Algebraic & Geometric Topology 8 (2008) 729–756

sl(2) tangle homology with a parameter and singular cobordisms

CARMEN LIVIA CAPRAU

We construct a bigraded cohomology theory which depends on one parameter a, and whose graded Euler characteristic is the quantum $\mathfrak{sl}(2)$ link invariant. We follow Bar-Natan's approach to tangles on one side, and Khovanov's $\mathfrak{sl}(3)$ theory for foams on the other side. Our theory is properly functorial under tangle cobordisms, and a version of the Khovanov $\mathfrak{sl}(2)$ invariant (or Lee's modification of it) corresponds to $a = 0$ (or $a = 1$). In particular, the construction naturally resolves the sign ambiguity in the functoriality of Khovanov's $\mathfrak{sl}(2)$ theory.

57M27, 57M25; 18G60

1 Introduction

Khovanov classified in [12] all possible Frobenius systems of rank two which give rise to link homologies via his construction in [9], and showed that there is a universal one corresponding to $\mathbb{Z}[X, a, b]/(X^2 - bX - a)$ (refer to Kadison [7] for a definition of Frobenius systems). The original Khovanov homology categorifying the unnormalized Jones polynomial corresponds to the choice $a = b = 0$, while Lee's [13] variant of it corresponds to $a = 1, b = 0$. Bar-Natan [1] extended the Khovanov homology to tangles by using a setup with cobordisms modulo relations, which in particular leads to an improvement in computational efficiency (see [2]). It was independently proved by Bar-Natan [1], Jacobsson [6] and Khovanov [11] that the Khovanov homology is functorial for link cobordisms, in the sense that given a link cobordism $S \in \mathbb{R}^3 \times [0, 1]$ between links L_1 and L_2, there is an induced map between their Khovanov homologies $\mathrm{Kh}(L_1)$ and $\mathrm{Kh}(L_2)$, well-defined up to overall minus sign, under ambient isotopy of S relative to ∂S.

In [10], Khovanov showed how to construct a link homology theory whose graded Euler characteristic is the $\mathfrak{sl}(3)$ link invariant. Instead of $(1+1)$–dimensional cobordisms he uses *webs* and *foams* modulo a finite set of relations. Following his approach, Mackaay and Vaz [14] defined the universal $sl(3)$ link homology which depends on 3 parameters, and their theory arises from a Frobenius system corresponding to $\mathbb{Z}[X, a, b, c]/(X^3 - aX^2 - bX - c)$.

Published: 25 May 2008 DOI: 10.2140/agt.2008.8.729

We construct a bigraded $\mathfrak{sl}(2)$ cohomology theory for oriented tangles over $\mathbb{Z}[i][a]$, where a is a formal variable and i is the primitive fourth root of unity. The construction follows closely the work by Bar-Natan [1], with the main difference that we keep track of orientations. More precisely, we start from the oriented state model for the Jones polynomial and work with webs and foams modulo local relations, much as it is done in [10] and [14], but instead of using planar trivalent graphs we consider bivalent graphs. Consequently, our foams are 2–dimensional CW–complexes with singularities where two 2–cells are joining, as opposed to three 2–cells.

Restricting to the case of links, we obtain a categorification of the $\mathfrak{sl}(2)$ link invariant, and a geometric approach to the link cohomology theory corresponding to a Frobenius system given by $\mathbb{Z}[i][X,a]/(X^2-a)$. The advantage of working with the oriented state model for the Jones polynomial (as opposed to the classical approach) on one side, and of considering the fourth root of unity i in the ground ring on the other side, is that we obtain a cohomology theory that satisfies functoriality in the proper way, with no sign indeterminacy.

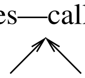
Adding the relation $a=0$ (or $a=1$), our construction yields a cohomology theory that is isomorphic to a version of the Khovanov homology theory (or Lee's variant of it). In particular, it naturally resolves the sign ambiguity in functoriality property of Khovanov's $\mathfrak{sl}(2)$ theory. For each oriented link L, there is an isomorphism

$$\mathcal{H}^{i,j}(L) \cong \mathrm{Kh}^{i,-j}(L^!) \otimes_{\mathbb{Z}} \mathbb{Z}[i]$$

where $L^!$ is the mirror image of L, and Kh is the homology theory defined in [9].

We note that our construction and main result for $a=0$ case are close to that by Clark, Morrison and Walker in [5], and that the two pieces of work were done independently. However, we borrowed from [5] the excellent idea of working with "homotopically isolated" objects when checking the functoriality property of our invariant, which makes the calculations much easier.

2 Webs and foams

Webs Let B be a finite set of points on a circle, such as the boundary ∂T of a tangle. A *web* with boundary B is a planar graph Γ properly embedded in a disk $\mathcal{D}^2$, with bivalent vertices—called *singular points*—near which the two arcs are oriented as follows: either  or . We also allow webs without vertices, thus closed oriented loops. Each singular point has a neighborhood homeomorphic to the letter V, and there is an ordering of the arcs corresponding to it; namely, the arc that goes in or goes out from the right is called *the preferred arc* of that singular point. Notice that this definition corresponds to the case when the arcs are oriented from south to north; otherwise, the word “right” above should be replaced by “left”. Two adjacent singular points are called *of the same type* if their common arc is either the preferred one or not. The middle arc in the picture is the preferred arc for both vertices, while in the drawing , the preferred arcs are those in the left and right. Otherwise, the singular points are called *of different type* (like these two). A *closed web* is a web with empty boundary.

There is a unique way to assign a Laurent polynomial $\langle\Gamma\rangle \in \mathbb{Z}[q, q^{-1}]$ to a closed web Γ, so that it satisfies the skein relations explained in Figure 1. Notice that this gives the oriented state model for the Jones polynomial, with $t^{1/2} = -q$ (see Kauffman [8, Section 6]). Note that for a k–component closed web, $\langle\Gamma\rangle = (q+q^{-1})^k$.

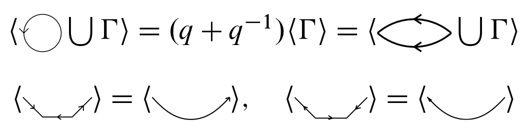


Figure 1: Web skein relations

Let L be a link in S^3. We fix a generic planar diagram D of L and form the sum over all closed web diagrams Γ which are obtained by replacing each crossing of D by one of the two pictures on the right side of Figure 2. We define $\langle D\rangle = \sum_\Gamma \pm q^{\alpha(\Gamma)}\langle\Gamma\rangle$, where the weights $\alpha(\Gamma)$ are determined by the rules given in Figure 2. If D_1 and D_2 are related by a Reidemeister move, then $\langle D_1\rangle = \langle D_2\rangle$. Consequently, we obtain an invariant of L, denoted by $P_2(L) := \langle D\rangle$.

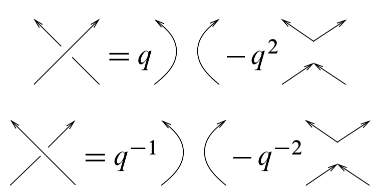
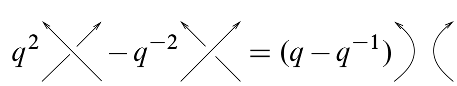


Figure 2: Decomposition of crossings

Excluding the rightmost terms in Figure 2, we obtain the following skein relation:



Therefore, our $P_2(L)$ is nothing else but the $\mathfrak{sl}(2)$ link invariant, or equivalently, the unnormalized Jones polynomial. Its categorification was introduce by Khovanov in [9].

Foams Let Γ_0 and Γ_1 be two webs with boundary points B. A *foam* is an abstract cobordism from Γ_0 to Γ_1, regarded up to boundary-preserving isotopies, which is a piecewise oriented 2–dimensional manifold S with boundary $\partial S = -\Gamma_1 \cup \Gamma_0 \cup B \times [0,1]$ and corners $B \times \{0\} \cup B \times \{1\}$, where the manifold $-\Gamma_1$ is Γ_1 with the opposite orientation. A cobordism between closed webs Γ_0 and Γ_1 is embedded in $\mathbb{R}^2 \times [0,1]$ and its boundary lies entirely in $\mathbb{R}^2 \times \{0,1\}$. We read foams as morphisms from bottom to top by convention, and we compose them by placing one on top the other.

Foams have *singular arcs* or *singular circles* where orientation disagrees. A point on a singular arc has a neighborhood homeomorphic to the product of the letter V and an interval. The facets of a foam are compatibly oriented near each singular arc or circle, and this compatibility induces orientations on singular arcs. Specifically, the orientation of singular arcs and circles is as in Figure 3, which shows examples of basic foams. For each singular arc, there is a preferred facet that it bounds, and each singular arc connects only singular points of the same type. Finally, if the preferred facet is at the left of the singular arc—where the concept of "left" and "right" is given by the orientation of the singular arc—we represent that arc by a continuous red curve. Otherwise, a dashed red curve is used. We remark that this notion of preferred side coincides with an ordering of the facets meeting at a singular arc (circle), and that it is a local property.

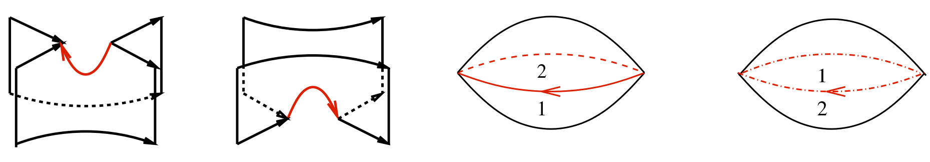


Figure 3: Singular saddles and ufo-foams

A cobordism from the empty web to itself gives rise to a foam with empty boundary, called *closed foam*. The last two drawings in Figure 3 show examples of closed foams, called *ufo*–foams. Notice the different ordering of their facets; in what follows we will always consider *ufo*–foams with the lower hemisphere as the preferred facet.

