1, x_2].$$

We also have that $\pi^*(c_1(\gamma_2)) = x_1 + x_2$ and $\pi^*(c_2(\gamma_2)) = x_1x_2$.

Proof We use Husemoller [13, 17.2.5 and 17.2.6]. Let $\lambda \to \mathbb{P}(\gamma_2)$ be the sub bundle of the pullback $\pi_*(\gamma_2)$ defined such that a point in the total space of λ is a pair

$(\{V_1 \subset V_2 \subset \mathbb{C}^{r+1}\}, v)$ where the complex dimension of V_i is i and $v \in V_1$. Let $\overline{\lambda}$ be the conjugate bundle of λ. Then, we have

$$H^*(\mathbb{P}(\gamma_2);\mathbb{Z}) \cong H^*(\mathbf{G}_2(\mathbb{C}^{r+1});\mathbb{Z})[c_1(\overline{\lambda})]/(c_1(\overline{\lambda})^2 + c_1(\gamma_2)c_1(\overline{\lambda}) + c_2(\gamma_2)).$$

Combining this with Theorem 3.1 we see that $H^*(\mathbb{P}(\gamma_2);\mathbb{Z})$ is generated by the three classes $\pi^*(c_1(\gamma_2))$, $\pi^*(c_2(\gamma_2))$ and $c_1(\lambda)$ with the three relations

$$\begin{aligned} &\phi_r\big(\pi^*(c_1(\gamma_2)), \pi^*(c_2(\gamma_2))\big) = 0, \\ &\phi_{r+1}\big(\pi^*(c_1(\gamma_2)), \pi^*(c_2(\gamma_2))\big) = 0, \\ &\pi^*(c_2(\gamma_2)) = c_1(\lambda)\pi^*(c_1(\gamma_2)) - c_1(\lambda)^2. \end{aligned}$$

Define $x_1 = c_1(\lambda)$ and $x_2 = \pi^*(c_1(\gamma_2)) - c_1(\lambda)$. Using the last of the above equations to eliminate $\pi^*(c_2(\gamma_2))$, we get that $H^*(\mathbb{P}(\gamma_2);\mathbb{Z})$ is generated by the classes x_1 and x_2 subject to the relations we get by substituting $\pi^*(c_1(\gamma_2))$ and $\pi^*(c_2(\gamma_2))$ expressed by x_1 and x_2 into ϕ_r and ϕ_{r+1}. Note that $\pi^*(c_1(\gamma_2)) = x_1 + x_2$ and that

$$\pi^* c_2(\gamma_2) = c_1(\lambda)\pi^*(c_1(\gamma_2)) - c_1(\lambda)^2 = x_1(x_1 + x_2) - x_1^2 = x_1 x_2.$$

So we put $Q_r(x_1, x_2) = (-1)^r \phi_r(x_1 + x_2, x_1 x_2)$. The two relations are polynomials Q_r, Q_{r+1} in x_1 and x_2. The polynomials Q_i are given inductively by substituting into the inductive definition of ϕ_i. The inductive formula becomes

$$Q_0 = 1, \quad Q_1 = x_1 + x_2, \quad Q_i = (x_1 + x_2)Q_{i-1} - x_1 x_2 Q_{i-2}.$$

It is easy to check that the polynomials

$$Q_i(x_1, x_2) = \sum_{j=0}^{i} x_1^j x_2^{i-j} = (x_1^{i+1} - x_2^{i+1})/(x_1 - x_2)$$

satisfy this inductive definition. □

Because of Lemma 2.6 we have the following result:

Corollary 3.3 *There is an isomorphism $H^*(\Delta_o(\mathbb{C}\mathrm{P}^r);\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/\{Q_r, Q_{r+1}\}$. The induced of the stabilization map $\Delta_o(\mathbb{C}\mathrm{P}^r) \hookrightarrow \Delta_o(\mathbb{C}\mathrm{P}^{r+1})$ maps x_1 and x_2 in $H^*(\Delta_o(\mathbb{C}\mathrm{P}^{r+1});\mathbb{Z})$ to the classes with the same names in $H^*(\Delta_o(\mathbb{C}\mathrm{P}^r);\mathbb{Z})$.*

Proof The statement about the stability of the classes x_1 and x_2 follows from the fact that they are defined using Chern classes of the bundles γ_2 and λ. These bundles pull back to bundles with the same names. □

We note that it is very easy to check whether a polynomial is in the ideal generated by Q_r and Q_{r+1}.

Lemma 3.4 *Let $P = \sum_{i=0}^{m} p_i x_1^i x_2^{m-i}$ be a homogeneous polynomial of degree m. Then P is contained in the ideal $I = (Q_r, Q_{r+1})$ if and only if $p_i = p_j$ for all i, j such that $m-r \leq i \leq r$ and $m-r \leq j \leq r$.*

Proof Since $x_1^{r+1} = Q_{r+1} - x_1 Q_r$, and similarly for x_2^r, the monomials $x_1^i x_2^{m-i}$ are contained in I if $0 \leq i \leq m-r-1$ or $r+1 \leq i \leq m$. It follows that $P \in I$ if and only if $\sum_{m-r \leq i \leq r} p_i x_1^i x_2^{m-i} \in I$. If $p_{m-r} = p_{m-r+1} = \cdots = p_r$, we see that P is congruent to Q_m modulo I, and $P \in I$.

To see that this condition also is necessary, let $J \subset I$ be the ideal generated by x_1^{r+1} and x_2^{r+1}. Then $x_1 Q_k = x_2 Q_k = Q_{k+1} \mod J$, so I is generated as an abelian group by J together with Q_k for $k \geq r$. So for any homogeneous polynomial $P \in I$ of degree m, there is a $\lambda \in \mathbb{Z}$ such that $P - \lambda Q_m \in J$. This completes the proof. □

Next we consider the space of parametrized geodesics $S(\tau(\mathbb{C}\mathrm{P}^r))$. We consider two fibration, namely the middle horizontal fibration in (4) and the obvious spherical fibration:

$$SO(3) \to S(\tau(\mathbb{C}\mathrm{P}^r)) \xrightarrow{q} \mathbf{G}_2(\mathbb{C}^{r+1}), \qquad S^{2r-1} \to S(\tau(\mathbb{C}\mathrm{P}^r)) \xrightarrow{\eta} \mathbb{C}\mathrm{P}^r.$$

Let γ_1 be the canonical bundle over $\mathbb{C}\mathrm{P}^r$, so that

$$H^*(\mathbb{C}\mathrm{P}^r; \mathbb{Z}) = \mathbb{Z}[c_1(\gamma_1)]/(c_1(\gamma_1)^{r+1}).$$

We have the following result (compare with the remark in the introduction of Astey, Gitler, Micha and Pastor [1]):

Lemma 3.5 *Put $t = \eta^*(c_1(\gamma_1))$. There is a class $\overline{\sigma} \in H^{2r+1}(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{Z})$ such that*

$$H^*(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{Z}) = \mathbb{Z}[t, \overline{\sigma}]/(t^{r+1}, \overline{\sigma}^2, (r+1)t^r, t^r \overline{\sigma}).$$

Furthermore, $q^*(c_1(\gamma_2)) = 2t$ and $q^*(c_2(\gamma_2)) = t^2$.

Proof We consider the Serre spectral sequence for the spherical fibration η:

$$E_2^{**} = \mathbb{Z}[c_1(\gamma_1), \sigma]/(c_1(\gamma_1)^{r+1}, \sigma^2) \Rightarrow H^*(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{Z}),$$

where $\deg(\sigma) = 2r-1$. For dimensional reasons, the only possibly non-trivial differential is $d_{2r}(\sigma)$ which is given by the Euler characteristic of $\mathbb{C}\mathrm{P}^r$. $d_{2r}(\sigma) = e(\eta) = (r+1)c_1(\gamma_1)^r$. For this, see Milnor [26, Corollary 11.12 and Theorem 12.2]. Let

$\overline{\sigma}$ be a class representing $c_1(\gamma_1)\sigma$. There cannot be either further differentials nor extensions for dimensional reasons. This finishes the cohomology computation.

Claim There is a bundle isomorphism $q^*(\gamma_2) \cong \eta^*(\gamma_1) \oplus \eta^*(\gamma_1)$.
To prove the claim, we use the diffeomorphism ϕ from Proposition 2.4. We have a commutative diagram

$$\begin{tikzcd} & \mathbb{C}\mathrm{P}^r \\ \mathbf{PV}_2(\mathbb{C}^{r+1}) \arrow[ur, "\mathrm{pr}_1"] \arrow[r, "\phi", "\cong"'] \arrow[dr, "p"'] & S(\tau(\mathbb{C}\mathrm{P}^r)) \arrow[u, "\eta"'] \arrow[d, "q"] \\ & \mathbf{G}_2(\mathbb{C}^{r+1}), \end{tikzcd}$$

where p is the canonical projection and pr_1 is given by projection on the first factor. We also have the projection on the second factor pr_2 and $\mathrm{pr}_1^*(\gamma_1) \cong \mathrm{pr}_2^*(\gamma_1)$. So it suffices to show that $p^*(\gamma_2) \cong \mathrm{pr}_1^*(\gamma_1) \oplus \mathrm{pr}_2^*(\gamma_1)$. But this isomorphism is obvious.

We can now compute the total Chern class as follows:

$$1+c_1(q^*(\gamma_2))+c_2(q^*(\gamma_2)) = (1+c_1(\eta^*(\gamma_1)))^2 = 1+2c_1(\eta^*(\gamma_1))+(c_1(\eta^*(\gamma_1)))^2.$$

The final statement of the lemma follows from the naturality of the Chern classes. □

Remark 3.6 Similar to the integral computation we can compute cohomology with coefficients in $\mathbb{F}_p = \mathbb{Z}/p$ for p prime. Let x be the mod p reduction of t. If $p \mid (r+1)$, there is a class $\sigma \in H^{2r-1}(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{F}_p)$ such that

$$H^*(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{F}_p) \cong \mathbb{F}_p[x,\sigma]/(x^{r+1}, \sigma^2).$$

Similarly, if $p \nmid (r+1)$, there is a class $\overline{\sigma} \in H^{2r+1}(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{F}_p)$ such that

$$H^*(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{F}_p) \cong \mathbb{F}_p[x,\overline{\sigma}]/(x^r, \overline{\sigma}^2).$$

Corollary 3.7 *The composition $S(\tau(\mathbb{C}\mathrm{P}^r)) \xrightarrow{q'} \Delta_o(\mathbb{C}\mathrm{P}^r) \xrightarrow{\psi} \mathbb{P}(\gamma_2)$ of the quotient map with the diffeomorphism satisfies*

$$(\psi \circ q')^*(x_1) = (\psi \circ q')^*(x_2) = \eta^*(c_1(\gamma_1)).$$

The kernel of the map $(\psi \circ q')^*$ is a copy of the integers, generated by $x_1 - x_2$.

Proof Put $f = \psi \circ q'$ and consider the composition

$$S(\tau(\mathbb{C}\mathrm{P}^r)) \xrightarrow{f} \mathbb{P}(\gamma_2) \xrightarrow{\pi} \mathbf{G}_2(\mathbb{C}^{r+1}),$$

which equals the map q from (4). By Lemma 3.5 we have that

$$(\pi \circ f)^*(c_1(\gamma_2)) = 2t \quad , \quad (\pi \circ f)^*(c_2(\gamma_2)) = t^2,$$

where $t = \eta^*(c_1(\gamma_1))$. According to Theorem 3.2 we also have that $\pi^*(c_1(\gamma_2)) = x_1 + x_2$ and $\pi^*(c_2(\gamma_2)) = x_1 x_2$. So we conclude that

$$f^*(x_1) + f^*(x_2) = 2t \quad , \quad f^*(x_1) f^*(x_2) = t^2.$$

If $r \geq 3$ one has $H^i(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{Z}) \cong \mathbb{Z}$ for $i = 2, 4$ generated by t and t^2 respectively. So these equations have the unique solution $f^*(x_1) = f^*(x_2) = t$ which proves the corollary in this case. If $i \leq 2$, we use the standard inclusion $\mathbb{C}^i \subset \mathbb{C}^3$ together with naturality to get the desired result. □

4 Borel cohomology of spaces of geodesics

In this section we compute the Borel cohomology of the spaces of geodesics $\mathbf{B}_n(\mathbb{C}\mathrm{P}^r)$ for $n \geq 1$. We use the sphere bundle description to do so.

Consider the space $S(\tau(\mathbb{C}\mathrm{P}^r))$ equipped with the free $\mathbb{T}$–action. Let $S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}$ denote $S(\tau(\mathbb{C}\mathrm{P}^r))$ where we have twisted the $\mathbb{T}$–action by composing it with the nth power map $\lambda_n\colon \mathbb{T} \to \mathbb{T}$; $z \mapsto z^n$. Write $C_n \subseteq \mathbb{T}$ for the cyclic group of order n. The map λ_n passes to an isomorphism $\mathbb{T}/C_n \to \mathbb{T}$ with inverse R_n given by sending z to an nth root of z. We write BC_n for $E\mathbb{T}/C_n$ with $\mathbb{T}$–action through $R_n\colon \mathbb{T} \to \mathbb{T}/C_n$.

The action of C_n on $S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}$ is trivial. This makes it possible to consider $S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}$ as a $\mathbb{T}/C_n$ space. We get homeomorphisms as follows:

$$E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)} \cong E\mathbb{T}/C_n \times_{\mathbb{T}/C_n} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)} \cong BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r)).$$

We can write $\mathbb{T} \to S(\tau(\mathbb{C}\mathrm{P}^r)) \to \Delta_o(\mathbb{C}\mathrm{P}^r)$ as a pullback of the universal $\mathbb{T}$–bundle $\mathbb{T} \to E\mathbb{T} \to B\mathbb{T}$ along a map $f\colon \Delta_o(\mathbb{C}\mathrm{P}^r) \to B\mathbb{T}$. By this pullback diagram we get a map of associated fibration sequences:

(6)
$$\begin{tikzcd}
BC_n \arrow[r] \arrow[d, equal] & BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r)) \arrow[r, "\rho"] \arrow[d] & \Delta_o(\mathbb{C}\mathrm{P}^r) \arrow[d, "f"] \\
BC_n \arrow[r] & B\mathbb{T} \arrow[r] & B\mathbb{T}.
\end{tikzcd}$$

Note that we have used the homotopy equivalence $\mathrm{pr}_1\colon BC_n \times_{\mathbb{T}} E\mathbb{T} \xrightarrow{\simeq} B\mathbb{T}$ in the middle of the lower part of the diagram. We can now compute the cohomology of the homotopy orbit spaces.

Theorem 4.1 *For any prime p one has*

$$H^*(E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}) \cong \begin{cases} \mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1}), & p \nmid n, \\ \mathbb{F}_p[u, x, \sigma]/(x^{r+1}, \sigma^2), & p \mid n,\ p \mid (r+1), \\ \mathbb{F}_p[u, x, \overline{\sigma}]/(x^r, \overline{\sigma}^2), & p \mid n,\ p \nmid (r+1), \end{cases}$$

where u, x, x_1, x_2 have degree 2 and $\deg(\sigma) = 2r-1$, $\deg(\overline{\sigma}) = 2r+1$.

Proof If p does not divide n, then the mod p cohomology of BC_n equals that of a point, and by an obvious spectral sequence argument the projection map ρ induces an isomorphism in $\mathbb{F}_p$ cohomology. The result follows by Corollary 3.3.

Now assume that p does divide n. Then one has $H^*(BC_n) \cong \mathbb{F}_p[v, w]/I_{n,p}$ where $I_{n,p}$ is the ideal defined by $I_{n,p} = (v^2 - w)$ if $p = 2$ and $4 \nmid n$ and $I_{n,p} = (v^2)$ otherwise. The degrees are $\deg(v) = 1$ and $\deg(w) = 2$.

The two fibrations of diagram (6) each give a spectral sequences, and the vertical maps induce a map of spectral sequences. Let us denote the spectral sequence derived from the lower row of the diagram by $E_r(I)$, the one derived from the upper row by $E_r(II)$ and the map by $f^*\colon E_r(I) \to E_r(II)$.

We have that $E_2(I) = \mathbb{F}_p[u] \otimes \mathbb{F}_p[v, w]/I_{n,p}$ where the only non-trivial differential is d_2, which is determined by $d_2 w = 0$, $d_2 v = u$ and the product structure. This follows since the inclusion of the fiber $BC_n \to B\mathbb{T}$ is given by the inclusion $C_n \subseteq \mathbb{T}$ and the spectral sequence converges to $H^*(B\mathbb{T})$ such that $E_\infty(I) = \mathbb{F}_p[w]$.

We compute the E_2 page of the other spectral sequence:

$$E_2(II) \cong H^*(\Delta_o(\mathbb{C}\mathrm{P}^r)) \otimes \mathbb{F}_p[v, w]/I_{n,p} \Rightarrow H^*(BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))).$$

The class w is a permanent cycle, since it is the image of a permanent cycle in the spectral sequence $E_r(I)$. The whole spectral sequence is generated by the permanent cycle w together with the classes in $E_2^{*,0}(II)$ and $E_2^{*,1}(II)$.

But this implies, for formal reasons, that the only possible non-trivial differential is d_2. Using the product structure we also see that this non-trivial differential is determined by $d_2(v)$, which by naturality equals the image $f^*(d_2(v)) = f^*(u)$.

We claim that $f^*(u) = \lambda(x_1 - x_2)$ for some $\lambda \in \mathbb{F}_p \setminus \{0\}$. To see this, we consider the fibration sequence

$$S(\tau(\mathbb{C}\mathrm{P}^r)) \longrightarrow \Delta_o(\mathbb{C}\mathrm{P}^r) \xrightarrow{f} B\mathbb{T}.$$

We have already investigated the involved spaces. According to Lemma 3.5 and Corollary 3.3, the corresponding spectral sequence for integral cohomology has the form

$$E_2 = \mathbb{Z}[u] \otimes \mathbb{Z}[t, \overline{\sigma}]/J \Rightarrow \mathbb{Z}[x_1, x_2]/(Q_r, Q_{r+1}),$$

where $J = (t^{r+1}, \overline{\sigma}^2, (r+1)t^r, t^r\overline{\sigma})$. In the notation, we do not distinguish between the classes $x_1, x_2 \in H^2(\mathbb{P}(\gamma_2); \mathbb{Z})$ and their pullbacks under $\psi\colon \Delta_o(\mathbb{C}\mathrm{P}^r) \xrightarrow{\psi} \mathbb{P}(\gamma_2)$

Consider the case $r \geq 2$ first. We have a short exact sequence

$$0 \longrightarrow H^2(B\mathbb{T}; \mathbb{Z}) \xrightarrow{f^*} H^2(\Delta_o(\mathbb{C}\mathrm{P}^r); \mathbb{Z}) \xrightarrow{(q')^*} H^2(S(\tau(\mathbb{C}\mathrm{P}^r)); \mathbb{Z}) \longrightarrow 0.$$

According to Corollary 3.7 the kernel of of $(q')^*$ is the free group generated by $x_1 - x_2$. It follows that $f^*(u) = \pm(x_1 - x_2)$. Using naturality on the canonical inclusion $\Delta_o(\mathbb{C}\mathrm{P}^1) \subset \Delta_o(\mathbb{C}\mathrm{P}^2)$, which is a $\mathbb{T}$–map, we see that this formula is also true for $r = 1$. So we have proved the claim.

We return to cohomology with $\mathbb{F}_p$ coefficients. Let K_r and C_r be the kernel and the cokernel of multiplication by the element $x_1 - x_2$ on $H^*(\Delta_o(\mathbb{C}\mathrm{P}^r))$. Then

$$E_3(II) = (\mathbb{F}_p[w] \otimes C_r) \oplus (\mathbb{F}_p[w] \otimes vK_r),$$

and the spectral sequence collapses from the E_3–page.

By Theorem 3.2 and the equation $Q_k(x_1, x_1) = (k+1)x_1^k$ we see that

$$C_r = \begin{cases} \mathbb{F}_p[x_1]/x_1^{r+1}, & p \mid (r+1), \\ \mathbb{F}_p[x_1]/x_1^r, & p \nmid (r+1). \end{cases}$$

The dimension of the cokernel agrees with the dimension of the kernel. So the kernel of multiplication by $x_1 - x_2$ is a vector space over $\mathbb{F}_p$ of dimension $r+1$ if $p \mid (r+1)$, and dimension r if $p \nmid (r+1)$.

We need more precise information about the kernel, since we want to compute the multiplicative structure of the spectral sequence $E_r(II)$. To determine the kernel, it is enough to exhibit as many linearly independent elements in the kernel as its known dimension. So it suffices to find $r+1$ non-trivial elements in pairwise different degrees if $p \mid (r+1)$, and r non trivial elements in pairwise different degrees if $p \nmid (r+1)$.

Consider the following homogeneous polynomial

$$a_k = x_1^k \sum_{i=0}^{r-1} (i+1) x_1^i x_2^{r-1-i}.$$

A computation shows that $(x_2 - x_1)a_k = x_1^k(Q_r(x_1,x_2) - (r+1)x_1^r)$. So if $p \mid (r+1)$, the elements a_k for $k \geq 0$ are all in the kernel of multiplication by $x_1 - x_2$. If $p \nmid (r+1)$ then a_k is in the kernel if $k \geq 1$, since $x_1^{r+1} = Q_{r+1} - x_2 Q_r$.

To show that the kernel is generated by these classes, we have to check that each a_k is non-trivial in $\mathbb{F}_p[x_1,x_2]/(Q_r, Q_{r+1})$, as long as $k \leq r$. We use Lemma 3.4 to do so. In a_k, the coefficient of $x_1^{k-1}x_2^r$ is 0 and the coefficient of $x_1^k x_2^{r-1}$ is 1. Furthermore, both $k-1$ and k lies between $(k+r-1)-r = k-1$ and r so by the lemma we conclude that $a_k \notin (Q_r, Q_{r+1})$ for $k \leq r$ as desired.

It follows that if $p \mid (r+1)$ then the $r+1$ elements $a_0, a_1, \dots, a_r$ form a basis for the kernel and if $p \nmid (r+1)$ then the r elements $a_1, a_2, \dots, a_r$ form a basis for the kernel.

By the explicit formula for a_k, we see that the kernel has basis $a_0, x_1 a_0, \dots, x_1^r a_0$ when $p \mid (r+1)$ and $a_1, x_1 a_1, \dots, x_1^{r-1} a_1$ when $p \nmid (r+1)$. Let σ represent a_0 when $p \mid (r+1)$ and and let $\overline{\sigma}$ represent a_1 when $p \nmid (r+1)$.

