Volume and homology of one-cusped hyperbolic 3–manifolds

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Let M be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp. If we assume that $\pi_1(M)$ has no subgroup isomorphic to a genus–2 surface group and that either (a) $\dim_{\mathbb{Z}_p} H_1(M;\mathbb{Z}_p) \geq 5$ for some prime p, or (b) $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 4$, and the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$ has dimension at most 1, then $\mathrm{vol} M > 5.06$. If we assume that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 7$ and that the compact core N of M contains a genus–2 closed incompressible surface, then $\mathrm{vol} M > 5.06$. Furthermore, if we assume only that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 7$, then $\mathrm{vol} M > 3.66$.

57M50; 57M27

1 Introduction

In this paper we shall prove:

Theorem 1.1 *Let M be a complete, finite-volume, orientable hyperbolic 3–manifold having exactly one cusp, such that $\pi_1(N)$ has no subgroup isomorphic to a genus–2 surface group. Suppose that either*

(a) $\dim_{\mathbb{Z}_p} H_1(N;\mathbb{Z}_p) \geq 5$ *for some prime p, or*

(b) $\dim_{\mathbb{Z}_2} H_1(N;\mathbb{Z}_2) \geq 4$*, and the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$ has dimension at most 1.*

Then
$$\mathrm{vol}\, M > 5.06.$$

Contrapositively, one can think of Theorem 1.1 as saying that if the one-cusped hyperbolic 3–manifold has volume at most 5.06, then $\dim_{\mathbb{Z}_p} H_1(N;\mathbb{Z}_p) \leq 4$ for every prime p; and that if in addition one assumes that the span of the image of the cup product has dimension at most 1, then $\dim_{\mathbb{Z}_2} H_1(N;\mathbb{Z}_2) \leq 3$.

We do not know whether these bounds on homology are sharp. The Weeks–Hodgson census [22] contains many examples of one-cusped manifolds with volume < 5.06

Published: 12 May 2008 DOI: 10.2140/agt.2008.8.343

for which $\dim_{\mathbb{Z}_2} H_1(N;\mathbb{Z}_2) = 3$. We have not calculated the cup product for these examples.

Theorem 1.1 is an immediate consequence of the following two results, Proposition 1.2 and Theorem 1.3.

Recall that the *rank* of a group Φ is the smallest cardinality of any generating set of Φ. A group Γ is said to be k*–free*, where k is a given integer, if every subgroup of Γ whose rank is at most k is a free group.

Proposition 1.2 *Let N be a compact, orientable 3–manifold whose boundary is a torus, and let $k \geq 2$ be an integer. Suppose that $\pi_1(N)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $k/2 < g < k$. In addition, suppose that either*

(a) $\dim_{\mathbb{Z}_p} H_1(N;\mathbb{Z}_p) \geq k+2$ *for some prime p, or*

(b) $\dim_{\mathbb{Z}_2} H_1(N;\mathbb{Z}_2) \geq k+1$, *and the subspace of $H^2(N;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2)\times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$ has dimension at most $k-2$.*

Then there is an infinite sequence of closed manifolds $(M_j)_{j\in\mathbb{N}}$, obtained by distinct Dehn fillings of N, such that $\pi_1(M_j)$ is k–free for every $j \geq 1$.

Theorem 1.3 *Let M be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp. Suppose that there is an infinite sequence of closed manifolds $(M_j)_{j\in\mathbb{N}}$, obtained by distinct Dehn fillings of the compact core of M, such that $\pi_1(M_j)$ is 3–free for every $j \geq 0$. Then*

$$\operatorname{vol} M > 5.06.$$

The proof of Proposition 1.2 will be given in Section 4. The proof of Theorem 1.3 will be given in Section 9.

In Section 10 we prove Proposition 10.1, which asserts that if a one-cusped (complete, finite-volume, orientable) hyperbolic manifold M satisfies $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 7$, then either $\operatorname{vol} M > 5.06$ or N contains a genus–2 (embedded) incompressible surface. We may think of this as saying that the restriction on surface subgroups in Theorem 1.1 can be relaxed at the expense of strengthening the lower bound on homology. The proof combines Theorem 1.3 with the results of Agol and the authors [2].

In work with DeBlois [10], the following strictly stronger version of Proposition 10.1 will be proved: If M is a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp, such that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 7$, then $\operatorname{vol} M > 5.06$. The proof of the stronger result takes Proposition 10.1 as a starting point, and involves a careful analysis of the case where M contains an incompressible surface of genus 2.

Our proof of Proposition 1.2 depends on the following result, Theorem 1.5, which is of independent interest and is related to a result of Oertel [17, Proposition 1.1].

Definition 1.4 Let N be a compact, orientable 3–manifold whose boundary is a torus, and let $g > 0$ be an integer. A slope (see 2.3) α in ∂N will be called a *genus–g singular boundary slope* if there exist a compact orientable surface F, with genus g and nonempty boundary, and a map $f\colon F \to N$, such that (i) $f_\sharp\colon \pi_1(F) \to \pi_1(N)$ and $f_\sharp\colon \pi_1(F,\partial F) \to \pi_1(N,\partial N)$ are injective, and (ii) f maps each component of ∂F homeomorphically onto a simple closed curve in ∂N representing the slope α.

(We show in Proposition 4.1 that injectivity of $f_\sharp\colon \pi_1(F) \to \pi_1(N)$ implies injectivity of $f_\sharp\colon \pi_1(F\partial F) \to \pi_1(N,\partial N)$.)

Theorem 1.5 *Let N be a compact, orientable 3–manifold whose boundary is a torus, let α be a slope (see 2.3) in ∂N, and let $g > 0$ be an integer. Suppose that the fundamental group of the manifold $N(\alpha)$ obtained from N by the surgery corresponding to α (see 2.3) has a subgroup isomorphic to a genus–g surface group. Then either $\pi_1(N)$ has a subgroup isomorphic to a genus–g surface group, or α is a singular genus–g boundary slope.*

The proof of Theorem 1.5 will also be given in Section 4.

A crucial ingredient in the proof of Theorem 1.3 is a result, Proposition 6.3, which asserts that if c is a suitably short geodesic in a hyperbolic 3–manifold M, and $\pi_1(M)$ is k–free for a given $k \geq 3$, then there exists a point $Q \in M$ which is far from c; the precise lower bound for the distance from Q to (a suitably chosen point of) c is a function of k and of the length l of c, which increases when one decreases l or increases k. For small l, this function is strictly greater than the lower bound for the radius of an embedded tube about c given by [4, Corollary 10.2]. Proposition 6.3 also asserts that Q may be taken to be a $(\log 3)$–thick point, in the sense that it is the center of an embedded hyperbolic ball of radius $(\log 3)/2$.

The proof of Proposition 6.3 is based on a new application of the Generalized $\log(2k-1)$ Theorem (Theorem 4.1 of Anderson, Canary, Culler and Shalen [4], which we combine with the Tameness Theorem of Agol [1] or Calagari and Gabai [9]), and may be regarded as an analogue of [4, Corollary 10.2] for $k > 2$.

We prove Proposition 6.3 in Section 6, after some algebraic preliminaries in Section 5. We deduce Proposition 6.3, via the topology of Margulis tubes, from a result, Proposition 6.1, which applies in a more general setting and which we expect to have other applications.

In Section 7, we combine Proposition 6.3 with geometric convergence techniques based on Thurston's Dehn filling theorem to prove a result, Theorem 7.3, which asserts that if M satisfies the hypotheses of Theorem 1.3, there is a $(\log 3)$–thick point Q which is suitably distant from a maximal standard neighborhood (2.4) of the cusp of M. Much of the general information about Dehn filling and geometric convergence that we needed is contained in our Proposition 7.2.

In Section 9 we deduce Theorem 1.3 from Theorem 7.3. This step requires the use of a stronger form of Böröckzy's results on density of sphere-packing [7] than has previously been used in the study of volumes of hyperbolic 3–manifolds. The needed background is explained in Section 8.

Section 2 is devoted to establishing some conventions that are used in the rest of the paper.

We are grateful to Rosemary Guzman for correcting a number of errors in an earlier version of the paper. This research was partially supported by NSF grants DMS-0204142 and DMS-0504975.

2 General conventions

2.1 A *manifold* may have a boundary. If M is a manifold, we shall denote the boundary of M by ∂M and its interior $M - \partial M$ by $\operatorname{int} M$.

In some of our topological results about manifolds of dimension ≤ 3, specifically those in Section 4, we do not specify a category. These results may be interpreted in the category in which manifolds are topological, PL or smooth, and submanifolds are respectively locally flat, PL or smooth; these three categories are equivalent in these low dimensions as far as classification is concerned. Some of the proofs in Section 3 are done in the PL category, but the applications to hyperbolic manifolds in later sections are carried out in the smooth category.

Definitions 2.2 A 3–manifold M will be termed *irreducible* if every 2–sphere in M bounds a ball in M. We shall say that M is *boundary-irreducible* if ∂M is π_1–injective in M, or equivalently if for every properly embedded disk $D \subset M$ the simple closed curve ∂D bounds a disk in ∂M. We shall say that a 3–manifold M is *simple* if (i) M is compact, connected, orientable, irreducible and boundary-irreducible; (ii) every non-cyclic abelian subgroup of $\pi_1(M)$ is carried (up to conjugacy) by a torus component of ∂M; and (iii) M is not a closed manifold with finite fundamental group, and is not homeomorphic to $S^1 \times S^1 \times [0,1]$.

2.3 If X is a 2–torus, we define a *slope* in X to be an unoriented isotopy class of nontrivial simple closed curves in X.

If N is a 3–manifold, X is a torus component of ∂N, and α is a slope in X, we shall denote by $N(\alpha)$ the manifold obtained from the disjoint union of N and $D^2 \times S^1$ by identifying $D^2 \times S^1$ with X via a homeomorphism that maps $D^2 \times \{1\}$ onto a curve with slope α. We shall say that $N(\alpha)$ is obtained from N by the *Dehn filling corresponding to* α.

2.4 If P and Q are points in a complete hyperbolic manifold M, we shall denote by $\mathrm{dist}_M(P,Q)$ the hyperbolic distance between P and Q (which may be thought of as the length of a shortest geodesic from P to Q, if $P \neq Q$).

Let X be a subset of a complete hyperbolic 3–manifold M. For any point $P \in M$ we define $\mathrm{dist}(P,X) = \inf_{Q \in X} \mathrm{dist}(P,Q)$. We have $\mathrm{dist}(P,X) = \mathrm{dist}(P,\bar{X})$, and $\mathrm{dist}(P,X) = 0$ if and only if $P \in \bar{X}$. If r is a real number, we shall denote by $N(r,X)$ the set of all points $P \in M$ such that $\mathrm{dist}(P,X) < r$. Note that $N(r,X) = \varnothing$ if $r \leq 0$. If Q is a point of M and r is a real number, we shall write $N(r,Q) = N(r,\{Q\})$.

By a *cylinder* in $\mathbb{H}^n$ we mean a set of the form $N(r,A)$, where r is a (strictly) positive real number and A is a line in $\mathbb{H}^n$.

Let M be a complete hyperbolic n–manifold. We shall often identify M with a quotient $\mathbb{H}^n/\Gamma$ where $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^n)$ is discrete and torsion-free. We define a *tube* (or, respectively, a *standard cusp neighborhood*) in M to be the image in M of a cylinder (or, respectively, a horoball) in $\mathbb{H}^n$ which is precisely invariant under Γ. Note that tubes and standard cusp neighborhoods are open subsets of M.

Every simple closed geodesic in M has a neighborhood which is a tube.

An orbit of points in the sphere at infinity which are fixed points of parabolic elements of Γ is said to define a *cusp* $\mathcal{K}$ of M. If z is a point of the given orbit, some horoball based at z is precisely invariant under Γ and thus defines a standard cusp neighborhood, which we may also refer to as a standard neighborhood of $\mathcal{K}$.

The tubes about a given simple geodesic c, or the standard neighborhoods of a given cusp $\mathcal{K}$, are totally ordered by inclusion, and their union is the unique tube about c, or respectively the unique maximal standard neighborhood of $\mathcal{K}$.

2.5 Let $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic n–manifold, and let $q\colon \mathbb{H}^n \to M$ denote the quotient map.

If λ is a positive number, we shall denote by $M_{\mathrm{thin}}(\lambda)$ the set of all points of M at which the injectivity radius is strictly less than $\lambda/2$. We shall denote by $\widetilde{M}_{\mathrm{thin}}(\lambda)$ the set $q^{-1}(M_{\mathrm{thin}}(\lambda))$. Equivalently, $\widetilde{M}_{\mathrm{thin}}(\lambda)$ is the set of all points $z \in \mathbb{H}^3$ such that $\mathrm{dist}(\gamma(z), z) < \lambda$ for some $\gamma \in \Gamma - \{1\}$.

We set $M_{\mathrm{thick}}(\lambda) = M - M_{\mathrm{thin}}(\lambda)$ and $\widetilde{M}_{\mathrm{thick}}(\lambda) = q^{-1}(M_{\mathrm{thick}}(\lambda)) = \mathbb{H}^n - \widetilde{M}_{\mathrm{thin}}(\lambda)$.

Each nontrivial element γ of Γ lies in a unique maximal abelian subgroup, which is the centralizer of γ. In particular, nontrivial elements which lie in distinct maximal abelian subgroups do not commute.

The nontrivial elements of a maximal abelian subgroup C either are all loxodromic isometries with a common axis, which we shall denote by A_C, or are all parabolic isometries with a common fixed point at infinity, which we shall denote by ω_C. If C is loxodromic it must be cyclic. We shall denote by $\mathcal{C}(\Gamma) = \mathcal{C}(M)$ the set of all maximal abelian subgroups of Γ. For any positive real number λ, we denote by $\mathcal{C}_\lambda(\Gamma) = \mathcal{C}_\lambda(M)$ the set of all subgroups $C \in \mathcal{C}(\Gamma)$ which are either parabolic or are generated by loxodromic elements of translation length $< \lambda$.