Foams can have dots that are allowed to move freely along the facet they belong to, but can't cross singular arcs or singular circles.

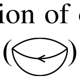
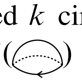
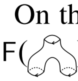
If B is a finite set of points on a circle, we denote by *Foams*(B) the category whose objects are webs with boundary B and whose morphisms are foams between such webs. If $B = \varnothing$, the corresponding category is denoted by *Foams*$(\varnothing)$.

2.1 A (1 + 1)–dimensional TQFT with dots

Let i be the primitive fourth root of unity and let $\mathbb{Z}[i][a]$ be the graded ring of polynomials in indeterminate a and Gaussian integer coefficients, with $\deg(1) = 0 = \deg(i)$ and $\deg(a) = 4$. Consider the commutative Frobenius ring $\mathcal{A} = \mathbb{Z}[i][a, X]/(X^2 - a)$ with trace map $\epsilon\colon \mathcal{A} \to \mathbb{Z}[i][a]$, $\epsilon(1) = 0$, $\epsilon(X) = 1$. Multiplication $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and comultiplication $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ are defined by the rules

$$\begin{cases} m(1 \otimes X) = m(X \otimes 1) = X \\ m(1 \otimes 1) = 1, \quad (X \otimes X) = a, \end{cases} \qquad \begin{cases} \Delta(1) = 1 \otimes X + X \otimes 1 \\ \Delta(X) = X \otimes X + a1 \otimes 1. \end{cases}$$

We make $\mathcal{A}$ graded by setting $\deg(1) = -1$ and $\deg(X) = 1$. The trace ϵ and unit $\iota\colon \mathbb{Z}[i][a] \to \mathcal{A}$, $\iota(1) = 1$ are maps of degree -1, while multiplication and comultiplication are maps of degree 1.

The Frobenius algebra $\mathcal{A}$ gives rise to a well-defined 2–dimensional TQFT, denoted here by F, from the category of oriented $(1+1)$–dimensional cobordisms to the category of graded $\mathbb{Z}[i][a]$–modules. F assigns $\mathbb{Z}[i][a]$ to the empty 1-manifold and $\mathcal{A}^{\otimes k}$ to the disjoint union of oriented k circles. On the generating morphisms, the functor is defined by: $\mathsf{F}($$) = \iota$, $\mathsf{F}($$) = \epsilon$, $\mathsf{F}($$) = m$ and $\mathsf{F}($$) = \Delta$.

A dot on a surface denotes multiplication by X endomorphism of $\mathcal{A}$. For example, the functor F applied to the 'cup' with a dot produces the map $\mathbb{Z}[i][a] \to \mathcal{A}$ which takes 1 to X. A twice dotted surface is the multiplication by $X^2 = a$ endomorphism of $\mathcal{A}$. Therefore, F(twice dotted surface) = $a\,\mathsf{F}$(surface with no dots). Dots can move freely on a connected component of an oriented surface. A singular cylinder (a cylinder with singular circles) may be regarded as the endomorphism defined in Figure 4.

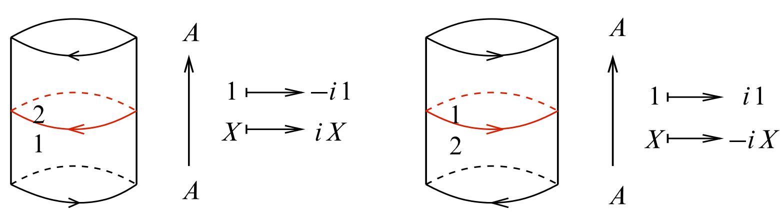
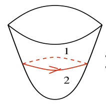
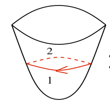


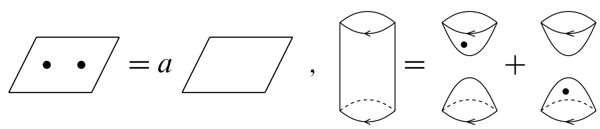
Figure 4: The meaning of singular arcs

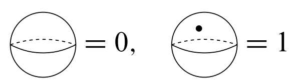
In particular,  $: \mathbb{Z}[i][a] \longrightarrow \mathcal{A}, 1 \to i$ and  $: \mathbb{Z}[i][a] \longrightarrow \mathcal{A}, 1 \to -i$.

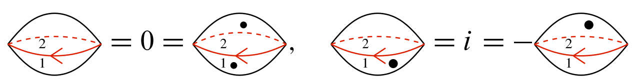
F extends to a functor from the category of dotted, singular 2–dimensional cobordisms to the category of graded $\mathbb{Z}[i][a]$–modules. The homomorphism $\mathsf{F}(S)$ associated with a cobordism S with d dots has degree given by the formula $\deg(S) = -\chi(S) + 2d$, where χ is the Euler characteristic of S. Note that F is degree-preserving.

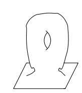
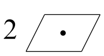
2.2 Local relations

We mod out the morphisms of the category *Foams* by the local relations ℓ = (2D, SF, S, UFO) below, and denote the corresponding quotient category by $\mathit{Foams}_{/\ell}$.

(2D)  (SF)

 (S)

 (UFO)

The surgery formula (SF) implies the genus reduction formula  $= 2$ . In particular we have  $= 2$,  $= 0$,  $= 8a$.

A closed foam S can be viewed as a morphism from the empty web to itself. By the relations ℓ, we assign to S an element $\mathcal{F}(S) \in \mathbb{Z}[i][a]$, called the *evaluation* of S. We view $\mathcal{F}$ as a functor from the category $\mathit{Foams}_{/\ell}(\varnothing)$ to the category of $\mathbb{Z}[i][a]$–modules.

Lemma 2.1 *The functors $\mathcal{F}$ and F behave similarly. In particular, F descends to a functor $\mathit{Foams}_{/\ell}(\varnothing) \to \mathbb{Z}[i][a]$–Mod.*

Proof We have already seen in Section 2.1 that F satisfies the (2D) relation. It only remains to show that F satisfies relations (S), (SF) and (UFO). That F satisfies the (SF) relation follows from $\mathrm{Id} = (\mathrm{m}(X) \circ \iota) \circ \epsilon + \iota \circ (\epsilon \circ \mathrm{m}(X))$, where $\mathrm{m}(X)$ stands for multiplication by X endomorphism of $\mathcal{A}$. The (S) relations follow from $\epsilon \circ \iota = 0$ and $\epsilon \circ \mathrm{m}(X) \circ \iota = 1$. A *ufo*–foam without dots is a cup with a clockwise oriented singular circle followed by a cap; we have: $1 \to i \to i\epsilon(1) = 0$. Moreover, the *ufo*–foam with a dot on each facet is a cup with a dot followed by a cylinder with a clockwise oriented singular circle and then followed by a cap with a dot. Composing these we obtain: $1 \xrightarrow{\mathrm{m}(X)\iota} X \longrightarrow i\,X \xrightarrow{\epsilon\,\mathrm{m}(X)} i\,a\epsilon(1) = 0$. The other two (UFO) relations are proved similarly. □

The proof of [10, Proposition 3] can be extended to our setting, to get the following lemma.

Lemma 2.2 *The set of local relations ℓ are consistent and determine uniquely the evaluation of every closed foam.*

Remark The evaluation of closed foams is multiplicative with respect to the disjoint union of closed foams: $\mathcal{F}(S_1 \cup S_2) = \mathcal{F}(S_1)\mathcal{F}(S_2)$. Moreover, if a closed foam S' is obtained from a closed foam S by reversing the ordering of the facets at a singular circle, then $\mathcal{F}(S') = -\mathcal{F}(S)$. The local relations ℓ imply a set of useful relations depicted in Figure 5, which establish the way of exchanging dots between two neighboring facets.

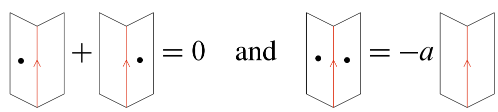
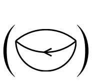
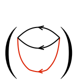
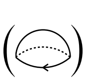
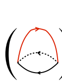


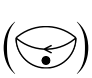
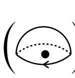
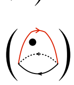
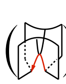
Figure 5: Exchanging dots between facets

Definition 2.3 For webs Γ, Γ', foams $S_i \in \mathrm{Hom}_{Foams_{/\ell}}(\Gamma, \Gamma')$ and $c_i \in \mathbb{Z}[i][a]$ we say that $\sum_i c_i S_i = 0$ if and only if $\sum_i c_i \mathcal{F}(V' S_i V) = 0$ holds for any foam $V \in \mathrm{Hom}_{Foams_{/\ell}}(\varnothing, \Gamma)$ and $V' \in \mathrm{Hom}_{Foams_{/\ell}}(\Gamma', \varnothing)$.

Definition 2.4 If $S \in Foams(B)$ is a foam with d dots we define the *grading* of S by $\deg(S) = -\chi(S) + \frac{1}{2}|B| + 2d$, where χ is the Euler characteristic and $|B|$ is the cardinality of B.

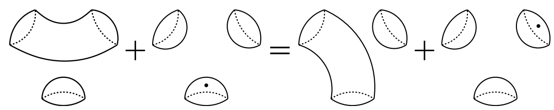
Note that the local relations ℓ are degree-preserving, and that given any composable foams S_1 and S_2, we have $\deg(S_1 S_2) = \deg(S_1) + \deg(S_2)$.