We obtain the E_3 term

$$E_3(II) = \begin{cases} \mathbb{F}_p[w, x_1, \sigma]/(x_1^{r+1}, \sigma^2), & p \mid (r+1), \\ \mathbb{F}_p[w, x_1, \overline{\sigma}]/(x_1^r, \overline{\sigma}^2), & p \nmid (r+1). \end{cases}$$

We already noted that there can be no nontrivial differentials beyond the second one, so $E_3(II) = E_\infty(II)$. □

Corollary 4.2 *If p divides n, then the mod p cohomology Serre spectral sequence for the fibration*

$$S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)} \xrightarrow{\quad i \quad} E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)} \xrightarrow{\quad \mathrm{pr}_1 \quad} B\mathbb{T}$$

collapses at the E_2 page. If p does not divide n, the inclusion i of the fiber induces a surjection in even degrees

$$i^*\colon H^{2*}(E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}) \twoheadrightarrow H^{2*}(S(\tau(\mathbb{C}\mathrm{P}^r))).$$

Proof Assume that p divides n. We can compute the E_2 page by Remark 3.6 and we have just computed the cohomology of the total space. It turns out that the E_2

page is isomorphic to the cohomology of the total space. So there is not room for any differentials.

Assume that p does not divide n. We only have to check that the classes x^j are in the image of i^*. But by the homeomorphisms we used for computing the cohomology of the Borel construction and Corollary 3.7 we have that $i^*(x_1) = x$ and $i^*(x_2) = x$ so the result follows. □

5 Equivariant vector bundles

In this section we collect results about the homotopy theory of $\mathbb{T}$–vector bundles which we will need later. For greater clarity, the level of generality will be greater than strictly necessary.

We are interested in the cohomology of the Borel construction on Thom spaces. Let G be a compact Lie group (we will only need the case $G = \mathbb{T}$). Assume that $\xi = (p\colon E \to X)$ is a G–vector-bundle over X in the sense of tom Dieck [30, I.9].

Lemma 5.1 *The map $EG \times_G p\colon EG \times_G E \to EG \times_G X$ is the projection of a vector bundle which we denote ξ_{hG}. There is a homeomorphism $\mathrm{Th}(\xi_{hG}) \cong EG_+ \wedge_G \mathrm{Th}(\xi)$. Moreover, if G is connected, then ξ_{hG} is oriented if and only if ξ is oriented when viewed as an ordinary vector bundle.*

Proof We have that $EG \times X$ is a free G–space. If this G–space is also strongly locally trivial, then [30, Proposition I.9.4] gives us that ξ_{hG} is a vector bundle. So we must show that $EG \times X$ has a tube around all of its point [30, page 46].

For $(e, x) \in EG \times X$ we have that $G/G_{(e,x)} = G$. The universal principal G–bundle $\pi\colon EG \to BG$ is locally trivial, so we have a neighborhood V around $\pi(e)$ and a trivialization $\phi\colon \pi^{-1}(V) \xrightarrow{\cong} V \times G$. We get a tube around (e, x) as follows:

$$\pi^{-1}(V) \times X \xrightarrow{\mathrm{pr}_1} \pi^{-1}(V) \xrightarrow{\phi} V \times G \xrightarrow{\mathrm{pr}_1} G.$$

Thus ξ_{hG} is a vector bundle as stated.

We may assume that there is a Riemannian metric on ξ which is invariant under the G action. We get a Riemannian metric on ξ_{hG} such that $D(\xi_{hG}) = EG \times_G D(\xi)$ and $S(\xi_{hG}) = EG \times_G S(\xi)$. So we have

$$\begin{aligned} \mathrm{Th}(\xi_{hG}) &= EG \times_G D(\xi)/EG \times_G S(\xi) \cong EG_+ \wedge_G (D(\xi)/S(\xi)) \\ &= EG_+ \wedge_G \mathrm{Th}(\xi). \end{aligned}$$

Regarding orientability, consider the fibration $G \to EG \times X \xrightarrow{\pi} EG \times_G X$. It is the pullback of $G \to EG \to BG$ along $\mathrm{pr}_1\colon EG \times_G X \to BG$. Furthermore, BG is 1–connected since G is connected so we have trivial coefficients in the E_2 pages of the Serre spectral sequences for π. Thus $H^1(EG \times_G X; \mathbb{F}_2) \to H^1(X; \mathbb{F}_2)$ is injective. The vector bundle ξ_{hG} is the pullback of ξ along $X \to EG \times_G X$. Since orientability is equivalent to the vanishing of the first Stiefel Whitney class the result follows. □

Corollary 5.2 *Let η be an n–dimensional $\mathbb{T}$–vector bundle over the $\mathbb{T}$–space X. Assume that η is oriented for $H^*(-;\mathbb{Z}/p)$. Then there is an isomorphism*

$$\widetilde{H}^*(\mathrm{Th}(\eta)_{h\mathbb{T}}; \mathbb{F}_p) \cong H^{*-n}(X_{h\mathbb{T}}; \mathbb{F}_p).$$

We are going to need a special case of the localization theorem for Borel cohomology. The set up for the theorem is the following.

Definition 5.3 A based space X homotopy equivalent to a $\mathbb{T}$ CW complex satisfies the localization finiteness condition if X contains only finitely many $\mathbb{T}$ orbit types, and X is finite dimensional.

Let Y be any $\mathbb{T}$ space with a fixed base point. The space $E\mathbb{T}_+ \wedge Y$ is equipped with a diagonal map $\widetilde{\Delta}\colon E\mathbb{T}_+ \wedge Y \to E\mathbb{T}_+ \wedge E\mathbb{T}_+ \wedge Y$ given by $\widetilde{\Delta}(s,y) = (s,s,y)$. After taking quotients with group actions, we obtain

$$\Delta_Y\colon E\mathbb{T}_+ \wedge_{\mathbb{T}} Y \to (B\mathbb{T}_+) \wedge (E\mathbb{T}_+ \wedge_{\mathbb{T}} Y).$$

This map induces a product

$$H^*(B\mathbb{T}) \otimes \widetilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y) \to \widetilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y),$$

which makes $\widetilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y)$ into a module over the ring $H^*(B\mathbb{T})$. A $\mathbb{T}$–equivariant map $f\colon Y_1 \to Y_2$ induces a module map by naturality. This module structure will pervade the whole theory.

One thing we can use it for is to localize. We invert the generator, to form the cohomology localized away from u as follows:

$$H^*(X_{h\mathbb{T}})\big[1/u\big] = H^*(X_{h\mathbb{T}}) \otimes_{\mathbb{F}_p[u]} \mathbb{F}_p[u,u^{-1}].$$

Theorem 5.4 *If X satisfies the localization finiteness condition, the inclusion $X^{C_p} \subseteq X$ induces an isomorphism of localized cohomology*

$$H^*(X_{h\mathbb{T}})[1/u] \xrightarrow{\quad\cong\quad} H^*((X^{C_p})_{h\mathbb{T}})[1/u].$$

Proof This is a special case of the localization theorem [30, Theorem III.4.2]. The parameters of the localization theorem, as given by tom Dieck, are chosen as follows: $G = \mathbb{T}$, the cohomology theory $H^*(-) = H^*(-;\mathbb{F}_p)$ and the set $S \subseteq H^*(BG) = H^*(B\mathbb{T};\mathbb{F}_p)$ is $S = \{1, u, u^2, \dots\}$. The subset $A \subseteq X$ is the empty set.

The statement of the localization theorem is to be interpreted as follows: The family $\mathcal{F}(S)$ of subgroups of $\mathbb{T}$ (defined in [30] after III.3.1) is the family of subgroups $C_n \subseteq \mathbb{T}$ with n not divisible by p. The set $X(\mathcal{F}) \subseteq X$ defined in [30, I.6.1], is the complement of the set X^{C_p} in X. But then $FX = X^{C_p}$, so that [30, Theorem III.4.2] indeed specializes to our theorem. $\square$

The application we want is the following: Let X be a space, satisfying the localization finiteness condition. Assume that the action is not effective, but that the subgroup $C_p \subseteq \mathbb{T}$ acts trivially. On the other hand, we assume that all isotropy groups of points in X are contained in $C_n \subseteq \mathbb{T}$ for some fixed n. Let ξ be a $\mathbb{T}$–vector bundle over X. We do not assume that the action of C_p on ξ is trivial. The fixed points of C_p forms a $\mathbb{T}$ subbundle $\xi^{C_p} \subseteq \xi$.

Theorem 5.5 *The following map induces an isomorphism on mod p cohomology localized away from u:*

$$E\mathbb{T}_+ \wedge_{\mathbb{T}} \mathrm{Th}(\xi^{C_p}) \to E\mathbb{T}_+ \wedge_{\mathbb{T}} \mathrm{Th}(\xi).$$

Proof The only possible isotropy groups of $\mathrm{Th}(\xi)$ are $\mathbb{T}$ itself (for the base point in the Thom space), and subgroups of C_n. In particular, there are finitely many orbit types. Since $\mathrm{Th}(\xi^{C_p}) = \mathrm{Th}(\xi)^{C_p}$, the result now follows by applying Theorem 5.4 on the $\mathbb{T}$–space $\mathrm{Th}(\xi)$. $\square$

6 The twisted action

In this section we prove a general result on the mod p Borel cohomology of a $\mathbb{T}$–space where the action has been twisted by the p–power map.

Let X be a $\mathbb{T}$–space with action map $\mu\colon \mathbb{T} \times X \to X$. We can twist this action by the power map $\theta_n\colon \mathbb{T} \to \mathbb{T}$; $\theta_n(z) = z^n$ and obtain another $\mathbb{T}$–space $X^{(n)}$. The underlying spaces of X and $X^{(n)}$ are equal, but the action map for $X^{(n)}$ is $\mu_n\colon \mathbb{T} \times X^{(n)} \to X^{(n)}$; $\mu_n(z, x) = \mu(\theta_n(z), x)$.

Lemma 6.1 *Let X be an $\mathbb{T}$–space. We have a pullback of fibration sequences which is natural in X as follows:*

$$\begin{tikzcd} BC_n \arrow[r] \arrow[d] & E\mathbb{T}\times_{\mathbb{T}} X^{(n)} \arrow[r, "g_n"] \arrow[d] & E\mathbb{T}\times_{\mathbb{T}} X \arrow[d] \\ BC_n \arrow[r] & B\mathbb{T} \arrow[r, "B\theta_n"] & B\mathbb{T}. \end{tikzcd}$$

Proof It is convenient to consider models of $E\mathbb{T}$ and $B\mathbb{T}$ which are realizations of simplicial topological abelian groups. So let $E\mathbb{T}=|E\mathbb{T}_\bullet|$ and $B\mathbb{T}=|B\mathbb{T}_\bullet|$, where $E\mathbb{T}_q$ is the $(q+1)$–fold Cartesian product of $\mathbb{T}$ by itself with

$$d_i(z_0,\dots,z_q)=(z_0,\dots,\widehat{z_i},\dots,z_q),\quad s_i(z_0,\dots,z_q)=(z_0,\dots,z_i,z_i,\dots,z_q)$$

and $B\mathbb{T}_q=E\mathbb{T}_q/\mathrm{diag}_q(\mathbb{T})$. The hat means that the element is left out.

We have simplicial maps $E(\theta_n)_\bullet$ and $B(\theta_n)_\bullet$ given by rising all elements in a tuple to the nth power. Since the kernel of $B(\theta_n)_q$ is exactly $(BC_n)_q$, the bottom row in the diagram is the realization of a short exact sequence of simplicial abelian groups, which means that it is a fibration. Since $E(\theta_n)$ is a map over $B(\theta_n)$ the right hand square in the diagram commutes when we put $g_n([e,x])=[E(\theta_n)(e),x]$. Note that this makes g_n well-defined since $E(\theta_n)(ez)=E(\theta_n)(e)z^n$. The map of the vertical fibers is the identity so the right hand square is a vertical pullback. It follows that it is also a horizontal pullback. □

Here is our main result on Borel cohomology of twisted actions.

Theorem 6.2 *For any prime p and $\mathbb{T}$–space X there is an isomorphism, natural in X:*

$$H^*(X)\otimes\mathbb{F}_p[u]\xrightarrow{\cong}H^*(E\mathbb{T}\times_{\mathbb{T}}X^{(p)}).$$

To prove the theorem, we first show that there exists a natural subgroup $\mathcal{F}_1^*(X)$ of $H^*((X^{(p)})_{h\mathbb{T}})$ with the property that the statement is true if we replace $H^*(X)$ by $\mathcal{F}_1^*(X)$. After we have constructed this subgroup, it is easy to identify it with $H^*(X)$.

The spectral sequence derived from the fibration of Lemma 6.1 defines a filtration

$$\mathcal{F}_0^*(X)\subseteq\mathcal{F}_1^*(X)\subseteq\cdots\subseteq H^*(E\mathbb{T}\times_{\mathbb{T}}X^{(p)})$$

such that $\mathcal{F}_i^*(X)/\mathcal{F}_{i-1}^*(X)\cong\bigoplus_{*\geq 0}E_\infty^{i,*}$. Each of these filtration subgroups is a functor on the category of $\mathbb{T}$–spaces. It is well known how to interpret the first group:

$$\mathcal{F}_0^*(X)=g_p^*(H^*(X_{h\mathbb{T}}))$$

We need to consider the next step in the filtration. This group does not have an equally simple description.

Definition 6.3 The group of low elements $\mathcal{F}_1^*(X) \subseteq H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(p)})$ is the subgroup consisting of those classes which in the Serre spectral sequence represents elements in $E_\infty^{*,0}$ or $E_\infty^{*,1}$.

We define a reduced version of the group in the obvious way. If X is a pointed $\mathbb{T}$–space, we have inclusions of groups, natural on such spaces

$$g_n^*(H^*(E\mathbb{T} \times_{\mathbb{T}} X, B\mathbb{T})) = \mathcal{F}_0^*(X,*) \subseteq \mathcal{F}_1^*(X,*) \subseteq H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(n)}, B\mathbb{T}).$$

Here is a general fact about the low elements corresponding to p–fold twisting.

Lemma 6.4 *$H^*(X_{h\mathbb{T}}^{(p)})$ is a free module over $H^*(B\mathbb{T}) = \mathbb{F}_p[u]$. The low elements $\mathcal{F}_1^*(X)$ form a basis for it as an $\mathbb{F}_p[u]$ module.*

Proof The module structure on $H^*(X_{h\mathbb{T}}^{(p)})$ is given by the map $\mathrm{pr}_1\colon X_{h\mathbb{T}}^{(p)} \to B\mathbb{T}$. The Serre spectral sequence associated to the upper fibration sequence in Lemma 6.1 has the form

$$E_2^{n,m}(X) \cong H^n(X_{h\mathbb{T}}) \otimes H^m(BC_p) \Rightarrow H^{n+m}(X^{(p)}).$$

Remember that $H^*(BC_p) \cong \mathbb{F}_p[v,w]/I_p$, where I_p is a principal ideal, generated by $v^2 - w$ if $p = 2$ and by v^2 otherwise.

The Serre spectral sequence of the lower fibration sequence in the lemma has $E_2^{*,*} = \mathbb{F}_p[u] \otimes \mathbb{F}_p[v,w]/I_p$ and converges towards $H^*(B\mathbb{T})$ so we must have $d_2(v) = u$ and $E_3(*) = E_\infty(*) = \mathbb{F}_p[w]$. Using this together with naturality in X of the spectral sequence we get that for any X the class $w \in E_2^{0,2}(X)$ is a permanent cycle. Furthermore,

$$E_2^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_2^{*,0} \oplus E_2^{*,1})$$

so it follows formally that $E_3^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_3^{*,0} \oplus E_3^{*,1})$. But then it also follows formally that higher differentials in this spectral sequence vanish, so that

$$E_\infty^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_\infty^{*,0} \oplus E_\infty^{*,1}).$$

The low elements are by definition a subspace of $H^*(X_{h\mathbb{T}}^{(p)})$. We can extend this inclusion to a unique map of $\mathbb{F}_p[u]$–modules

$$f\colon \mathbb{F}_p[u] \otimes \mathcal{F}_1^*(X) \to H^*(X_{h\mathbb{T}}^{(p)}).$$

This map sends $u^i \otimes \mathcal{F}_1^*(X)$ to $\mathcal{F}_i^*(X)$, and the map induces an isomorphism on the corresponding graded rings, by the above computation of $E_\infty^{*,j}$, and because $u \in H^2(B\mathbb{T})$ represents $w \in E_\infty^{0,2}$. It follows that f itself is an isomorphism. $\square$

Proof of Theorem 6.2 We have to exhibit a natural isomorphism $\mathcal{F}_1^*(X) \to H^*(X)$. From the spherical fibration

$$\mathbb{T} \longrightarrow E\mathbb{T} \times X^{(p)} \xrightarrow{\ i\ } E\mathbb{T} \times_{\mathbb{T}} X^{(p)}$$

we get a long exact Gysin sequence

$$\longrightarrow H^{*-2}(X_{h\mathbb{T}}^{(p)}) \xrightarrow{\cdot u} H^*(X_{h\mathbb{T}}^{(p)}) \xrightarrow{i^*} H^*(X^{(p)}) \longrightarrow H^{*-1}(X_{h\mathbb{T}}^{(p)}) \longrightarrow$$

(One can use the map $X \to *$ to see that the Euler class is u.) Since $H^*(X_{h\mathbb{T}}^{(p)})$ is a free $\mathbb{F}_p[u]$ module, multiplication by u is injective, and the long exact sequences breaks up into short exact sequences. So via Lemma 6.4 we get the short exact sequence

$$0 \longrightarrow \mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X) \xrightarrow{\cdot u} \mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X) \xrightarrow{i^*} H^*(X^{(p)}) \longrightarrow 0.$$

Thus, i^* factors through $\big(\mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X)\big)/\mathrm{im}(\cdot u) = \mathcal{F}_i^*(X)$ and gives the desired isomorphism. $\square$

7 Morse theory on LM and the iteration map

In this section we recall some results on Morse theory for the energy integral on the Hilbert manifold model of the free loop space. For details on this we refer to Klingenberg [18], especially to Chapter 1. Furthermore, we introduce the n–fold iteration map on the free loop space and prove a localization result in mod p cohomology for the p–fold iteration map.

Let M be a compact Riemannian manifold. In this section we use the Hilbert manifold model of the free loop space. We denote this Hilbert manifold by LM. An element in LM is a closed curve $\gamma\colon \mathbb{T} \to M$ of class H^1. The Hilbert manifold model is homotopy equivalent to the usual continuous mapping space model.

Let $T_\gamma LM$ denote the tangent space of LM at γ. It consists of the vector fields along γ of class H^1, and it is a real Hilbert space with inner product

$$\langle \xi, \eta\rangle_1 = \int_{\mathbb{T}} \big(\langle \xi(z), \eta(z)\rangle + \langle \nabla\xi(z), \nabla\eta(z)\rangle\big) dz.$$

Here ∇ denotes the covariant differentiation along the curve defined by the Levi–Civita connection of the Riemannian manifold M. The inner product defines a Riemannian metric on the Hilbert manifold LM.

The energy integral (or energy function)

$$E\colon LM \to \mathbb{R}; \quad E(\gamma) = \int_{\mathbb{T}} |\gamma'(z)|^2 dz$$

is a smooth function on LM. Let $-\mathrm{grad}\, E$ be the negative gradient vector field of E with respect to $\langle \cdot, \cdot \rangle_1$. The critical points for E are precisely the closed geodesic on M.

Let γ be one of these geodesics. The Hessian of the energy function is a quadratic form $D^2 E_\gamma$ on the tangent space $T_\gamma LM$. This form determines a self adjoint operator $A_\gamma\colon T_\gamma LM \to T_\gamma LM$ by the equation

$$D^2 E_\gamma(\xi, \eta) = \langle A_\gamma(\xi), \eta \rangle_1.$$

The operator A_γ is the sum of the identity with a compact operator, so there are at most a finite number of negative eigenvalues, each corresponding to a finite dimensional vector space of eigenvectors of A_γ. The kernel of A_γ, which is also finite dimensional, consists of the periodic Jacobi fields along γ.

Now let $N(\lambda)$ be the space of critical points of E with energy level λ. The negative bundle $\mu^-(\lambda)$ over $N(\lambda)$ is the vector bundle whose fiber at γ is the vector space spanned by the eigenvectors belonging to negative eigenvalues of A_γ. Similarly, $\mu^0(\lambda)$ and $\mu^+(\lambda)$ are the vector bundles with fibers spanned by the eigenvectors corresponding to the eigenvalue 0 and the positive eigenvalues respectively.

It is known that $-\mathrm{grad}\, E$ satisfy condition (C) of Palais and Smale so that one can do Morse theory on LM if some additional non-degeneracy condition is satisfied. For us the so called Bott non-degeneracy condition is the relevant one. It requires firstly that for each critical value λ the space $N(\lambda)$ is a compact submanifold of LM and secondly that for each $\gamma \in N(\lambda)$ the restriction of the operator A_γ to the complement $(T_\gamma N(\lambda))^\perp$ of $T_\gamma N(\lambda)$ in $T_\gamma LM$ is invertible. The Bott non-degeneracy condition is a strong assumption on the metric of M, but for instance the symmetric spaces satisfy this, according to Ziller [34, Theorem 2].

Let the critical values of the energy function be $0 = \lambda_0 < \lambda_1 < \dots$. Consider the filtration of LM given by $\mathcal{F}(\lambda_i) = E^{-1}(-\infty, \lambda_i)$. This filtration is equivariant with respect to the action of the circle.

The tangent bundle of LM restricted to $N(\lambda_i)$ splits $\mathbb{T}$–equivariantly into a sum of three bundles.

$$TLM|_{N(\lambda_i)} \cong \mu^-(\lambda_i) \oplus \mu^0(\lambda_i) \oplus \mu^+(\lambda_i).$$

Assume that the Bott non-degeneracy condition holds. Then the standard Morse theory argument can be carried through equivariantly on the Hilbert manifold LM. This was done by Klingenberg. For an account of this work see [18, Section 2.4], especially Theorem 2.4.10. The statement of this theorem implies that we have an equivariant homotopy equivalence

$$\mathcal{F}(\lambda_n)/\mathcal{F}(\lambda_{n-1}) \simeq \mathrm{Th}(\mu^-(\lambda_n)).$$

Next we examine the iteration map. The power map $\theta_n\colon \mathbb{T} \to \mathbb{T}$; $\theta_n(z) = z^n$ gives us an n–fold iteration map

$$P_n\colon LM \to LM; \quad P_n(\gamma) = \gamma \circ \theta_n.$$

Just like other versions of the iteration map, this map is not equivariant with respect to the $\mathbb{T}$–action, but becomes equivariant if we twist the action. That is, the formula defines an equivariant map $P_n\colon (LM)^{(n)} \to LM$.