Let γ be an element of Γ, and let λ be a positive real number. We shall denote by $Z_\lambda(\gamma)$ the set of points $z \in \mathbb{H}^3$ such that $\mathrm{dist}(z, \gamma(z)) < \lambda$. If γ is parabolic then $Z_\lambda(\gamma)$ is a horoball based at the fixed point of γ. If γ is loxodromic and the translation length l of γ is at least λ then $Z_\lambda(\gamma) = \varnothing$. If γ is loxodromic and $l < \lambda$ then $Z_\lambda(\gamma)$ is a cylinder about the axis of γ. Its radius $R_\gamma(\lambda)$ is given explicitly by

$$R_\gamma(\lambda) = \operatorname{arcsinh}\left(\left(\frac{\cosh\lambda - \cosh l}{\cosh l - \cos\theta}\right)^{1/2}\right), \tag{2.5.1}$$

where θ denotes the twist angle of γ. This follows from the formula

$$\cosh \mathrm{dist}(z, \gamma\cdot z) = \cosh l + (\sinh^2 r)(\cosh l - \cos\theta) \tag{2.5.2}$$

which holds for any point $z \in \mathbb{H}^3$ at a distance r from the axis of a loxodromic isometry γ with translation length l and twist angle θ.

It follows from the definitions that

$$\widetilde{M}_{\mathrm{thin}}(\lambda) = \bigcup_{1\neq\gamma\in\Gamma} Z_\lambda(\gamma).$$

It follows from our description of the sets $Z_\lambda(\gamma)$ that the set $Z_\lambda(C) \doteq \bigcup_{1\neq\gamma\in C} Z_\lambda(\gamma)$ is empty for every $C \in \mathcal{C}(\Gamma) - \mathcal{C}_\lambda(\Gamma)$; is a cylinder about A_C for every loxodromic

$C \in \mathcal{C}_\lambda(\Gamma)$; and is a horoball based at ω_C for every parabolic C. Furthermore, we have

$$\widetilde{M}_{\mathrm{thin}}(\lambda) = \bigcup_{C \in \mathcal{C}_\lambda(\Gamma)} Z_\lambda(C). \tag{2.5.3}$$

We define a *Margulis number* for M to be a positive real number λ such that for any two subgroups $C, C' \in \mathcal{C}_\lambda(\Gamma)$ we have $Z_\lambda(C) \cap Z_\lambda(C') = \varnothing$. If λ is a Margulis number for M, the components of $M_{\mathrm{thin}}(\lambda)$ are tubes and standard cusp neighborhoods.

Since for any $\lambda > 0$ the set $\widetilde{M}_{\mathrm{thin}}(\lambda)$ is a union of sets of the form $Z_\lambda(\gamma)$, each of which is precisely invariant under a maximal abelian subgroup, λ is a Margulis number if and only if the fundamental group of each component of $M_{\mathrm{thin}}(\lambda)$ has abelian image in $\pi_1(M)$.

2.6 Note also that if λ is a Margulis number then $\widetilde{M}_{\mathrm{thick}}(\lambda)$ is connected, since the union in (2.5.3) is disjoint. In particular it follows that $M_{\mathrm{thick}}(\lambda)$ is connected.

2.7 It follows from [11, Theorem 10.3], together with the main result of [1] or [9], that if M is a complete, orientable hyperbolic 3–manifold without cusps and $\pi_1(M)$ is 2–free, then $\log 3$ is Margulis number for M.

3 Cup products and homology of covering spaces

The main result of this section is Proposition 3.5, which is a variation on [2, Lemma 6.3].

3.1 As in [2], if X is a topological space, we will set $\mathrm{rk}_2 X = \dim_{\mathbb{Z}_2} H_1(X; \mathbb{Z}_2)$.

3.2 As in [2, 6.1], if N is a normal subgroup of a group G, we shall denote by $G\#N$ the subgroup of G generated by all elements of the form $gag^{-1}a^{-1}b^2$ with $g \in G$ and $a, b \in N$.

Also recall from [2, 6.1] the five term exact sequence due to Stallings [21]. If N is a normal subgroup of a group G and if $Q = G/N$, then there is an exact sequence

$$\begin{aligned} H_2(G;\mathbb{Z}_2) \longrightarrow H_2(Q;\mathbb{Z}_2) \longrightarrow N/(G\#N) \\ \longrightarrow H_1(G;\mathbb{Z}_2) \longrightarrow H_1(Q;\mathbb{Z}_2) \longrightarrow 0, \end{aligned} \tag{3.2.1}$$

in which the maps $H_1(G;\mathbb{Z}_2) \longrightarrow H_1(Q;\mathbb{Z}_2)$ and $H_2(G;\mathbb{Z}_2) \longrightarrow H_2(Q;\mathbb{Z}_2)$ are induced by the natural homomorphism from G to $Q = G/N$.

3.3 As in [2, 6.2], for any group Γ, we define subgroups Γ_d of Γ recursively for $d \geq 0$, by setting $\Gamma_0 = \Gamma$ and $\Gamma_{d+1} = \Gamma \# \Gamma_d$. We regard Γ_d / Γ_{d+1} as a $\mathbb{Z}_2$–vector space.

The following result is a variation on [20, Lemma 1.3].

Lemma 3.4 *Suppose that M is a closed, aspherical 3–manifold. Let t denote the rank of the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$. Set $\Gamma = \pi_1(M)$ and $r = \mathrm{rk}_2 M$. Then the $\mathbb{Z}_2$–vector space Γ_1 / Γ_2 has dimension exactly $(r(r+1)/2) - t$.*

Proof The group $Q = \Gamma / \Gamma_1$ is an elementary 2–group of rank r. If we regard Q as a direct product $C_1 \times \cdots \times C_r$, where each C_i is cyclic of order 2, then the Künneth theorem for cohomology with field coefficients gives an identification

$$H^2(Q;\mathbb{Z}_2) = \bigoplus_{D \in I} V_D,$$

where I is the set of all r–tuples of non-negative integers whose sum is 2, and

$$V_D = H^{d_1}(C_1;\mathbb{Z}_2) \otimes \cdots \otimes H^{d_r}(C_r;\mathbb{Z}_2)$$

for every $D = (d_1, \dots, d_r) \in I$. We have

$$I = \{D_{hk} : 1 \leq h < k \leq r\} \cup \{E_h : 1 \leq h \leq r\},$$

where the r–tuples D_{hk} and E_h are defined by $D_{hk} = (\delta_{ih} + \delta_{ik})_{1 \leq i \leq r}$ and $D_h = (\delta_{ih})_{1 \leq i \leq r}$. In particular we have $\dim H^2(Q;\mathbb{Z}_2) = |I| = r(r+1)/2$.

Note that when $1 \leq h < k \leq r$, the generator of $V_{D_{hk}}$ is the cup product of the generators of $H^1(C_h;\mathbb{Z}_2)$ and $H^1(C_k;\mathbb{Z}_2)$; and that when $1 \leq h \leq r$, the generator of V_{D_h} is the cup product of the generator of $H^1(C_h;\mathbb{Z}_2)$ with itself. (We remark that the second statement would not hold if we were working over $\mathbb{Z}_p$ for an odd prime p, since the cup product of a generator of $H^1(C_h;\mathbb{Z}_p)$ with itself would vanish.) Hence if $c\colon H^1(Q;\mathbb{Z}_2) \otimes H^1(Q;\mathbb{Z}_2) \to H^2(Q;\mathbb{Z}_2)$ is the map defined by cup product, the image of c contains all the $V_{D_{hk}}$ and all the V_{D_h}, so that c is surjective.

There is a natural commutative diagram

$$\begin{array}{ccc}
H^1(Q;\mathbb{Z}_2) \otimes H^1(Q;\mathbb{Z}_2) & \xrightarrow{c} & H^2(Q;\mathbb{Z}_2) \\
\Big\downarrow j & & \Big\downarrow \beta^* \\
H^1(\Gamma;\mathbb{Z}_2) \otimes H^1(\Gamma;\mathbb{Z}_2) & \xrightarrow{\bar{c}} & H^2(\Gamma;\mathbb{Z}_2)
\end{array}$$

in which the map $\overline{c}$ is defined by the cup product in the $\mathbb{Z}_2$–cohomology of Γ, and j is the isomorphism induced by the quotient map $\Gamma \to Q = \Gamma/\Gamma_1$. Since M is aspherical, the $\mathbb{Z}_2$–cohomology ring of Γ is isomorphic to that of M. The definition of t therefore implies that the image of $\overline{c}$ is a $\mathbb{Z}_2$–vector space of dimension t. Since we have shown that c is surjective, and since j is an isomorphism, it follows that the linear map β^*, which is induced by the quotient map $\Gamma \to Q$, has rank t. Since $\mathbb{Z}_2$ is a field, it follows that the homomorphism $\beta\colon H_2(\Gamma;\mathbb{Z}_2) \to H_2(Q;\mathbb{Z}_2)$ induced by the quotient map also has rank t.

Applying (3.2.1) with $G = \Gamma$ and $N = \Gamma_1$, so that $Q = \Gamma/\Gamma_1$ is the rank–r elementary 2–group defined above and $N/(G\#N) = \Gamma_1/\Gamma_2$, we obtain an exact sequence

$$H_2(\Gamma;\mathbb{Z}_2) \xrightarrow{\beta} H_2(Q;\mathbb{Z}_2) \longrightarrow \Gamma_1/\Gamma_2 \longrightarrow H_1(\Gamma;\mathbb{Z}_2) \xrightarrow{\alpha} \Gamma/\Gamma_1 \longrightarrow 0\ .$$

But we have seen that β has rank t, and it is clear that α, which is induced by the quotient map $\Gamma \to Q = \Gamma/\Gamma_1$, is an isomorphism. Hence by exactness we have

$$\begin{aligned}\dim_{\mathbb{Z}_2}(\Gamma_1/\Gamma_2) &= \dim_{\mathbb{Z}_2}(H_2(Q;\mathbb{Z}_2)) - t \\ &= \dim_{\mathbb{Z}_2}(H^2(Q;\mathbb{Z}_2)) - t = r(r+1)/2 - t.\end{aligned}$$ □

Proposition 3.5 *Suppose that M is a closed, aspherical 3–manifold. Let t denote the rank of the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$. Set $r = \mathrm{rk}_2 M$. Then for any integer $m \geq 0$ and any regular covering $\widetilde{M}$ of M with covering group $(\mathbb{Z}_2)^m$, we have*

$$\mathrm{rk}_2 \widetilde{M} \geq (m+1)r - \frac{m(m+1)}{2} - t.$$

Proof We set $\Gamma = \pi_1(M)$. According to Lemma 3.4, the $\mathbb{Z}_2$–vector space Γ_1/Γ_2 has dimension $r(r+1)/2 - t$.

Let N denote the subgroup of Γ corresponding to the regular covering space $\widetilde{M}$. We have $\Gamma/N \cong (\mathbb{Z}_2)^m$. Hence may write $N = E\Gamma_1$ for some $(r-m)$–generator subgroup E of Γ. It now follows from [20, Lemma 1.4] that

$$\begin{aligned}\mathrm{rk}_2 \widetilde{M} &= \dim H_1(E\Gamma_1;\mathbb{Z}_2) \\ &\geq \dim(\Gamma_1/\Gamma_2) - \frac{(r-m)(r-m-1)}{2} \\ &= \frac{r(r+1)}{2} - t - \frac{(r-m)(r-m-1)}{2} \\ &= (m+1)r - \frac{m(m+1)}{2} - t.\end{aligned}$$ □

4 Dehn filling and k-freeness

This section contains the proofs of Theorem 1.5 and Proposition 1.2, which were stated in Section 1, the Introduction.

Proposition 4.1 *Suppose that N is a compact 3–manifold whose boundary is a torus, that F is a compact orientable 2–manifold with $\chi(F) < 0$, and that $f\colon F \to N$ is a map such that $f(\partial F) \subset \partial N$, and such that $f_\sharp\colon \pi_1(F) \to \pi_1(N)$ is injective. Then $f_\sharp\colon \pi_1(F, \partial F) \to \pi_1(N, \partial N)$ is also injective.*

Proof Suppose, to the contrary, that there is a path a in F which is not homotopic rel endpoints into ∂F, but that the path $f(a)$ is homotopic rel boundary into ∂N. The path a determines a map $h\colon G \to F$ where G is a certain connected graph with three edges and two vertices. If a has both endpoints in the same component b of ∂F then G is a "theta" graph; the restriction of h to one edge is the path a and the restriction to the union of the other two edges is a homeomorphism onto b. If a has endpoints contained in two components b_1 and b_2 of F then G is an "eyeglass" graph; the restriction of h to the separating edge is the path a and the restriction to the union of the other two edges is a homeomorphism onto $b_1 \cup b_2$. Since F has genus $g > 0$, the map h induces an injection on π_1. But the composition $f \circ h$ is homotopic into ∂N, which is a torus, and hence the induced map on π_1 is not injective. It follows that $f_\sharp\colon \pi_1(F) \to \pi_1(N)$ is not injective, contradicting the hypothesis. □

Proof of Theorem 1.5 Set $M = N(\alpha)$. It follows from the hypothesis that M has a connected covering space $p\colon \widetilde{M} \to M$ such that $\pi_1(\widetilde{M})$ is isomorphic to the fundamental group of a closed genus–g orientable surface Σ_g. The compact core [19] of $\widetilde{M}$ is a compact, orientable, irreducible 3–manifold K with $\pi_1(K) \cong \pi_1(\Sigma_g)$. Hence K is homeomorphic to $\Sigma_g \times [-1, 1]$ by [14, Theorem 10.6]. In particular, there is a genus–g surface $S \subset \widetilde{M}$ such that

(1) the inclusion induces an isomorphism from $\pi_1(S)$ to $\pi_1(\widetilde{M})$.