Example deg () = deg () = deg () = deg () = −1,

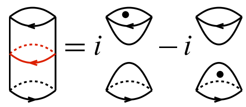
deg () = deg () = deg () = deg () = 1. Also deg () = 1.

With the previous definition at hand, the category *Foams* is graded, and so is $Foam_{/\ell}$.

Lemma 2.5 *The following relations hold in* $Foams_{/\ell}$*:*



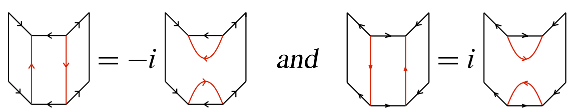
(3C)



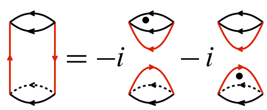
(RSC)

Proof These follow from relation (SF). Applying a surgery on each tube in (3C) we end up with the same combination of foams in both sides of the identity. Similarly, doing surgeries above and below the singular circle of the left-hand side in (RSC) and then using the (UFO) relations, we get the right-hand side of (RSC). □

Lemma 2.6 *The following relations hold in* $Foams_{/\ell}$ *:*



(CI)

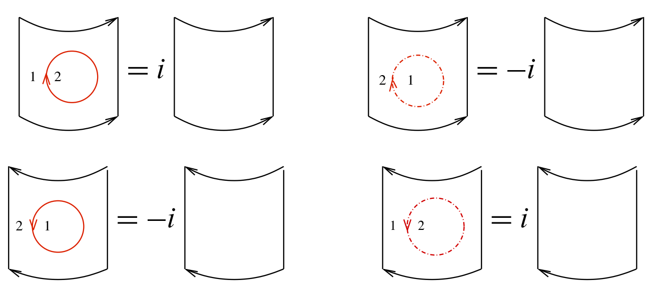


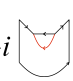
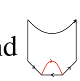
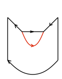
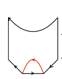
(CN)

where the dots in (CN) are on preferred facets, the facets in the back.

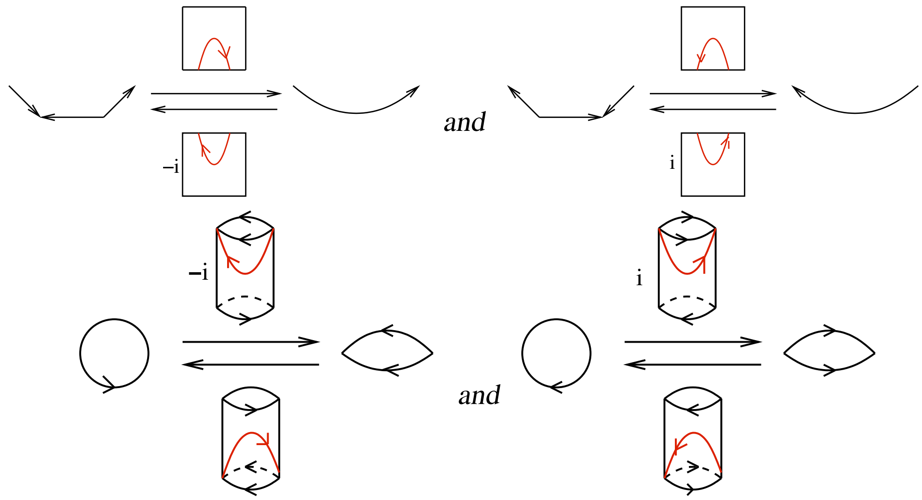
Proof (CI) and (CN) are proved similarly as in [10, Proposition 8], using our local relations ℓ. We let the details to the reader. □

Lemma 2.7 *The following relations hold in* $Foams_{/\ell}$*.*



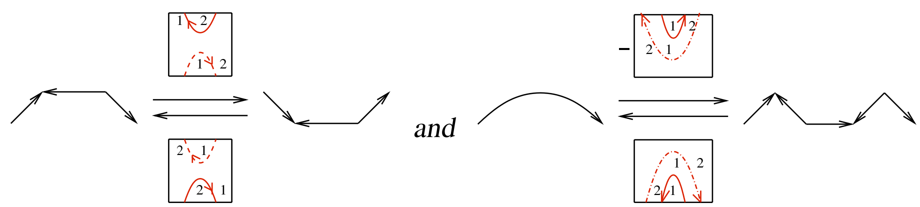
From the above identities and those in (CI), we obtain that $-i$  and  (as well as i  and ) are mutually inverse isomorphisms in $Foam_{/\ell}$.

Corollary 2.8 *The following isomorphisms hold in the category* $\mathit{Foam}_{/\ell}$.



The previous corollary says that we can 'remove' or 'create' pairs of adjacent singular points of the same type. From relations (CI) and Lemma 2.7 we also have the following corollary.

Corollary 2.9 *The following isomorphisms hold in the category* $\mathit{Foam}_{/\ell}$.



3 From tangles to formal complexes

We start with a generic tangle diagram T with boundary points B, and denote by n_+ and n_- the number of positive and negative crossings in T. We replace each crossing by one of its two resolutions (see Figure 2) and form a commutative n–dimensional cube of resolutions (where $n = n_+ + n_-$), similar to the one in Bar-Natan's work [1]. The chain objects are finite formal direct sums of webs and differentials are matrices of foams. The construction of $[T]$ is explained in Figure 6, where the numbers $-1, 0$ and 1 under resolutions indicate the cohomological degrees, and $\{m\}$ is the grading shift by m.

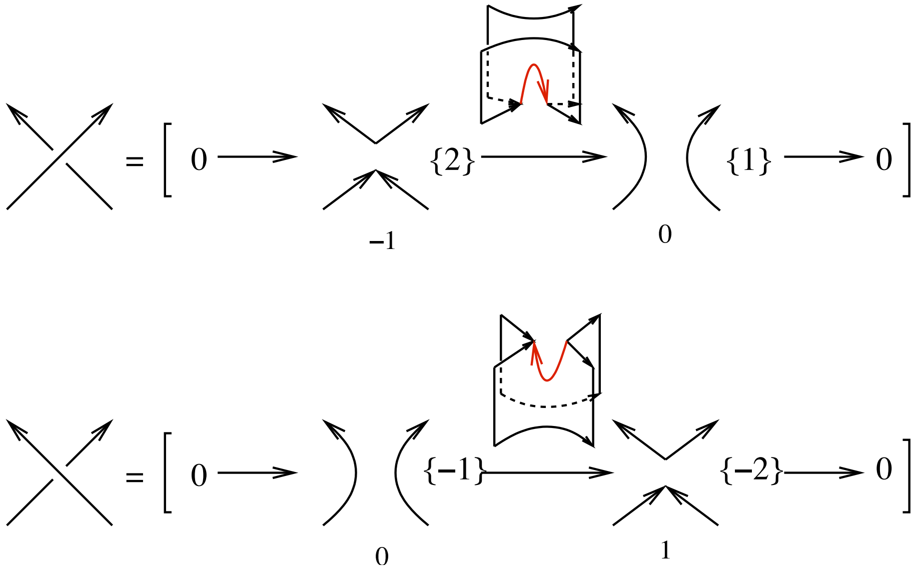


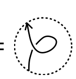
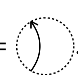
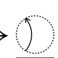
Figure 6: Constructing the chain complex [T]

We borrow some notations from Bar-Natan [1] and denote the category of complexes over *Foams* by Kom(Mat(*Foams*)) and its 'modulo homotopy' subcategory by $\mathrm{Kom}_{/h}$(Mat(*Foams*)). Moreover, we define *Kof*:=Kom(Mat($Foams_{/\ell}$)) and $Kof_{/h}$:= $\mathrm{Kom}_{/h}$(Mat($Foams_{/\ell}$)). We remark that the later ones are analogous to Bar-Natan's Kob = Kom(Mat($\mathrm{Cob}^3_{/l}$)) and $\mathrm{Kob}_{/h}$ = $\mathrm{Kom}_{/h}$ (Mat($\mathrm{Cob}^3_{/l}$)). All these categories are graded by degree.

3.1 Invariance under the Reidemeister moves

Theorem 3.1 *The chain complex $[T]$, regarded as an object in $Kof_{/h}$, is invariant under the Reidemeister moves.*

Proof We work diagrammatically and show the invariance under Reidemeister moves for the small tangles representing these moves, using the local relations ℓ and the identities from Lemma 2.6 and Lemma 2.7. The invariance under Reidemeister moves within larger tangles follows from Bar-Natan's discussion [1] on planar algebras and 'canopolies'.

Reidemeister 1a Consider diagrams $D_1 =$  and $D' =$ . We give the homotopy equivalence between complexes $[D'] = (0 \longrightarrow$  $\longrightarrow 0)$ and $[D_1] = (0 \longrightarrow$

$\{-1\} \longrightarrow$ $\{-2\} \longrightarrow 0)$ in Figure 7 (underlined objects are at the cohomological degree zero).