The map preserves energy up to a constant factor, $E(P_n(\gamma)) = n^2 E(\gamma)$. That is, it induces an equivariant map of energy filtration $\mathcal{F}(a) = E^{-1}(-\infty, a)$ as follows:

$$P_n\colon (\mathcal{F}(a))^{(n)} \to \mathcal{F}(n^2 a)$$

We also get an equivariant map of quotients

$$P_n\colon (\mathcal{F}(a)/\mathcal{F}(a-\epsilon))^{(n)} \to \mathcal{F}(n^2a)/\mathcal{F}(n^2a - n^2\epsilon).$$

One technical problem is that the iteration map is not an (injective) isometry with respect to the Riemannian metric $\langle \cdot, \cdot \rangle_1$ above. To study the properties of P_n it will be convenient to consider a modified inner product on $T_\gamma LM$. Let

$$\langle \xi, \eta \rangle_{c,1} = \int_{\mathbb{T}} \big(\langle \xi(z), \eta(z) \rangle + c \langle \nabla\xi(z), \nabla\eta(z) \rangle \big) dz.$$

where c is a constant. The differential of the iteration map is given by

$$D_\gamma(P_n)\colon T_\gamma LM \to T_{P_n(\gamma)} LM; \quad D_\gamma(P_n)(\xi) = \xi \circ \theta_n.$$

So we have $\nabla(D_\gamma(P_n)\xi) = n\nabla(\xi) \circ \theta_n = nD_\gamma(P_n)(\nabla\xi)$ and

$$\langle D_\gamma(P_n)\xi, D_\gamma(P_n)\eta \rangle_1 = \int_{\mathbb{T}} \langle \xi(z^n), \eta(z^n) \rangle + n^2 \langle \nabla(\xi)(z^n), \nabla(\eta)(z^n) \rangle dz = \langle \xi, \eta \rangle_{n^2,1}.$$

This means that even if the iteration map does not preserve the inner product, it becomes an isometry if we replace the inner product on the target by a suitably modified inner product.

The metric on M determines an exponential map $\exp_p\colon T_pM \to M$ for each $p \in M$. This induces an exponential map

$$\widetilde{\exp_\gamma}\colon T_\gamma LM \to LM; \quad \widetilde{\exp_\gamma}(\xi)(t) = \exp_{\gamma(t)}(\xi(t)).$$

This is not the exponential map derived from the Hilbert metric on LM, but it has the advantage that it is compatible with the iteration map in the sense that the following diagram commutes:

$$\begin{tikzcd} T_\gamma LM \arrow[r, "D_\gamma(P_n)"] \arrow[d, "\widetilde{\exp_\gamma}"] & T_{P_n(\gamma)} LM \arrow[d, "\widetilde{\exp_{P_n(\gamma)}}"] \\ LM \arrow[r, "P_n"] & LM. \end{tikzcd}$$

The exponential map is a diffeomorphism from a neighborhood of 0 in $T_\gamma LM$ to a neighborhood of γ in LM.

Obviously the operator A_γ depends not just on the Hessian of the energy integral, but also on the metric on LM. Since we are considering modifications of the inner product, we also have to consider the corresponding modifications of the self adjoint operator. To emphasize this dependence, for a real vector space V with inner product $\langle\, ,\, \rangle_\alpha$ and with a quadratic form Q, we define the operator $A(\langle\, ,\, \rangle_\alpha, Q)$ by the property that $Q(\xi,\eta) = \langle A(\langle\, ,\, \rangle_\alpha, Q)\xi, \eta\rangle_\alpha$.

Using the modified inner product $\langle\, ,\, \rangle_{c,1}$ we obtain a modified negative bundle μ_c^- etc. This bundle is not the same as μ^-. But on the other hand, the difference between these two bundles is not so dramatic, since the orthogonal projection from μ_c^- to μ^- with respect to the standard inner product defines an isomorphism of bundles.

Lemma 7.1 *The iteration map induces a bijection $P_n\colon N(a) \to (N(an^2))^{C_n}$. The group C_n acts on the fiber $(\mu^-)_{P_n\gamma}$, and for any $\gamma \in N(a)$ the differential of the iteration map induces an isomorphism of vector spaces*

$$D_\gamma(P_n)\colon (\mu^-_{n^2})_\gamma \xrightarrow{\cong} ((\mu^-)_{P_n\gamma})^{C_n}.$$

Proof The property of being a geodesic is a local property of a curve, so $P_n\gamma$ is a geodesic if and only if γ is a geodesic. It follows that $P_n(N(a)) \subseteq (N(n^2a))^{C_n}$. On the other hand, if $\eta \in (N(n^2a))^{C_n}$ we can write $\eta = P_n\gamma$ for some $\gamma \in LM$. But

γ has to be a geodesic, since η is, and $E(\gamma) = E(P_n\gamma)/n^2 = a$, which means that $\gamma \in N(a)$. Thus, P_n induces a bijection as stated.

Now we look at the tangent map. We first claim that this map induces an isomorphism

$$D_\gamma(P_n)\colon T_\gamma LM \xrightarrow{\ \cong\ } (T_{P_n\gamma}LM)^{C_n}.$$

This is clear, since $P_n\gamma$ is periodic of period n, and any n–periodic vector field along $P_n\gamma$ is the image under the differential of the iteration map of a vector field on γ.

We must show that this isomorphism restricts to an isomorphism as stated. The groups C_n acts on $V = T_{P_n\gamma}LM$, let us say that a generator $\sigma \in C_n$ acts by

$$(\sigma\xi)(z) = \xi(z\zeta_n) \in T_{P_n\gamma(z\zeta_n)}M = T_{P_n\gamma(z)}M; \quad \zeta_n = e^{2\pi i/n}.$$

Let $N \in \mathbb{C}[C_n]$ be the sum $\frac{1}{n}\sum_{k=0}^{n-1}\sigma^k$. This N acts as an idempotent on V, and so does $\mathbf{Id} - N$. The inner product is manifestly invariant under the action on C_n, so a simple calculation shows that N is self adjoint such that N and $\mathbf{Id} - N$ are orthogonal idempotents. Moreover, the idempotents commute with the action of the group C_n. So V splits as a C_n representation into an orthogonal sum $NV \oplus (\mathbf{Id} - N)V$.

The energy function is also invariant under the group action, so by the same argument as we just used for the inner product, the quadratic form $(V, D^2_{P_n\gamma}(E)$ splits as a direct sum $(NV, D^2_{P_n\gamma}(E)) \oplus ((\mathbf{Id} - N)V, D^2_{P_n\gamma}(E))$.

Let $A = A(\langle\,,\,\rangle_1, D^2_{P_n\gamma}(E))$. Since both the inner product and the quadratic form split as direct sums, the linear endomorphism $A\colon V \to V$ is a direct sum of two self adjoint endomorphisms

$$A|_{NV}\colon NV \to NV, \quad A|_{(\mathbf{Id}-N)V}\colon (\mathbf{Id}-N)V \to (\mathbf{Id}-N)V.$$

It follows that the subspace of V generated by negative eigenvectors equals the direct sum of the negative eigenvector spaces of $A|_{NV}$ and $A|_{(\mathbf{Id}-N)V}$.

We claim that $NV = V^{C_n}$ and $((\mathbf{Id} - N)V)^{C_n} = 0$. This follows since $\sigma N = N$ so that $NV \subseteq V^{C_n}$ and if $\sigma(\mathbf{Id} - N)\xi = (\mathbf{Id} - N)\xi$ then $\sigma\xi = \xi$ so that $(\mathbf{Id} - N)\xi = 0$.

To finish the proof, we have to show that $D_\gamma(P_n)(\mu^-_{n^2})_\gamma$ equals the negative eigenvector space of $A|_{NV}$.

But $D_\gamma(P_n)$ is an isometry (with respect to the modified inner product), and it preserves D^2E up to multiplication by n^2. So have a commutative diagram

$$\begin{tikzcd}
T_\gamma LM \arrow[r, "D_\gamma P_n"] \arrow[d, "n^2 A(\langle\,,\rangle_{n^2,1}, D^2_\gamma E)"'] & T_{P_n\gamma} LM \arrow[d, "A(\langle\,,\rangle_1, D^2_{P_n\gamma} E)"] \\
T_\gamma LM \arrow[r, "D_\gamma P_n"] & T_{P_n\gamma} LM
\end{tikzcd}$$

by which we get the desired result. □

Assume that $N(a) \subseteq LM$ is a non-degenerate critical manifold of energy a. Also assume that $P_p(N(a)) \subseteq N(p^2a)$ is a non-degenerate critical manifold, and that there are no other critical points at energy levels in the intervals $(a-\epsilon, a)$ and $(p^2(a-\epsilon), p^2a)$.

The Whitney sum $\mu(a) = \mu^-(a) \oplus \mu^+(a)$ is the normal bundle of the submanifold $N(a) \subseteq LM$ because we are assuming that $N(a)$ satisfies Bott's non-degeneracy condition.

Theorem 7.2 *The p–fold iteration map induces an isomorphism in cohomology localized away from u as follows:*

$$\widetilde{H}^*((\mathcal{F}(a)/\mathcal{F}(a-\epsilon))_{h\mathbb{T}})[1/u] \cong \widetilde{H}^*((\mathcal{F}(p^2a)/\mathcal{F}(p^2(a-\epsilon)))^{(p)}_{h\mathbb{T}})[1/u].$$

Proof We have a commutative diagram (possibly after decreasing ϵ)

$$\begin{tikzcd}
(D(\mu^-_{p^2}(a)), S(\mu^-_{p^2}(a))) \arrow[r, "DP_p"] \arrow[d, "\exp^\sim"] & (D(\mu^-(p^2a)), S(\mu^-(p^2a))) \arrow[d, "\exp^\sim"] \\
(\mathcal{F}(a), \mathcal{F}(a-\epsilon)) \arrow[r, "P_p"] & (\mathcal{F}(p^2a), \mathcal{F}(p^2(a-\epsilon))).
\end{tikzcd}$$

According to Theorem 5.4 the upper horizontal map induces an isomorphism in mod p cohomology localized away from u.

The exponential map in the right column induces a $\mathbb{T}$ homotopy equivalence on quotients by [18, Theorem 2.4.10]. This is the Hilbert manifold version of the fundamental theorem of Morse theory. So to prove the theorem, we need to see that the left vertical map induces a $\mathbb{T}$ equivariant homotopy equivalence on quotients.

But this is true for the same reason. The proof of [18, Theorem 2.4.10] does not use the explicit form of the metric on the Hilbert manifold LM, but only the non-degeneracy of D^2E. The statement of the variation of the theorem using LM with the modified metric $\langle\, , \rangle_{1/p^2,1}$ is exactly the statement that the left vertical map is a homotopy equivalence. $\square$

Remark 7.3 The theorem and its proof are also valid if a is not a critical value. In this case the interpretation of the statement is that if λ is a critical value of the energy function, such that λ/p^2 is not a critical value, then for $\epsilon > 0$ sufficiently small,

$$\widetilde{H}^*((\mathcal{F}(\lambda)/\mathcal{F}(\lambda-\epsilon))^{(p)}_{h\mathbb{T}})\,[1/u] = 0.$$

Remark 7.4 Since P_p is a homeomorphism whose image is the C_p fixed points, we could try to use Theorem 5.4 directly on the inclusion $\mathcal{F}(p^2a)^{C_p} \subseteq \mathcal{F}(p^2a)$, cleverly bypassing this entire section. The problem with this approach is that you need a finite dimensionality condition for the localization statement Theorem 5.4 to be true. The only role of Morse theory in the proof of Theorem 7.2 is to reduce the infinite dimensional situation to a finite dimensional one.

It seems likely that this reduction can be done in greater generality, and that the non-degeneracy condition in Theorem 7.2 is far stronger than necessary.

8 The Morse spectral sequences

In this section we set up the Morse spectral sequences coming from the energy filtration of the free loop space. We find a relationship between two of these spectral sequences based on the localization result in the previous section.

Let M be a compact Riemannian manifold and assume that the Bott non-degeneracy condition holds. Let $0 = \lambda_0 < \lambda_1 < \dots$ be the critical values of the energy function and consider the energy filtration of LM given by $\mathcal{F}(\lambda_i) = E^{-1}(-\infty, \lambda_i)$. Note that the filtration is $\mathbb{T}$–equivariant. This filtration induces spectral sequences of various forms in cohomology. We call these spectral sequences Morse spectral sequences.

We will consider cohomology of the homotopy orbits $LM_{h\mathbb{T}}$. We are also going to consider the twisted action. This is not because we have a particular interest in $H^*(LM^{(p)}_{h\mathbb{T}})$ in itself. As we will see, these groups are easy to compute anyhow. The purpose of considering them is rather to be able to use a comparison argument to obtain information about the Morse spectral sequence of $H^*(LM_{h\mathbb{T}})$. The abstract situation is as follows:

Theorem 8.1 *There are three spectral sequences*

$$\{E_r^{n,m}(\mathcal{M})(LM)\} \Rightarrow H^{n+m}(LM),$$
$$\{E_r^{n,m}(\mathcal{M})(LM_{h\mathbb{T}})\} \Rightarrow H^{n+m}(LM_{h\mathbb{T}}),$$
$$\{E_r^{n,m}(\mathcal{M})(LM_{h\mathbb{T}}^{(p)})\} \Rightarrow H^{n+m}(LM_{h\mathbb{T}}^{(p)}).$$

The E_1 pages are given by

$$\widetilde{H}^m(\mathrm{Th}(\mu^-(\lambda_n)))), \quad \widetilde{H}^m(\mathrm{Th}(\mu^-(\lambda_n)_{h\mathbb{T}}))), \quad \widetilde{H}^m(\mathrm{Th}(\mu^-(\lambda_n)_{h\mathbb{T}}^{(p)})))$$

respectively. We have a natural isomorphism of spectral sequences

$$E_*(\mathcal{M})(LM_{h\mathbb{T}}^{(p)}) \cong \mathbb{F}_p[u] \otimes_{\mathbb{F}_p} E_*(\mathcal{M})(LM)$$

as a module over $H^*(B\mathbb{T}) \cong \mathbb{F}_p[u]$.

Proof Since the energy filtration $\mathcal{F}(\lambda_0) \subseteq \mathcal{F}(\lambda_1) \subseteq \dots$ with union LM is $\mathbb{T}$–equivariant, it induces filtrations of the spaces LM, $LM_{h\mathbb{T}}$ and $LM_{h\mathbb{T}}^{(p)}$. These filtrations give rise to three exact couples which gives the three spectral sequences. The spectral sequences all converge (strongly). One can use [29, Chapter 9, 1.6] or Boardman [2, Theorem 12.6] and the remark following [2, Theorem 7.1] to see this. The equivariant homotopy equivalence $\mathcal{F}(\lambda_n)/\mathcal{F}(\lambda_{n-1}) \simeq \mathrm{Th}(\mu^-(\lambda_n))$ implies that the E_1 pages are as stated.

The functorial isomorphism of $H(B\mathbb{T};\mathbb{F}_p)$–modules from Theorem 6.2 imply the natural isomorphism of spectral sequences. □

We next have to discuss the p–fold iteration map. If there exists a periodic geodesic of energy λ, its p–fold iterate is a geodesic of energy $p^2\lambda$. So the sequence $p^2\lambda_0 < p^2\lambda_1 < p^2\lambda_2 < \dots$ is a subsequence of the sequence $\lambda_0 < \lambda_1 < \lambda_2 < \dots$. The p–fold iteration map induces an equivariant map of filtrations:

$$\begin{array}{ccccccc}
\mathcal{F}(\lambda_0)^{(p)} & \longrightarrow & \mathcal{F}(\lambda_1)^{(p)} & \longrightarrow & \cdots & \longrightarrow & LM^{(p)} \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathcal{F}(p^2\lambda_0) & \longrightarrow & \mathcal{F}(p^2\lambda_1) & \longrightarrow & \cdots & \longrightarrow & LM.
\end{array}$$

Lemma 8.2 *Let $p^2\lambda_{i-1} = \lambda_j < \lambda_{j+1} < \cdots < \lambda_k = p^2\lambda_i$ be the critical values of the energy function in the interval $[p^2\lambda_{i-1}, p^2\lambda_i]$. Then*

$$H^*(\mathcal{F}(\lambda_\ell)_{h\mathbb{T}}, \mathcal{F}(\lambda_{\ell-1})_{h\mathbb{T}}; \mathbb{F}_p)\,[1/u] = \begin{cases} 0 & \ell \neq k, \\ H^*(\mathcal{F}(\lambda_i), \mathcal{F}(\lambda_{i-1}); \mathbb{F}_p) \otimes \mathbb{F}_p[u, u^{-1}] & \ell = k. \end{cases}$$

The isomorphism is induced by the iteration map followed by the isomorphism of Theorem 6.2.

Proof This follows from Theorem 7.2 and Remark 7.3. □

Theorem 8.3 *Up to a re-indexing of the columns in the spectral sequence the p–fold iteration map induces a natural isomorphisms of spectral sequences*

$$E_*(\mathcal{M})(LM_{h\mathbb{T}})[1/u] \cong E_*(\mathcal{M})(LM) \otimes \mathbb{F}_p[u, u^{-1}].$$

Proof We have to explain the re-indexing. Let $j(i)$ be the non-negative number such that $p^2\lambda_i = \lambda_{j(i)}$, and $k(i)$ the largest number $s \le i$ such that $s = j(t)$ for some t. So by definition, $k(i) \le i$.

Besides the filtration $\mathcal{F}_i := \mathcal{F}(\lambda_i)$ we consider two derived filtrations of LM;

$$\mathcal{F}'_i := \mathcal{F}_{j(i)} \quad , \quad \mathcal{F}''_i := \mathcal{F}_{k(i)}.$$

The filtrations define spectral sequences

$$E_*(\mathcal{M})'(LM_{h\mathbb{T}}) \Rightarrow H^*(LM_{h\mathbb{T}}) \quad , \quad E_*(\mathcal{M})''(LM_{h\mathbb{T}}) \Rightarrow H^*(LM_{h\mathbb{T}}).$$

The p fold iteration map $P := P_p$ induces a map of filtrations $P\colon \mathcal{F}_i \to \mathcal{F}_{j(i)}$, so we have a ladder

$$\begin{tikzcd} (\mathcal{F}_0)^{(p)}_{h\mathbb{T}} \arrow[r] \arrow[d, "P"] & (\mathcal{F}_1)^{(p)}_{h\mathbb{T}} \arrow[r] \arrow[d, "P"] & \cdots \arrow[r] & LM^{(p)}_{h\mathbb{T}} \arrow[d, "P"] \\ (\mathcal{F}'_0)_{h\mathbb{T}} \arrow[r] & (\mathcal{F}'_1)_{h\mathbb{T}} \arrow[r] & \cdots \arrow[r] & LM_{h\mathbb{T}}. \end{tikzcd}$$

According to Theorem 7.2 we get an isomorphism of spectral sequences

$$P^*\colon E_*(\mathcal{M})'(LM_{h\mathbb{T}})[1/u] \xrightarrow{\cong} E_*(\mathcal{M})(LM^{(p)}_{h\mathbb{T}})[1/u].$$

Note that the spectral sequence on the right hand side can be rewritten by the isomorphism in Theorem 8.1.

The filtration $\mathcal{F}''_i$ is just a trivial reindexing of the filtration $\mathcal{F}'_i$. We have that

$$\mathcal{F}'_0 = \mathcal{F}''_0 = \mathcal{F}''_1 = \cdots = \mathcal{F}''_{j(1)-1} \subseteq \mathcal{F}'_1 = \mathcal{F}''_{j(1)} = \mathcal{F}''_{j(1)+1} = \ldots$$

So the two spectral sequences

$$E_*(\mathcal{M})''(LM_{h\mathbb{T}})[1/u] \quad , \quad E_*(\mathcal{M})'(LM_{h\mathbb{T}})[1/u]$$

are the same up to a re-indexing of the columns.

Finally, since $\mathcal{F}''_i = \mathcal{F}_{k(i)} \subseteq \mathcal{F}_i$ we have an inclusion of filtrations, which induces a map of spectral sequences

$$E_*(\mathcal{M})''(LM_{h\mathbb{T}})[1/u] \longrightarrow E_*(\mathcal{M})(LM_{h\mathbb{T}})[1/u].$$

According to Lemma 8.2 this map is an isomorphism of E_1 pages, and thus of spectral sequences. This proves the theorem. □

9 The Morse spectral sequences for $\mathbb{C}\mathrm{P}^r$

We now turn to the special case $M = \mathbb{C}\mathrm{P}^r$ with the usual symmetric space (Fubini–Study) metric. We know that this space satisfies the conditions of Section 8, in particular Theorem 8.3 is valid. We are going to study the Morse spectral sequence for $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. Using Theorem 8.3 we can pin down the possible structure of the differentials, but we cannot completely determine them. In particular, using only differential geometry, we can not prove that the differentials are non-trivial. But we will make an effort to obtain as much information as possible about the Morse spectral sequence with the information we already have available at this point.

It will turn out in Section 12 that there are non-trivial differentials in the Morse spectral sequence. In that section we are going to compare the Morse spectral sequence to a purely homotopy theoretical spectral sequence. This will not quite solve the differentials, but it will at least determine the dimension of $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ as a vector space over $\mathbb{F}_p$.

Theorem 9.1 *The Morse spectral sequence $E_*^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a spectral sequence of $H^*(B\mathbb{T}) = \mathbb{F}_p[u]$ modules converging towards $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. In case $p \mid (r+1)$ the E_1 page is given by*

$$\begin{aligned}
E_1^{0,*} &= \mathbb{F}_p[u,x]/(x^{r+1}),\\
E_1^{pm+k,*} &= \alpha_{pm+k}\mathbb{F}_p[x_1,x_2]/(Q_r,Q_{r+1}), \quad 0 \le m, \quad 1 \le k \le p-1,\\
E_1^{pm,*} &= \alpha_{pm}\mathbb{F}_p[x,u]/(x^{r+1}) \oplus \zeta_{pm}\mathbb{F}_p[x,u]/(x^{r+1}), \quad 1 \le m.
\end{aligned}$$

In case $p \nmid (r+1)$ the E_1 page is given by

$$\begin{aligned}
E_1^{0,*} &= \mathbb{F}_p[u,x]/(x^{r+1}),\\
E_1^{pm+k,*} &= \alpha_{pm+k}\mathbb{F}_p[x_1,x_2]/(Q_r,Q_{r+1}), \quad 0 \le m, \quad 1 \le k \le p-1,\\
E_1^{pm,*} &= \alpha_{pm}\mathbb{F}_p[x,u]/(x^{r}) \oplus \overline{\zeta}_{pm}\mathbb{F}_p[x,u]/(x^{r}), \quad 1 \le m.
\end{aligned}$$

The total degree of the element $\alpha_{pm+k}x_1^i x_2^j$ is $2r(pm+k-1)+2i+2j+1$ and the filtration degree is $pm+k$, where $1 \le k \le p-1$. The generators which are free $\mathbb{F}_p[u]$–module generators have total degree and filtration degree as follows:

class	*case*	*total degree*	*filtration*
$\alpha_{pm}x^i$	all r, $0 \le i \le r-1$	$2r(pm-1)+2i+1$	pm
$\alpha_{pm}x^r$	$p \mid (r+1)$	$2r(pm-1)+2r+1$	pm
$\zeta_{pm}x^i$	$p \mid (r+1)$ and $0 \le i \le r$	$2rpm+2i$	pm
$\bar{\zeta}_{pm}x^i$	$p \nmid (r+1)$ and $0 \le i \le r-1$	$2rpm+2+2i$	pm

Proof The sequence of critical values for the energy function is given by $\lambda_n = n^2$. So in order to determine the E_1 page we have to compute $\widetilde{H}^*(\mathrm{Th}(\mu^-(n^2))_{h\mathbb{T}})$ for each n.