We may write $N(\alpha) = N \cup T$, where T is a solid torus with $T \cap M = \partial T = \partial M$, and the meridian disks of T represent the slope α. Set $\widetilde{T} = p^{-1}(T)$ and $\widetilde{N} = p^{-1}(N)$. We may choose S so that

(2) each component of $S \cap \widetilde{T}$ is a meridian disk in $\widetilde{T}$.

We let m denote the number of components of $S \cap \widetilde{T}$, and we suppose S to be chosen, among all genus–g closed surfaces in $\widetilde{M}$ for which (1) and (2) hold, so that m is as small as possible. If $m = 0$ then $h(S) \subset \widetilde{N}$, and so the fundamental group of the connected covering space $\widetilde{N}$ has a subgroup isomorphic to a genus–g surface group. It follows that $\pi_1(N)$ also has such a subgroup in this case.

Now suppose that $m > 0$. Set $F = S \cap \widetilde{N}$, so that F is a compact orientable surface of genus g, properly embedded in $\widetilde{N}$, with $\partial F \neq \varnothing$. If the inclusion homomorphism $\pi_1(F) \to \pi_1(\widetilde{N})$ has a nontrivial kernel, it follows from Dehn's lemma and the loop theorem that there is a disk $D_1 \subset \widetilde{N}$ such that $D_1 \cap F = \partial D_1$, and such that ∂D_1 does not bound a disk in F. However, it follows from (1) that ∂D_1 does bound a disk $D \subset S$, which must contain at least one component of $S \cap \widetilde{T}$. If we set $S_1 = \overline{S - D} \cup D_1$, then (1) and (2) still hold when S is replaced by S_1, but $S_1 \cap \widetilde{T}$ has at most $m-1$ components. This contradiction to the minimality of m proves that $\pi_1(F) \to \pi_1(\widetilde{N})$ is injective.

Hence if we set $f = p|F\colon F \to N$, then $f_\sharp\colon \pi_1(F) \to \pi_1(N)$ is injective. But since $g > 0$ we have $\chi(F) < 0$, and it therefore follows from Proposition 4.1 that $f_\sharp\colon \pi_1(F, \partial F) \to \pi_1(N, \partial N)$ is also injective. Thus condition (i) of Definition 1.4 holds. Since $F = S \cap \widetilde{N}$, condition (ii) of Definition 1.4 follows from (2). Hence α is a genus–g singular boundary slope in this case. $\square$

Remark 4.2 While Theorem 1.5 is similar to [17, Proposition 1.1], neither of the two results contains the other. Oertel's result assumes a weaker hypothesis on the closed surface F, since it only requires that essential *simple* closed curves and arcs on S have homotopically nontrivial images. On the other hand, the conclusion of [17, Proposition 1.1] is also weaker since it does not imply the injectivity of the induced map on relative fundamental groups.

Proposition 4.3 *Let M be a simple 2.2, closed, orientable 3–manifold, and let $k \geq 2$ be an integer. Suppose that for some prime p we have $\dim_{\mathbb{Z}_p} H_1(M;\mathbb{Z}_p) \geq k+2$, and that $\pi_1(M)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $k/2 < g < k$. Then $\pi_1(M)$ is k–free.*

Proof This is the same result as [20, Proposition 1.8, case (b)], except that in the latter result, instead of assuming only that $\pi_1(M)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $k/2 < g < k$, one makes the superficially stronger assumption that $\pi_1(M)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $0 < g < k$. However, if $\pi_1(M)$ does have a subgroup G isomorphic to a genus–g surface group for some integer g with $0 < g < k$, then we must have $g > 1$ since M is simple; and G has a subgroup H of index

$$d = \left[\frac{k-2}{g-1}\right] \geq 1,$$

where $[\cdot]$ is the greatest integer function. It follows that $H \leq \pi_1(M)$ is a genus-g' surface group, where $g' = d(g-1)+1$ satisfies $k/2 < g' < k$. This contradicts the hypothesis of Proposition 4.3. $\square$

Proposition 4.4 *Let M be a simple, closed, orientable 3–manifold, let $k \geq 2$ be an integer, and suppose that $\pi_1(M)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $k/2 < g < k$. Suppose also that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq k+1$, and the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$ has dimension at most $k-1$. Then $\pi_1(M)$ is k–free.*

Proof Set $r = \dim_{\mathbb{Z}_p} H_1(M;\mathbb{Z}_p)$ and suppose that Φ is a subgroup of $\pi_1(M)$ whose rank is at most k. Since $r \geq k+1$, some index–2 subgroup $\widetilde{\Gamma}$ of $\pi_1(M)$ contains Φ. Let $\widetilde{M}$ denote the covering space of M defined by $\widetilde{\Gamma}$. It follows from the case $m=1$ of Proposition 3.5 that

$$\dim_{\mathbb{Z}_2} H_1(\widetilde{M};\mathbb{Z}_2) \geq 2r-1-t,$$

where t denotes the rank of the subspace of $H^2(M;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(M;\mathbb{Z}_2) \times H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$. Since by hypothesis we have $t \leq k-1$, it follows that

$$\dim_{\mathbb{Z}_2} H_1(\widetilde{M};\mathbb{Z}_2) \geq 2r-k \geq k+2.$$

It therefore follows from Proposition 4.3 that $\widetilde{\Gamma} \cong \pi_1(\widetilde{M})$ is k–free. Hence Φ is a free group. $\square$

Proof of Proposition 1.2 If α is any slope on ∂N, let us denote by $N(\alpha)$ the closed manifold obtained from N by the Dehn filling corresponding to α.

Let us fix a prime p such that either

(a) $\dim_{\mathbb{Z}_p} H_1(N;\mathbb{Z}_p) \geq k+2$ for some prime p, or

(b) $p=2$, the $\mathbb{Z}_2$–vector space $H_1(N;\mathbb{Z}_2)$ has dimension at least $k+1$, and the subspace of $H^2(N;\mathbb{Z}_2)$ spanned by the image of the cup product $H^1(N;\mathbb{Z}_2) \times H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z}_2)$ has rank at most $k-2$.

Let us fix a basis $\{\lambda,\mu\}$ of $H_1(\partial N;\mathbb{Z})$ such that λ lies in the kernel of the natural homomorphism $H_1(\partial N;\mathbb{Z}) \to H_1(N;\mathbb{Z}_p)$. For every integer n, let α_n denote the slope defined by the primitive class $\lambda + pn\mu \in H_1(\partial N;\mathbb{Z})$. If $i\colon N \to N(\alpha_n)$ denotes the inclusion homomorphism, then $i_*\colon H_1(N;\mathbb{Z}_p) \to H_1(N(\alpha_n);\mathbb{Z}_p)$ and

$i^*\colon H^1(N(\alpha_n);\mathbb{Z}_p) \to H^1(N;\mathbb{Z}_p)$ are isomorphisms, whereas $i^*\colon H^2(N(\alpha_n);\mathbb{Z}_p) \to H^2(N;\mathbb{Z}_p)$ is surjective and has a 1–dimensional kernel.

It follows from [12, Theorem 1], for any given integer $g > 0$ there are only finitely many genus–g singular boundary slopes in ∂N. Hence there is an integer $n_0 \geq 0$ such that for every n with $|n| \geq n_0$ and every g with $k/2 < g < k$, the slope α_n fails to be a genus–g singular boundary slope. Since by hypothesis $\pi_1(N)$ has no subgroup isomorphic to a genus–g surface group for any integer g with $k/2 < g < k$, it follows from Theorem 1.5 that when $|n| \geq n_0$ and $k/2 < g < k$, the group $\pi(N(\alpha))$ has no subgroup isomorphic to a genus–g surface group.

If (a) holds, then

$$\dim_{\mathbb{Z}_p} H_1(N(\alpha_n);\mathbb{Z}_p) = \dim_{\mathbb{Z}_p} H_1(N;\mathbb{Z}_p) \geq k+2$$

for every n. It therefore follows from Proposition 4.3 (with $M = N(\alpha_n)$) that $\pi_1(N(\alpha_n))$ is k–free whenever $|n| \geq n_0$.

Now suppose that (b) holds. Then

$$\dim_{\mathbb{Z}_2} H_1(N(\alpha_n);\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H_1(N;\mathbb{Z}_2) \geq k+1$$

for every n. Consider the commutative diagram

$$\begin{tikzcd}
H^1(N(\alpha_n);\mathbb{Z}_2) \otimes H^1(N(\alpha_n);\mathbb{Z}_2) \arrow[r, "c_n"] \arrow[d, "i^*\otimes i^*"] & H^2(N(\alpha_n);\mathbb{Z}_2) \arrow[d, "i^*"] \\
H^1(N;\mathbb{Z}_2) \otimes H^1(N;\mathbb{Z}_2) \arrow[r, "c"] & H^2(N;\mathbb{Z}_2)
\end{tikzcd}$$

in which the homomorphisms c and c_n are defined by cup product. Since $i^* \otimes i^*$ is an isomorphism, we have

$$c(H^1(N;\mathbb{Z}_2) \otimes H^1(N;\mathbb{Z}_2)) = i^* \circ c_n(H^1(N(\alpha_n);\mathbb{Z}_2) \otimes H^1(N(\alpha_n);\mathbb{Z}_2)).$$

Since i^* is surjective and has a 1–dimensional kernel, it follows that

$$\begin{aligned}\dim_{\mathbb{Z}_2} & c(H^1(N;\mathbb{Z}_2) \otimes H^1(N;\mathbb{Z}_2)) \\ &= \dim_{\mathbb{Z}_2} c_n(H^1(N(\alpha_n);\mathbb{Z}_2) \otimes H^1(N(\alpha_n);\mathbb{Z}_2)) - 1.\end{aligned}$$

Condition (b) gives

$$\dim_{\mathbb{Z}_2} c(H^1(N;\mathbb{Z}_2) \otimes H^1(N;\mathbb{Z}_2)) \leq k-2,$$

and hence $$c_n(H^1(N(\alpha_n);\mathbb{Z}_2) \otimes H^1(N(\alpha_n);\mathbb{Z}_2)) \leq k-1.$$

It now follows from Proposition 4.4 (with $M = N(\alpha_n)$) that $\pi_1(N(\alpha_n))$ is k–free whenever $|n| \geq n_0$. □

Remark 4.5 In the proof given above, we quoted Theorem 1 of Hass, Wang and Zhou's paper [12] for the fact that there are only finitely many genus–g singular boundary slopes in ∂N for any given $g > 0$. Restricting the genus is essential here: there exist examples [5] of compact, orientable, irreducible 3–manifolds whose boundary is a single torus, in which there are infinitely many singular boundary slopes.

The result proved by Hatcher in [13] implies that if one considers only embedded surfaces, rather than singular ones, a finiteness result holds without restricting genus. A slope β in ∂N, where N is a compact, orientable, irreducible 3–manifold whose boundary is a torus, is called a *boundary slope* if there is a connected orientable surface F with nonempty boundary, properly embedded in N and not boundary-parallel, such that the inclusion homomorphism $\pi_1(F) \to \pi_1(N)$ is injective and all components of ∂F represent the slope β. The main theorem of [13] implies that there are only finitely many boundary slopes in ∂N.

5 k-free groups and generating sets

This brief section provides the algebra needed for the material in the following sections.

We shall denote by $|X|$ the cardinality of a set X. If S is a subset of a group Γ we shall denote by $\langle S\rangle$ the subgroup of Γ generated by S.

We shall say that a subset S of a group Γ is *independent* (or that the elements of S are independent) if $\langle S\rangle$ is a free group with basis S.

5.1 It is a basic fact in the theory of free groups [16, Vol 2, p 59] that a finite set $S \subset \Gamma$ is independent if and only if $\langle S\rangle$ is free of rank $|S|$.

Proposition 5.2 *Let k be a positive integer. Suppose that Γ is a finitely generated group which is k–free but not free, that X is a generating set for Γ, and that T is a finite independent subset of X with $|T| \leq k$. Then there is an independent subset S of X such that $T \subset S$ and $|S| = k$.*

Proof Set $m = |T|$. If $m = k$ we may take $S = T$. If $m < k$ we shall show that there is an independent subset T' of Γ such that $T \subset T' \subset X$ and $|T'| = m + 1$. The result will then follow at once by induction.

Let us write $X = \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are distinct and are indexed in such a way that $T = \{x_1, \ldots, x_m\}$. For each j with $0 \leq j \leq n$, let A_j denote the subgroup of Γ generated by $x_1, \dots, x_j$. Since $A_n = \Gamma$ is k–free but not free, it must have rank $> k$. On the other hand, since T is independent, $A_m = \langle T \rangle$ is a free group of rank $m < k$. Hence there is an index p with $m \le p < n$ such that A_p has rank $\le m$ and A_{p+1} has rank $> m$. Set $\xi = x_{p+1}$. Since $A_{p+1} = \langle A_p, \xi \rangle$ we must have $\operatorname{rank} A_{p+1} \le 1 + \operatorname{rank} A_p$. Hence $\operatorname{rank} A_p = m$ and $\operatorname{rank} A_{p+1} = m + 1$.