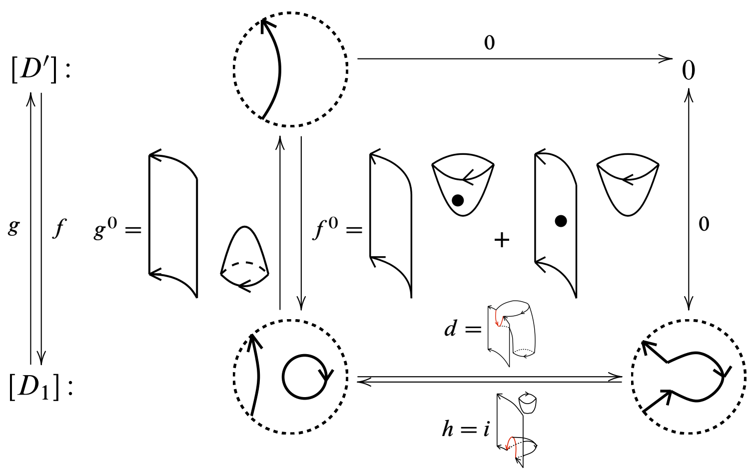


Figure 7: Invariance under Reidemeister 1a

f and g are morphisms, since $df^0 = 0$ (it follows from the first relation of Figure 5). Using relations (S) we have $g^0 f^0 = \mathrm{Id}()$. From (CI) we obtain $dh = \mathrm{Id}()$, while $f^0 g^0 + hd = \mathrm{Id}()$ follows from relation (SF) and Lemma 2.7. Thus $[D_1] \sim [D']$.

Reidemeister 1b Consider diagrams $D_2 =$ and $D' =$. The homotopy equivalence between complexes $[D_2] = (0 \longrightarrow$ $\{2\} \xrightarrow{d}$ $\{1\} \longrightarrow 0)$ and $[D'] = (0 \longrightarrow$ $\longrightarrow 0)$ is given in Figure 8.

$g^0 f^0 = \mathrm{Id}()$ follows from (S), and $g^0 d = 0$ is implied by the first identity in Figure 5. From (CI) we have $hd = \mathrm{Id}()$, while $f^0 g^0 + dh = \mathrm{Id}()$ follows from relation (SF), the first relation in Lemma 2.7 and identities given in Figure 5. Hence $[D_2] \sim [D']$.

Reidemeister 2a Consider diagrams $D =$ and $D' =$. We give the homotopy equivalence between $[D]$ and $[D']$ in Figure 9. The following identities hold.

- $d_1^{-1} + g_2^0 d_2^{-1} = 0, \quad d_1^0 + d_2^0 f_2^0 = 0$ (it uses isotopies).
- $h^0 d_2^{-1} = \mathrm{Id}(), \quad d_2^0 h^1 = \mathrm{Id}()$ (it uses isotopies).

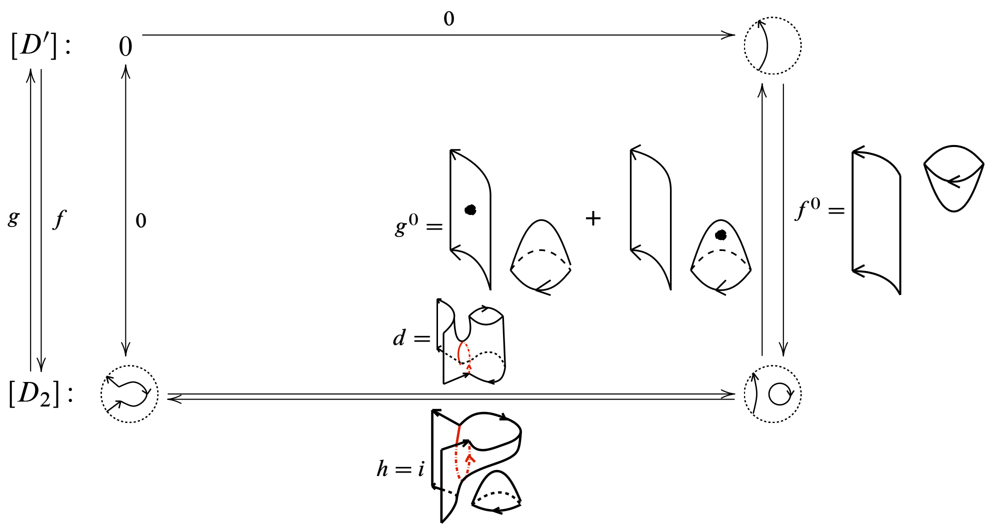


Figure 8: Invariance under Reidemeister 1b

- $f_2^0 g_2^0 + d_2^{-1} h^0 + h^1 d_2^0 = \mathrm{Id}()$, $g_2^0 f_2^0 = 0$ (by (CN) and (UFO), respectively).

Reidemeister 2b Consider diagrams $D =$ and $D' =$. Figure 10 explains the homotopy equivalence between the formal complexes $[D_2]$ and $[D']$. We have the following.

- $g_1^0 d_1^{-1} + g_2^0 d_2^{-1} = 0$, $d_1^0 f_1^0 + d_2^0 f_2^0 = 0$ (it uses isotopies).
- $h_2^0 d_2^{-1} = \mathrm{Id}()$, $d_2^0 h_2^1 = \mathrm{Id}()$, $f_1^0 g_1^0 = \mathrm{Id}()$ (by relations (CI)).
- $f_2^0 g_2^0 + d_2^{-1} h_2^0 + h_2^1 d_2^0 = \mathrm{Id}()$ (by relation (SF) and Lemma 2.7).
- $g_1^0 f_1^0 + g_2^0 f_2^0 = \mathrm{Id}()$ (by relations (S) and Lemma 2.7). Thus $[D] \sim [D']$.

The proof of invariance under Reidemeister 2 moves shows, in particular, that the morphisms $g\colon \to$ and $g\colon \to$ are strong deformation retracts.

We will prove the invariance under the Reidemeister 3 move using mapping cones and strong deformation retracts, in Bar-Natan's spirit [1], but before we proceed, we need to show a few moves involving tangles with singular points.

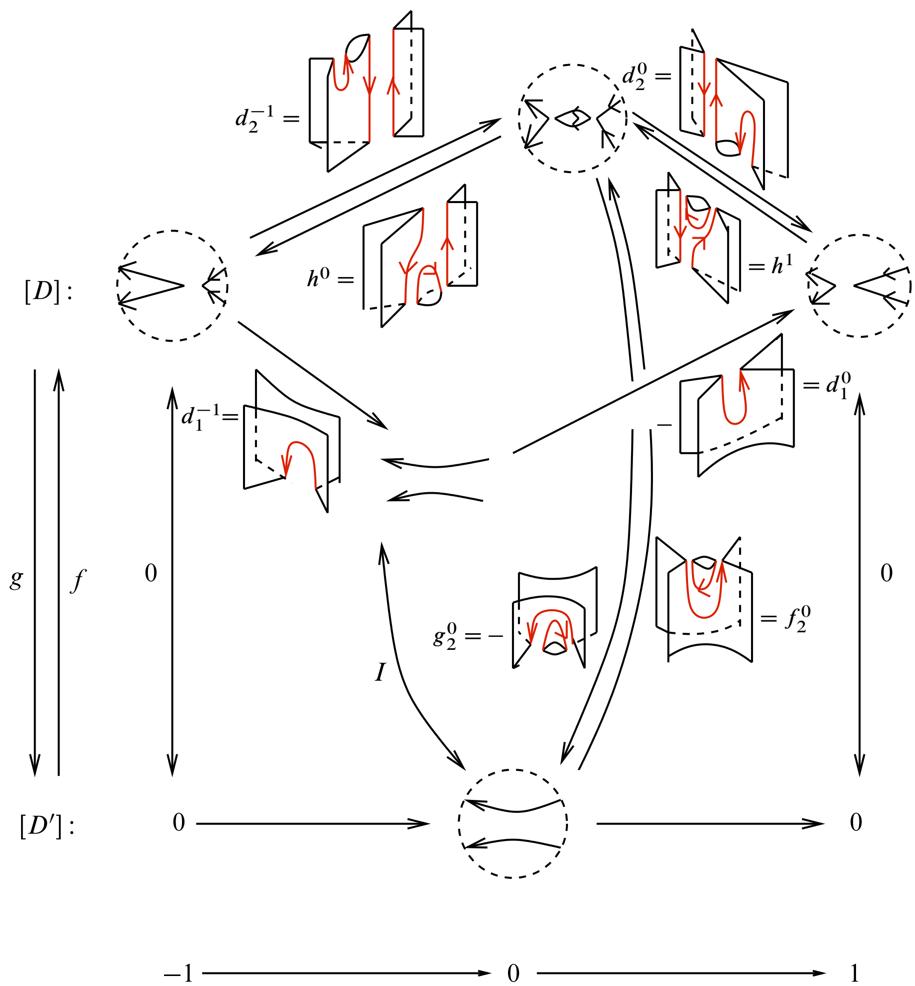
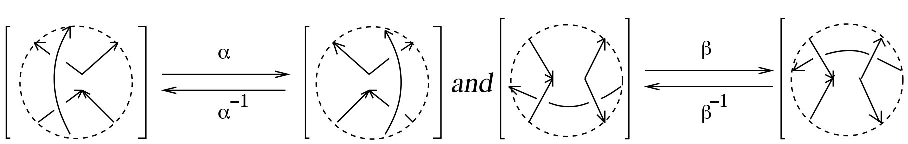