The case $n=0$ is special. The critical manifold $N(0)$ consists of the constant curves, so it is diffeomorphic to $\mathbb{C}\mathrm{P}^r$ itself, with trivial action of $\mathbb{T}$. Since the constant curves are the absolute minima of the energy function, the negative bundle is the trivial bundle. We find

$$E_1^{0,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) = H^*(B\mathbb{T}\times\mathbb{C}\mathrm{P}^r) = H^*(B\mathbb{T})\otimes H^*(\mathbb{C}\mathrm{P}^r).$$

This is what the theorem states for $E_1^{0,*}$.

Now consider $n>0$. The negative bundle $\mu^-(n^2)$ is a $\mathbb{T}$–vector bundle over $N(n^2)\cong S(\tau(\mathbb{C}\mathrm{P}^r))^{(n)}$. We know from Theorem 8.1 that

$$E_1^{n,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \cong \widetilde{H}^*(\mathrm{Th}(\mu^-(n^2))_{h\mathbb{T}}).$$

Recall from [4] that if we forget the $\mathbb{T}$–action, then $\mu^-(n^2)$ equals the vector bundle $(n-1)\eta^*(\tau(\mathbb{C}\mathrm{P}^r))\oplus\epsilon$ over $S(\tau(\mathbb{C}\mathrm{P}^r))$. In particular, this means that $\mu^-(n^2)$ is orientable. So by Lemma 5.1, the vector bundle $\mu^-(n^2)_{h\mathbb{T}}$ is orientable and $\mathrm{Th}(\mu^-(n^2))_{h\mathbb{T}}\cong\mathrm{Th}(\mu^-(n^2)_{h\mathbb{T}})$. The vector bundle $\mu^-(n^2)_{h\mathbb{T}}$ is of the same dimension as $\mu^-(n^2)$, that is of dimension $2r(n-1)+1$. So by the Thom isomorphism we find that

$$E_1^{n,*}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \cong \widetilde{H}^*(\mathrm{Th}(\mu^-(n^2)_{h\mathbb{T}})) \cong H^{*-(2r(n-1)+1)}(N(n^2)_{h\mathbb{T}}).$$

The theorem now follows from the computation of the Borel cohomology of the space of geodesics in Theorem 4.1. □

Remark 9.2 The symbol $\alpha_n x^i$ refers to the cup product in a critical manifold. The precise meaning is that it is the Thom isomorphism of the bundle μ_n^- applied to the class x^i. In particular, the class α_n is the Thom class itself. The product is not defined on the cohomology of the Thom space of the negative bundle. It is also not defined in the spectral sequence. So strictly speaking, the product notation is improper. Products with u on the other hand are genuine products, which are defined in the spectral sequence.

We also consider the non-equivariant case.

Lemma 9.3 *Consider the Morse spectral sequence*

$$\{E_r^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)\} \Rightarrow H^*(L\mathbb{C}\mathrm{P}^r).$$

We have the following formulas for its E_1 page:

$$E_1^{n,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r) = \begin{cases} \mathbb{F}_p[x]/(x^{r+1}) & ,n=0, \\ \alpha_n \mathbb{F}_p[x,\sigma]/(x^{r+1},\sigma^2) & ,n \geq 1, \quad p \mid (r+1), \\ \alpha_n \mathbb{F}_p[x,\overline{\sigma}]/(x^r,\overline{\sigma}^2) & ,n \geq 1, \quad p \nmid (r+1). \end{cases}$$

The total degree of α_n is $2r(n-1)+1$, the total degree of σ is $2r-1$ and the total degree of $\overline{\sigma}$ is $2r+1$. If $i \geq rn+1$ and $p \mid (r+1)$ or if $i \geq rn$ and $p \nmid (r+1)$ then $E_1^{n,2i+1-n}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r) = 0$ for all n. Finally, the canonical map

$$E_1^{n,2j+1-n}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to E_1^{n,2j+1-n}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)$$

is surjective for all n and all j.

Proof Most of this follows directly from Remark 3.6. In $\mathbb{F}_p[x,\sigma]/(x^{r+1},\sigma^2)$, the class of highest even degree is x^r of degree $2r$ and in $\mathbb{F}_p[x,\overline{\sigma}]/(x^r,\overline{\sigma}^2)$ it is x^{r-1} of degree $2r-2$. The stated vanishing result follows from this observation. The final surjectivity statement follows from the surjectivity result in Corollary 4.2. □

A basic fact, which we will use again and again, is the following:

Theorem 9.4 *The Morse spectral sequence for non-equivariant cohomology collapses.*

This theorem follows for instance from the stable splitting of $L\mathbb{C}\mathrm{P}^r$ that we constructed in [4]. But the theorem definitely does not belong to us. To prove the basic fact we only need a splitting on homology level, and such a splitting was known to Ziller [34].

Corollary 9.5 *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq L\mathbb{C}\mathrm{P}^r$ be the energy filtration and let $m \geq 0$. If $p \mid (r+1)$, the odd degree cohomology $H^{\mathrm{odd}}(\mathcal{F}_m)$ is a vector space of dimension $(r+1)m$. If $p \nmid (r+1)$, its dimension is rm.*

Proof According to Theorem 9.4 the spectral sequence $\{E_r^{*,*}\} = \{E_r^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)\}$ collapses, so that

$$H^{\mathrm{odd}}(\mathcal{F}_m) \cong \bigoplus_{i,n} E_\infty^{n,2i+1-n} \cong \bigoplus_{i,n} E_1^{n,2i+1-n},$$

where the direct sums are taken over pairs n,i such that $0 \leq n \leq m$ and $n \leq 2i+1$. By Lemma 9.3 we have that $E_1^{0,2i+1} = 0$, and for a fixed n with $n \geq 1$ we have that

$$\sum_{n \leq 2i+1} \dim(E_1^{n,2i+1-n}) = \begin{cases} r+1 & \text{if } p \mid (r+1), \\ r & \text{if } p \nmid (r+1), \end{cases}$$

independently of n. It follows that

$$\dim H^{\mathrm{odd}}(\mathcal{F}_m) = \sum_{1 \leq n \leq m} \sum_{n \leq 2i+1} \dim(E_1^{n,2i+1-n}) = \begin{cases} (r+1)m & \text{if } p \mid (r+1), \\ rm & \text{if } p \nmid (r+1). \end{cases}$$

which completes the proof. □

Later it will be convenient to be able to estimate the sheer size of the Morse spectral sequence using a coarse dimension counting. So let's get that over with.

Lemma 9.6 *In case $p \mid (r+1)$, the Poincaré series of $E_1^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is*

$$\frac{1-t^{2r+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

In case $p \nmid (r+1)$, the Poincaré series of $E_1^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is

$$\frac{1-t^{2r+2}-t^{2pr}+t^{2pr+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

Proof In Theorem 9.1 the E_1 term is written as a direct sum of three terms. We compute the Poincaré series of each term, and add. First note that the Poincaré series of the algebra $\mathbb{F}_p[x_1,x_2]/(Q_r,Q_{r+1})$ is

$$\frac{1-t^{2r}}{1-t^2}\,\frac{1-t^{2r+2}}{1-t^2}$$

as one sees by writing down a basis of monomials.

Here is the case $p \mid (r+1)$:

$$\frac{1}{1-t^2}\frac{1-t^{2r+2}}{1-t^2} + \frac{t}{1-t^{2pr}}\frac{1-t^{2r(p-1)}}{1-t^{2r}}\frac{1-t^{2r}}{1-t^2}\frac{1-t^{2r+2}}{1-t^2} + \frac{1}{1-t^2}\frac{1-t^{2r+2}}{1-t^2}\frac{t^{2r(p-1)+1}+t^{2rp}}{1-t^{2pr}},$$

which equals

$$\frac{1-t^{2r+2}}{(1-t^2)^2(1-t^{2pr})}\big[(1-t^{2pr})+(t-t^{2r(p-1)+1})+(t^{2pr}+t^{2r(p-1)+1})\big]=$$
$$\frac{(1-t^{2r+2})(1+t)}{(1-t^2)^2(1-t^{2pr})}=\frac{1-t^{2r+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

In case $p\nmid(r+1)$, the last term in the direct sum is changed and thus the Poincaré series equals the sum

$$\frac{1}{1-t^2}\frac{1-t^{2r+2}}{1-t^2}+\frac{t}{1-t^{2pr}}\frac{1-t^{2r(p-1)}}{1-t^{2r}}\frac{1-t^{2r}}{1-t^2}\frac{1-t^{2r+2}}{1-t^2}+$$
$$\frac{1}{1-t^2}\frac{1-t^{2r}}{1-t^2}\frac{t^{2r(p-1)+1}+t^{2pr+2}}{1-t^{2pr}},$$

which equals

$$\frac{1}{(1-t^2)^2(1-t^{2pr})}\Big[(1-t^{2r+2}-t^{2pr}+t^{2pr+2r+2})+(t-t^{2(p-1)r+1}-t^{2r+3}+t^{2pr+3})$$
$$+(t^{2(p-1)r+1}+t^{2pr+2}-t^{2pr+1}-t^{2pr+2r+2})\Big]$$
$$=\frac{1-t^{2r+2}-t^{2pr}+t^{2pr+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

This completes the proof. □

Now that we have completed the census, we will investigate what we can say about the algebraic properties of the Morse spectral sequence for $\mathbb{C}\mathrm{P}^r$.

Lemma 9.7 *The classes $\zeta_{pm}x^i$, $\overline{\zeta}_{pm}x^i$, $\alpha_{pm}x^i$ and $\alpha_n x_1^i$ are not in the image of any differential.*

Proof The inclusion map $i\colon L\mathbb{C}\mathrm{P}^r\to L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}$ induces a map of filtrations, and a map of cohomology Morse spectral sequences. We know from Theorem 9.4 that the target spectral sequence collapses.

But on the E^1 level, the classes $\zeta_{pm}x^i$ and $\overline{\zeta}_{pm}x^i$ survive to the classes represented by the Thom isomorphism applied to σx^i respectively $\overline{\sigma}x^i$, according to Corollary 4.2. These classes are not in the image of a differential (since all differentials vanish). So by naturality, the classes $\zeta_{pm}x^i$ and $\overline{\zeta}_{pm}x^i$ cannot be in the image of a differential either.

Similarly, the classes $\alpha_{pm}x^i$ and $\alpha_n x_1^i$ are mapped non-trivially according to Corollary 4.2 and the result follows. □

Lemma 9.8 *In the Morse spectral sequence $E_*^{*,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ every non-trivial differential starts in an even total degree.*

Proof We have to do some counting, but the strategy of the proof is simple. First we note that it is enough to consider the generators of E_1, secondly we see that except for a very few special cases these generators cannot map non-trivially for dimensional reasons. Thirdly we use Lemma 9.7 to dispose of the remaining cases.

The elements of odd degree in the spectral sequence are of the form $\alpha_{pm}x^iu^j$ or $\alpha_n x_1^i x_2^j$. Because the spectral sequence is a spectral sequence of $\mathbb{F}_p[u]$ modules, to prove the lemma it suffices to show the special case that all differentials vanish on the generators $\alpha_{pm}x^i$ and $\alpha_n x_1^i$.

We consider a class $d_s(\alpha_{pm}x^i)$. We wish to prove that it is trivial. This class has filtration $pm+s$ and total degree $2r(pm-1)+2i+2$. We ask, when does there exist a non trivial class of this filtration and total degree? To figure this out, we inspect the table of Theorem 9.1, and realize the following facts.

- There exist non-trivial classes in $E_1(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}};\mathbb{F}_p)$ of total even degree, and of filtration n if and only if n is divisible by p.
- If n is divisible by p there is a unique such class of filtration n with lowest possible even dimension. In case p divides $r+1$ it is ζ_n of dimension $2rn$, if p does not divide $r+1$ it is $\overline{\zeta}_n$ of dimension $2rn+2$.

The class $d_s(\alpha_{pm}x^i)$ has total even degree, so if it were non-trivial it would to have degree at least equal to the lowest possible degree of such a class. That is

$$2r(pm-1)+2i+2 \geq \begin{cases} 2r(pm+s), & p \mid (r+1), \\ 2r(pm+s)+2, & p \nmid (r+1). \end{cases}$$

If p does not divide $r+1$, we have the two inequalities $i \geq r(s+1)$ and $r-1 \geq i$. This system has no solution with $s \geq 1$. If p divides $r+1$, we have the two inequalities $i+1 \geq r(s+1)$ and $r \geq i$. Now there is the unique solution $s=r=i=1$. That is, we would have that $d_1(\alpha_{pm-1}x) = \lambda\zeta_{pm}$ for a scalar $\lambda \neq 0$. But this contradicts Lemma 9.7.

A similar argument shows that $d_s(\alpha_n x_1^i) = 0$. □

At this point, to a certain extent we can make a shortcut in the theory. Using the elementary method as in the proof of Lemma 9.8 we can prove the following statement (Theorem 9.9). The term "elementary" should be read as "without using Theorem 8.3". Together with the purely homotopy theoretical analysis of the Serre spectral sequence (which we discuss in detail in Section 11), Theorem 9.9 is sufficient to compute the cohomology $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ in the special case $p \mid (r+1)$. It does not seem possible to cover the case $p \nmid (r+1)$ using a similar method. It is precisely because of this that we need Theorem 8.3.

Theorem 9.9 *If p divides $r+1$ then there is a class $\widehat{\zeta}_{pm} \in H^{2pmr}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ such that the restriction $i^*(\widehat{\zeta}_{pm}) \in H^{2pmr}(L\mathbb{C}\mathrm{P}^r)$ is non-trivial.*

Proof The argument is very similar to the proof of Lemma 9.8. We show that the class $\zeta_{pm} \in E_1^{pm,(2r-1)pm}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ of Theorem 9.1 survives to E_∞.

Suppose that $d_s(\zeta_{pm}) = \alpha_{pm+s}y$ for some nontrivial scalar y. Equality of total degrees gives the relation

$$2rpm+1 = \deg(\zeta_{pm})+1 = \deg(\alpha_{pm+s}y) = 2r(pm+s-1)+\deg(y)+1 \geq 2rpm+1.$$

So we conclude that $s=1$, $\deg(y)=0$, and $d_1(\zeta_{pm}) = \lambda\alpha_{pm+1}$ for a non-trivial scalar λ. But this contradicts Lemma 9.7. That is, ζ_{pm} is a permanent cycle. Furthermore, ζ_{pm} is not in the image of any differential by Lemma 9.7 , so it survives to E_∞.

We claim that any class $\widehat{\zeta}_{pn} \in H^{2pnr}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}};\mathbb{F}_p)$ representing ζ_{pm} will satisfy the property of the theorem. By naturality, ζ_{pm} maps nontrivially by i^* to the E_1 page of the Morse spectral sequence converging to $H^*(L\mathbb{C}\mathrm{P}^r;\mathbb{F}_p)$. The theorem follows, again since this spectral sequence collapses. □

In order to prove this sections final result we need a lemma on non-negatively graded modules over the graded ring $\mathbb{F}_p[u]$ (graded by $\deg(u)=2$). We say that the graded module M is trivial in degrees strictly greater than n if $M^i = 0$ for all $i > n$.

Let us also say that a graded $\mathbb{F}_p[u]$–module is generated in degrees less or equal to m if there is a set of generators which have degrees less or equal to m. Because M is bounded from below, a set of elements $\{x_\alpha\}$ generate M exactly if the reductions $[x_\alpha]$ span the $\mathbb{F}_p$–vector space M/uM.

Lemma 9.10 *Let $f\colon M \to N$ be a degree preserving map of non-negatively graded $\mathbb{F}_p[u]$ modules. Assume that M is generated in degrees less or equal to m, and that N is trivial in degrees strictly greater than n. Then the kernel of f is generated in degrees less or equal to $\max(m,n)$.*

Proof Assume without loss of generality that f is surjective. Multiplication by u defines a map of the short exact sequence $0 \to \ker(f) \to M \to N \to 0$ to itself. The snake lemma produces an exact sequence

$$\ker(u\colon N \to N) \longrightarrow \ker(f)/u\ker(f) \longrightarrow M/uM \longrightarrow N/uN \longrightarrow 0.$$

That M is generated in degrees less or equal to m implies that the graded vector space M/uM is zero in degrees strictly greater than m. The exact sequence proves that $\ker(f)/u\ker(f)$ is trivial in degrees greater than $\max(m,n)$. Pick a set of spanning classes $\{[x_\alpha]\}$ in $\ker(f)/u\ker(f)$, and lift them arbitrarily to classes $\{x_\alpha\}$ in $\ker(f)$ of degree less or equal to $\max(m,n)$. These classes generate $\ker(f)$. □

We are now ready to apply our general theorem on the localized Morse spectral sequence to our particular specimen.

Theorem 9.11 *The spectral sequence $E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ collapses from the E_p page.*

Proof Recall from Lemma 9.8 that all non-trivial differentials are defined on classes in even total degree. It is easy to see from Theorem 9.1 that the only classes of even total degree sit in filtrations divisible by p. Let $X_s^{pm,*} \subseteq E_s^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ be the subgroup of elements of even total degree. This is a module over the ring $\mathbb{F}_p[u]$, and for all s one has

$$X_{s+1}^{pm,*} = \ker\left(d_s\colon X_s^{pm,*} \to E_s^{pm+s,*}\right).$$

Claim If $1 \le s \le p-1$, then the $\mathbb{F}_p[u]$–module $X_{s+1}^{pm,*}$ is generated by elements in degrees less than $2r(pm+s+1)-2$.

The claim follows from Lemma 9.10 and induction over s. By Theorem 9.1 we see that $X_1^{pm,*}$ in all cases is generated in degrees less or equal to $2pmr+2r$. $E_s^{pm+s,*}$ is a quotient of the $\mathbb{F}_p[u]$–module $E_1^{pm+s,*}$. Using Theorem 9.1 we see that for $1 \le s \le p-1$ this last module is trivial in dimensions strictly greater than $2r(pm+s)+2r-1$. Now we use Lemma 9.10 on the differential

$$d_1\colon X_1^{pm,*} \to E_1^{pm+1,*}.$$

The differential raises total degree by 1. Taking this into account, Lemma 9.10 shows that X_1^{pm} is generated in degree less than $\max(2pmr+2r, 2r(pm+1)+2r-2) = 2r(pm+2)-2$, which proves the claim for $s=1$. Using Lemma 9.10 inductively on d_s for $2 \le s \le p-1$ proves the claim.

In particular, $X_p^{mp,*}$ is generated in degrees less or equal to $2r(pm+p)-2$. The basic Theorem 9.4 says that the Morse spectral sequence $E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)$ collapses from the E_1 page. Because of Theorem 8.3 this implies that the localized spectral sequence $E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})[1/u]$ collapses from the E_1 page.

From our computation in Theorem 9.1 we can read off that the localization map

$$E_1^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \longrightarrow E_1^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})\,[1/u]$$

is injective. By naturality, this means that no non-trivial differentials are arriving at $E_1^{pm,*}(\mathcal{M})(L\mathbb{C}P^r_{h\mathbb{T}})$. So in particular, the d_p differential on $X_p^{mp,*}$ is trivial, and $X_{p+1}^{mp,*} = X_p^{mp,*}$ is generated in degrees less than $2r(pm+p)-2$.

We claim that all higher differentials d_s vanish on $X_{p+1}^{pm,*}$. Assume to the contrary that s is the smallest $s \geq p+1$ which does not vanish on $X_s^{pm,*}$. Since $X_s^{pm,*} = X_{p+1}^{pm,*}$, we know that $X_s^{pm,*}$ is generated in degrees less or equal to $2r(pm+p)-2$.

It is enough to prove that the differential d_s vanishes on the generators. A generator of $X_{p+1}^{pm,*}$ will be mapped by d_s to a class in some $E_{p+1}^{pm+s,*}$ of total degree less or equal to $2r(pm+p)-1$. We have to show that there is no such non-trivial class.

But the non-zero class of lowest degree in $E_{p+1}^{pm+s,*}$ is α_{pm+s} which has degree $2r(pm+s-1)+1$. Since we are assuming that $s \geq p+1$, this degree is larger than or equal to $2r(pm+p)+1$. This finishes the proof. $\square$

10 Cohomology of the free loop space

The cohomology algebra $H^*(\mathbb{C}P^r)$ (as well as $H^*(\mathbb{H}P^r)$) has been computed by Menichi [25] and the action differential on this algebra can be found by Connes' exact sequence. We write down a slightly more general result, and postpone its proof to the first appendix (the proof is based on methods from [5]). The extra generality is not needed for the main results in this paper, but we would like to have it for later reference.

Theorem 10.1 *Let p be a prime. Assume that X is a 1–connected space with mod p homology of finite type. Assume also that*

$$H^*(X) = \mathbb{F}_p[x]/(x^{r+1}),$$

where the degree $|x|$ of x is even and $r \geq 1$. Put $\alpha = |x|$ and $\rho = (r+1)\alpha - 2$.
1) If $p \mid (r+1)$ then there is an algebra isomorphism

$$H^*(LX) \cong \mathbb{F}_p[x]/(x^{r+1}) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega],$$

where $|x| = \alpha$, $|\mathbf{d}x| = \alpha - 1$, $|\gamma_i(\omega)| = \rho i$. The action differential is given by

$$d\colon H^*(LX) \to H^*(LX); \quad d(x) = \mathbf{d}x, \quad d(\mathbf{d}x) = 0, \quad d(\gamma_i(\omega)) = 0.$$

2) If $p \nmid (r+1)$ then there is an algebra isomorphism

$$H^*(LX) \cong \mathbb{F}_p[a_i, b_i | i \geq 0]/I,$$

where I is the ideal generated by the following elements for $i, j \geq 0$:

$$a_i a_j, \quad b_i b_j - \tbinom{i+j}{i} b_0 b_{i+j}, \quad b_i a_j - \tbinom{i+j}{i} b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

The degrees of the generators are $|a_i| = \rho i + \alpha - 1$ and $|b_i| = \rho i + \alpha$. In particular, the dimensions of $H^{2k}(LX)$ and $H^{2k-1}(LX)$ are the same. The action differential is given by

$$d\colon H^*(LX) \to H^*(LX); \quad d(a_i) = \kappa_i b_0^{r-1} b_i, \quad d(b_i) = ((r+1)i+1)a_i.$$

where $\kappa_i = 0$ unless $\alpha = 2$, $p = 2$, r is even and i is odd.