Since Γ is k–free and $m < k$, the subgroups A_p and A_{p+1} are free. Let $y_1, \dots, y_m$ be a basis for A_p. Then the rank–$(m+1)$ free group A_{p+1} is generated by the elements $y_1, \dots, y_m, \xi$, which must therefore form a basis for A_{p+1} (see 5.1). This shows that A_{p+1} is a free product $A_p * \langle \xi \rangle$, and that the factor $\langle \xi \rangle$ is infinite cyclic. Hence the subgroup $\langle A_m, \xi \rangle$ of A_{p+1} is a free product $A_m * \langle \xi \rangle$. Since $T = \{x_1, \dots, x_m\}$ freely generates A_m it follows that the set $T' = T \cup \{\xi\}$ has cardinality $m + 1$ and freely generates A_{p+1}. In particular, T' is independent. $\square$

6 Distant points and k-freeness

The two main results of this section, Proposition 6.1 and Proposition 6.3, were discussed in Section 1, the Introduction.

Proposition 6.1 *Let k and m be integers $k \ge 2$ and $0 \le m \le k$. Suppose that $M = \mathbb{H}^3/\Gamma$ is a closed, orientable hyperbolic 3–manifold such that $\pi_1(M) \cong \Gamma$ is k–free. Let $q\colon \mathbb{H}^3 \to M$ denote the quotient map. Suppose that $N \subset M$ is a closed set such that $q^{-1}(N)$ is connected. Let P be a point of M, let $\xi_1, \dots, \xi_m$ be independent elements of $\pi_1(M, P)$ represented by loops $\ell_1, \dots, \ell_m$ based at P, and let λ_j denote the length of ℓ_j. Then there is a point $Q \in N$ such that $\rho = \operatorname{dist}_M(P, Q)$ satisfies*

$$\frac{k-m}{1+e^{2\rho}} + \sum_{j=1}^{m} \frac{1}{1+e^{\lambda_j}} \le \frac{1}{2}.$$

Proof Set
$$\rho = \max\nolimits_{Q \in N} \operatorname{dist}_M(P, Q).$$

Fix a point $\widetilde{P} \in \mathbb{H}^3$ with $q(\widetilde{P}) = P$. Then $N \subset q(B)$, where B denotes the open ball of radius ρ centered at $\widetilde{P}$. Hence if we set $\widetilde{N} = q^{-1}(N)$, we have

$$\widetilde{N} \subset \Gamma \cdot B \doteq \bigcup_{\gamma \in \Gamma} \gamma(B).$$

Let X_0 denote the set of all elements $\eta \in \Gamma$ such that $\operatorname{dist}(\eta(\widetilde{P}), \widetilde{P}) < 2\rho$. Note that the triangle inequality gives $\eta \in X_0$ for any $\eta \in \Gamma$ such that $\eta(B) \cap B \neq \varnothing$.

We claim that X_0 generates Γ. To show this, we set $\Gamma_0 = \langle X_0 \rangle \leq \Gamma$, and choose a system Y of left coset representatives for Γ_0 in Γ. For each $y \in Y$, we set $\widetilde{N}_y = \widetilde{N} \cap y\Gamma_0 \cdot B$, where

$$y\Gamma_0 \cdot B = \bigcup_{\gamma \in \Gamma} y\gamma(B).$$

Then
$$\widetilde{N} = \bigcup_{y \in Y} \widetilde{N}_y.$$

If y and y' are distinct elements of Y such that $\widetilde{N}_y \cap \widetilde{N}_{y'} \neq \varnothing$, then for some $\gamma, \gamma' \in \Gamma_0$ we have $y\gamma(B) \cap y'\gamma'(B) \neq \varnothing$, so that $B \cap \gamma^{-1}y^{-1}y'\gamma'(B) \neq \varnothing$. It follows that $\gamma^{-1}y^{-1}y'\gamma' \in \Gamma_0$ and hence that $y^{-1}y' \in \Gamma_0$, a contradiction since y and y' represent distinct left cosets of Γ_0. This shows that the sets $\widetilde{N}_y$ are pairwise disjoint as y ranges over Y. Since these sets are open in the subspace topology of the connected set $\widetilde{N}$, we must have $\widetilde{N}_y = \widetilde{N}$ for some $y \in Y$. Since $P \in N$ it follows that $\widetilde{N}_y$ contains the Γ–orbit of $\widetilde{P}$; and as Γ acts freely on $\mathbb{H}^3$, this implies that $\Gamma_0 = \Gamma$, as claimed.

By hypothesis there exist m independent elements $\xi_1, \dots, \xi_m$ of $\pi_1(M, P)$ which are represented by piecewise smooth loops of respective lengths $\lambda_1, \dots, \lambda_m$ based at P. If we choose a point $\widetilde{P} \in q^{-1}(P)$, we may regard $(\mathbb{H}^3, \widetilde{P})$ as a based covering space of (M, P), and identify tgroup Γ with $\pi_1(M, P)$. Then for $j = 1, \dots, m$ we have $\operatorname{dist}(\widetilde{P}, \xi_j(\widetilde{P})) \leq \lambda_j$.

Let us set $X = X_0 \cup \{\xi_1, \dots, \xi_m\}$. We wish to apply Proposition 5.2, taking k and Γ to be defined as above, and taking $T = \{\xi_1, \dots, \xi_m\}$ and $X = T \cup X_0$. By hypothesis the finitely generated group $\Gamma \cong \pi_1(M)$ is k–free. To see that it is not free, we need only observe that $H_3(\Gamma; \mathbb{Z})$ has rank 1 because M is orientable, closed and aspherical. Since X_0 generates Γ, in particular the set $X = \{\xi_1, \dots, \xi_m\} \cup X_0$ generates Γ. We have $T \subset X$ by definition, and T is independent by hypothesis. Hence Proposition 5.2 gives an independent subset S of Γ such that $T \subset S \subset X$ and $|S| = k$. We may write $S = \{\xi_1, \dots, \xi_k\}$, where $\xi_{m+1}, \dots, \xi_k$ are elements of X_0.

Since S is independent, it follows from [4, Theorem 6.1], together with the main result of [1] or [9], that

$$\sum_{j=1}^{k} \frac{1}{1 + e^{d_j}} \leq \frac{1}{2},$$

where $d_j = \operatorname{dist}(\widetilde{P}, \xi_j \cdot \widetilde{P})$ for $j = 1, \dots, k$.

Since $\xi_{m+1},\dots,\xi_k$ belong to X_0, we have $d_j \le 2\rho$ for $j = m+1,\dots,k$. We have seen that $d_j \le \lambda_j$ for $j = 1,\dots,m$. Hence

$$\frac{k-m}{1+e^{2\rho}} + \sum_{j=1}^{m} \frac{1}{1+e^{\lambda_j}} \le \frac{1}{2},$$

and the proposition is proved. □

Corollary 6.2 *Let k and m be integers $k \ge 2$ and $0 \le m \le k$. Suppose that $M = \mathbb{H}^3/\Gamma$ is a closed, orientable hyperbolic 3–manifold such that $\pi_1(M) \cong \Gamma$ is k–free, and let μ be a Margulis number for M. Let P be a point of M, let $\xi_1,\dots,\xi_m$ be independent elements of $\pi_1(M,P)$ represented by loops $\ell_1,\dots,\ell_m$ based at P, and let λ_j denote the length of ℓ_j. Then there is a point $Q \in M_{\text{thick}}(\mu)$ such that $\rho = \operatorname{dist}_M(P,Q)$ satisfies*

$$\frac{k-m}{1+e^{2\rho}} + \sum_{j=1}^{m} \frac{1}{1+e^{\lambda_j}} \le \frac{1}{2}.$$

Proof Write $M = \mathbb{H}^3/\Gamma$, and let $q\colon \mathbb{H}^3 \to M$ denote the quotient map. As we pointed out in 2.6, the set $\widetilde{M}_{\text{thick}}(\mu) = q^{-1}(M_{\text{thick}}(\mu))$ is connected. Hence the assertions of the corollary follow from Proposition 6.1 if we take $N = M_{\text{thick}}(\mu)$. □

Proposition 6.3 *Let $k \ge 2$ be an integer, let M be a closed, orientable hyperbolic 3–manifold, and suppose that $\pi_1(M)$ is k–free. Let μ be a Margulis number for M. Let c be a closed geodesic in M, let l denote the length of c, and suppose that $l < \mu$. Let R denote the radius of the maximal (embedded) tube about c, and set*

$$R' = \frac{1}{2}\operatorname{arccosh}\big(\cosh(2R)\cosh\big(\tfrac{l}{2}\big)\big).$$

Then there exist points $P \in c$ and $Q \in M_{\text{thick}}(\mu)$ such that $\rho = \operatorname{dist}_M(P,Q)$ satisfies

$$\frac{k-2}{1+e^{2\rho}} + \frac{1}{1+e^{l}} + \frac{1}{1+e^{2R'}} \le \frac{1}{2}.$$

Proof Let us choose a component A of $q^{-1}(c)$. Then A is the axis of a loxodromic isometry $\gamma_0 \in \Gamma$, which has translation length l and generates a maximal cyclic subgroup C_0 of Γ.

Since R is the radius of a maximal tube about c, there is an element η of $\Gamma - C_0$ such that the open cylinders $N(R,A)$ and $N(R,\eta\cdot A)$ are disjoint but have intersecting closures. Hence the minimum distance between the lines A and $\eta\cdot A$ is $2R$. Let

$\widetilde{P}$ and $\widetilde{P}'$ denote the respective points of intersection of $\eta \cdot A$ and A with their common perpendicular, so that $\operatorname{dist}(\widetilde{P}, \widetilde{P}') = 2R$. Let $P, P' \in M$ denote the images of $\widetilde{P}$ and $\widetilde{P}'$ under the quotient map $\mathbb{H}^3 \to M$. Since $\eta^{-1} \cdot \widetilde{P} \in A$, the C_0–orbit of $\eta^{-1} \cdot \widetilde{P}$ contains a point $\widetilde{P}'' = \gamma_0^m \eta^{-1} \cdot \widetilde{P}$ such that $\operatorname{dist}(\widetilde{P}', \widetilde{P}'') \le l/2$. Setting $d = \operatorname{dist}(\widetilde{P}, \widetilde{P}'')$, we have

$$\cosh d = \cosh(2R) \cosh \operatorname{dist}(\widetilde{P}', \widetilde{P}'') \le \cosh(2R) \cosh(l/2)$$

and hence $d \le 2R'$.

We regard $(\mathbb{H}^3, \widetilde{P}'')$ as a based covering space of (M, P), and identify the deck transformation group Γ with $\pi_1(M, P)$. Since $\eta \notin C_0$, the elements $\eta\gamma_0^{-m}$ and γ_0 of $\pi_1(M, P)$ do not commute, and since Γ is in particular 2–free, they are independent. Since $\operatorname{dist}(\widetilde{P}'', \widetilde{P}) = d$ and $\operatorname{dist}(\widetilde{P}'', \gamma_0 \cdot \widetilde{P}'') = l$, we may apply Corollary refbig radius corollary with $m = 2$, $\xi_1 = \gamma_0$, $\xi_2 = \eta\gamma_0^{-m}$, $\lambda_1 = l$ and $\lambda_2 = d$, to obtain a point $Q \in M_{\text{thick}}(\mu)$ such that $\rho = \operatorname{dist}_M(P, Q)$ satisfies

$$\frac{1}{2} \ge \frac{k-2}{1+e^{2\rho}} + \frac{1}{1+e^{l}} + \frac{1}{1+e^{d}} \ge \frac{k-2}{1+e^{2\rho}} + \frac{1}{1+e^{l}} + \frac{1}{1+e^{2R'}}. \qquad \square$$

7 Distant points from maximal cusp neighborhoods

We begin by reviewing a version of W Thurston's Dehn Surgery Theorem and proving a result, Proposition 7.2, which is a consequence of the proof of Thurston's theorem. The proofs of Thurston's result that have appeared vary in the level of detail as well as in the strength of the statement, but most of them are based on the same ideas. For the sake of definiteness we refer the reader to the proof given by Petronio and Porti in [18] which seems to contain the most complete and elementary treatment of a version of Thurston's theorem that is adequate for our purposes.

Suppose that M is a complete, finite-volume, orientable hyperbolic manifold with cusps and that M' is a closed manifold obtained by Dehn filling the compact core of M. Following [6], we shall say that M' is a *hyperbolic Dehn filling* of M provided that M' admits a hyperbolic structure in which the core curves of the filling solid tori are isotopic to geodesics, which we shall call *core geodesics*.

While the general versions of Thurston's Theorem apply to finite-volume hyperbolic 3–manifolds with arbitrarily many cusps, we are only concerned here with the case of one cusp. We record the following statement, which follows from any of the various versions of Thurston's Theorem.

Theorem 7.1 (Thurston) *Let M be a one-cusped complete, finite-volume, orientable hyperbolic 3–manifold. Then all but finitely many Dehn fillings of the compact core of M are hyperbolic Dehn fillings of M.*

Here is the result whose proof is extracted from that of Thurston's theorem. The deepest conclusions are (3) and (4). Conclusion (3) was used in the proof of [3, Lemma 4.3]. While we presume that all of the conclusions are well known to experts in geometric convergence, conclusion (4) was new to us. In the proof we have attempted to follow the terminology used in [15] as closely as possible.