Figure 9: Invariance under Reidemeister 2a

Moves involving singular points

Lemma 3.2 *The following isomorphisms hold in the category $Kof_{/h}$.*



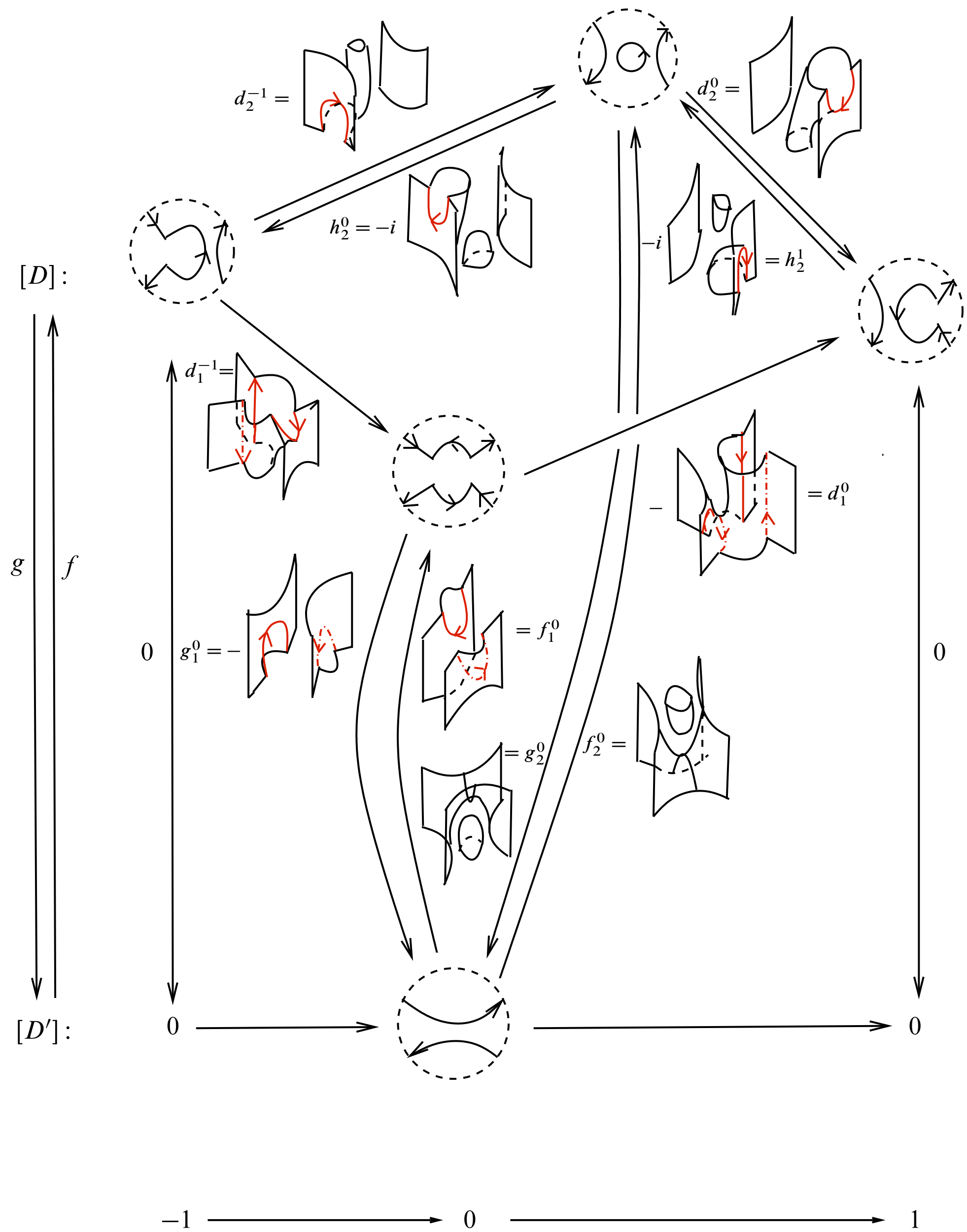


Figure 10: Invariance under Reidemeister 2b

Proof The isomorphisms of the corresponding chain complexes are given in Figure 11 and Figure 12. One can easily verify, using Corollary 2.8 and Corollary 2.9, that the maps α and α^{-1}, as well as β and β^{-1}, are mutually inverse isomorphisms. □

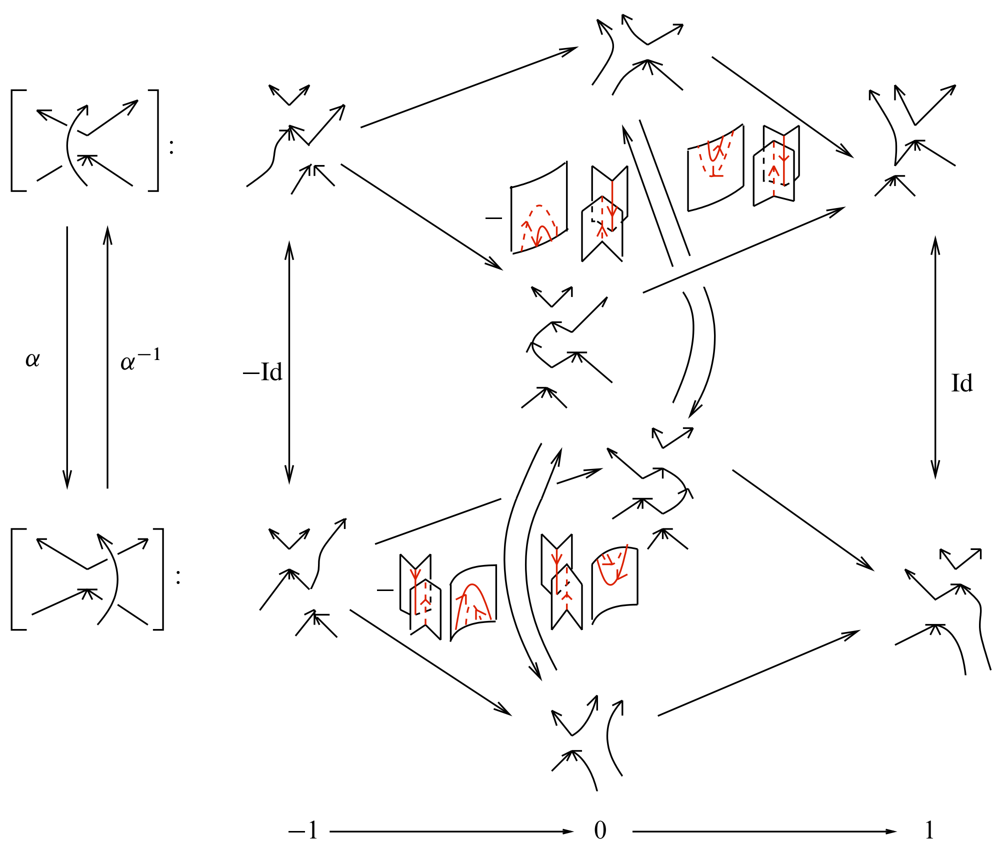
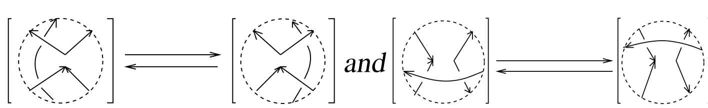


Figure 11: Isomorphism α

Lemma 3.3 *The associated chain complexes corresponding to the diagrams that differ in a circular region, as in the figure below, are isomorphic in the category $Kof_{/h}$.*



Given a morphism of complexes ψ, we denote its mapping cone by $\mathbf{M}(\psi)$.

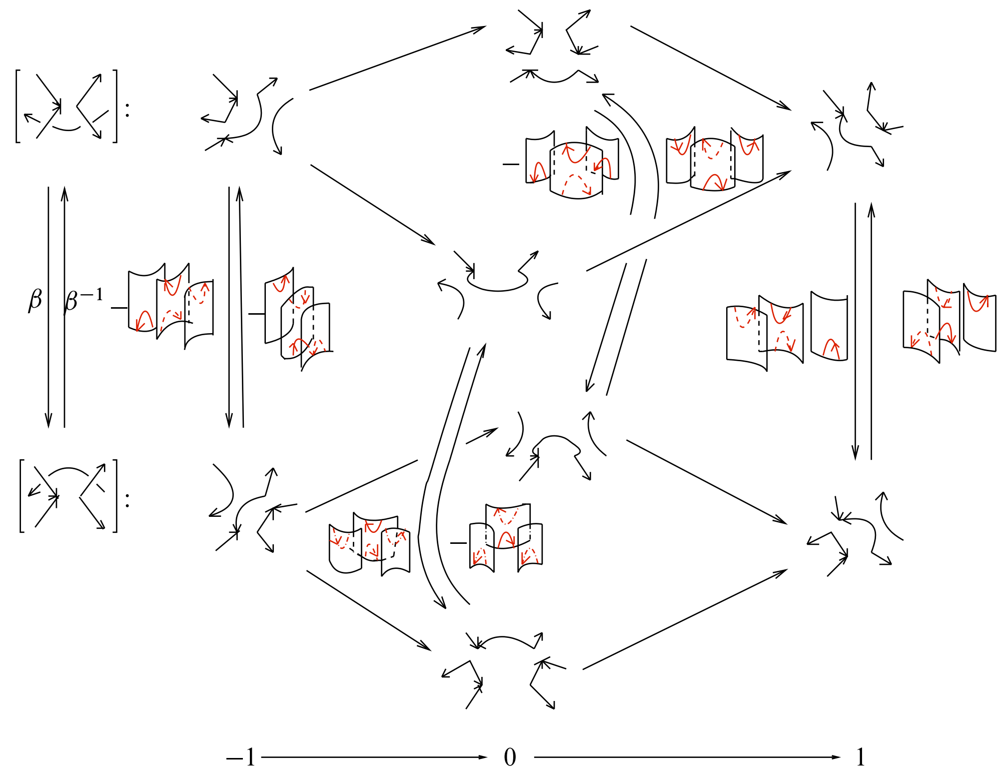
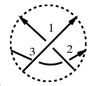
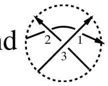
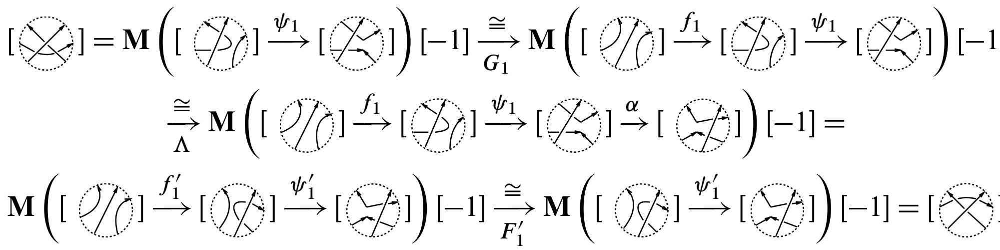