Remark 10.2 When $p \nmid (r+1)$ we have the following formula for $0 \le i$, $1 \le j \le r$:

$$d(b_0^{j-1} b_i) = ((r+1)i + j) b_0^{j-1} a_i.$$

We will show later (in Corollary 12.8) that $\kappa_i = 0$. So we have $d(b_0^{j-1} a_i) = 0$.

We will later want to know how big the cokernel of the action differential is, so we do the counting now. We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of natural numbers.

Definition 10.3 Let p be a prime and let $r, \alpha \in \mathbb{N}$ with α even. Put $\rho = (r+1)\alpha - 2$ and let

$$\chi_p(s) = \begin{cases} 0 & \text{if } p \mid s, \\ 1 & \text{if } p \nmid s. \end{cases}$$

We define two subsets of $\mathbb{N}$ as follows:

$$\mathcal{I}F(r,p,\alpha) = \{\rho i + \alpha j \mid \chi_p(r+1) \le j \le r, 0 \le i \text{ and } p \mid ((r+1)i + j)\} \setminus \{0\},$$
$$\mathcal{I}T(r,p,\alpha) = \{\rho i + \alpha j \mid \chi_p(r+1) \le j \le r, 0 \le i \text{ and } p \nmid ((r+1)i + j)\}.$$

The notation $\mathcal{I}F$ refers to an *index* set for *free* generators and $\mathcal{I}T$ refers to an *index* set for *transfer* generators. This choice of notation will make sense later on.

Lemma 10.4 *Whether a natural number k is contained in $\mathcal{I}F(r,p,\alpha)$ respectively in $\mathcal{I}T(r,p,\alpha)$ only depends on the congruence class of k modulo ρp. Furthermore,*

$$\mathcal{I}F(r,p,\alpha) \cap \mathcal{I}T(r,p,\alpha) = \begin{cases} 2r\mathbb{N} & \text{if } p \mid (r+1) \text{ and } \alpha = 2, \\ \varnothing & \text{otherwise,} \end{cases}$$
$$\mathcal{I}F(r,p,2) \cup \mathcal{I}T(r,p,2) = 2\mathbb{N}.$$

If a number is in $\mathcal{I}F(r,p,\alpha)$ or in $\mathcal{I}T(r,p,\alpha)$, there is a unique choice of numbers i, j that displays it as such. If $p \mid (r+1)$, the set $\{0 < 2k \le \rho p m \mid 2k \in \mathcal{I}F(p,r,\alpha)\}$ has $m(r+1)$ elements. If $p \nmid (r+1)$, the set has mr elements.

Proof Note that $0 \notin \mathbb{N}$. We warm up by first ignoring the congruence conditions. Assume that $i, i' \in \mathbb{Z}$ and that $\chi_p(r+1) \leq j, j' \leq r$. We make two claims about this situation.

(1) If $i' > i$ and $\rho i + \alpha j = \rho i' + \alpha j'$, then $\alpha = 2$, $p \mid (r+1)$, $i' = i+1$, $j' = 0$ and $j = r$.

(2) If $\rho i + \alpha j > 0$, then $i \geq 0$.

To prove the first claim, note that $\alpha(j - j') = \rho(i' - i) \geq \rho$. So

$$j - j' \geq \frac{\rho}{\alpha} = r + \frac{\alpha - 2}{\alpha}.$$

Since $0 \leq j, j' \leq r$ and $\alpha \geq 2$, this is only possible if $j = r$, $j' = 0$, $\alpha = 2$. But this implies that $i' = i+1$ and by assumption, if $j' = 0$, then $p \mid (r+1)$.

To prove the second claim, note that

$$i > -\frac{\alpha j}{\rho} \geq -\frac{\alpha r}{\rho} = -\frac{\alpha r}{\alpha(r+1) - 2} \geq -\frac{\alpha r}{\alpha r} = -1.$$

We now prove the lemma. First assume that k and $k + m\rho p$ are natural numbers. Assume also that $k \in \mathcal{IF}(r, p, \alpha)$. We can write $k = \rho i + \alpha j$, where i, j satisfy the appropriate conditions. Then $k + m\rho p = \rho(i + mp) + \alpha j$, and the pair $(i + mp, j)$ satisfies the same congruence conditions, and conditions on j. This proves that $k + m\rho p \in \mathcal{IF}(r, p, \alpha)$, if we know that $i + mp \geq 0$. But this follows from claim 2. together with our assumption that $k + m\rho p$ is a natural number.

The same argument shows that $\mathcal{IT}(r, p, \alpha)$ is also a union of congruence classes of natural numbers.

If $x \in \mathcal{IF}(r, p, \alpha) \cap \mathcal{IT}(r, p, \alpha)$, it must be possible to write x in two different ways in the form $x = \rho i + \alpha j$.

By the first claim, we get that the only possible way this can happen is that $\alpha = 2$, $p \mid (r+1)$ and $x = 2ri + 2r = 2r(i+1) + 0$. This proves that $\mathcal{IF}(r, p, \alpha) \cap \mathcal{IT}(r, p, \alpha)$ is empty unless $p \mid (r+1)$ and $\alpha = 2$. It also shows that $\mathcal{IF}(r, p, 2) \cap \mathcal{IT}(r, p, 2) \subseteq 2r\mathbb{N}$. We have to show that if $p \mid (r+1)$ then $2r\mathbb{N} \subseteq \mathcal{IF}(r, p, 2) \cap \mathcal{IT}(r, p, 2)$. In this case we write $2rm = \rho i + \alpha j = \rho i' + \alpha j'$ for $(i, j) = (m, 0)$ and $(i', j') = (m-1, r)$. This proves the claim, since $p \mid j$ but $p \nmid j'$.

We have $\mathcal{IF}(r, p, 2) \cup \mathcal{IT}(r, p, 2) = \{2(ri + j) \mid \chi_p(r+1) \leq j \leq r, 0 \leq i\}$, which equals $2\mathbb{N}$ as stated. The uniqueness statement on i, j follows directly from claim 1.

To prove the final statement about the number of elements, it is enough to show that the number of congruence classes in $\mathcal{IF}(r, p, \alpha)$ modulo ρp is $r+1$ respectively r. In case $p \mid (r+1)$ the congruence classes of $2k$ are the classes of form $\rho i + \alpha p j'$ for $0 \le i < p$ and $0 \le j' < (r+1)/p$, and there are clearly $p((r+1)/p) = r+1$ of those. In case $p \nmid (r+1)$, each j uniquely determines a congruence class $i(j)$ modulo p such that $(r+1)i(j) + j \equiv 0 \mod p$. That is, each j with $1 \le j \le r$ uniquely determines an i, $0 \le i < p$ such that $(r+1)i + j \equiv 0 \mod p$. So there are exactly r pairs (i, j) qualifying, and there are r congruence classes in $\mathcal{IF}(r, p, \alpha)$. □

Example 10.5 Let $\alpha = 2$, $r = 2$. Then $\rho = 4$ and

$$\begin{aligned}
\mathcal{IF}(2,2,2) &= \{4 + 8m, 6 + 8m \mid m \ge 0\},\\
\mathcal{IF}(2,3,2) &= 4\mathbb{N},\\
\mathcal{IF}(2,5,2) &= \{8 + 20m, 14 + 20m \mid m \ge 0\},\\
\mathcal{IF}(2,7,2) &= \{10 + 28m, 20 + 28m \mid m \ge 0\}.
\end{aligned}$$

Lemma 10.6 *Let X be as in Theorem 10.1, $k \in \mathbb{N}$ and put $H^* = H^*(LX)$.*

(1) The kernel of the action differential $d\colon H^{2k} \to H^{2k-1}$ is either a trivial or a one dimensional vector space. It is non-trivial if and only if $2k \in \mathcal{IF}(r, p, \alpha)$.

(2) The image of $d\colon H^{2k} \to H^{2k-1}$ is either a trivial or a one dimensional vector space. It is non-trivial if and only if $2k \in \mathcal{IT}(r, p, \alpha)$.

(3) The cokernel of $d\colon H^{2k} \to H^{2k-1}$ is either a trivial or a one dimensional vector space. In case $p \nmid (r+1)$, it is non-trivial if and only if $2k \in \mathcal{IF}(r, p, \alpha)$. In case $p \mid (r+1)$ it is non-trivial if and only if either $2k \in \mathcal{IF}(r, p, \alpha)$ and $\rho \nmid 2k$ or if $k > 1$ and $2k \equiv 2 \mod \rho$.

(4) The kernel of the map $d\colon \bigoplus_{2 \le 2k \le \rho p m} H^{2k} \to \bigoplus_{1 \le 2k-1 \le \rho p m - 1} H^{2k-1}$ is a vector space of dimension mr if $p \nmid (r+1)$, and of dimension $m(r+1)$ if $p \mid (r+1)$.

(5) The cokernel of the map $d\colon \bigoplus_{2 \le 2k \le \rho p m} H^{2k} \to \bigoplus_{1 \le 2k-1 \le \rho p m - 1} H^{2k-1}$ is a vector space of dimension rm when $p \nmid (r+1)$.

(6) The cokernel of the map $d\colon \bigoplus_{2 \le 2k \le \rho p m + 2} H^{2k} \to \bigoplus_{1 \le 2k-1 \le \rho p m + 1} H^{2k-1}$ is a vector space of dimension $(r+1)m$ when $p \mid (r+1)$.

Proof For the action differential $d\colon H^{2k} \to H^{2k-1}$ we have the equation

$$\begin{aligned}
\dim \operatorname{coker}(d) &= \dim H^{2k-1} - \dim \operatorname{im}(d)\\
&= \dim H^{2k-1} - (\dim H^{2k} - \dim \ker(d)) \qquad (7)\\
&= -(\dim H^{2k} - \dim H^{2k-1}) + \dim \ker(d).
\end{aligned}$$

We first consider the case $p \nmid (r+1)$. The even part H^{even} has basis $\{b_0^{j-1} b_i\}$ and the odd part H^{odd} has basis $\{b_0^{j-1} a_i\}$ where $0 \le i$, $1 \le j \le r$. The basis elements sit in degrees $\rho i + \alpha j$ and $\rho i + \alpha j - 1$ respectively. The kernel of the action differential $d\colon H^{\text{even}} \to H^{\text{odd}}$ is generated by those $b_0^{j-1} b_i$ for which $p \mid ((r+1)i + j)$ and its image of those $b_0^{j-1} a_i$ for which $p \nmid ((r+1)i + j)$. Combining this with the uniqueness statement for i, j in Lemma 10.4 we see that (1). and (2). are valid. Furthermore we see that $\dim H^{2k} = \dim H^{2k-1}$ which combined with (7) gives us (3).

Next we consider the case $p \mid (r+1)$. Here H^{even} has basis $\{x^j \gamma_i(\omega)\}$ and H^{odd} has basis $\{x^j dx \gamma_i(\omega)\}$ where $0 \le i$, $0 \le j \le r$. The basis elements sit in degrees $\rho i + \alpha j$ and $\rho i + \alpha j + \alpha - 1$ respectively. The kernel for the action differential $d\colon H^{\text{even}} \to H^{\text{odd}}$ is generated by those $x^j \gamma_i(\omega)$ for which $p \mid j$ and the image is generated by those $x^{j-1} dx \gamma_i(\omega)$ for which $p \nmid j$. As above Lemma 10.4 give us that (1) and (2) are valid.

We now prove (3). One checks it directly for $(r, p, \alpha) = (1, 2, 2)$. Assume that $(r, p, \alpha) \neq (1, 2, 2)$ which implies $\rho > 2$. By a counting argument we find the following:

$$\dim H^{2k} - \dim H^{2k-1} = \begin{cases} 1 & \text{if } \rho \mid 2k, \\ -1 & \text{if } \rho \mid (2k-2) \text{ and } k > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We combine (8) and (7) in order to prove statement (3).

If $\rho \mid 2k$ we have $2k \in \mathcal{IF}(r, p, \alpha)$ so the dimension of the cokernel becomes 0. Assume that $\rho \mid (2k-2)$ and $k > 1$. We claim that this implies that $2k \notin \mathcal{IF}(r, p, \alpha)$. For if $2k \in \mathcal{IF}(r, p, \alpha)$ we have that $2k = \rho i + \alpha j$ where $0 \le i$, $0 \le j \le r$ and $p \mid j$. It follows that $2 \equiv 2k \equiv \alpha j \mod \rho$. Since $\rho > 2$, we cannot have $j = 0$. We conclude that $1 \le p \le j$. On the other hand, $0 \equiv 2k - 2 = \rho i + \alpha j - 2 \equiv \alpha j - 2 \mod \rho$, so $\rho \le \alpha j - 2$. Now we have our contradiction, finishing the proof of (3) since $\alpha j - 2 \le \alpha r - 2 < \rho$.

We have reduced the last three statements of the lemma to statements about the sets $\mathcal{IT}(r, p, \alpha)$ and $\mathcal{IF}(r, p, \alpha)$. We see that (4). is equivalent to the statement that the set $\{0 < 2k \le \rho p m | 2k \in \mathcal{IF}(r, p, \alpha)\}$ has rm elements if $p \nmid (r+1)$ and $(r+1)m$ elements if $p \mid (r+1)$. But this is exactly the content of the last statement of Lemma 10.4. Statement (5) follows from statement (4) and (7).

Finally, we prove statement (6). One verifies it directly for $(r,p,\alpha)=(1,2,2)$. Assume that $(r,p,\alpha)\neq(1,2,2)$. Formula (8) gives us that

$$\sum_{2\leq 2k\leq \rho pm+2}(\dim H^{2k}-\dim H^{2k-1})=0.$$

So by (7), (4) and (1) it suffices to check that $\rho pm+2\notin\mathcal{I}F(r,p,\alpha)$. This follows from Lemma 10.4 since $\rho pm+2\equiv 2 \mod \rho p$. □

We define two formal power series in $\mathbb{Z}[\![t]\!]$ by

$$P_{\mathcal{I}T(r,p,\alpha)}(t)=\sum_{n\in\mathcal{I}T(r,p,\alpha)}t^n,\qquad P_{\mathcal{I}F(r,p,\alpha)}(t)=\sum_{n\in\mathcal{I}F(r,p,\alpha)}t^n.$$

Note that the constant terms are zero in both series. Furthermore, if the numbers κ_i in Theorem 10.1 are zero for all i, then $P_{\mathcal{I}T(r,p,\alpha)}(t)=tP_{\mathrm{im}(d)}(t)$ where $P_{\mathrm{im}(d)}(t)$ is the Poincaré series for the image of the action differential. By Lemma 10.4 we find the following result:

Lemma 10.7 *When $\alpha=2$ such that $\rho=2r$ we have that*

$$P_{\mathcal{I}T(r,p,2)}(t)+P_{\mathcal{I}F(r,p,2)}(t)=\frac{t^2}{1-t^2}+(1-\chi_p(r+1))\frac{t^{2r}}{1-t^{2r}}.$$

11 The E_3 page of the Serre spectral sequence

In this section we write down the E_3 page of the Serre spectral sequence for the free loop space on a space whose cohomology is a truncated polynomial algebra on one generator. Then we compute the Poincaré series of the E_3 page. Finally, we use the $\mathbb{T}$–transfer map to show that some classes must survive to E_∞.

Let Y be a $\mathbb{T}$–space with $H_*(Y)$ of finite type. Write $q\colon E\mathbb{T}\times Y\to E\mathbb{T}\times_{\mathbb{T}}Y$ for the quotient map. As described in Appendix B there is a $\mathbb{T}$–transfer map τ such that the composite $q^*\circ\tau$ equals the action differential:

$$d\colon H^{*+1}(Y)\xrightarrow{\tau}H^*(Y_{h\mathbb{T}})\xrightarrow{q^*}H^*(Y).$$

Since $B\mathbb{T}$ is 1–connected and $H^*(B\mathbb{T})=\mathbb{F}_p[u]$ where $|u|=2$, the Serre spectral sequence for the fibration $Y\to Y_{h\mathbb{T}}\to B\mathbb{T}$ has the following form:

$$E_2^{*,*}=\mathbb{F}_p[u]\otimes H^*(Y)\Rightarrow H^*(Y_{h\mathbb{T}}).$$

The d_2 differential is determined by the action differential since $d_2 y = u dy$ for all $y \in H^*(Y)$. Thus, the E_3 term has the following form:

$$E_3^{*,*} = \mathrm{im}(d) \oplus (\mathbb{F}_p[u] \otimes H(d)),$$

where $\mathrm{im}(d)$ and $H(d)$ denotes the image and the homology of the action differential respectively.

Proposition 11.1 *The subspace $\mathrm{im}(d) \subseteq E_3^{*,*}$ survives to $E_\infty^{*,*}$. For any $a \in H^*(Y)$ one has that $\tau(a) \in H^*(Y_{h\mathbb{T}})$ represents $da \in E_\infty^{0,*}$ and that $\mathrm{pr}_1^*(u)\tau(a) = 0$ in $H^*(Y_{h\mathbb{T}})$, where $\mathrm{pr}_1\colon Y_{h\mathbb{T}} \to B\mathbb{T}$ denotes the projection on the first factor.*

Proof There are two commutative diagrams

$$\begin{tikzcd}
H^*(Y) \arrow[r, "\tau"] \arrow[dr, "d"'] & H^*(Y_{h\mathbb{T}}) \arrow[r, two heads] \arrow[d, "q^*"'] & E_\infty^{0,*} \arrow[d, hook] \\
& & E_3^{0,*} \arrow[d, hook] \\
& H^*(Y) \arrow[r, equal] & E_2^{0,*}
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(Y_{h\mathbb{T}}) \arrow[r, "q^*"] & H^*(E\mathbb{T} \times Y) \\
H^*(B\mathbb{T}) \arrow[r] \arrow[u, "\mathrm{pr}_1^*"] & H^*(E\mathbb{T}) \arrow[u, "\mathrm{pr}_1^*"']
\end{tikzcd}$$

Assume that $a \in H^*(Y)$ has $da \neq 0$. The diagram to the left shows that $\tau(a) \neq 0$ and that da survives to E_∞ and is represented by $\tau(a) \in H^*(Y_{h\mathbb{T}})$. The diagram to the right shows that $q^* \circ \mathrm{pr}_1^*(u) = 0$ since $E\mathbb{T}$ is contractible. By Frobenius reciprocity $\mathrm{pr}_1^*(u)\tau(b) = \tau(q^*(\mathrm{pr}_1^*(u))b) = 0$. □

We now take $Y = LX$. By the result in the previous section we can compute the E_3 term of the Serre spectral sequence.

Proposition 11.2 *Let p be a prime. Assume that X is a 1–connected space with mod p homology of finite type. Assume also that*

$$H^*(X; \mathbb{F}_p) = \mathbb{F}_p[x]/(x^{r+1}),$$

where $\alpha = |x|$ is even and $r \geq 1$. Put $\rho = (r+1)\alpha - 2$.

(1) *If $p \mid (r+1)$ then*

$$E_3^{*,*} \cong \big(\mathbb{F}_p[u, \phi, q, \delta_0, \delta_1, \dots, \delta_{p-2}]/I\big) \otimes \Gamma[\omega],$$

where I is the ideal

$$I = (\phi^{(r+1)/p},\ q^2,\ \delta_j u,\ \delta_j q,\ \delta_j \delta_k\ |0 \le j \le p-2,\ 0 \le k \le p-2).$$

The bidegrees are $\|u\| = (2,0)$, $\|\phi\| = (0, p\alpha)$, $\|q\| = (0, p\alpha - 1)$, $\|\gamma_i(\omega)\| = (0, \rho i)$ *and* $\|\delta_j\| = (0, j\alpha + \alpha - 1)$. *The generators are represented by elements in the* E_2 *term as follows:*

$$u = [u], \quad \phi = [x^p], \quad q = [x^{p-1} dx], \quad \delta_j = [x^j dx], \quad \gamma_i(\omega) = [\gamma_i(\omega)].$$

(2) *If* $p \nmid (r+1)$ *and the numbers* κ_t *from Theorem 10.1 are zero for all* t, *then*

$$E_3^{*,*} \cong$$
$$\mathbb{F}_p\big[v_i^{(k)}, w_i^{(k)}, \tau_i^{(h)}, u | 1 \le k \le r,\, p \mid ((r+1)i + k),\, 1 \le h \le r,\, p \nmid ((r+1)i + h),\, 0 \le i\big]/I,$$

where I *is the ideal generated by the elements*

$$\begin{aligned} &\tau_i^{(h)} u, \quad \tau_i^{(h)} w_j^{(\ell)}, \quad \tau_i^{(h)} \tau_j^{(m)}, \quad w_i^{(k)} w_j^{(\ell)}, \\ &v_i^{(k)} v_j^{(\ell)} - \epsilon_r(k+\ell)\tbinom{i+j}{i} v_{i+j}^{(k+\ell)}, \\ &v_i^{(k)} w_j^{(\ell)} - \epsilon_r(k+\ell)\tbinom{i+j}{i} w_{i+j}^{(k+\ell)}, \\ &v_i^{(k)} \tau_j^{(h)} - \epsilon_r(k+h)\tbinom{i+j}{i} \tau_{i+j}^{(k+h)}. \end{aligned}$$

Here the number $\epsilon_r(s)$ *equals* 1 *if* $1 \le s \le r$ *and* 0 *otherwise. The bidegrees of the generators are*

$$\|v_i^{(k)}\| = (0, \rho i + \alpha k), \qquad \|w_i^{(k)}\| = (0, \rho i + \alpha k - 1),$$
$$\|\tau_i^{(h)}\| = (0, \rho i + \alpha h - 1), \qquad \|u\| = (2,0),$$

and the generators are represented by elements in the E_2 *term as follows:*

$$v_i^{(k)} = [b_0^{k-1} b_i], \quad w_i^{(k)} = [b_0^{k-1} a_i], \quad \tau_i^{(h)} = [b_0^{h-1} a_i], \quad u = [u].$$

Proof (1) Assume that $p \mid (r+1)$ such that $r + 1 = mp$ for some $m \ge 1$. By the Künneth formula we have that

$$H(d) = H\big(\mathbb{F}_p[x]/(x^{mp}) \otimes \Lambda(dx); d\big) \otimes \Gamma[\omega].$$

Since $d(x^j) = j x^{j-1} dx$ and $d(x^j dx) = 0$, the kernel and image of the differential on $\mathbb{F}_p[x]/(x^{mp}) \otimes \Lambda(dx)$ has the following $\mathbb{F}_p$–bases:

$$\{x^{kp}, x^j dx | 0 \le k \le m-1, 0 \le j \le mp-1\}, \qquad \{x^{j-1} dx | 1 \le j \le mp-1,\, p \nmid j\}.$$

It follows that $H(d) = \mathbb{F}_p[\phi]/(\phi^m) \otimes \Lambda(q) \otimes \Gamma[\omega]$, where $\phi = x^p$ and $q = x^{p-1}dx$. We can get the basis elements in $\mathrm{im}(d)$ by multiplying the elements in $\{x^k dx | 0 \le k \le p-2\}$ by the elements $\gamma_i(\omega)$ and powers of ϕ. The result follows.