Proposition 7.2 *Let M_∞ be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp and let $(M_j)_{j\in\mathbb{N}}$ be a sequence of distinct hyperbolic Dehn fillings of M_∞. Let T_j be a maximal tube about the core geodesic for the Dehn filling M_j and let H be a maximal standard neighborhood of the cusp in M. Then*

(1) $\operatorname{vol} M_j \to \operatorname{vol} M_\infty$ *as* $j \to \infty$*;*

(2) *the length of the core geodesic of M_j tends to 0 as $j \to \infty$;*

(3) $\operatorname{vol} T_j \to \operatorname{vol} H$ *as* $j \to \infty$*;*

(4) *if a given positive number λ has the property that it is a Margulis number for each M_j, then λ is also a Margulis number for M_∞.*

Proof To start, we recall some of the statements proved in [18] in the course of proving Thurston's theorem. We let U denote the open unit disk in $\mathbb{C}$ and let $\rho_0\colon \pi_1(M_\infty) \to PSL_2(\mathbb{C})$ be a discrete faithful representation. By a *hyperbolic ideal tetrahedron* we mean the convex hull in $\mathbb{H}^3$ of a 4–tuple of distinct points on the sphere at infinity of $\mathbb{H}^3$. (We call these points vertices, although they are not contained in the tetrahedron.) We say that a hyperbolic ideal tetrahedron is *flat* if it is contained in a plane. If the vertices of a non-flat hyperbolic ideal tetrahedron are given an ordering, then the cross-ratio of the 4–tuple of vertices has positive imaginary part if and only if the ordering is consistent with the orientation that the tetrahedron inherits from hyperbolic space. There is a smooth function, which we shall denote $V(w)$, that is defined on $\mathbb{C} - \{0, 1\}$ and has the property that if the vertices of a hyperbolic ideal tetrahedron Δ are ordered, and if w is the cross-ratio of the 4–tuple of vertices, then $V(w) = \operatorname{vol}\Delta$ if $\operatorname{Im} w \geq 0$ and $V(w) = -\operatorname{vol}\Delta$ if $\operatorname{Im} w \leq 0$.

For $j \in \mathbb{N} \cup \{\infty\}$ we give M_j the orientation that it inherits as a quotient of $\mathbb{H}^3$.

Let $\widetilde{M}_\infty$ and $\widetilde{M}_j$ denote the universal covers of M_∞ and M_j respectively. We will set $\Gamma = \pi_1(M_\infty)$.

The results that we need from [18] are summarized below:

- For some $\pi_1(M)$–equivariant topological ideal triangulation $\mathcal{T}$ of $\widetilde{M}$ there exists a ρ_0–equivariant developing map $D_0\colon \widetilde{M} \to \mathbb{H}^3$ that maps each ideal 3–simplex of $\mathcal{T}$ to a hyperbolic ideal tetrahedron and preserves the orientation of each ideal 3–simplex which does not have a flat image. Moreover D_0 is a homeomorphism on the union of the interiors of the ideal 3–simplices with non-flat image.
- ρ_0 lies in a continuous family (with respect to the complex topology on the representation variety) of representations $\rho_z\colon \Gamma \to PSL_2(\mathbb{C})$, for $z \in U$. This family has the property that, given an arbitrarily small neighborhood O of 0, all but finitely many Dehn fillings on M are realized as hyperbolic Dehn fillings by hyperbolic manifolds of the form $\mathbb{H}^3/\rho_z(\Gamma)$, for $z \in O$.
- for each $z \in U$ there is a ρ_z–equivariant developing map $D_z\colon \widetilde{M} \to \mathbb{H}^3$ which sends each tetrahedron in $\mathcal{T}$ to a hyperbolic ideal tetrahedron. Moreover the cross-ratios of the image tetrahedra vary continuously with z.

According to the second result above we may assume, after deleting finitely many terms from the sequence (M_j), that there are complex numbers z_j ∈U such that $M_j = \mathbb{H}^3/\rho_{z_j}(\Gamma)$ for $j \in \mathbb{N}$.

Let $\Delta_1, \dots, \Delta_n$ be a system of distinct representatives for the Γ orbits of ideal 3–simplices in $\mathcal{T}$. Order the vertices of each Δ_i in a way that is consistent with the orientation inherited from M_∞. For $i \in \{1, \dots, n\}$ and $z \in U$ let $w_i(z)$ denote the cross-ratio of the vertices of $D_z(\Delta_i)$, with the ordering determined by the ordering of the vertices of Δ_i. The results above then imply that the function

$$\mathcal{V}(z) = \sum_{i=1}^{n} V(w_i(z))$$

has the property that $V(z_j) = \operatorname{vol} M_j$. Since $\mathcal{V}$ is continuous, this implies conclusion (1).

The continuity of the family ρ_z implies, in the terminology of [15], that the Kleinian groups $\Gamma_j = \rho_{z_j}(\Gamma)$ converge algebraically to the Kleinian group $\Gamma_\infty = \rho_0(\Gamma)$. In particular, there are surjective homomorphisms $\phi_j\colon \Gamma_\infty \to \Gamma_j$ such that each element γ of Γ_∞ is the limit of the sequence $(\phi_j(\gamma))$. For each $j \in \mathbb{N} \cup \{\infty\}$ we shall let $p_j\colon \mathbb{H}^3 \to M_j$ denote the quotient map.

We fix a base point in the boundary of a compact core N of M, and let $\Lambda \le \Gamma$ denote the image of $\pi_1(\partial N)$ under the inclusion homomorphism. Set $\Lambda_\infty = \rho_0(\Lambda)$ and $\Lambda_j = \rho_j(\Lambda)$ for $j \in \mathbb{N}$. Thus Λ_∞ is a maximal parabolic subgroup of Γ_∞ and the groups Λ_j for $j \in \mathbb{N}$ are cyclic groups generated by a loxodromic isometry.

To prove conclusion (2), for $j \in \mathbb{N}$ we let l_j denote the length of the core geodesic of M_j for $j \in \mathbb{N}$ and let g_j denote a loxodromic isometry of translation length l_j that generates Λ_j.

Let us fix, arbitrarily, a nontrivial element γ_0 of Λ. For every $j \in \mathbb{N}$ we have $\rho_{z_j}(\gamma_0) = g_j^{m_j}$ for some $m_j \in \mathbb{Z}$. Since the Dehn fillings in the sequence are distinct, there is at most one index j for which $\rho_{z_j}(\gamma_0)$ is the identity. In particular we have $m_j \neq 0$ for large j, so that $\rho_{z_j}(\gamma_0)$ is loxodromic with translation length $|m_j| l_j \geq l_j$.

On the other hand, algebraic convergence implies that $\rho_{z_j}(\gamma_0)$ approaches the parabolic isometry $\rho_{z_j}(\gamma_0)$ as $j \to \infty$. In particular, $(\text{trace}\,\rho_{z_j}(\gamma_0))^2 \to 4$, so that the translation length of $\rho_{z_j}(\gamma_0)$ approaches 0. Hence $l_j \to 0$, which is conclusion (2).

Our proofs of conclusions (3) and (4) depend on the techniques of [15]. It follows from [15, Proposition 3.8] that algebraic convergence implies polyhedral convergence, for the sequence of non-elementary groups $(\Gamma_j)_{j \in \mathbb{N}}$ as well as for the sequence of elementary groups $(\Lambda_j)_{j \in \mathbb{N}}$ (see [15, Remark 3.10]). We will describe in some detail what is meant by polyhedral convergence of the sequence (Γ_j), and introduce some notation that will be used later. An analogous description applies to the sequence (Λ_j). Consider a point $\widetilde{P} \in \mathbb{H}^3$, chosen from the dense set of points whose associated Dirichlet domains for the groups Γ_∞ and Γ_j, $j \in \mathbb{N}$ are generic in the sense of [15]. For $j \in \mathbb{N} \cup \{\infty\}$ let us set $P_j = p_j(\widetilde{P})$, and $W_j(r) = \overline{N(r, P_j)} \subset M_j$ for each $r > 0$. We denote by Z_j the Dirichlet domain for Γ_j centered at $\widetilde{P}$. The truncated Dirichlet domain $Z'_j = Z_j \cap N(r, \widetilde{P})$ (a polyhedral set having some faces that are planar and some that lie in a sphere) is a fundamental domain for the action of Γ_j on $p_j^{-1}(W_j(r))$. To say that $(\Gamma_j)_{j \in \mathbb{N}}$ converges polyhedrally to Γ_∞ means that for any sufficiently large positive number r, say for $r \geq r_1$, there exists $J_r \in \mathbb{N}$ such that for $j > J_r$ the polyhedral sets Z'_j and Z'_∞ have isomorphic combinatorial structures, with face-pairings that correspond under the homomorphism ϕ_j. Moreover, for $j > J_r$, there are positive numbers ϵ_j tending to 0 and homeomorphisms $g_j \colon Z'_\infty \to Z'_j$ which preserve vertices, edges and faces, commute with the face-pairings, and satisfy $\text{dist}(x, g_j(x)) < \epsilon_j$.

It follows from [15, Proposition 4.7] that when $r \geq r_1$, the set $\overline{M_\infty - W_\infty(r)}$ is a closed subset of M_∞ that is homeomorphic to $S^1 \times S^1 \times [1, \infty)$, and that for $j > J_r$ the set $M_j - W_j(r)$ is a topological regular neighborhood of the core geodesic of the Dehn filling.

When $r \geq r_1$ it follows from polyhedral convergence that for $j > J_r$ there exist bi-Lipschitz homeomorphisms $f_{j,r} \colon W_\infty(r) \to W_j(r)$ whose Lipschitz constants tend to 1 as $j \to \infty$. (In general we shall write f_j for $f_{j,r}$, the choice of r being clear from the context.) We may take the homeomorphisms f_j to have the following property:

for any $x \in W_\infty(r)$ *there exist* $\tilde{x}_\infty \in p_\infty^{-1}(x)$ *and* $\tilde{x}_j \in p_j^{-1}(f_j(x))$
with $\operatorname{dist}(\tilde{x}_\infty, \tilde{x}_j) < \epsilon_j$.

Given a subgroup G of Γ_j for some $j \in \mathbb{N} \cup \{\infty\}$ we define a function $\mu_G\colon M_j \to \mathbb{R}$ by $\mu_G(x) = \min\{\operatorname{dist}(\tilde{x}, \gamma(\tilde{x})\}$ where the minimum is taken over all elements γ of G and all points $\tilde{x} \in p_j^{-1}(x)$. Note that $\mu_{\Gamma_j}(x)$ is twice the injectivity radius of M_j at $p_j(x)$.

We now claim:

7.2.1 *For any* $r \geq r_1$ *and for any point* $x \in W_\infty(r)$ *we have* $\mu_{\Gamma_j}(f_j(x)) \to \mu_{\Gamma_\infty}(x)$ *and* $\mu_{\Lambda_j}(f_j(x)) \to \mu_{\Lambda_\infty}(x)$ *as* $j \to \infty$.

To prove the first assertion of 7.2.1, let $r \geq r_1$ be given, and let x be a given point in $W_\infty(r)$. Let $\tilde{x}$ and $\tilde{x}_j$ be as above. Then, for any $\gamma \in \Gamma_\infty - \{1\}$, we have $\tilde{x}_j \to \tilde{x}_\infty$ and $\phi_j(\gamma) \to \gamma$ as $j \to \infty$. Thus $\operatorname{dist}(\tilde{x}_j, \phi_j(\gamma)(\tilde{x}_j)) \to \operatorname{dist}(\tilde{x}_\infty, \gamma(\tilde{x}_\infty))$ as $j \to \infty$. It follows that $\phi_j(\gamma) \neq 1$ for sufficiently large j. Hence if we choose $\gamma \in \Gamma_\infty - \{1\}$ so as to minimize $\operatorname{dist}(\tilde{x}_\infty, \gamma(\tilde{x}_\infty))$ then for any $\epsilon > 0$ we have

$$\mu_{\Gamma_\infty}(x) = \operatorname{dist}(\tilde{x}_\infty, \gamma(\tilde{x}_\infty)) \geq \operatorname{dist}(\tilde{x}_j, \phi_j(\gamma)(\tilde{x}_j)) - \epsilon \geq \mu_{\Gamma_j}(f_j(x)) - \epsilon.$$

It follows that $\mu_{\Gamma_\infty}(x) \geq \limsup \mu_{\Gamma_j}(f_j(x))$.

Next we shall show that $\mu_{\Gamma_\infty}(x) \leq \liminf \mu_{\Gamma_j}(f_j(x))$. It will simplify the notation if we assume, as we may by passing to a subsequence, that the sequence $\big(\mu_{\Gamma_j}(f_j(x))\big)$ converges. For each $j \in \mathbb{N}$ choose $\gamma_j \in \Gamma_\infty$ such that $\mu_{\Gamma_j}(f_j(x)) = \operatorname{dist}(\tilde{x}_j, \phi_j(\gamma_j)(\tilde{x}_j))$. For sufficiently large j we have $\mu_{\Gamma_j}(f_j(x)) < 1 + \mu_{\Gamma_\infty}(x)$. In particular, this shows that the set $\{\operatorname{dist}(\tilde{x}_j, \phi_j(\gamma_j)(\tilde{x}_j)) \mid j \in \mathbb{N}\}$ is bounded and hence that the set $\{\phi(\gamma_j) \mid j \in \mathbb{N}\}$ is a bounded subset of $PSL(2, \mathbb{C})$. We now appeal to [15, Proposition 3.10] which states that a sequence of non-elementary Kleinian groups converges polyhedrally if and only if it converges geometrically, and that the geometric and polyhedral limits are the same whenever they exist. Thus Γ_∞ is the geometric limit of (Γ_n). Now the definition of geometric convergence [15, Definition 3.2] implies in particular that if we are given any subsequence Γ_{j_k} of (Γ_j) and elements $g_{j_k} \in \Gamma_{j_k}$ such that $g_{j_k} \to h \in PSL(2, \mathbb{C})$, then h must be an element of Γ_∞. Since $\{\phi(\gamma_j) \mid j \in \mathbb{N}\}$ is bounded we may choose a subsequence (γ_{j_k}) of (γ_j) such that $\phi(\gamma_{j_k}) \to h \in \Gamma_\infty$. We then have

$$\mu_{\Gamma_{j_k}}(f_{j_k}(x)) = \operatorname{dist}(\tilde{x}_{j_k}, \phi_{j_k}(\gamma_{j_k})(\tilde{x}_{j_k})) \to \operatorname{dist}(\tilde{x}, \gamma(x)) \geq \mu_{\Gamma_\infty}(x).$$

Thus, under our assumption that $(\mu_{\Gamma_j}(f_j(x))$ converges, we have shown that $\mu_{\Gamma_\infty}(x) \leq \lim_{j \to \infty} \mu_{\Gamma_j}(f_j(x))$. Dropping this assumption, we conclude in the general case that

$\mu_{\Gamma_\infty}(x) \le \liminf_{j\to\infty} \mu_{\Gamma_j}(f_j(x))$, as required to complete the proof of the first assertion of 7.2.1.