Figure 12: Isomorphism β

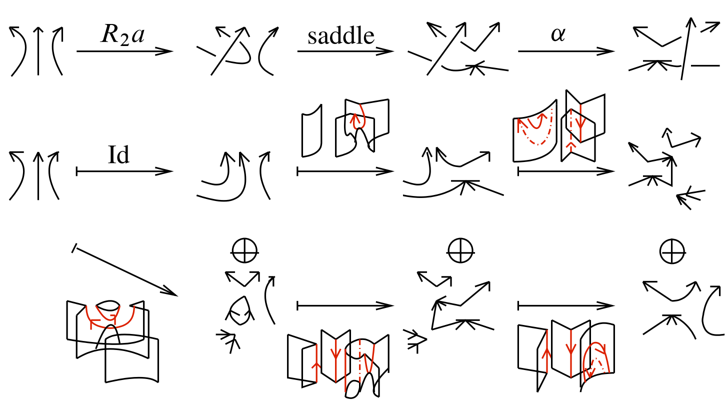
Lemma 3.4 *$[\;] = \mathbf{M}([\;] \longrightarrow [\;)\,(\;])$ and $[\;] = \mathbf{M}([\;)\,(\;] \longrightarrow [\;])[-1]$, where $[s]$ is the shift operator that shifts complexes s steps to the left; that is, if C^i is the chain object in the i th position of some complex C, then C^{s+i} is the chain object in the i th position of $C[s]$.*

Reidemeister 3 Each side of the Reidemeister move R3 can be realized as the mapping cone over the morphism switching between the two resolutions of a crossing. Using our moves with singular points, we are ready to apply the 'categorified Kauffman trick'. For this, we use that the cone construction is invariant under composition with isomorphisms. On the other hand, it was shown in [1] that the mapping cone construction is invariant, up to homotopy, under composition with strong deformation retracts, and under composition with inclusions in strong deformation retracts.

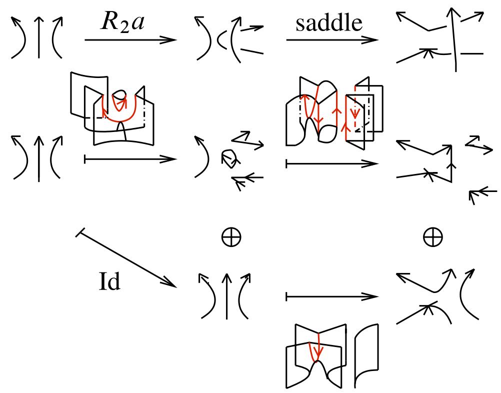
Consider the two tangles  and , and the mapping cones corresponding to crossings labeled 2. We have:

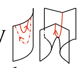
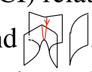


Morphisms f_1 and f'_1 are the inclusions in the strong deformation retracts g_1 and g'_1 from the proof of invariance under R2 moves, and $F'_1 = \begin{pmatrix} f'_1 & 0 \\ 0 & I \end{pmatrix}$, $G_1 = \begin{pmatrix} g_1 & 0 \\ \psi_1 h_1 & I \end{pmatrix}$ for some homotopy h_1. Moreover, $\Lambda = \begin{pmatrix} I & 0 \\ 0 & \alpha \end{pmatrix}$, where α is the isomorphism from Lemma 3.2. We are left to show that the third and fourth morphisms above are the same. These are analyzed below (starting with the third one).



and



Composing and applying the first (CI) relation, we obtain that both morphisms have the same components, namely  and . The other oriented versions of Reidemeister 3 are done similarly (see the author's work [3] for more details). □

3.2 Functoriality

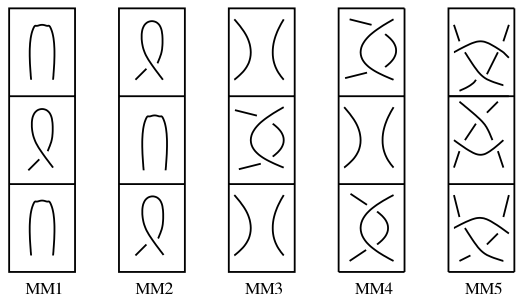
The categories *Foams* and $Foams_{/\ell}$ are examples of canopolies, as well as are the categories *Kof*, $Kof_{/h}$ and Cob^4 (for a definition of canopolies see [1]). Cob^4 is the category of cobordisms between oriented tangle diagrams, and is generated by the cobordisms corresponding to the Reidemeister moves and the Morse moves: birth or death of an oriented circle, and oriented saddles (regarded as sitting in $4D$).

Theorem 3.5 *There is a degree preserving canopoly morphisms $\mathcal{L}\colon Cob^4_{/i} \to Kof_{/h}$ from the canopoly of up to isotopy cobordisms in the 4–dimensional space between oriented tangle diagrams to the canopoly of formal complexes between them, up to homotopy.*

Proof We define a degree-preserving functor $\mathcal{L}\colon Cob^4 \to Kof_{/h}$ which associates to a tangle diagram T the formal chain complex $[T]$. To each Reidemeister move it associates the chain morphism inducing the homotopy equivalence between the complexes associated to the initial and final frame of the corresponding move (as constructed in the proof of Theorem 3.1). Moreover, the Morse moves induce morphisms between the one step corresponding formal complexes, interpreted in a skein-theoretic sense where each symbol represents a small neighborhood within a larger context.

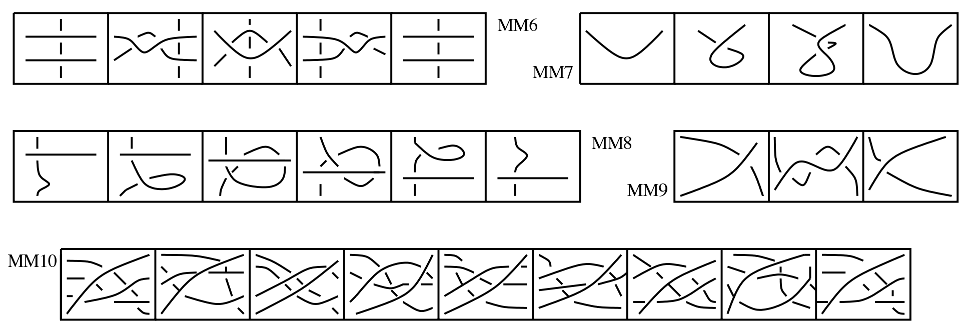
We show that $\mathcal{L}$ descends to a functor (denoted by the same symbol) $\mathcal{L}\colon Cob^4_{/i} \to Kof_{/h}$. For this, we need to verify that $\mathcal{L}$ respects the relations in the kernel of the map $Cob^4 \to Cob^4_{/i}$, namely the *movie moves* of Carter and Saito [4]. Specifically, we need to check that the morphisms of complexes corresponding to each movie are homotopic to identity morphism (for type I and II of movie moves) or to each other (for type III).

Type I: Reidemeister moves and inverses



These are equivalent to identity clips. The morphisms obtained by applying $\mathcal{L}$ are homotopic to identity (it follows from Theorem 3.1), since the induced maps between two successive frames are a homotopy equivalence and its inverse.

Type II: Reversible circular clips



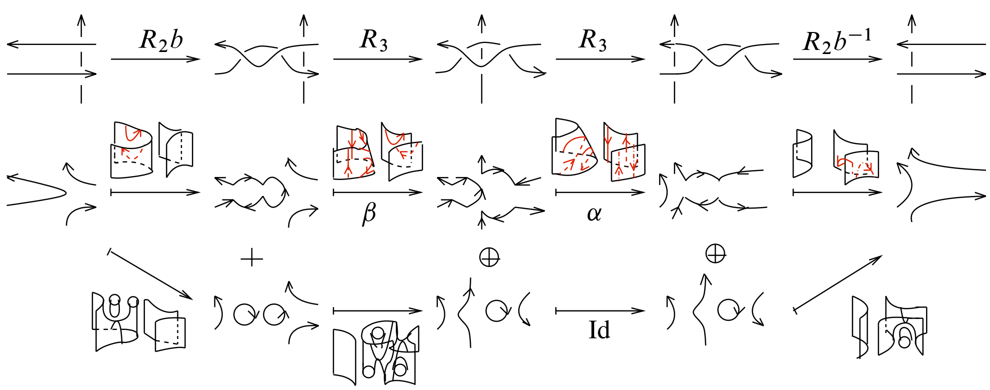
These circular clips have the same initial and final frames and are equivalent to identity. Our goal is to show that, at the level of chain complexes, the associated morphisms are homotopy equivalent to the identity morphism. We do this in two steps.