(2) Assume that $p \nmid (r+1)$. By Remark 10.2 we see that $\ker(d)$, $\mathrm{im}(d)$, $H(d)$ has respective $\mathbb{F}_p$–bases as follows:

$$\begin{array}{ll} b_0^{j-1} a_i,\ b_0^{k-1} b_i & \text{for } p \mid ((r+1)i + k), \\ b_0^{h-1} a_i & \text{for } p \nmid ((r+1)i + h), \\ [b_0^{k-1} b_i],\ [b_0^{k-1} a_i] & \text{for } p \mid ((r+1)i + k), \end{array}$$

where $0 \le i$ and $1 \le h, k, j \le r$. The result follows. □

Corollary 11.3 *If X satisfies the hypothesis of Proposition 11.2, then the Poincaré series for the E_3 term of the Serre spectral sequence is given by the following when $p \mid (r+1)$:*

$$\frac{1 - t^{(r+1)\alpha}}{(1 - t^{p\alpha})(1 - t^{(r+1)\alpha - 2})} \cdot \Big(\frac{1 + t^{p\alpha - 1}}{1 - t^2} + \frac{t^{\alpha - 1} - t^{p\alpha - 1}}{1 - t^{\alpha}} \Big),$$

and by the following when $p \nmid (r+1)$:

$$\frac{1}{1 - t^2} \Big(1 + P_{\mathcal{I}F(r,p,\alpha)}(t) + \frac{1}{t} P_{\mathcal{I}F(r,p,\alpha)}(t) \Big) + \frac{1}{t} P_{\mathcal{I}T(r,p,\alpha)}(t).$$

By Proposition 11.1 and Proposition 11.2 we have

Corollary 11.4 *If X satisfies the hypothesis of Proposition 11.2 then,*

(1) *If $p \mid (r+1)$ then the element $\gamma_i(\omega)\phi^j \delta_k \in E_3$ survives to E_∞ and is represented by $\tau(x^{pj+k+1}\gamma_i(\omega))$ for $0 \le i$, $0 \le j < (r+1)/p$ and $0 \le k \le p-2$.*

(2) *If $p \nmid (r+1)$ then the generator $\tau_i^{(h)} \in E_3$ survives to E_∞ and is represented by $\tau(b_0^{h-1} b_i)$ for $1 \le h \le r$ with $p \nmid ((r+1)i + h)$ and $0 \le i$.*

12 Comparing the two spectral sequences.

In this section we will complete the investigation of the Morse spectral sequence $E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ and prove the main result of this paper.

We write $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq L\mathbb{C}\mathrm{P}^r$ for the energy filtration. So far, the main structural facts which we proved in Section 9 are the following:

SF(1) The classes of even total degree are concentrated in $\bigoplus_m E_*^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ (Theorem 9.1).

SF(2) $E_*^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a free $\mathbb{F}_p[u]$ module. If $p \nmid n$, then $E_*^{n,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a finite dimensional vector space (Theorem 9.1).

SF(3) Every non-trivial differential goes from even total degree to odd total degree (Lemma 9.8).

SF(4) The spectral sequence collapses from the E_p page (Theorem 9.11).

Remark 12.1 It follows from SF(3) that the inclusion $j\colon (\mathcal{F}_n)_{h\mathbb{T}} \hookrightarrow L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}$ induces a surjective map $j^*\colon H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to H^{\mathrm{odd}}((\mathcal{F}_n)_{h\mathbb{T}})$ for all $n \geq 0$.

By SF(1) and SF(4) we see that the only possibly non-trivial differentials in the spectral sequence are

$$E_s^{pm,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{d_s} E_s^{pm+s,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \cong E_1^{pm+s,*}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$$

for $1 \leq s \leq p-1$.

We are also going to use the non-equivariant spectral sequence $E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)$. From Lemma 9.3 we have the following structural facts:

SF(5) $E_1^{n,2i+1-n}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r) = 0$ if $p \mid (r+1)$ and $i \geq rn+1$ or if $p \nmid (r+1)$ and $i \geq rn$.

SF(6) The map $E_1^{\mathrm{odd}}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to E_1^{\mathrm{odd}}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)$ is surjective.

Remark 12.2 By SF(6) and SF(3) the map $E_\infty^{\mathrm{odd}}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to E_\infty^{\mathrm{odd}}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r)$ is a surjection. A filtration argument shows that if a map in some degree induces a surjective map on the E_∞ pages, then it also induces a surjective map on the cohomology of the targets of the spectral sequences. So $H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r)$ is also surjective.

Our plan for computing $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ goes as follows. First, we concentrate on the odd part of $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. The sum of the odd dimensional cohomology groups $H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a submodule of $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ over $H^*(B\mathbb{T})$, since this ring is concentrated in even degrees. We will list a set of elements in this module, and use the above properties of the spectral sequences to show that these elements are generators. We also give the relations satisfied by these elements, thus computing the odd part of the Borel cohomology.

To determine the odd part of the cohomology is not quite the same as to determine the differentials in the Morse spectral sequence, but in our situation it is close enough. We can use the knowledge of the odd cohomology together with the spectral sequences to determine the even dimensional cohomology.

The first step in this program is to find elements in $H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ which can serve as generators. We use the results on the transfer map to find the right elements.

Consider the $\mathbb{T}$ transfer map τ in the context of the Morse filtration of $L\mathbb{C}\mathrm{P}^r$. Let $i\colon L\mathbb{C}\mathrm{P}^r \to L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}$ be the inclusion. It follows from Theorem B.1 that the composite

$$H^{*+1}(L\mathbb{C}\mathrm{P}^r) \xrightarrow{\tau} H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{i^*} H^*(L\mathbb{C}\mathrm{P}^r)$$

equals the action differential d.

We can now choose one half of our generators. This bunch of generators come with the relation that the generators are annihilated by multiplication by u.

Lemma 12.3 *There is a graded subgroup $\mathcal{T}^* \subseteq H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ such that*

(1) $u\mathcal{T}^* = 0$.

(2) The map $i^*\colon H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \to H^*(L\mathbb{C}\mathrm{P}^r)$ restricts to an injective map on $\mathcal{T}^*$.

(3) The subgroup $i^*(\mathcal{T}^*) \subseteq H^*(L\mathbb{C}\mathrm{P}^r)$ agrees with the image of the composite map

$$i^* \circ \tau\colon H^{*+1}(L\mathbb{C}\mathrm{P}^r) \to H^*(L\mathbb{C}\mathrm{P}^r).$$

Proof We chose a graded subgroup $\overline{\mathcal{T}}^{*+1} \subseteq \widetilde{H}^{*+1}(L\mathbb{C}\mathrm{P}^r)$ which maps isomorphically to the image of d. That is, we chose (arbitrarily) a splitting of the surjective map $d\colon H^{*+1}(L\mathbb{C}\mathrm{P}^r) \to \mathrm{im}(d)^*$. Then $\mathcal{T}^* = \tau(\overline{\mathcal{T}}^{*+1}) \subseteq H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a subgroup which by its definition satisfies the second and third property. It also satisfies the first property, since $u\tau = 0$ by Theorem B.1. □

Our second bunch of generators is not involved with any relations.

Lemma 12.4 *There is a graded subgroup $\mathcal{U}^* \subseteq H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ such that the composite*

$$\mathcal{T}^* \oplus \mathcal{U}^* \longrightarrow H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{i^*} H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r)$$

is an isomorphism. In addition to this, the restriction

$$\mathcal{U}^{2i+1} \longrightarrow H^{2i+1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{j^*} H^{2i+1}((\mathcal{F}_{pm})_{h\mathbb{T}})$$

is trivial if either $p \nmid (r+1)$ and $i \geq rpm$, or $p \mid (r+1)$ and $i \geq rpm+1$.

Proof To construct $\mathcal{U}^*$, we make a choice $\overline{\mathcal{U}}^* \subseteq H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r)$ of a complementary subgroup of $i^*(\mathcal{T})$, so that we have a direct sum decomposition of vector spaces $H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r) \cong i^*(\mathcal{T}^*) \oplus \overline{\mathcal{U}}^*$. We intend to find $\mathcal{U}^* \subseteq H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ such that i^* maps this subgroup isomorphically to $\overline{\mathcal{U}}^*$ and such that the statement for the restriction map holds.

According to the long exact sequence of Theorem B.1, the following diagram has exact rows. Because of Remark 12.1 the left and middle vertical maps are surjections, because of Remark 12.2 the upper right horizontal map is a surjection.

$$\begin{tikzcd}
H^{2i-1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \arrow[r, "u"] \arrow[d, two heads] & H^{2i+1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \arrow[r, two heads, "i^*"] \arrow[d, two heads] & H^{2i+1}(L\mathbb{C}\mathrm{P}^r) \arrow[d] \\
H^{2i-1}((\mathcal{F}_{pm})_{h\mathbb{T}}) \arrow[r, "u"] & H^{2i+1}((\mathcal{F}_{pm})_{h\mathbb{T}}) \arrow[r] & H^{2i+1}(\mathcal{F}_{pm}).
\end{tikzcd}$$

Assume that $p \nmid (r+1)$ and $i \geq rpm$, or $p \mid (r+1)$ and $i \geq rpm+1$. By SF(5) this implies that $H^{2i+1}(\mathcal{F}_n, \mathcal{F}_{n-1}) = 0$ for $0 \leq n \leq pm$ such that $H^{2i+1}(\mathcal{F}_{pm}) = 0$. Thus $\overline{\mathcal{U}}^{2i+1}$ is contained in the kernel of the right vertical map.

The rest is a diagram chase. By the surjectivity of the upper right map $\overline{\mathcal{U}}^*$ is the isomorphic image of a subgroup of $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. The degree $2i+1$ part of this subgroup might not itself be in the kernel of the middle vertical map, but using the surjectivity of the left vertical map, we can replace it with a subgroup $\mathcal{U}^{2i+1} \subseteq H^{2i+1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ which also maps isomorphically to $\overline{\mathcal{U}}^*$, and such that $\mathcal{U}^{2i+1}$ is in the kernel of the middle vertical map. □

Remark 12.5 It follows by (5) and (6) of Lemma 10.6, that if $p \nmid (r+1)$, the dimension of the group $\bigoplus_{1 \leq 2k-1 \leq 2rpm-1} \mathcal{U}^{2k-1}$ is rm. If $p \mid (r+1)$ the dimension of the group $\bigoplus_{1 \leq 2k-1 \leq 2rpm+1} \mathcal{U}^{2k-1}$ is $(r+1)m$.

Theorem 12.6 *There is a map of $\mathbb{F}_p[u]$–modules*

$$h_1 \oplus h_2 \colon (\mathbb{F}_p[u] \otimes \mathcal{U}^*) \oplus \mathcal{T}^* \to H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$$

which is an isomorphism of $\mathbb{F}_p[u]$–modules.

Proof We can extend the inclusion of $\mathcal{U}^*$ in a unique way to an $\mathbb{F}_p[u]$–linear map $h_1 \colon \mathbb{F}_p[u] \otimes \mathcal{U}^* \to H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. Because of Lemma 12.3 the inclusion of $\mathcal{T}^*$ is already an $\mathbb{F}_p[u]$–linear map $h_2 \colon \mathcal{T}^* \hookrightarrow H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$.

Lemma 12.4 and the exact sequence

$$H^{2i-1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{u} H^{2i+1}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \xrightarrow{i^*} H^{2i+1}(L\mathbb{C}\mathrm{P}^r)$$

shows that $h_1 \oplus h_2$ is surjective on indecomposables. But then it is a surjective $\mathbb{F}_p[u]$–linear map, and it suffices to show that it is also injective.

The main step is to prove injectivity of the localized map $(h_1 \oplus h_2)[1/u]$. Because of the vanishing statement of Lemma 12.4 we have a commutative diagram as follows, where $\chi_p(r+1)$ is the number defined in Definition 10.3:

$$\begin{tikzcd} \mathbb{F}_p[u] \otimes \bigoplus_{0 \le i} \mathcal{U}^{2i+1} \arrow[r, "h_1"] \arrow[d, "\mathrm{id} \otimes pr"] & H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}}) \arrow[d, two heads, "j^*"] \\ \mathbb{F}_p[u] \otimes \bigoplus_{0 \le i \le rpm - \chi_p(r+1)} \mathcal{U}^{2i+1} \arrow[r, "\overline{h}_1"] & H^{\mathrm{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}}). \end{tikzcd}$$

The map j^* is surjective by Remark 12.1.

We localize by inverting u. Because all modules are of finite type, this localization agrees with tensoring with $\mathbb{F}_p[u, u^{-1}]$ over $\mathbb{F}_p[u]$. Since $h_1 \oplus h_2$ is surjective, and the localization of h_2 vanishes, we know that

$$h_1[1/u]\colon \mathbb{F}_p[u, u^{-1}] \otimes \mathcal{U}^* \longrightarrow H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})[1/u]$$

is surjective. It follows by the diagram that the map

$$\overline{h}_1[1/u]\colon \mathbb{F}_p[u, u^{-1}] \otimes \bigoplus_{0 \le i \le rpm - \chi_p(r+1)} \mathcal{U}^{2\mathrm{i}+1} \longrightarrow H^{\mathrm{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}})[1/u]$$

is also surjective. We know from Remark 12.5 that as abstract module, the domain space is given by

$$\mathbb{F}_p[u, u^{-1}] \otimes \bigoplus_{0 \le i \le rpm - \chi_p(r+1)} \mathcal{U}^{2\mathrm{i}+1} \cong \begin{cases} \mathbb{F}_p[u, u^{-1}]^{\oplus (r+1)m} & \text{if } p \mid (r+1) \\ \mathbb{F}_p[u, u^{-1}]^{\oplus rm} & \text{if } p \nmid (r+1). \end{cases}$$

The target space is determined by Theorem 8.3 and Corollary 9.5. By the proof of Theorem 8.3 the re-indexing is given by multiplying the filtration degree by p. We find that

$$H^{\mathrm{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}})\,[1/u] \cong \begin{cases} \mathbb{F}_p[u, u^{-1}]^{\oplus (r+1)m} & \text{if } p \mid (r+1) \\ \mathbb{F}_p[u, u^{-1}]^{\oplus rm} & \text{if } p \nmid (r+1). \end{cases}$$

The punch line is that $h_1[1/u]$ is a surjective map between two finitely generated, free $\mathbb{F}_p[u, u^{-1}]$–modules of the same rank. But then the map has to be an isomorphism.

This proves the injectivity of the localization h_1. To get the injectivity of $h_1 \oplus h_2$, we consider an element $(c, t) \in \ker(h_1 \oplus h_2)$. We have to prove that this element is trivial. Since the localization of t vanishes, the localization of c is in the kernel of the localization of h_1. This localized map is injective, so the localization of c vanishes. But the canonical map $\mathbb{F}_p[u]\otimes\mathcal{U}^*\to\mathbb{F}_p[u,u^{-1}]\otimes\mathcal{U}^*$ is injective, so c itself vanishes, and t is in the kernel of h_2. But h_2 is injective on $\mathcal{T}$, which proves that $t=0$. □

We can finally complete the main calculation of the paper. Recall the index sets $\mathcal{I}T(r,p,2)$ and $\mathcal{I}F(r,p,2)$ from Definition 10.3. We need a small perturbation as follows:

$$\mathcal{I}F'(r,p,2)=\begin{cases}(\mathcal{I}F(r,p,2)\setminus 2r\mathbb{N})\cup(2+2r\mathbb{N}) & \text{if } p\mid(r+1),\\ \mathcal{I}F(r,p,2) & \text{if } p\nmid(r+1).\end{cases}$$

This makes sense, since $2r\mathbb{N}\subseteq\mathcal{I}F(r,p,2)$ by Lemma 10.4 when $p\mid(r+1)$.

Theorem 12.7 *As a graded $\mathbb{F}_p[u]$–module, $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}};\mathbb{F}_p)$ is isomorphic to the direct sum*

$$\mathbb{F}_p[u]\oplus\bigoplus_{2k\in\mathcal{I}F(r,p,2)}\mathbb{F}_p[u]f_{2k}\oplus\bigoplus_{2k\in\mathcal{I}F'(r,p,2)}\mathbb{F}_p[u]f_{2k-1}\oplus\bigoplus_{2k\in\mathcal{I}T(r,p,2)}(\mathbb{F}_p[u]/(u))t_{2k-1}.$$

Here the lower index denotes the degree of the generator.

Proof In Theorem 12.6 we proved the formula for the group of odd degree elements. We have to show that $H^{\mathrm{even}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a free $\mathbb{F}_p[u]$–module, with generators in the stated degrees. Put $E_r^{**}=E_r^{**}(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. From SF(1) and SF(2) we see that E_1^{even} is a free $\mathbb{F}_p[u]$–module. The degrees of the generators can be read off Theorem 9.1. We see that

$$E_1^{(0,*)(\mathrm{even})}\cong\oplus_{0\le 2i\le 2r}\mathbb{F}_p[u]x_{2i},$$
$$E_1^{(pm,*)(\mathrm{even})}\cong\begin{cases}\oplus_{2rpm\le 2i\le 2r(pm+1)}\mathbb{F}_p[u]x_{2i} & \text{if } p\mid(r+1),\\ \oplus_{2rpm+2\le 2i\le 2r(pm+1)}\mathbb{F}_p[u]x_{2i} & \text{if } p\nmid(r+1).\end{cases}$$

Because of SF(4) together with SF(2) we see that $E_\infty^{(pm,*)(\mathrm{even})}$ is a submodule of finite index in $E_1^{(pm,*)(\mathrm{even})}$. By abstract structure theory of $\mathbb{F}_p[u]$–modules, it follows that $E_\infty^{(pm,*)(\mathrm{even})}$ is a free module on certain generators. If we can filter a graded module with quotient which are free modules, the original module was also a free module. The generators are in the same degrees as the generators for the direct sum of the quotients.

What is left is to figure out the degrees of the generators of the free module $E_\infty^{(pm,*)(\mathrm{even})}$. It would be best if we could compute the differentials. Unfortunately, we cannot do this. What we can do, is to compute the dimension of E_∞^{even} in each degree, and recover the degrees of the generators from this.

To do this, we consider Poincaré series. Let $P(t)$, $P^{\mathrm{even}}(t)$ and $P^{\mathrm{odd}}(t)$ be the Poincaré series of $H^*(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$, $H^{\mathrm{even}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ and $H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ respectively. Similarly, let $P_r(t)$, $P_r^{\mathrm{even}}(t)$ and $P_r^{\mathrm{odd}}(t)$ be the Poincaré series of E_r, the even total degree part of E_r and the odd total degree part E_r respectively.

The series $P_1^{\mathrm{even}}(t)$ and $P_1^{\mathrm{odd}}(t)$ can be recovered from $P_1(t)$ as the sum of the even respectively the sum of the odd monomials occurring in it. From the computation in Lemma 9.6, we recall that if $p \mid (r+1)$, then

$$P_1(t) = \frac{1}{1-t} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}$$

If follows that

$$P_1^{\mathrm{even}}(t) = \frac{1}{1-t^2} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}, \quad P_1^{\mathrm{odd}}(t) = \frac{t}{1-t^2} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}.$$

The important thing here is that $P_1^{\mathrm{odd}}(t) = tP_1^{\mathrm{even}}(t)$. This means that the dimensions of the group of classes of total degree $2i$ in E_1 agrees with the dimension of the classes of total degree $2i+1$. Now recall that by SF(3), all differentials in the spectral sequence go from even total degree to odd total degree. It follows that for any r, including $r=\infty$, we have that $P_r^{\mathrm{odd}}(t) = tP_r^{\mathrm{even}}(t)$, and also that $P^{\mathrm{odd}}(t) = tP^{\mathrm{even}}(t)$.

In case $p \nmid (r+1)$ the formula for $P_1(t)$ in Lemma 9.6 is different, but the argument above still applies, so that $P^{\mathrm{odd}}(t) = tP^{\mathrm{even}}(t)$ in both cases.

We can use Theorem 12.6 to give an expression for $P^{\mathrm{odd}}(t) = P_\infty^{\mathrm{odd}}(t)$. Using the notation of Definition 10.3, we see that

$$tP^{\mathrm{odd}}(t) = P_{\mathcal{I}T(r,p,2)}(t) + \frac{1}{1-t^2} P_{\mathcal{I}F'(r,p,2)}(t).$$

By definition of $\mathcal{I}F'(r,p,2)$, we have

$$P_{\mathcal{I}F'(r,p,2)}(t) = P_{\mathcal{I}F(r,p,2)}(t) - (1-\chi_p(r+1))\frac{t^{2r}(1-t^2)}{1-t^{2r}}, \tag{9}$$

which we insert above and find

$$tP^{\mathrm{odd}}(t) = P_{\mathcal{I}T(r,p,2)}(t) + \frac{1}{1-t^2} P_{\mathcal{I}F(r,p,2)}(t) - (1-\chi_p(r+1))\frac{t^{2r}}{1-t^{2r}}.$$

Rewriting the first two terms, and using the result $P^{\mathrm{odd}}(t) = tP^{\mathrm{even}}(t)$, we see that

$$\begin{aligned} t^2 P^{\mathrm{even}}(t) &= tP^{\mathrm{odd}}(t) \\ &= P_{\mathcal{I}T(r,p,2)}(t) + P_{\mathcal{I}F(r,p,2)}(t) + \frac{t^2}{1-t^2} P_{\mathcal{I}F(r,p,2)}(t) - (1-\chi_p(r+1))\frac{t^{2r}}{1-t^{2r}}. \end{aligned}$$

We rewrite the sum of the first two terms using Lemma 10.7 and obtain

$$t^2 P^{\mathrm{even}}(t) = \frac{t^2}{1-t^2} + \frac{t^2}{1-t^2} P_{\mathcal{I}F(r,p,2)}(t),$$

which completes the proof. □

Corollary 12.8 *The numbers κ_i in Theorem 10.1 are always zero.*

Proof In Theorem 10.1 this is proved for $\alpha \geq 4$, and in some cases also when $\alpha = 2$. So assume that $\alpha = 2$. Then an obstruction argument shows that X is homotopy equivalent to $\mathbb{C}\mathrm{P}^r$. So we can without loss of generality assume that $X = \mathbb{C}\mathrm{P}^r$. Since the action differential factors over the transfer map, it sufficient to show that the transfer map

$$\tau\colon H^{\mathrm{odd}}(L\mathbb{C}\mathrm{P}^r) \to H^{\mathrm{even}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$$

is the zero map. The image of the transfer map is annihilated by multiplication by u (Theorem B.1). According to Theorem 12.7 we also have that $H^{\mathrm{even}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$ is a free module over $\mathbb{F}_p[u]$, so multiplication by u is injective. The result follows. □

We can now give an other version of our main result. Recall that

$$IF(r,p,2) = \{2(ri+j) \mid 0 \leq i,\ \chi_p(r+1) \leq j \leq r,\ p \mid ((r+1)i+j)\} \setminus \{0\},$$

where $\chi_p(s)$ equals 0 when p divides s and 1 when p does not divide s.