The proof of the second assertion of 7.2.1 is almost the same as the proof of the first assertion, with the sequence (Γ_j) replaced by the sequence (Λ_j). The only difference occurs in the step where we applied [15, Proposition 3.10], which applies only to a sequence of nonelementary Kleinian groups. For the proof of the second assertion, we appeal instead to [15, Remark 3.10], which gives the same conclusion for the special sequence (Λ_j) of elementary Kleinian groups. Thus 7.2.1 is proved.

For the rest of the argument, to unify the notation, we set $T_\infty = H \subset M_\infty$. We next claim:

7.2.2 *For each $j \in \mathbb{N} \cup \{\infty\}$, the function μ_{Λ_j} takes a constant value I_j on the frontier of T_j. On the open set T_j, μ_{Λ_j} is a monotone decreasing function of distance from the frontier.*

To prove 7.2.2 when $j \in \mathbb{N}$ we may apply the formula (2.5.2), which implies that for $z \in p_j^{-1}(T_j)$ we have

$$\cosh \mu_{\Lambda_j}(z) = \min_{n\in\mathbb{N}} (\cosh nl + (\sinh^2 r)(\cosh nl - \cos n\theta)), \tag{7.2.3}$$

where l and θ denote the twist angle of a generator of Λ_j and r is the distance from z to the axis of Λ_j. Since the factor $\cosh nl - \cos n\theta$ increases monotonically with n for large n, the minimum over $\mathbb{N}$ in (7.2.3) may be replaced by a minimum over a finite subset of $\mathbb{N}$. Thus $\mu_{\Lambda_j}(z)$ is the minimum of a finite collection of smooth, monotone increasing functions of r, and is therefore monotone increasing and piecewise smooth. This proves 7.2.2 for $j \in \mathbb{N}$; for $j = \infty$ it is similar but easier.

It follows from 7.2.2 that with the exception of the core curve of T_j for $j \in \mathbb{N}$, the level sets of the μ_{Λ_j} for $j \in \mathbb{N} \cup \{\infty\}$ are smooth embedded tori. For any $r \ge r_1$ and any $j \in \mathbb{N} \cup \{\infty\}$ we may characterize the submanifold $T_j' = T_j \cap W_j(r)$ as the set $\{x \in W_j(r) \mid \mu_{\Lambda_j}(x) < I_j\}$. The frontier of T_j' in $W_j(r)$ is a singular torus (with self-tangencies) which is a component of the level set $\{x \mid \mu_{\Lambda_\infty}(x) = I_j\}$. Since the preimage of T_j' in $\widetilde{M}_j$ has two distinct components whose closures meet, there exists an element of Γ_j which does not commute with the image of $\pi_1(T_j') \to \Gamma_j$, but is contained in the image of $\pi_1(\overline{T_j'}) \to \Gamma_j$. It follows that:

7.2.4 *The image of $\pi_1(\overline{T_j'})$ in $\pi_1(W_j(r))$ is not carried (up to conjugacy) by the frontier torus $\partial W_j(r)$.*

For the proof of (3), we consider any subsequence (M_{j_n}) of (M_j) such that the manifolds $X_n = f_{j_n}^{-1}(\overline{T'_{j_n}})$, which are contained in the compact set $W_\infty(r_1)$, converge in the Hausdorff topology. The limit is then a compact connected subset X_∞ of $W_\infty(r_1)$ which contains the frontier of $W_\infty(r_1)$. Moreover it follows from 7.2.1 that $X_\infty = \{x \mid \mu_{\Lambda_\infty}(x) \leq I\}$, where $I = \lim_{n\to\infty} I_{j_n}$.

Next we will show that $I = I_\infty$, and hence that $X_\infty = \overline{T'_\infty}$. If $I < I_\infty$ then X_∞ is a submanifold contained in the interior of T'_∞ and hence, for small $\epsilon > 0$, the image of $\pi_1(N(\epsilon, X_\infty))$ is carried by $\partial W_\infty(r)$. However, for sufficiently large j, we have that $X_j \subset N(\epsilon, X_\infty)$ and hence that the image of $\pi_1(X_j)$ is carried by $\partial W_\infty(r)$. Since f_j is a homeomorphism and $f_j(X_j) = T'_j$, this contradicts 7.2.4. If $I > I_\infty$ then, for sufficiently large j, $f_j(\overline{T'_\infty})$ is contained in the interior of T'_j. Thus the image of $\pi_1(f_j(\overline{T'_\infty}))$ is carried by $\partial W_j(r)$, which again contradicts 7.2.4.

Since the manifolds X_n are converging in the Hausdorff topology to $\overline{T'_\infty}$ we have $\operatorname{vol} X_n \to \operatorname{vol} T'_\infty$. Moreover, since the Lipschitz constants of the homeomorphisms f_{j_n} are converging to 1, we also have $\operatorname{vol} T'_{j_n} \to \operatorname{vol} T'_\infty$ and $\operatorname{vol}(W_{j_n}(r_1) - T'_{j_n}) \to \operatorname{vol}(W_\infty(r_1) - T'_\infty)$ as $n \to \infty$. Thus

$$\operatorname{vol}(M_{j_n} - T_{j_n}) = \operatorname{vol}(W_{j_n}(r_1) - T'_{j_n}) \to \operatorname{vol}(W_\infty(r_1) - T'_\infty) = \operatorname{vol}(M_\infty - T_\infty).$$

In view of conclusion (1) it follows that $\lim_{n\to\infty} \operatorname{vol} T_{j_n} = \operatorname{vol} T_\infty$.

Thus we have shown that for any subsequence (M_{j_n}) of (M_j) such that (X_n) converges in the Hausdorff topology, we have $\lim_{n\to\infty} \operatorname{vol} T_{j_n} = \operatorname{vol} T_\infty$. Since every subsequence of (M_j) contains a subsequence (M_{j_n}) of this type, conclusion (3) follows.

Assume, for the proof of conclusion (4), that λ is a Margulis number for each M_j. It suffices to show that $\lambda - 2\delta$ is a Margulis number for M_∞ for any given δ with $0 < \delta \leq \delta_0$.

For a sufficiently large $r > r_1$, the frontier torus of $W_\infty(r)$ is contained in the λ–thin part of M_∞. We fix a number $r = r_2$ with this property, and denote the frontier torus of $W_\infty(r_2)$ by F. We consider a positive number δ which is small enough to ensure that F is contained in the $(\lambda - 2\delta)$-thin part of M_∞, which we shall denote by Θ.

By 7.2.1 and the compactness of W_∞ we may fix $K \in \mathbb{N}$ so that $\operatorname{inj}_{M_\infty}(x) \geq \operatorname{inj}_{M_K}(f_K(x)) - \delta$ for all $x \in W_\infty(r_2)$. Hence if Ψ denotes the λ–thin part of M_K, we have $f_K(\Theta \cap W_\infty(r_2)) \subset \Psi$, so that Θ is contained in the set $Q = f_K^{-1}(\Psi \cap W_K(r_2)) \cup (M_\infty - W_\infty(r_2)) \subset M_\infty$.

We are required to show that $\lambda - 2\delta$ is a Margulis number for M_∞. According to the discussion in 2.5, this is equivalent to showing that if θ is a component of Θ, then

$\pi_1(\theta)$ has abelian image in $\Gamma\pi_1(M_\infty)$. Hence it suffices to show that every component of Q has abelian fundamental group.

We have seen that $V_+ = \overline{M_\infty - W_\infty(r_2)}$ is homeomorphic to $S^1 \times S^1 \times [1, \infty)$, so that in particular V_+ is connected. We have $F \subset \Theta \cap W_\infty(r_2) \subset f_K^{-1}(\Psi \cap W_K(r_2))$. If V_- denotes the component of $f_K^{-1}(\Psi \cap W_K(r_2))$ containing F, then $V_+ \cup V_-$ is one component of Q, and every other component has the form $f_K^{-1}(\psi)$, where ψ is a component of Ψ contained in $\operatorname{int} W_K(r_2)$. Any component of the latter type is homeomorphic to a component of Ψ, and is therefore a solid torus since λ is a Margulis number for M_K. In particular such a component has an abelian fundamental group.

To describe the topology of $V_+ \cup V_-$, we consider the component ψ_0 of Ψ containing $f_K(V_-)$. Since λ is a Margulis number for M_K, the set ψ_0 is a tube about some closed geodesic C_0 in M_K. But $f_K(F) \subset \psi_0$ is the frontier of $M_K - W_K(r_2)$, which we have seen is a topological regular neighborhood of the core geodesic C in M_K. We must therefore have $C = C_0$. As $M_K - W_K(r_2)$ and ψ_0 are open regular neighborhoods of C_0, with $M_K - W_K(r_2) \subset \psi_0$, the set $\psi_0 \cap W_K(r_2) = f_K(V_-)$ is homeomorphic to $S^1 \times S^1 \times (0, 1]$, and hence so is V_-. Since $V_+ \cong S^1 \times S^1 \times [1, \infty)$, and $\partial V_+ = \partial V_-$, it follows that $V_+ \cup V_- \cong S^1 \times S^1 \times (0, \infty)$; in particular, $\pi_1(V_+ \cup V_-)$ is abelian. □

The proof of the following result combines Propositions 6.3 and 7.2.

Theorem 7.3 *Let M be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp, and let $k \geq 3$ be an integer. Suppose that there is an infinite sequence of closed manifolds $(M_j)_{j \in \mathbb{N}}$, obtained by distinct Dehn fillings of the compact core of M, such that $\pi_1(M_j)$ is k–free for every $j \geq 0$. Let H denote the maximal standard neighborhood (see 2.4) of the cusp of M. Suppose that* $\operatorname{vol} H < 2\pi$. *Then there exist a real number β with $1 < \beta < 2$ and a point $Q \in M_{\text{thick}}(\log 3)$ such that*

(1) $\operatorname{vol} H \geq \pi\beta$, *and*

(2) $\operatorname{dist}(Q, \bar{H}) \geq -\frac{1}{2} \log \left(\dfrac{\beta - 1}{k - 2} \right)$.

Proof According to Theorem 7.1, all but finitely many of the M_j are hyperbolic Dehn fillings of M; hence we may assume, after passing to a subsequence, that all the M_j are hyperbolic Dehn fillings.

Let c_j denote the core geodesic of the Dehn filling M_j and set $l_j = \operatorname{length} c_j$. According to Proposition 7.2 we have $l_j \to 0$ as $j \to \infty$.

Since $\pi_1(M_j)$ is in particular 2–free for each M_j, the number $\log 3$ is a Margulis number for each M_j (see 2.7).

After passing to a smaller subsequence we may assume that $l_j < \log 3$ for every j. Let T_j denote the maximal (embedded) tube about c_j, let R_j denote its radius, and set

$$R'_j = \frac{1}{2}\operatorname{arccosh}\big(\cosh(2R_j)\cosh\big(\frac{l_j}{2}\big)\big).$$

Then according to Proposition 6.3 there exist points $P_j \in c_j$ and $Q_j \in M_j$ such that $\rho = \operatorname{dist}_M(P_j, Q_j)$ satisfies

$$\frac{k-2}{1+e^{2\rho}} + \frac{1}{1+e^{l_j}} + \frac{1}{1+e^{2R'_j}} \leq \frac{1}{2}. \tag{7.3.1}$$

Furthermore, it follows from Proposition 6.3 that we may assume that our subsequence has been chosen so that Q_j lies in the $(\log 3)$–thick part of M_j for every j.

From the definition of the R'_j we see that $R_j \leq R'_j \leq R_j + (l_j/4)$. Since $l_j \to 0$, it follows that $R_j - R'_j \to 0$ as $j \to \infty$.

From (7.3.1) it follows in particular that

$$\frac{1}{1+e^{l_j}} + \frac{1}{1+e^{2R'_j}} \leq \frac{1}{2}.$$

Let us set
$$r_j = \frac{1}{2}\log\left(\frac{e^{l_j}+3}{e^{l_j}-1}\right),$$

so that

$$\frac{1}{1+e^{l_j}} + \frac{1}{1+e^{2r_j}} = \frac{1}{2}. \tag{7.3.2}$$

Then we have $R'_j \geq r_j$ for every j. In particular, $R'_j \to \infty$, and hence $R_j \to \infty$, as $j \to \infty$.

We set $h'_j = R'_j - r_j \geq 0$, and $h_j = R_j - r_j$. Note that $h'_j - h_j \to 0$ as $j \to \infty$.