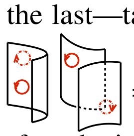
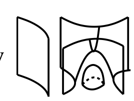
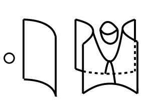
Lemma 3.6 *For each* $6 \le i \le 10$, $\mathcal{L}(MM_i)$ *is homotopy equivalent to* $i^k\,\mathrm{Id}$, *where* $k \in \{0, 1, 2, 3\}$.

Proof One can prove this as Bar-Natan does in [1, Lemmas 8.6–8.9], by showing that the space of degree 0 automorphisms of the complexes corresponding to the tangles in the initial (and final) frames appearing in the type II movie moves is 1–dimensional. The main difference with [1] is that we use our local relations ℓ, and homotopy equivalences constructed in Section 3.1. □

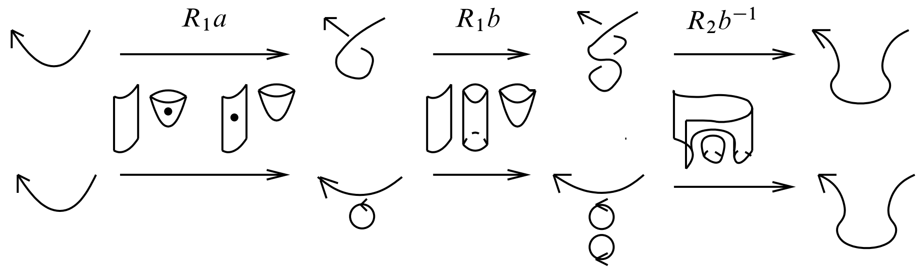
Next step is to show that actually $\mathcal{L}(MM_i)$ is homotopy equivalent to identity, for each $6 \le i \le 10$. There are many oriented representatives that one needs to check for each movie move, but we approach here only one of them for each particular movie, and we refer the reader to author's exposition in [3], for more details. We choose a direct summand in the chain complex associated to the first frame of the clip, having the property that has no homotopies in or out, and we observe its image under the clip. This is the method used by Clark, Morrison and Walker in [5], from where we recall the following results (which hold in our case as well). If A is a resolution in a formal complex $[T]$, so that A does not contain closed webs and is not connected by differentials to resolutions containing closed webs, then A is *homotopically isolated*; that is, for any homotopy h, the restriction of $dh + hd$ to A is zero. Moreover, if f and g are chain maps so that $f \sim cg$ for some constant c, and if f and g agree on some homotopically isolated object A, then $f \sim g$.

MM6 Let's have a look at the following oriented representative of MM6:

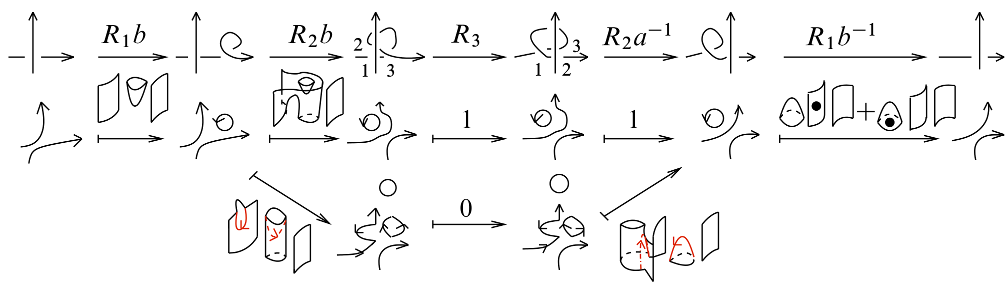


We start at the height zero object of the complex associated to the first—which is also the last—tangle of the clip. Composing the maps above, we obtain in the first row  $= (-i)^4\,\mathrm{Id} = \mathrm{Id}$, and in the second row  $\circ$  $= 0$. Therefore the induced chain map is the identity.

MM7 We consider the case with a negative crossing in the second frame. After composing the morphisms and applying relations (S), the corresponding cobordism is a vertical "curtain", thus the morphism is the identity.

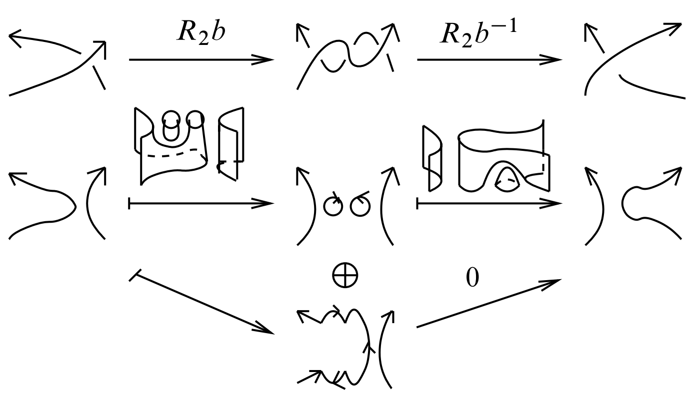


MM8 Let's consider the case when the Reidemeister move R1 introduces a positive crossing.

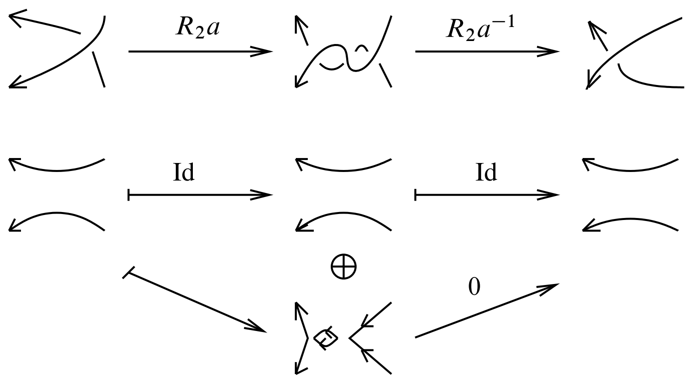


From the proof of invariance under R3 move we know that the map in the lower row above is the zero map, while in the upper row is the identity map.

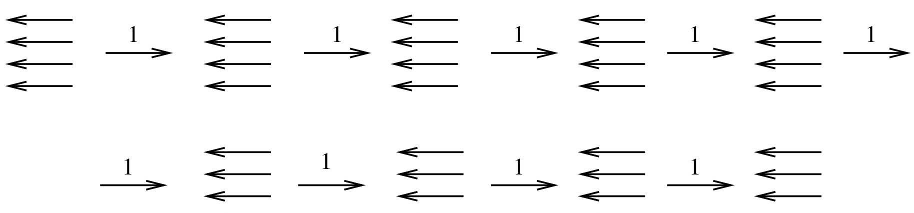
MM9 Below we considered two oriented representatives of the movie move; we can easily see that after composition, we obtain in both cases the identity map.



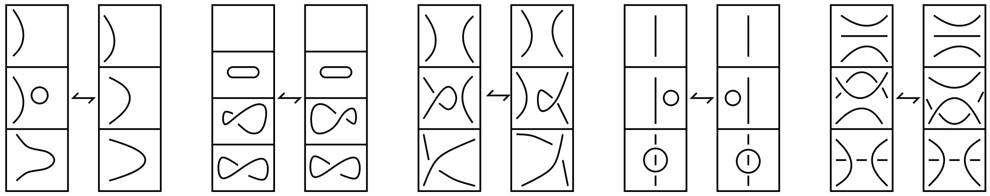
and



MM10 Let's assume that we have oriented all strings in this movie move from right to left, thus each crossing is negative. We pick the complete oriented resolution—each crossing was given the oriented resolution— at each step, the map from this resolution to the similar one in the next complex is the identity. Moreover, at each stage, there are no other maps from other resolutions going into this oriented one. Therefore, this representative of MM10 movie move induces the identity morphism.

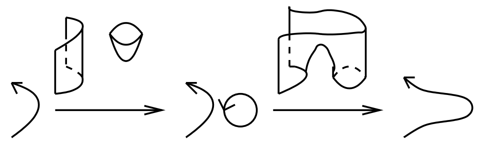


Type III: Non-reversible clips

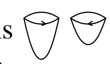
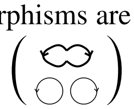
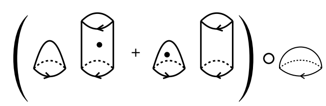
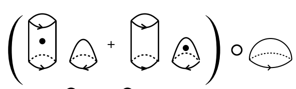


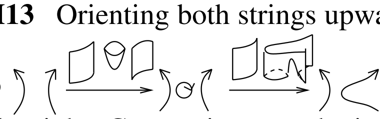
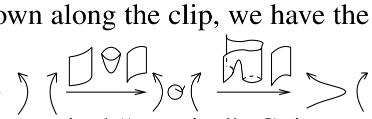
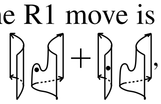
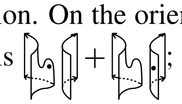
Each pair of a type III clip should produce the same morphisms when read from top to bottom or from bottom to top. As before, we show only one oriented representative for each movie move, but we point out that the others give similar results (see [3]).

MM11

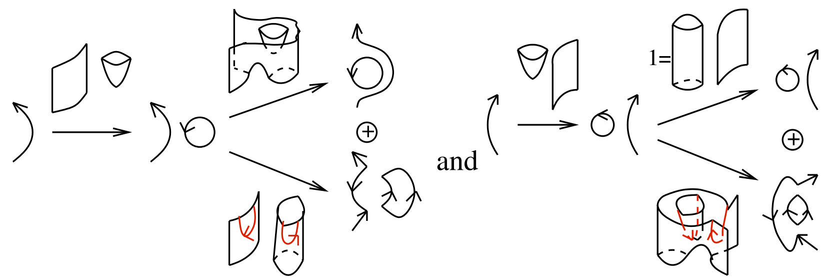


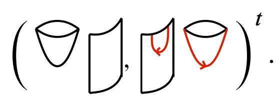
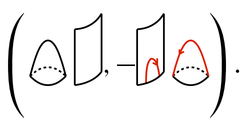
Going down along the left side of MM11 we get the morphism which is the composition of the two maps in the above row; but this composition is isotopic to the cobordism obtained by going down along the right side of MM11. Going up along the clip, we just need to turn all these cobordisms upside down. Reversing the orientation of the string, the induced maps are the same as those we just obtained.