Theorem 12.9 *Let $\{E_*\}$ be the mod p Serre spectral sequence for the fibration sequence $L\mathbb{C}\mathrm{P}^r \to (L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}} \to B\mathbb{T}$. That is*

$$E_2^{*,*} = H^*(B\mathbb{T};\mathbb{F}_p) \otimes H^*(L\mathbb{C}\mathrm{P}^r;\mathbb{F}_p) \Rightarrow H^*((L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}};\mathbb{F}_p).$$

For any positive integer r and any prime p one has that $E_3 = E_\infty$. Furthermore, the Poincaré series $P_{r,p}(t)$ for $H^*((L\mathbb{C}\mathrm{P}^r)_{h\mathbb{T}};\mathbb{F}_p)$ is given by

$$P_{r,p}(t) = \frac{1}{1-t}\Big(1 + \sum_{k \in \mathcal{I}F(r,p,2)} t^k\Big).$$

If p divides $r+1$ we can rewrite this as

$$P_{r,p}(t) = \frac{1-t^{2(r+1)}}{(1-t)(1-t^{2r})(1-t^{2p})}.$$

Remark 12.10 We have described the $\mathbb{F}_p$–algebra structure of the E_∞ page in Proposition 11.2 with $\alpha = 2$.

Proof We first use Theorem 12.7 to prove that the Poincaré series is as stated. By this theorem we have that

$$P_{r,p}(t) = \frac{1}{1-t^2}\Big(1 + P_{\mathcal{I}F(r,p,2)}(t) + \frac{1}{t}P_{\mathcal{I}F'(r,p,2)}(t)\Big) + \frac{1}{t}P_{\mathcal{I}T(r,p,2)}(t). \tag{10}$$

By using equation (9) and Lemma 10.7 we can rewrite this as

$$P_{r,p}(t) = \frac{1}{1-t^2}\Big(1 + P_{\mathcal{I}F(r,p,2)}(t) + \frac{1}{t}P_{\mathcal{I}F(r,p,2)}(t)\Big) - \frac{1}{t}P_{\mathcal{I}F(r,p,2)}(t) + \frac{t}{1-t^2}.$$

The desired result follows by a small reduction.

When p divides $r+1$, we can write the index set as

$$\mathcal{I}F(r,p,2) = \{2(ri+pm) \mid 0 \le i,\ 0 \le m \le (r+1)/p - 1\} \setminus \{0\}.$$

The last formula for $P_{r,p}(t)$ follows.

Since Corollary 12.8 holds, we have a formula for the Poincaré series of the E_3 term in Corollary 11.3 with $\alpha = 2$. This series is the same as $P_{r,t}(t)$ when p does not divide $r+1$ by (10). When p divides $r+1$ the two series also agree by our last formula for $P_{r,p}(t)$. Thus, the Serre spectral sequence collapses from the E_3 page. □

The proof of the main theorem involves the existence of non-trivial differentials in the Morse spectral sequence. This fact has a certain geometrical content which we describe below for $r \ge 2$. By similar methods one can get the same result for $r = 1$, but we will not go into this here.

Corollary 12.11 *For any $r \ge 2$ and any $n \ge 2$ there is a trajectory of loops on $\mathbb{C}\mathrm{P}^r$ which converges in positive time towards a geodesic with period n and in negative time towards a non-constant geodesic with period $n+1$.*

Proof Assume that there are no such trajectories. Then the geometric map

$$(\mathcal{F}_{n+1}/\mathcal{F}_n)_{h\mathbb{T}} \to \Sigma(\mathcal{F}_n)_{h\mathbb{T}} \to \Sigma(\mathcal{F}_n/\mathcal{F}_{n-1})_{h\mathbb{T}},$$

which induces the d_1 differential in the Morse spectral sequence, is nullhomotopic. So in the mod p Morse spectral sequence $E_* = E_*(\mathcal{M})(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$, we have that $d_1 = 0$: $E_1^{n,*} \to E_1^{n+1,*}$ for any prime p. We intend to show that you can always chose a prime p, such that this d_1 must be non trivial, leading us to the desired contradiction.

Since $n \ge 2$, there is a prime p that divides n, say $n = pm$. We actually claim that any such p will do. Assume to the contrary, that d^1 is trivial on $E_1^{pm,*}$. The generators of the $\mathbb{F}_p[u]$–module $E_1^{pm,*}$ have degrees less or equal to $2r(pm+1)$.

Since the lowest class in $E_1^{pm+s,*}$ is in degree $(2r)(pm+s-1)+1$, the only possible non-trivial differential originating in $E_*^{pm,*}$ would be a d_2 hitting the non-trivial class in $E_2^{pm+2,2r(pm+1)-pm-1}$. But comparison to the non-equivariant spectral sequence shows that this class is a permanent cycle, so every class in $E_1^{pm,*}$ is a permanent cycle. (cf. the proof of Theorem 9.11).

But this means that every generator of the free $\mathbb{F}_p[u]$–module $E_1^{(pm,*)\mathrm{even}}$ corresponds to a generator in the free $\mathbb{F}_p[u]$–module $H^{\mathrm{even}}(L\mathbb{C}\mathrm{P}^r_{h\mathbb{T}})$. So this modules has a generator in every degree $2k$ where $2rpm+2 \le 2k \le 2r(pm+1)$. In case $p \mid (r+1)$, in addition to this, it has a generator in degree $2k = 2rpm$. We have listed the generators of this module in Theorem 12.7, so this says that all these numbers $2k$ are contained in $\mathcal{I}F'(r,p,2)$.

The rest of the proof is just very elementary number theory. As usual, there are two possible cases. We first deal with the case $p \mid (r+1)$. According to the definition of $\mathcal{I}F'(r,p,2)$ just prior to Theorem 12.7, this set does not contain any numbers divisible by $2r$. So we cannot have that $2rpm \in \mathcal{I}F'(r,p,2)$, contradicting the assumption.

So now assume that $p \nmid (r+1)$. We first show that there are not many pairs of consecutive even numbers in $\mathcal{I}F(r,p,2) = \mathcal{I}F'(r,p,2)$. Assume that both $2k$ and $2k+2$ are contained in $\mathcal{I}F(r,p,2)$. As in Definition 10.3 we find four numbers i_1, i_2, j_1, j_2 pairwise satisfying the conditions mentioned in that definition, and such that $2k = 2ri_1 + 2j_1$ and $2k+2 = 2ri_2 + 2j_2$. Then $2 = 2r(i_2 - i_1) + 2(j_2 - j_1)$. Since $| j_1 - j_2 | \le 2(r-1)$ it follows that either $i_1 = i_2$ and $j_2 = j_1 + 1$, or $i_2 = i_1 + 1$, $j_2 = 0$ and $j_1 = r-1$. The first possibility together with the congruence condition leads to $p \mid ((r+1)i_1 + j_1)$ and $p \mid ((r+1)i_1 + j_1 + 1)$, which is a contradiction.

The second possibility leads to $p \mid ((r+1)i_1 + r - 1)$ and $p \mid ((r+1)(i_1+1))$. Subtracting, we get that $p \mid 2$, that is $p = 2$. So the only possibility for $\{2k, 2k+2\} \in \mathcal{I}F(r,p,2)$ is that $p = 2$, r is even, $j_1 = r-1$ is odd and $k = ri_1 + j_1$ is odd. In particular, we can never have that $\{2k, 2k+2, 2k+4\} \in \mathcal{I}F(r,p,2)$. Since the set of k such that $2rpm + 2 \le 2k \le 2rp(m+1)$ is a set of r consecutive numbers, we have that $r = 2$.

We are now reduced to showing that $\mathcal{I}F(2,2,2)$ does not contain two numbers of the form $8m+2, 8m+4$. We already noted that if $2k, 2k+2 \in \mathcal{I}F(r,p,2)$, then k is odd. So if $2k = 8m+4$, this particular k does not qualify. The proof is complete. □

Appendix A Derived functors at odd primes

In this appendix we prove Theorem 10.1 by homotopy theoretical methods. The main result we are aiming for is a computation of the action of the circle on the cohomology.

It seems hard to obtain this using Morse theory methods, since the action will cause non-trivial interaction between the layers in the Morse filtration. In Section 12 we show that this computation has consequences for the Morse spectral sequence $E_*(\mathcal{M})(L\mathbb{C}P^r_{h\mathbb{T}})$, and essentially solves its differentials.

First we make a preliminary algebraic computation, which we will need for the homotopy theory calculation. Let as before p be a prime number. Write $T_r(x)$ for the truncated polynomial algebra $\mathbb{F}_p[x]/(x^{r+1})$, where $r \geq 1$ and the degree of the generator $|x|$ is a positive even number. In [6] we defined the derived functors $H_*(T_r(x); \overline{\Omega})$, and in the case $p = 2$ we computed them. In this section we extend this calculation to compute these derived functors for odd p. We will use definitions and notation from [6].

Assume that p is an odd prime. Recall the functor $\overline{\Omega}\colon \mathcal{F} \to \mathcal{A}lg$ from [5]. The cohomology with $\mathbb{F}_p$ coefficients of a space defines an object in $\mathcal{F}$, and $\mathcal{A}lg$ is the category of non-negatively graded algebras A over $\mathbb{F}_p$ such that $a^p = a$ for $a \in A^0$. We view $T_n(x)$ as an object in $\mathcal{F}$ with $\lambda = 0$ and $\beta = 0$. Note that by definition of $\mathcal{F}$, any polynomial algebra $\mathbb{F}_p[z_i | i \in I]$ on even dimensional generators is a free object in $\mathcal{F}$. This special type of free object has trivial λ and β. By a similar argument as in [6, Theorem 2.1] we find the following result:

Theorem A.1 *For odd primes p, there is an almost free simplicial resolution $R_\bullet \in s\mathcal{F}$ of the object $T_r(x) \in \mathcal{F}$ as follows: $R_q = \mathbb{F}_p[x, y_1, \dots, y_q]$ for $q \geq 0$ where $|y_i| = (r+1)|x|$. The face and degeneracy maps are given by $s_i(x) = x$, $d_i(x) = x$,*

$$s_i(y_j) = \begin{cases} y_j & , i \geq j, \\ y_{j+1} & , i < j, \end{cases}$$

$$d_i(y_j) = \begin{cases} x^{r+1} & , i = 0, j = 1, \\ y_{j-1} & , i < j, j > 1, \\ y_j & , i \geq j, j < q, \\ 0 & , i = q, j = q. \end{cases}$$

Using this resolution, the derived functors can be computed as the homotopy groups $H_i(T_r(x); \overline{\Omega}) = \pi_i \overline{\Omega}(R_\bullet)$. It is convenient to use the normalized chain complex $N_*(\overline{\Omega} R_\bullet)$ with $N_i(\overline{\Omega} R_\bullet) = \cap_{j=1}^i \ker(d_j)$ and differential d_0 for this purpose.

The de Rham differential on $\overline{\Omega} R_\bullet$ is not the only simplicial derivation. There is is another one which turns out to be useful.

Lemma A.2 *There is a well defined derivation $\theta\colon \overline{\Omega} R_q \to \overline{\Omega} R_q$ of degree 1 for each $q \geq 0$ which satisfies $\theta(ab) = \theta(a)b + (-1)^{|a|} a\theta(b)$ for all $a, b \in \overline{\Omega} R_q$ and is defined by the following for $1 \leq j \leq q$:*

$$\theta(x) = 0, \quad \theta(y_j) = 0, \quad \theta(\mathbf{d}x) = x, \quad \theta(\mathbf{d}y_j) = (r+1)y_j.$$

One has that $\theta \circ \theta = 0$. Furthermore, θ commutes with the simplicial face and degeneracy maps and hence defines a simplicial derivation $\theta\colon \overline{\Omega} R_\bullet \to \overline{\Omega} R_\bullet$.

Proof We have that $\overline{\Omega} R_q = \mathbb{F}_p[x, y_1, \dots, y_q] \otimes \Lambda(\mathbf{d}x, \mathbf{d}y_1, \dots, \mathbf{d}y_q)$. By the derivation property we see that $\theta((\mathbf{d}x)^2) = 0$ and $\theta((\mathbf{d}y_j)^2) = 0$ so θ is well defined. The derivation property for θ also implies that $\theta \circ \theta$ is a derivation of degree two. So $\theta \circ \theta$ is zero since it maps the algebra generators to zero.

One checks that $\theta(s_i z) = s_i \theta(z)$ and $\theta(d_i z) = d_i \theta(z)$ for each algebra generator z by direct computations. The most interesting case goes as follows:

$$\begin{aligned}\theta(d_0(\mathbf{d}y_1)) &= \theta(\mathbf{d}(x^{r+1})) = \theta((r+1)x^r \mathbf{d}x) \\ &= (r+1)x^{r+1} = d_0((r+1)y_1) = d_0\theta(\mathbf{d}y_1).\end{aligned}$$ □

Here is a complete computation of the derived functors.

Theorem A.3 *If p is a prime such that $p \mid (r+1)$, Then there is an isomorphism of bigraded $\mathbb{F}_p$–algebras*

$$H_*(T_r(x); \overline{\Omega}) \cong T_r(x) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega],$$

where $\|x\| = (0, |x|)$, $\|\mathbf{d}x\| = (0, |x|-1)$, $\|\gamma_i(\omega)\| = (i, i((r+1)|x|-1))$. The algebra generators are represented by cycles in the normalized chain complex $N_*(\overline{\Omega} R_\bullet)$ as follows: $x = [x]$, $\mathbf{d}x = [\mathbf{d}x]$, $\gamma_i(\omega) = [\mathbf{d}y_1 \dots \mathbf{d}y_i]$. The de Rham differential induces the map

$$\mathbf{d}_*\colon H_*(T_r(x); \overline{\Omega}) \to H_*(T_r(x); \overline{\Omega}); \quad x \mapsto \mathbf{d}x, \quad \mathbf{d}x \mapsto 0, \quad \gamma_i(\omega) \mapsto 0.$$

If p is a prime such that $p \nmid (r+1)$, then there is an isomorphism of bigraded $\mathbb{F}_p$–algebras

$$H_*(T_r(x); \overline{\Omega}) \cong \mathbb{F}_p[a_i, b_i | i \geq 0]/I_r,$$

where I_r is the ideal generated by the following elements for $i, j \geq 0$:

$$a_i a_j, \quad b_i b_j - \tbinom{i+j}{i} b_0 b_{i+j}, \quad a_i b_j - \tbinom{i+j}{i} b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

Here $\|a_i\| = (i, i((r+1)|x|-1) + |x| - 1)$ and $\|b_i\| = \|a_i\| + (0,1)$. The algebra generators are represented by cycles in the normalized chain complex $N_*(\overline{\Omega} R_\bullet)$ as *follows:* $a_i = [\mathbf{d}x\mathbf{d}y_1 \dots \mathbf{d}y_i]$, $b_i = [\theta(\mathbf{d}x\mathbf{d}y_1 \dots \mathbf{d}y_i)]$. *The de Rham differential induces the map*

$$\mathbf{d}_*\colon H_*(T_r(x);\overline{\Omega}) \to H_*(T_r(x);\overline{\Omega});\quad a_i \mapsto 0,\quad b_i \mapsto (1+(r+1)i)a_i.$$

Remark A.4 Explicitly, the cycle that represents b_i is

$$\theta(\mathbf{d}x\mathbf{d}y_1 \dots \mathbf{d}y_i) = x\mathbf{d}y_1 \dots \mathbf{d}y_i + (r+1)\mathbf{d}x \sum_{k=1}^{i} (-1)^k y_k \mathbf{d}y_1 \dots \widehat{\mathbf{d}y_k} \dots \mathbf{d}y_i.$$

Proof For $p=2$ these results were proved in [6] so assume that p is an odd prime. We first compute the derived functors as $\mathbb{F}_p$–vector spaces.

We have that $T_r(x)$ is the pushout of the diagram $\mathbb{F}_p \leftarrow \mathbb{F}_p[y] \to \mathbb{F}_p[x]$ in $\mathcal{F}$ where $y \mapsto x^{r+1}$. Since $\overline{\Omega}$ commutes with colimits (see Ottosen [28, Appendix]) and $\mathbb{F}_p[x]$ is a free module over $\mathbb{F}_p[y]$, [5, Proposition 6.3] gives us a Quillen spectral sequence

$$E^2_{i,j} = \operatorname{Tor}_i^{H_*(\mathbb{F}_p[y];\overline{\Omega})}(\mathbb{F}_p, H_*(\mathbb{F}_p[x];\overline{\Omega}))_j \Rightarrow H_{i+j}(T_r(x);\overline{\Omega}).$$

Since polynomial algebras on even dimensional generators are free objects in $\mathcal{F}$, we see that $E^2_{i,j}=0$ for $j>0$ and that

$$H_i(T_r(x);\overline{\Omega}) \cong E^2_{i,0} \cong \operatorname{Tor}_i^{\overline{\Omega}(\mathbb{F}_p[y])}(\mathbb{F}_p, \overline{\Omega}(\mathbb{F}_p[x])).$$

Note that $\overline{\Omega}(\mathbb{F}_p[y]) = \mathbb{F}_p[y]\otimes\Lambda(\mathbf{d}y)$ is the free graded commutative algebra on $\{y, \mathbf{d}y\}$ so we have the Koszul resolution (K_*, ∂) of $\mathbb{F}_p$ by free $\overline{\Omega}(\mathbb{F}_p[y])$–modules:

$$K_* = \Lambda(v)\otimes\Gamma[w]\otimes\overline{\Omega}(\mathbb{F}_p[y]);\quad \partial v = y,\quad \partial\gamma_i(w) = \gamma_{i-1}(w)\mathbf{d}y,$$

where $v \in K_1$ and $\gamma_i(w) \in K_i$. In order to compute the group $\operatorname{Tor}_i^{\overline{\Omega}(\mathbb{F}_p[y])}(\mathbb{F}_p, \overline{\Omega}(\mathbb{F}_p[x]))$ we tensor this Koszul resolution with $\overline{\Omega}(\mathbb{F}_p[x])$ over $\overline{\Omega}(\mathbb{F}_p[y])$ and get a chain complex (C_*, ∂) with

$$C_* = \Lambda(v)\otimes\Gamma[w]\otimes\overline{\Omega}(\mathbb{F}_p[x]),\quad \partial v = x^{r+1},\quad \partial\gamma_i(w) = (r+1)\gamma_{i-1}(w)x^r\mathbf{d}x.$$

Computing the homology of this chain complex, we find that if $p \mid (r+1)$ then

$$H_*(T_r(x);\overline{\Omega}) \cong T_r(x)\otimes\Lambda(\mathbf{d}x)\otimes\Gamma[w].$$

If $p \nmid (r+1)$ we compute that $H_0(T_r(x);\overline{\Omega}) \cong T_r(x)\otimes\Lambda(\mathbf{d}x)/(x^r\mathbf{d}x)$. For $i>0$, $H_i(T_r(x);\overline{\Omega})$ is a sum of two copies of $T_r(x)/(x^r)$, the generators being $\mathbf{d}x\gamma_i(w)$ and $x\gamma_i(w) - (r+1)\mathbf{d}xv\gamma_{i-1}(w)$. This completes the computation of the derived functors as $\mathbb{F}_p$–vector spaces.

We now check that the listed representatives are indeed cycles in the chain complex $N_*(\overline{\Omega} R_\bullet)$. Define elements in $\overline{\Omega} R_i$ using the derivation θ as follows:

$$\omega_i = \mathbf{d}y_1 \dots \mathbf{d}y_i, \quad \alpha_i = \mathbf{d}x\omega_i, \quad \beta_i = \theta(\alpha_i).$$

Here by convention $\omega_0 = 1$ such that $\alpha_0 = \mathbf{d}x$ and $\beta_0 = x$. We have that $d_j\omega_i = 0$ for $0 < j \le i$ since $(\mathbf{d}y_j)^2 = 0$ and $d_i y_i = 0$. Furthermore, $d_0\omega_i = (r+1)x^r\mathbf{d}x\omega_{i-1}$. Thus ω_i is a cycle when $p \mid (r+1)$ and α_i, β_i are cycles when $p \nmid (r+1)$ as stated.

By a similar argument as the one presented in [6, Theorem 2.5] one checks that

$$[x^j(\mathbf{d}x)^\epsilon\omega_i], \quad 0 \le j \le n, \quad \epsilon \in \{0,1\}$$

are linearly independent in $H_i(N_*\overline{\Omega} R_\bullet)$ when $p \mid (r+1)$ and that

$$[\beta_0^j\alpha_i], [\beta_0^j\beta_i], \quad 0 \le j \le r-1$$

are linearly independent in $H_i(N_*\overline{\Omega} R_\bullet)$ when $p \nmid (r+1)$. By a dimension count, these two sets are then vector spaces bases.

We now prove that the algebra structure of $H_*(T_r(x); \overline{\Omega})$ is as stated. The algebra structure comes from the chain map

$$\rho\colon C_*(\overline{\Omega} R) \otimes C_*(\overline{\Omega} R) \xrightarrow{g} C_*(\overline{\Omega} R \otimes \overline{\Omega} R) \xrightarrow{C_*(m)} C_*(\overline{\Omega} R)$$

where g is the Eilenberg–Mac Lane shuffle map (see Mac Lane [20]) and m the multiplication map on $\overline{\Omega} R_\bullet$. By $C_*(V)$ for a simplicial $\mathbb{F}_p$–vector space V, we mean the chain complex with $C_i(V) = V_i$ and differential $\sum_{j=0}^i (-1)^j d_j$. So we have the formula

$$\rho(v_i \otimes w_j) = \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} s_\nu(v_i) s_\mu(w_j)$$

where the sum is over all (i,j) shuffles (μ,ν) of the set $\{0,1,\dots,i+j-1\}$ and where $\epsilon(\mu) = \sum_{k=1}^i (\mu_k - (k-1))$.