According to Proposition 7.2 we have

$$\lim_{j\to\infty} \operatorname{vol} T_j = \operatorname{vol} H. \tag{7.3.3}$$

On the other hand, we have $\operatorname{vol} T_j = \pi l_j \sinh^2 R_j$. Since R_j tends to infinity with j, it follows that

$$\frac{\operatorname{vol} T_j}{l_j e^{2R_j}} \to \frac{\pi}{4}. \tag{7.3.4}$$

We have $R_j = r_j + h_j$. From the definition of r_j, and the fact that $l_j \to 0$, we find that $e^{2r_j} l_j \to 4$. Combining these observations with (7.3.3) and (7.3.4), we deduce that $\lim_{j\to\infty} \pi e^{2h_j}$ exists and that

$$\lim_{j\to\infty} \pi e^{2h_j} = \lim_{j\to\infty} \operatorname{vol} T_j = \operatorname{vol}(H).$$

Hence $\alpha = \lim_{j\to\infty} h_j$ exists, and if we set $\beta = e^{2\alpha}$ we have

$$\operatorname{vol} H = \beta\pi. \tag{7.3.5}$$

Since $h'_j - h_j \to 0$ we have $h'_j \to \alpha$ as $j \to \infty$. In particular, since $h'_j \geq 0$, we have $\alpha \geq 0$ and hence

$$\beta \geq 1. \tag{7.3.6}$$

The hypothesis $\operatorname{vol} H < 2\pi$, with (7.3.5), gives

$$\beta < 2. \tag{7.3.7}$$

By comparing (7.3.1) and (7.3.2) we find that

$$\frac{k-2}{1+e^{2\rho_j}} + \frac{1}{1+e^{2R'_j}} \leq \frac{1}{1+e^{2r_j}},$$

which gives

$$(k-2)\frac{1+e^{2R'_j}}{1+e^{2\rho_j}} \leq \frac{1+e^{2R'_j}}{1+e^{2r_j}} - 1. \tag{7.3.8}$$

As $j \to \infty$, since $R'_j - r_j = h'_j \to \alpha$, the right hand side of (7.3.8) approaches $e^{2\alpha} - 1 = \beta - 1$. Hence

$$(k-2)\limsup e^{2(R'_j - \rho_j)} \leq \beta - 1.$$

Since $R_j \leq R'_j$ it follows that

$$(k-2)\limsup e^{2(R_j - \rho_j)} \leq \beta - 1.$$

Recalling that $\beta < 2$ by (7.3.7), we find that

$$0 < \liminf(\rho_j - R_j) \leq \infty.$$

Hence after passing to a subsequence we may assume that $\rho_j - R_j \to \gamma$, where $\gamma = +\infty$ if $\beta = 1$ and

$$\gamma \geq -\frac{1}{2}\log\left(\frac{\beta-1}{k-2}\right) \qquad \text{if} \qquad \beta > 1. \tag{7.3.9}$$

In particular, since $\beta < 2$ and $k \geq 3$, we have $\gamma > 0$.

For every j set $d_j = \operatorname{dist}(Q_j, \bar{T}_j)$ (so that, a priori, we have $d_j \geq 0$, with equality if and only if $Q_j \in \bar{T}_j$). Since $T_j = N(R_j, c_j)$, and c_j has diameter at most $l_j/2$, we have $\rho_j = \operatorname{dist}(P_j, Q_j) \leq d_j + R_j + (l_j/2)$, and hence

(7.3.10) $$d_j \geq \rho_j - R_j - (l_j/2).$$

In (7.3.10) we have $\rho_j - R_j \to \gamma$ and $l_j \to 0$. Setting $\delta = \gamma/2$ if $0 < \gamma < \infty$ and $\delta = 1$ if $\gamma = \infty$, we deduce that $d_j \geq \delta$ for large j, ie $Q_j \in M_j - N(\delta, \bar{T}_j)$ for large j. In view of the properties of the maps f_j it then follows that $f_j(Q_j) \in M - H$ for every sufficiently large j. As $M - H$ is compact, we may assume after passing to a subsequence that the sequence $(f_j(Q_j))$ converges to a point $Q \in M - H$. Furthermore, we have $d_j \to \operatorname{dist}(Q, H)$ as $j \to \infty$. Taking limits in (7.3.10) we deduce that

(7.3.11) $$\gamma \leq \operatorname{dist}(Q, H).$$

In particular $\gamma < \infty$, so that the definition of γ gives $\beta > 1$, and by (7.3.9) we have $\gamma \geq -(1/2)\log((\beta-1)/(k-2))$. It now follows from (7.3.11) that

$$\operatorname{dist}(Q, \bar{H}) \geq -\frac{1}{2}\log\left(\frac{\beta-1}{k-2}\right).$$

This is conclusion (2) of the theorem. Conclusion (1) is included in (7.3.5). This completes the proof of Theorem 7.3. □

8 Volume and density

8.1 We shall define functions $B(r)$, $a_2(r)$, $h_2(r)$, $a_3(r)$, $h_3(r)$, $\beta(r)$, $\tau(r)$ and $d(r)$ for $r > 0$. Each of these will be defined by an analytic formula accompanied by a geometric interpretation. For $n = 2, 3$ we let $\Delta_n(r)$ denote a regular hyperbolic simplex in $\mathbb{H}^n$ with sides of length $2r$.

- $B(r) = \pi(\sinh(2r) - 2r)$ is the volume of a ball of radius r in $\mathbb{H}^3$;
- $a_2(r) = \operatorname{arccosh}(\cosh(2r)/\cosh(r)$ is the altitude of $\Delta_2(r)$;
- $h_2(r) = \operatorname{arctanh}((\cosh a_2(r)\cosh(r) - 1)/\sinh a_2(r)\cosh r)$ is the distance from a vertex of $\Delta_2(r)$ to its barycenter;
- $a_3(r) = \operatorname{arccosh}(\cosh(2r)/\cosh(h_2(r)))$ is the altitude of $\Delta_3(r)$;
- $h_3(r) = \operatorname{arctanh}((\cosh a_3(r)\cosh h_2(r) - 1)/(\sinh a_3(r)\cosh h_2(r)))$ is the distance from a vertex of $\Delta_3(r)$ to its barycenter;

- $\beta(r) = \operatorname{arcsec}(\operatorname{sech}(2r) + 2)$ is a dihedral angle of $\Delta_3(r)$;
- $\tau(r) = 3\int_{\beta(r)}^{\operatorname{arcsec} 3} \operatorname{arcsech}((\sec t) - 2)dt$ is the volume of $\Delta_3(r)$;
- $d(r) = (3\beta(r) - \pi)(\sinh(2r) - 2r)/\tau(r)$ is the ratio of $\operatorname{vol}(W \cap \Delta_3(r))$ to $\operatorname{vol} \Delta_3(r)$, where W is the union of four mutually tangent balls of equal radii centered at the vertices of $\Delta_3(r)$.

It is shown in [8] that $d(r)$ is a monotonically increasing function for $r > 0$. We set $d(\infty) = \lim_{r\to\infty} d(r) = .853\ldots$

If P is a point in a complete hyperbolic manifold M, and if the injectivity radius at P is at least r then it is immediate that $\operatorname{vol} N(r, P) = B(r)$. Less trivially, we have:

Proposition 8.2 *Suppose r is a positive number, that P is a point in a hyperbolic manifold M, and that the injectivity radius of M at P is at least r. Then* $\operatorname{vol}(N(h_3(r), P)) \geq B(r)/d(r)$.

Proof Set $J = N(r, P)$ and $W = N(h_3(r), P)$. Then J is isometric to a ball of radius r in $\mathbb{H}^3$, since the injectivity radius of M at P is at least r. If we write $M = \mathbb{H}^3/\Gamma$ where $\Gamma \leq \operatorname{Isom}_+(\mathbb{H}^3)$ is discrete and torsion-free, and if $q\colon \mathbb{H}^3 \to M$ denotes the quotient map, then each component of $q^{-1}(J)$ is a hyperbolic ball of radius r. These components form a sphere-packing in the sense of [7]. Let us fix any component $\widetilde{J}$ of $q^{-1}(J)$, and let $\widetilde{W} \subset \mathbb{H}^3$ denote the ball of radius $h_3(r)$ with the same center as $\widetilde{J}$. The Voronoĭ polytope Z for $\widetilde{J}$, whose interior consists of all the points which are nearer to the center of $\widetilde{J}$ than to the center of any other sphere of the packing, is a Dirichlet domain for Γ. In the proof of the main result on p 259 of [7] it is shown that $(\operatorname{vol} \widetilde{J})/\operatorname{vol}(Z \cap \widetilde{W}) \leq d(r)$. But $\operatorname{vol} \widetilde{J} = B(r)$; and since q maps $\widetilde{W}$ into W, and is one-to-one on the interior of Z, we have $\operatorname{vol}(Z \cap \widetilde{W}) \leq \operatorname{vol} W$. Hence $B(r)/(\operatorname{vol} W) \leq d(r)$. □

Proposition 8.3 *Suppose that H is a standard neighborhood of a cusp in a finite-volume hyperbolic 3–manifold M, and set*

$$N = N(\log\sqrt{\tfrac{3}{2}}, \bar{H}) \subset M.$$

Then
$$\frac{\operatorname{vol} H}{\operatorname{vol} N} \leq d(\infty).$$

Proof This is essentially a restatement of [7, Theorem 4]. In [7, Section 6] it is remarked that the proof of the main theorem (about sphere packings) applies without change to horoball packings. A stronger version of [7, Theorem 4], analogous to the statement which was used in the previous proposition, follows from this proof in the same way. However, it must be formulated slightly differently since $h_3(r) \to \infty$ as $r \to \infty$. For this reformulation we consider the function $k_3(r) \doteq h_3(r) - r$. Geometrically, the value of $k_3(r)$ can be described as the distance from the barycenter of $\Delta_3(r)$ to any of the four spheres of radius r centered at a vertex of $\Delta_3(r)$. Note that if P is a vertex of $\Delta_3(r)$ and B is a ball of radius r centered at P then $N(h_3(r), P) = N(k_3(r), B)$.

If Δ is a regular ideal hyperbolic simplex then there is a unique family of four "standard" horoballs which meet the sphere S^∞ at the vertices of Δ and are each tangent to the other three. By the barycenter of Δ we shall mean the unique common fixed point of the 12 symmetries of Δ, which permute the standard horoballs. It is clear that the quantity $k_3(\infty) = \lim_{r\to\infty} k_3(r)$ is the distance from the barycenter of Δ to any of these four horoballs. Working in the upper half-space model, with coordinates (z, t), where $z \in \mathbb{C}$ and $t \in \mathbb{R}_+$, it is straightforward to check, for the regular ideal hyperbolic tetrahedron with vertices $(0,0)$, $(1,0)$, $((1+i\sqrt{3})/2, 0)$ and ∞, that the barycenter is the point $(3+i\sqrt{3}/(6), \sqrt{2}/3)$ and that the standard horoball containing ∞ is bounded by the plane $t = 1$. Thus we have $k_3(\infty) = \log\sqrt{3}/2$.

The components of the pre-image of H in $\mathbb{H}^3$ form a horoball packing to which [7, Theorem 4] applies, and a Ford domain Z for one of the horoballs is a Voronoĭ polyhedron for the packing. The theorem states that $\operatorname{vol} H/\operatorname{vol} Z \le d(\infty)$. However, the proof indicated actually shows that $\operatorname{vol} H/\operatorname{vol}(Z \cap N(k_3(\infty), H) \le d(\infty)$. Since $k_3(\infty) = \log\sqrt{3}/2$, this is equivalent to the statement that $\operatorname{vol} H/\operatorname{vol} N \le d(\infty)$. □

9 The main estimate

This section contains the proof of Theorem 1.3, which was stated in the Introduction.

Lemma 9.1 *Let M be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp. Suppose that there is an infinite sequence of closed manifolds $(M_j)_{j\in\mathbb{N}}$, obtained by distinct Dehn fillings of the compact core of M, such that $\pi_1(M_j)$ is 2–free for every $j \ge 0$. Then $\log 3$ is a Margulis number for M.*

Proof According to Theorem 7.1, we may assume, after removing finitely many terms from the sequence (M_j), that the M_j are all hyperbolic. According to 2.7, the number $\log 3$ is a Margulis number for each M_j, and it therefore follows from Proposition 7.2 that $\log 3$ is a Margulis number for M. □

Proof of Theorem 1.3 Let H denote the maximal standard neighborhood of the cusp of M. If $\operatorname{vol} H \geq 2\pi$ the conclusion is obvious, and so we shall assume $\operatorname{vol} H < 2\pi$. According to Theorem 7.3 we may fix a real number β with $1 < \beta < 2$ and a point $Q \in M_{\mathrm{thick}}(\log 3)$ such that

(1) $\operatorname{vol} H \geq \pi\beta$, and

(2) $\operatorname{dist}(Q, \bar{H}) \geq -(\log(\beta - 1))/2$.

Let us set
$$N = N(\log \sqrt{\tfrac{3}{2}}, \bar{H}) \subset M.$$

According to Proposition 8.3, we have $\operatorname{vol} H / \operatorname{vol} N \leq d(\infty)$, and by (1) it follows that
$$\operatorname{vol} N \geq \pi\beta/d(\infty).$$

It follows from (2) that
$$\operatorname{dist}(Q, \bar{N}) \geq -\frac{\log(\beta-1)}{2} - \frac{\log(3/2)}{2},$$
ie
$$\operatorname{dist}(Q, \bar{N}) \geq -\frac{1}{2}\log\left(\frac{3}{2}(\beta-1)\right).$$
Setting
$$r_\beta = -\frac{1}{2}\log\left(\frac{3}{2}(\beta-1)\right)$$
and
$$N_1 = N(r_\beta, Q)$$
we deduce that $N \cap N_1 = \varnothing$, and hence that

$$\operatorname{vol} M \geq \frac{\pi\beta}{d(\infty)} + \operatorname{vol} N_1. \tag{9.1.1}$$

We shall now distinguish several cases depending on the size of r_β. To simplify the notation we shall set
$$C_1 = \exp\left(h_3\left(\frac{\log 3}{2}\right)\right) = 1.931\ldots$$
and
$$C_2 = \frac{B((\log 3)/2)}{d((\log 3)/2)} = 0.929\ldots$$

Case I $r_\beta \leq 0$.

In this case $N_1 = \varnothing$. However, we have $\beta \geq 5/3$, hence $\pi\beta/d(\infty) > 6$, so the conclusion follows from (9.1.1).

Case II $0 < r_\beta \le (\log 3)/2$.

In this case we have $5/3 > \beta \ge 11/9$. Since $Q \in M_{\text{thick}}(\log 3)$, the set N_1 is a hyperbolic ball of radius r_β. In the notation of 8.1 we have $\operatorname{vol} N_1 = B(r_\beta) = \pi(\sinh(2r_\beta) - 2r_\beta)$, and (9.1.1) becomes

$$\operatorname{vol} M \ge \pi\left(\frac{\beta}{d(\infty)} - \frac{3}{4}(\beta-1) - \frac{1}{3}(\beta-1)^{-1} + \log\left(\frac{3}{2}(\beta-1)\right)\right);$$

that is, we have $\operatorname{vol} M \ge f(\beta-1)$, where the function $f(x)$ is defined for $x > 0$ by

$$f(x) = \pi\left(\frac{x+1}{d(\infty)} - \frac{3}{4}x + \frac{1}{3}x^{-1} + \log\left(\frac{3}{2}x\right)\right).$$

We have $f(x) \to \infty$ as x approaches 0 or ∞, and $f'(x) = \pi q(1/x)$, where q is the quadratic polynomial

$$q(y) = -\frac{1}{3}y^2 + y + \left(\frac{1}{d_\infty} - \frac{3}{4}\right).$$

As the only zero of $q(y)$ with $y > 0$ is at $y = (3/2)(1 + \sqrt{4/(3d_\infty)})$, the least value of $f(x)$ on $(0, \infty)$ is

$$f\left(\frac{2}{3(1 + \sqrt{4/(3d_\infty)})}\right) > 5.06.$$

In particular it follows that $\operatorname{vol} M > 5.06$ in this case.

Case III $(\log 3)/2 < r_\beta \le h_3((\log 3)/2)$.

In this case $11/9 > \beta \ge 1 + \frac{2}{3}C_1^{-2} = 1.178\ldots$. We have $N_1 \supset N((\log 3)/2, Q)$ and, since $Q \in M_{\text{thick}}(\log 3)$, the set $N((\log 3)/2, Q)$ is a hyperbolic ball of radius $(\log 3)/2$. Hence

$$\operatorname{vol} N_1 \ge B((\log 3)/2) = 0.737\ldots$$

We also have $\beta \ge 1.178$, and so (9.1.1) gives

$$\operatorname{vol} M \ge \frac{1.178\pi}{d(\infty)} + .737\ldots. > 5.07.$$

Case IV $r_\beta > h_3((\log 3)/2)$.

In this case we have $\beta < 1 + \frac{2}{3}C_1^{-2}$. We set

$$s_\beta = \frac{1}{2}(r_\beta - h_3((\log 3)/2).$$

Lemma 9.1 guarantees that $\log 3$ is a Margulis number for M. Hence some component X of the $(\log 3)$–thin part of $M_{\mathrm{thin}}(\log 3)$ is a standard neighborhood of the cusp of M. Since H is the unique maximal standard neighborhood of the cusp, we have $X \subset H$. It follows that the frontier Y of H is contained in $M_{\mathrm{thick}}(\log 3)$.

Since $\log 3$ is a Margulis number for M, the set $M_{\mathrm{thick}}(\log 3)$ is connected (see 2.5). Define a continuous function $\Delta\colon M_{\mathrm{thick}}(\log 3) \to \mathbb{R}$ by $\Delta(x) = \mathrm{dist}_M(x, Q)$. Then $\Delta(Q) = 0$. For any point $y \in Y \subset M_{\mathrm{thick}}(\log 3)$ we have $\Delta(y) \geq \mathrm{dist}(Q, H) \geq -\log(\beta - 1)/2 > r_\beta$. By the connectedness of $M_{\mathrm{thick}}(\log 3)$ we have $[0, r_\beta] \subset \Delta(M_{\mathrm{thick}}(\log 3))$. Since the definition of s_β and the hypothesis of Case IV give that $0 < 2s_\beta + h_3((\log 3)/2) < r_\beta$, we may choose a point $R \in M_{\mathrm{thick}}(\log 3)$ such that $\Delta(R) = 2s_\beta + h_3((\log 3)/2)$, ie $\mathrm{dist}(R, Q) = 2s_\beta + h_3((\log 3)/2)$. This implies on the one hand that $N_2 = N(2s_\beta, R)$ is disjoint from $W = N(h_3((\log 3)/2), Q)$, and on the other hand that

$$N_2 \subset N(2s_\beta + h_3((\log 3)/2), Q) = N(r_\beta, Q) = N_1.$$

Hence

$$\text{(9.1.2)} \qquad \mathrm{vol}\, N_1 \geq \mathrm{vol}\, N_2 + \mathrm{vol}\, W.$$

According to Proposition 8.2, we have

$$\text{(9.1.3)} \qquad \mathrm{vol}\, W \geq C_2.$$

Combining (9.1.1), (9.1.2) and (9.1.3), we obtain

$$\text{(9.1.4)} \qquad \mathrm{vol}\, M > C_2 + \frac{\pi\beta}{d(\infty)} + \mathrm{vol}\, N_2.$$

We have two sub-cases.

Case IV(A) $h_3((\log 3)/2) < r_\beta \leq h_3((\log 3)/2) + \log 3$.

In this sub-case we have $1 + \frac{2}{3}C_1^{-2} > \beta \geq 1 + \frac{2}{9}C_1^{-2} = 1.05\ldots$. Also, since $r_\beta \leq h_3((\log 3)/2) + \log 3$, we have $s_\beta \leq (\log 3)/2$.

Since $R \in M_{\mathrm{thick}}(\log 3)$, the set $N_2 = N(s_\beta, R)$ contains a hyperbolic ball of radius s_β. Hence $\mathrm{vol}\, N_2 \geq B(s_\beta)$, and (9.1.4) gives

$$\text{(9.1.5)} \qquad \mathrm{vol}\, M > C_2 + \frac{\pi\beta}{d(\infty)} + B(s_\beta).$$

Recalling that $B(s_\beta) = \pi(\sinh(2s_\beta) - 2s_\beta)$, that $s_\beta = (1/2)(r_\beta - h_3((\log 3)/2)$, and that $r_\beta = -(1/2)\log\big(\frac{3}{2}(\beta-1)\big)$, we obtain the expression

$$B(s_\beta) = \frac{\pi}{2}\left(C_1^{-1}\big(\frac{3}{2}(\beta-1)\big)^{-1/2} - C_1\big(\frac{3}{2}(\beta-1)\big)^{1/2} + \log\big(\frac{3}{2}(\beta-1)\big) + 2\log C_1\right).$$

We may therefore rewrite (9.1.5) as $\operatorname{vol} M \geq g(\beta-1)$, where $g(x)$ is defined for $x > 0$ by

$$g(x) = \frac{\pi}{2}\left(C_1^{-1}\big(\frac{3}{2}x\big)^{-1/2} - C_1\big(\frac{3}{2}x\big)^{1/2} + \log\big(\frac{3}{2}x\big) + 2\log C_1 + \frac{2}{d(\infty)}(x+1)\right) + C_2.$$

Note that for $x > 0$ we have

$$g'(x) = \frac{\pi}{2}c\left(\sqrt{\frac{3}{2x}}\right),$$

where c denotes the cubic polynomial

$$c(y) = -\frac{3}{4}C_1^{-1}y^3 + \frac{2}{3}y^2 - \frac{3}{4}C_1 y + \frac{2}{d_\infty}.$$

Writing
$$c(y) = -\frac{3}{4}C_1^{-1}y^2(y-A) - \frac{3}{4}C_1(y-B),$$

where $A = (8/9)C_1 = 1.7\ldots$ and $B = 8/(3d_\infty C_1) = 1.6\ldots$, makes it evident that $c(y) < 0$ for $y > 2$. Hence $g(x)$ decreases monotonically on the interval $0 < x < 3/8$. Since we are in Case IV we have $\beta - 1 < (2/3)C_1^{-2} = 0.178\ldots < 3/8$ and hence

$$\operatorname{vol} M \geq g(\beta-1) > g\big(\frac{2}{3}C_1^{-2}\big) = \frac{\pi}{d(\infty)}(1 + \frac{2}{3}C_1^{-2}) + C_2 = 5.26\ldots$$

Case IV(B) $r_\beta > h_3((\log 3)/2) + \log 3$.

In this sub-case we have $\beta < 1 + \frac{2}{9}C_1^{-2}$, and $s_\beta > (\log 3)/2$. Since $R \in M_{\text{thick}}(\log 3)$, the set $N_2 = N(s_\beta, R)$ contains a hyperbolic ball of radius $(\log 3)/2$. Hence $\operatorname{vol} N_2 \geq B((\log 3)/2)$, and (9.1.4) gives

$$\operatorname{vol} M > C_2 + \frac{\pi\beta}{d(\infty)} + B((\log 3)/2) > C_2 + \frac{\pi}{d(\infty)} + B((\log 3)/2) = 5.34\ldots.$$

Thus the desired lower bound is established in all cases. □

10 Relaxing the restriction on surface subgroups

In this section we prove Proposition 10.1, which was discussed in the introduction to the paper.

Proposition 10.1 *Let M be a complete, finite-volume, orientable hyperbolic manifold having exactly one cusp, such that $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geq 7$. Then either*

(a) $\operatorname{vol} M > 5.06$, *or*

(b) *M contains a genus–2 connected incompressible surface.*

Proof Let us fix a basis $\{\lambda, \mu\}$ of $H_1(\partial N;\mathbb{Z})$ such that λ lies in the kernel of the natural homomorphism $H_1(\partial N;\mathbb{Z}) \to H_1(N;\mathbb{Z}_2)$. For every integer n, let α_n denote the slope defined by the primitive class $\lambda + 2n\mu \in H_1(\partial N;\mathbb{Z})$, and let M_n denote the closed manifold obtained from N by the Dehn filling corresponding to α_n. If $i\colon N \to M_n$ denotes the inclusion homomorphism, then $i_*\colon H_1(N;\mathbb{Z}_2) \to H_1(M_n;\mathbb{Z}_2)$ is an isomorphism for every n. In particular we have $\dim_{\mathbb{Z}_2} H_1(M_n;\mathbb{Z}_2) \geq 7$.

We first consider the case in which there is an infinite sequence of distinct integers $(n_j)_{j\in\mathbb{N}}$ such that $\pi_1(M_{n_j})$ is 3–free for every $j \geq 0$. In this case it follows immediately from Theorem 1.3 that $\operatorname{vol} M > 5.06$. This is conclusion (a) of the present proposition.

We shall therefore assume for the rest of the proof that there is an integer $n_0 > 0$ such that $\pi_1(M_n)$ fails to be 3–free whenever $|n| \geq n_0$. Since $\dim_{\mathbb{Z}_2} H_1(M_n;\mathbb{Z}_2)$ is in particular ≥ 5, it follows from Proposition 4.3 that $\pi_1(M_n)$ has a subgroup isomorphic to a genus–2 surface group whenever $|n| \geq n_0$. Since $\dim_{\mathbb{Z}_2} H_1(M_n;\mathbb{Z}_2) \geq 7$, it then follows from the case $g = 2$ of [2, Theorem 8.13] that for each n with $|n| \geq n_0$, the manifold M_n contains a connected incompressible closed surface of genus 2.

The main theorem of [13] implies that there are only finitely many boundary slopes (see Remark 4.5) in ∂N. Hence there is an integer $n_1 \geq n_0$ such that α_n is not a boundary slope. We may write $M_{n_1} = N \cup T$, where T is a solid torus with $T \cap N = \partial T = \partial N$, and the meridian disks of T represent the slope α_{n_1}.

Let $Y \subset M_{n_1}$ be a surface such that

(1) Y is orientable, incompressible, connected and closed, and $\operatorname{genus}(Y) = 2$.

We may choose Y so that

(2) each component of $Y \cap T$ is a meridian disk in T.

We let m denote the number of components of $Y \cap T$, and we suppose Y to be chosen, among all surfaces in M for which (1) and (2) hold, so that m is as small as possible.

Suppose that $m > 0$. Set $F = Y \cap N$, so that F is a compact orientable surface of genus 2, properly embedded in N, with $\partial F \neq \varnothing$. Since ∂N is a torus, F cannot be boundary-parallel. If the inclusion homomorphism $\pi_1(F) \to \pi_1(N)$ were injective then α_{n_1} would be a boundary slope, in contradiction to our choice of n_1. On the other hand, if $\pi_1(F) \to \pi_1(N)$ has a nontrivial kernel, it follows from Dehn's lemma and the loop theorem that there is a disk $D_1 \subset N$ such that $D_1 \cap F = \partial D_1$, and such that ∂D_1 does not bound a disk in F. However, since Y is incompressible in M_{n_1}, the curve ∂D_1 does bound a disk $D \subset Y$, which must contain at least one component of $Y \cap T$. If we set $Y_1 = (Y - D) \cup D_1$, then (1) and (2) still hold when Y is replaced by Y_1, but $Y_1 \cap T$ has at most $m - 1$ components. This contradicts the minimality of m. Hence we must have $m = 0$. Thus $Y \subset N$ is a genus–2 connected incompressible surface, and conclusion (b) of the proposition holds. □

As we mentioned in the Introduction, in [10] it will be shown that the hypothesis of Proposition 10.1 implies conclusion (a). The proof of this stronger result uses Proposition 10.1.

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Received: 24 August 2007 Revised: 3 February 2007