[6, Lemma 2.4] still holds for $\mathbb{F}_p$–coefficients. (There is a small misprint in the lemma: There should be a hat over ν_j in the index set of the last formula.) So we find that

$$\rho(\omega_i \otimes \omega_j) = \tbinom{i+j}{i}\omega_{i+j}.$$

By this formula and Remark A.4 it follows directly that

$$\rho(\alpha_i \otimes \alpha_j) = 0 \quad , \quad \rho(\alpha_i \otimes \beta_j) = \tbinom{i+j}{i}\beta_0\alpha_{i+j}.$$

Since the degeneracy maps commute with θ, there is a commutative diagram as follows where $A_\bullet = \overline{\Omega} R_\bullet$:

$$\begin{tikzcd}
A_i \otimes A_j \arrow[r, "g"] \arrow[d, "\theta\otimes 1+(-1)^{i+j} 1\otimes\theta"] & A_{i+j} \otimes A_{i+j} \arrow[r, "m"] \arrow[d, "\theta\otimes 1+(-1)^{i+j} 1\otimes\theta"] & A_{i+j} \arrow[d, "\theta"] \\
A_i \otimes A_j \arrow[r, "g"] & A_{i+j} \otimes A_{i+j} \arrow[r, "m"] & A_{i+j}
\end{tikzcd}$$

By mapping $\alpha_i \otimes \theta(\alpha_j)$ both ways around one finds that

$$\rho(\beta_i \otimes \beta_j) = \tbinom{i+j}{i}\beta_0\beta_{i+j}.$$

Thus the algebra structure is as stated. It follows directly form the formulas for the representing cycles, that the de Rham map is as stated. □

Proof of Theorem 10.1 According to [5] there is a strongly convergent second quadrant spectral sequence of cohomology type

$$E_2^{-i,j} = H_i(H^*(X);\overline{\Omega})^j \Rightarrow H^*(LX),$$

where the E_2 page is the derived functors which we computed in Theorem A.3. The spectral sequence is a spectral sequence of algebras, and the induced of the de Rham differential $\mathbf{d}_*$ on E_2 corresponds to the action differential d on $H^*(LX)$.

There are two distinct cases: $p \mid (r+1)$ and $p \nmid (r+1)$. In both cases, the E^2 page does not admit any differentials because of its distribution of zeros, so $E_\infty \cong E_2$ as a vector space over $\mathbb{F}_p$. The theorem can be paraphrased as that the E_2 page is isomorphic to $H^*(LX)$ as an algebra, and that also the action differential agrees. We have to look close to exclude the possibility of multiplicative extensions as well as extension problems concerning the action differential. Write $|\cdot|$ for the total degree in the spectral sequence.

Case 1 Assume that $p \mid (r+1)$. Since x and $\mathbf{d}x$ lie in $H_0(T_r(x);\overline{\Omega})$ they have unique representatives in $H^*(LX)$ which satisfy $x^{r+1}=0$, $(\mathbf{d}x)^2=0$ and $d(x)=\mathbf{d}x$. So the possible extension questions are if the relations $\gamma_i^p = 0$ and $d\gamma_i = 0$ are valid. Let us denote a class in $H^*(LX)$ representing $\gamma_i \in E_\infty$ by the symbol $\overline{\gamma}_i$.

We look at $\overline{\gamma}_i^p$. We know that $\overline{\gamma}_i^p = 0$ up to classes of the same total degree and strictly higher filtration. The class γ_{ip} has the same total degree, but also the same filtration degree as γ_i^p. Using that $|\gamma_{j+1}| = |x^r\gamma_j| + \alpha - 2$, we see that if $\alpha \geq 4$ this is the only class of the same total degree as γ_i^p, so that independently of the choice of $\overline{\gamma}_i$, we indeed have that $\overline{\gamma}_i^p = 0$.

Similarly $d\gamma_i = 0$ up to classes of the same total degree and of strictly higher filtration. But if $\alpha \geq 4$, using that $|d\gamma_{j+1}| = |x^{r-1}\mathbf{d}x\gamma_j| + \alpha - 2$ we see that there is no such class. So $d\overline{\gamma}_i = 0$.

Now we consider the slightly more complicated case $\alpha = 2$. In this case the class $x^r\gamma_j$ has the same total degree as γ_{j+1} and strictly higher filtration. It is the only such class. Similarly, $x^{r-1}\mathbf{d}x\gamma_j$ is the only class of same total degree as $d\gamma_{j+1}$ and of strictly higher filtration.

The filtration of $H^* = H^*(LX)$ has the form

$$H^* \supseteq \cdots \supseteq F^{-i}H^* \supseteq F^{-i+1}H^* \supseteq \cdots \supseteq F^0H^* \supseteq F^1H^* = 0.$$

We now show that we can choose $\overline{\gamma}_i \in F^{-i}H^*$ such that $[\overline{\gamma}_i] = \gamma_i \in E_\infty^{-i,*}$ and $d\overline{\gamma}_i = 0$. For $i = 0$ the unit $\overline{\gamma}_0 = 1$ has these properties. Assume that $\overline{\gamma}_j$ has been chosen with these properties for $1 \leq j < i$. Choose $\gamma_i' \in F^{-i}H^*$ such that $[\gamma_i'] = \gamma_i \in E_\infty^{-i,*}$. We have $[d\gamma_i'] = 0$ so $d\gamma_i' = kx^{r-1}dx\overline{\gamma}_{i-1}$ for some $k \in \mathbb{F}_p$. Put $\overline{\gamma}_i = \gamma_i' - \frac{k}{r}x^r\overline{\gamma}_{i-1}$. Then, $[\overline{\gamma}_i] = \gamma_i \in E_\infty^{-i,*}$ and $d\overline{\gamma}_i = d\gamma_i' - kx^{r-1}dx\overline{\gamma}_{i-1} = 0$.

We claim that these representatives satisfy $(\overline{\gamma}_{p^i})^p = 0$ for $i \geq 0$. From the spectral sequence we see that $(\overline{\gamma}_{p^i})^p = cx^r\overline{\gamma}_{p^{i+1}-1}$ for some $c \in \mathbb{F}_p$. We apply the action differential on this equation and find

$$0 = d((\overline{\gamma}_{p^i})^p) = crx^{r-1}dx\overline{\gamma}_{p^{i+1}-1},$$

and since $r \equiv -1 \mod p$ it follows that $c = 0$.

So there is a well defined algebra map

$$\phi\colon \mathbb{F}_p[x]/(x^{r+1}) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega] \to H^*(LX),$$

which maps $\gamma_i(\omega)$ to $\overline{\gamma}_i$. We can filter the domain such that ϕ becomes a map of filtered algebras. Since ϕ is an isomorphism on associated graded objects, it is itself an isomorphism.

Case 2 Assume that $p \nmid (r+1)$. The classes and multiplicative relations in E_2 are as follows:

class or relation	type	total degree/parity	filtration
$b_0^i a_j, \quad 0 \le i \le r-1$	class	$\rho j + \alpha(i+1) - 1$, odd	$-j$
$b_0^i b_j, \quad 0 \le i \le r-1$	class	$\rho j + \alpha(i+1)$, even	$-j$
$b_i a_j - \binom{i+j}{i} b_0 a_{i+j}$	relation	$\rho(i+j) + 2\alpha - 1$, odd	$-i-j$
$b_0^r a_j$	relation	$\rho j + \alpha(r+1) - 1$, odd	$-j$
$a_i a_j$	relation	$\rho(i+j) + 2\alpha - 2$, even	$-i-j$
$b_i b_j - \binom{i+j}{i} b_0 b_{i+j}$	relation	$\rho(i+j) + 2\alpha$, even	$-i-j$
$b_0^r b_j$	relation	$\rho j + \alpha(r+1)$, even	$-j$

We note that if $0 \le i \le r-1$, then

$$0 < \alpha \le \alpha(i+1) \le \alpha r \le \rho.$$

From this and the list of classes above it follows easily that there is at most one class in each total degree. A differential in the spectral sequence raises total degree by one, and it strictly increases the filtration. The unique class of total degree one higher than a_j is b_j, which has the same filtration as a_j, so that a_j is a permanent cycle. If $\alpha \ge 4$, there is no non-trivial class of degree one higher than b_j. If $\alpha = 2$, the unique class of degree one higher than b_j is $b_0 a_j$ which again has the same filtration so b_j is also a permanent cycle.

Each of the relations given above is true for any lifting of the generators in E_∞ to generators of $H^*(LX)$ up to classes of the same total degree and strictly higher filtration. We claim that in each case, there are no non-trivial such classes.

If $\alpha \ge 4$, there are no nontrivial classes of the same dimension as $b_0^r a_j$ or $b_0^r b_j^r$. In case $\alpha = 2$, there are the unique classes a_{j+1} respectively b_{j+1}. But these have lower filtration than the relation, so they cannot contribute to extensions.

If $\alpha \ge 4$ there is no class which has the same total degree as the relation $a_i a_j$. In case $\alpha = 2$, there is the unique possibility b_{i+j}, which does not matter anyway since its filtration is too low.

Also, the relations $b_i a_j - \binom{i+j}{i} b_0 a_{i+j}$ and $b_i b_j - \binom{i+j}{i} b_0 b_{i+j}$ have the same total degree as $b_0 a_{i+j}$ respectively $b_0 b_{i+j}$. By filtration check, there can be no extension problems.

Finally we have to consider the action differential. The filtration argument says that db_i is as stated. We have to argue that $da_i = 0$. If $\alpha \ge 4$, there is no non-trivial possibility for da_i. If $\alpha = 2$, the class da_i has the same total degree as $b_{i-1} b_0^{r-1}$. Because d is a differential, $d(a_i) = 0$ unless

$$0 = db_i = (1 + (r+1)i) a_i,$$

and

$$0 = d(b_{i-1}b_0^{r-1}) = (1+(r+1)(i-1))b_0^{r-1}a_{i-1}+(r-1)b_{i-1}b_0^{r-2}a_0 = ((r+1)i-1)b_0^{r-1}a_{i-1}.$$

Solving this, we get that $2 = 0$ (that is $p = 2$), that r is even and that i is odd. □

Appendix B The circle transfer map

In this appendix we describe an elementary construction of a $\mathbb{T}$ transfer map τ. There are several discussions of transfer maps in the literature, and also of extensive refinements and generalizations. See for example Madsen and Schlichtkrull [21] or Mann, Miller and Miller [23]. For our purposes, we need only a rather coarse version. We give a simple, self contained construction close to the one we gave in [3].

Theorem B.1 *Let X be a based $\mathbb{T}$–CW complex such that the action of $\mathbb{T}$ is free away from the base point. Write $q\colon X \to X/\mathbb{T}$ for the canonical projection. Let R be a principal ideal domain and M an R–module. There is a linear map which is natural in X and M as follows:*

$$\tau\colon \widetilde{H}^n(X;M) \to \widetilde{H}^{n-1}(X/\mathbb{T};M).$$

It is the connecting homomorphism in a long exact Gysin sequence

$$\cdots \longrightarrow \widetilde{H}^{n-2}(X/\mathbb{T};M) \xrightarrow{\gamma} \widetilde{H}^n(X/\mathbb{T};M) \xrightarrow{q^*} \widetilde{H}^n(X;M) \xrightarrow{\tau} \widetilde{H}^{n-1}(X/\mathbb{T};M) \longrightarrow \cdots$$

in particular, $\tau \circ q^* = 0$. Assume that $M = R$. Then Frobenius reciprocity holds

$$\tau(aq^*(b)) = \tau(a)b,$$

and $q^* \circ \tau = d$ where d denotes the action differential.

Remark B.2 We have $H^*(B\mathbb{T};R) = R[u]$ where $\deg u = 2$ which gives us a class $\mathrm{pr}_1^*(u) \in H^2(X_{h\mathbb{T}};R)$. The map γ in the Gysin sequence is given by multiplication by this class:

$$\begin{array}{ccc}
H^{n-2}(X_{h\mathbb{T}}, B\mathbb{T};M) & \xrightarrow{\mathrm{pr}_1^*(u)\cdot} & H^n(X_{h\mathbb{T}}, B\mathbb{T};M) \\
\cong\uparrow & & \cong\uparrow \\
\widetilde{H}^{n-2}(X/\mathbb{T};M) & \xrightarrow{\gamma} & \widetilde{H}^n(X/\mathbb{T};M).
\end{array}$$

Proof The key point is to compare X to $E\mathbb{T}\times X/E\mathbb{T}$. So we start by considering the spherical fibration

$$\mathbb{T} \longrightarrow E\mathbb{T}\times X \xrightarrow{Q} E\mathbb{T}\times_{\mathbb{T}} X,$$

together with the two subspaces $E\mathbb{T}=E\mathbb{T}\times *$ of $E\mathbb{T}\times X$ and $B\mathbb{T}=E\mathbb{T}\times_{\mathbb{T}} *$ of $E\mathbb{T}\times_{\mathbb{T}} X$ with $Q^{-1}(B\mathbb{T})=E\mathbb{T}$.

The fibration Q is a pullback of the fibration $\mathbb{T}\to E\mathbb{T}\to B\mathbb{T}$ along the projection map pr_2. Since $B\mathbb{T}$ is 1–connected it follows that Q is orientable. Thus there is a relative Gysin sequence for Q (see Whitehead [33, VII.5.12]). We let $\widetilde{\tau}$ be the connecting homomorphism in this sequence.

Let X_0 denote X with trivial $\mathbb{T}$ action. Pick a point $e\in E\mathbb{T}$ and define the $\mathbb{T}$ map $\theta\colon \mathbb{T}\times X_0\to E\mathbb{T}\times X$ by $(z,x)\mapsto(ez^{-1},zx)$. We have a commutative diagram

$$\begin{tikzcd}
(\mathbb{T}\times X_0,\varnothing) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\theta"] & (X,\varnothing) \arrow[d] \\
(E\mathbb{T}\times X, E\mathbb{T}) \arrow[r, "Q"] & (E\mathbb{T}\times_{\mathbb{T}} X, B\mathbb{T}).
\end{tikzcd}$$

By naturality of the Gysin sequence we have the following diagram, where the cohomology groups have coefficients in M:

$$\begin{tikzcd}
\arrow[r] & H^{n-2}(X_{h\mathbb{T}},B\mathbb{T}) \arrow[r, "u\cdot"] \arrow[d, "Q^*"] & H^n(X_{h\mathbb{T}},B\mathbb{T}) \arrow[r, "Q^*"] \arrow[d, "Q^*"] & \widetilde{H}^n(X) \arrow[r, "\widetilde{\tau}"] \arrow[d, "\theta^*"] & H^{n-1}(X_{h\mathbb{T}},B\mathbb{T}) \arrow[r, "u\cdot"] \arrow[d, "Q^*"] & {} \\
\arrow[r] & H^{n-2}(X) \arrow[r, "0"] & H^n(X) \arrow[r, "\mathrm{pr}_2^*"] & H^n(\mathbb{T}\times X) \arrow[r, "\overline{\tau}"] & H^{n-1}(X) \arrow[r, "0"] & {}
\end{tikzcd}$$

By the upper sequence, $\widetilde{\tau}\circ Q^*=0$. Write $\eta\colon \mathbb{T}\times X\to X$ for the action map. $\eta*$ and θ^* are the same in positive degrees. Assume that $M=R$ Then $\eta^*(y)=1\otimes y+v\otimes dy$ where v has degree 1. By the diagram $\overline{\tau}(1\otimes a)=0$ and $\overline{\tau}(v\otimes b)=b$ such that $Q^*\circ\widetilde{\tau}=\overline{\tau}\circ\eta^*=d$. Finally, we have Frobenius reciprocity for $\widetilde{\tau}$ by [33, VII.5.16].

The map $\mathrm{pr}_2\colon E\mathbb{T}\times X/E\mathbb{T}\to X$ is a homotopy equivalence if we restrict it to the fixed points for any closed subgroup of $\mathbb{T}$. So it is a $\mathbb{T}$–equivariant homotopy equivalence, see for instance tom Dieck [30, II.2.7]. Thus, we have a natural isomorphism

$$\mathrm{pr}_2^*\colon H^*(X/\mathbb{T},*;M)\to H^*(X_{h\mathbb{T}},B\mathbb{T};M).$$

The result follows. □

Appendix C Rational coefficients

In this appendix we use Theorem 10.1 to give an alternative computation of cohomology and equivariant cohomology of free loop spaces with rational coefficients. The result of the computation is very classical and can easily be obtained by rational homotopy theoretic techniques.

Proposition C.1 *Let X be a 1–connected space with $H_*(X;\mathbb{Z})$ of finite type and assume that*

$$H^*(X;\mathbb{Z}) = \mathbb{Z}[x]/(x^{r+1}),$$

where $\alpha = \deg(x)$ is even and $r \geq 1$. Put $\rho = (r+1)\alpha - 2$. Then,

$$H^k(LX;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & k \in \{0\} \cup \{\rho i + \alpha j, \rho i + \alpha j - 1 | 0 \leq i, 1 \leq j \leq r\}, \\ 0, & \textit{otherwise.} \end{cases}$$

The action differential $d\colon H^k(LX;\mathbb{Q}) \to H^{k-1}(LX;\mathbb{Q})$ is an isomorphism when $k \in \{\rho i + \alpha j | 0 \leq i, 1 \leq j \leq r\}$ and zero otherwise.

Proof By the Serre spectral sequence for the fibration $\Omega X \to PX \to X$ we see that $H_*(\Omega X;\mathbb{Z})$ is of finite type. The Serre spectral sequence for $\Omega X \to LX \to X$ then gives us that $H_*(LX;\mathbb{Z})$ is of finite type.

Consider the universal coefficient sequence where A is an abelian group:

$$0 \longrightarrow \mathrm{Ext}(H_{k-1}(LX;\mathbb{Z}), A) \longrightarrow H^k(LX;A) \longrightarrow \mathrm{Hom}(H_k(LX;\mathbb{Z}), A) \longrightarrow 0.$$

By choosing $A = \mathbb{Z}/p$ for p sufficiently large, we obtain that the Ext group is zero and that we can apply Theorem 10.1 part 2). Thus,

$$H_k(LX;\mathbb{Z})/T_k(LX) \cong \begin{cases} \mathbb{Z}, & k \in \{0\} \cup \{\rho i + \alpha j, \rho i + \alpha j - 1 | 0 \leq i, 1 \leq j \leq r\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $T_k(LX)$ denotes the torsion subgroup of $H_k(LX;\mathbb{Z})$. We then choose $A = \mathbb{Q}$ and obtain the stated result for $H^k(LX;\mathbb{Q})$.

Let $\eta\colon \mathbb{T} \times LX \to LX$ denote the action map. By definition of the action differential d we have that $\eta^*(y) = 1 \otimes y + v \otimes dy$ where $\deg(v) = 1$ for $\mathbb{Z}/p$ or $\mathbb{Q}$ coefficients. There is also the projection $\mathrm{pr}_2\colon \mathbb{T} \times LX \to LX$ with $\mathrm{pr}_2^*(y) = 1 \otimes y$.

By Theorem 10.1 we have $b_0^{j-1} b_i$ in $H^*(LX;\mathbb{Z}/p)$ with $\deg(b_0^{j-1} b_i) = \rho i + \alpha j$ and

$$d(b_0^{j-1} b_i) = (j + (r+1)i) b_0^{j-1} a_i$$

for $0 \le i$, $1 \le j \le r$. Fix such i and j and put $k = \rho i + \alpha j$.

We use the universal coefficients sequence with abelian group A again. By naturality we get commutative diagrams for η and for pr_2. Choose $A = \mathbb{Z}/p$ where p is so large that the Ext groups vanish, $p > r+1$ and $p > j + (r+1)i$. Then,

$$0 \neq \eta_* - (\mathrm{pr}_2)_*\colon H_k(\mathbb{T} \times LX;\mathbb{Z})/T_k(\mathbb{T} \times LX) \to H_k(LX;\mathbb{Z})/T_k(LX).$$

We then take $A = \mathbb{Q}$ and find that $0 \neq \eta^* - \mathrm{pr}_2^*$ in degree k for $\mathbb{Q}$ coefficients. Thus, the action differential $d\colon H^k(LX;\mathbb{Q}) \to H^{k-1}(LX;\mathbb{Q})$ is an isomorphism as stated. Since $d \circ d = 0$ the vanishing statement for d follows. □

The ring structure of $H^*(LX;\mathbb{Q})$ was first computed by Švarc in [32]. Combining his computation with the proposition above we obtain the following result:

Proposition C.2 *Let X be as in Proposition C.1. Then,*

$$H^*(LX;\mathbb{Q}) = \mathbb{Q}[a_i, b_i | i \ge 0]/I,$$

where I is the ideal generated by the following elements for $i, j \ge 0$:

$$a_i a_j, \quad b_i b_j - b_0 b_{i+j}, \quad b_i a_j - b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

The degrees of the generators are $|a_i| = \rho i + \alpha - 1$ and $|b_i| = \rho i + \alpha$. Furthermore,

$$H^*(LX_{h\mathbb{T}};\mathbb{Q}) = \big(\mathbb{Q}[u] \otimes \mathbb{Q}[w_i^{(j)} | 0 \le i, 1 \le j \le r]\big)/J,$$

where J is the ideal generated by all the products $w_i^{(j)} w_k^{(\ell)}$, $u w_i^{(j)}$ for $0 \le i$, $0 \le k$, $1 \le j \le r$, $1 \le \ell \le r$. The degrees of the generators are $|w_i^{(j)}| = \rho i + \alpha j - 1$ and $|u| = 2$.

Proof The result of Švarc's computation can be found in Klein [15, Section 6]. Performing the substitutions $a_i = g_{i+1}$ for $i \ge 0$ and $b_0 = x$, $b_i = h_i$ for $i \ge 1$ in [15, Theorems 6.2 and 6.3], we obtain the stated description of the cohomology of LX.

Consider the Serre spectral sequence for the homotopy orbit space. Note that the elements u^k for $k \ge 0$ are not hit by any differentials since we may factor $\mathrm{id}_{B\mathbb{T}}$ as $B\mathbb{T} \to LX_{h\mathbb{T}} \to B\mathbb{T}$, where we use a constant loop to define the first map and the second map is the projection pr_1.

The d_2 differential is given by the action differential d. By Proposition C.1 we see that only the elements of the form $b_0^{j-1} a_i$ in $E_2^{0,*}$ and u^k in $E_2^{*,0}$ survive to the E_3 page and that $E_3 = E_\infty$. The unique class $w_i^{(j)}$ in $H^*(LX_{h\mathbb{T}};\mathbb{Q})$ representing $b_0^{j-1} a_i$ satisfies $(w_i^{(j)})^2 = 0$ since it has odd degree. Thus any product $w_i^{(j)} w_k^{(\ell)}$ has finite multiplicative order and hence cannot equal a nonzero constant times a power of u. Hence the multiplicative structure is as stated. □

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