MM12 Going down along the left side of MM12 we get the morphism $\varnothing \to$ , which from the proof of invariance under Reidemeister 1 move is . Similarly, going down along the right side we get the morphisms . But these two morphisms are the same, up to isotopy. On the other hand, going up we get the morphism  $\longrightarrow \varnothing$, which on the first component is the zero map on both sides, and on the second component is  on the left side of the clip and  on the right. Up to isotopy, these cobordisms are just .

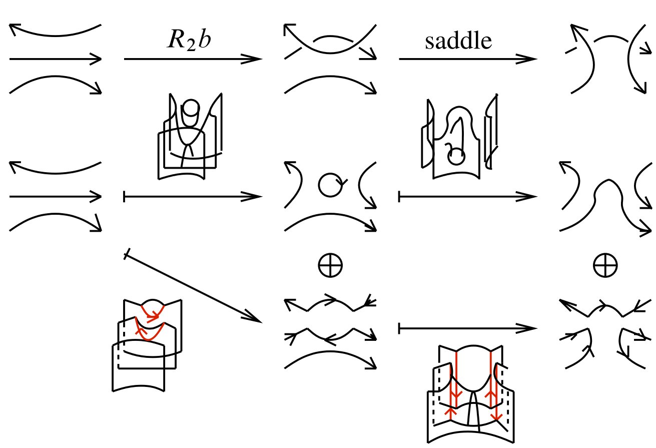
MM13 Orienting both strings upwards and going down along the clip, we have the map  on the left, and  on the right. Composing, we obtain in both cases two vertical "curtains". Going up both maps are zero on the singular resolution, since the chain map corresponding to the R1 move is zero on this resolution. On the oriented resolution, the map on the left is , while on the right is ; these are the same, up to isotopy.

MM14 Consider the oriented representative for MM14 in which the vertical string in oriented upwards and the circle is oriented counterclockwise. From the proof of invariance under Reidemeister 2, we know that going down along MM14, we have on the left and right, respectively, the following maps:

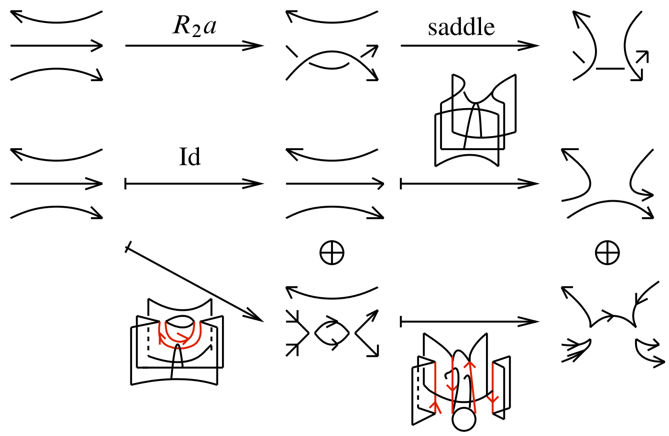


Both maps are equal to . Going up we then obtain .

MM15 We pick the oriented representative given below. Going down we have on the left and right, respectively, the following:



and



These maps are the same, namely, the first component is a saddle involving the two upper strands, and the second component is a singular saddle involving the lower two strands. Going up, the corresponding maps are the previous saddles turned upside-down, with a minus sign on the second component. □

4 Web homology and the algebraic invariant

Let Γ_0 be a fixed web with boundary B (note that if $B = \varnothing$, we choose Γ_0 to be the empty web diagram). We define a functor $\mathcal{F}_{\Gamma_0}\colon \mathit{Foams}_{/\ell}(B) \to \mathbb{Z}[i][a]$–Mod, which extends to $\mathit{Kof} = \mathrm{Kom}(\mathrm{Mat}(\mathit{Foams}_{/\ell}))$.

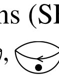
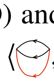
Definition 4.1 Given $\Gamma \in \mathit{Foams}_{/\ell}(B)$, we define $\mathcal{F}_{\Gamma_0}(\Gamma) = \mathrm{Hom}_{\mathit{Foam}_{/\ell}(B)}(\Gamma_0, \Gamma)$, and call it the 'homology' of Γ. For a foam $S \in \mathrm{Hom}_{\mathit{Foam}_{/\ell}(B)}(\Gamma', \Gamma'')$ we define the $\mathbb{Z}[i][a]$–linear map

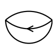
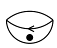
$$\mathcal{F}_{\Gamma_0}(S)\colon \mathrm{Hom}_{\mathit{Foam}_{/l}(B)}(\Gamma_0, \Gamma') \to \mathrm{Hom}_{\mathit{Foam}_{/\ell}(B)}(\Gamma_0, \Gamma'')$$

given by composition. This homomorphism has degree equal to $\deg(S)$.

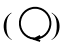
By definition, the functor associates to the empty web the ground ring $\mathbb{Z}[i][a]$. Moreover, $\mathcal{F}_{\Gamma_0}(\Gamma_1 \cup \Gamma_2) \cong \mathcal{F}_{\Gamma_0}(\Gamma_1) \otimes_{\mathbb{Z}[i][a]} \mathcal{F}_{\Gamma_0}(\Gamma_2)$, for any disjoint union of webs Γ_1, Γ_2.

4.1 Web homology skein relations

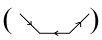
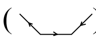
Consider $B = \varnothing$ and $\Gamma_0 = \varnothing$. Firstly, by relations (SF) and (CN) and secondly, by (S), (UFO) and (2D) we have $\mathcal{F}_\varnothing($$) = V = \langle$, $\rangle_{\mathbb{Z}[i][a]}$. Likewise, $\mathcal{F}_\varnothing($$) = V' = \langle$, $\rangle_{\mathbb{Z}[i][a]}$ (where we fix the dot, once and for all, say on the back facet).

Note that we can identify V (or V') with $\mathcal{A}$ via the canonical isomorphism $V \cong \mathcal{A}$,  $\to 1$,  $\to X$ (or $V' \cong \mathcal{A}$,  $\to 1$,  $\to X$).

Lemma 4.2 *There are canonical isomorphisms of graded abelian groups, that mimic the web skein relations given in Figure 1.*

(1) $\mathcal{F}_{\varnothing}($  $) \cong \mathcal{A} \cong \mathcal{F}_{\varnothing}($  $)$

(2) $\mathcal{F}_{\Gamma_0}(\Gamma \cup$  $) \cong \mathcal{F}_{\Gamma_0}(\Gamma) \otimes_{\mathbb{Z}[i][a]} \mathcal{A} \cong \mathcal{F}_{\Gamma_0}(\Gamma \cup$  $)$

(3) $\mathcal{F}_{\Gamma_0}($  $) \cong \mathcal{F}_{\Gamma_0}($  $)$ *and* $\mathcal{F}_{\Gamma_0}($  $) \cong \mathcal{F}_{\Gamma_0}($  $)$.

In particular, $\mathcal{F}_{\varnothing}(\Gamma)$ is a free $\mathbb{Z}[i][a]$–module of graded rank $\langle \Gamma \rangle$.

Proof These follow from the isomorphisms $V \cong \mathcal{A} \cong V'$ described above, relations (SF) and (CN) and Corollary 2.8. □

Corollary 4.3 *The functor $\mathcal{F}_{\varnothing}$ is the same as the functor F defined in Section 2.1.*

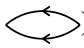
$\mathcal{F}$ extends to a functor $\mathcal{F}\colon Kof \to \mathbb{Z}[i][a]$–Mod. For any tangle diagram $T, \mathcal{F}([T])$ is an ordinary complex, and applying the functor to all homotopies we obtain that $\mathcal{F}([T])$ is an invariant of the tangle T, up to homotopy. The isomorphism class of the homology $H(\mathcal{F}([T]))$ is a bigraded invariant of T, which we denote by $\mathcal{H}(T)$.

It is clear from construction that for the case of links, the graded Euler characteristic of the complex $\mathcal{F}([L])$ equals $P_2(L)$, the quantum $\mathfrak{sl}(2)$ polynomial of L. In other words,

$$P_2(L) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rk}(\mathcal{H}^{i,j}(L)).$$

4.2 Relationship with Khovanov's $\mathfrak{sl}(2)$ invariant

We show now that adding the relation $a = 0$ (or $a = 1$) and considering closed tangles, thus knots and links, our invariant is isomorphic to a version of the original Khovanov's invariant (or Lee's modification of it), after the latter is tensored with $\mathbb{Z}[i]$. Since our construction is properly functorial under link cobordisms (relative to boundary), it naturally resolves the sign ambiguity in the functoriality property of the Khovanov homology. Note that Clark, Morrison and Walker [5] obtained a similar result as ours for the $a = 0$ case.

Let's consider a link diagram L and its corresponding formal complex $[L]$. Each resolution of L is a collection of webs (with an even number of vertices) and oriented loops. Applying the isomorphisms from the end of Section 2.2, we can 'erase' pairs of adjacent singular points of the same type, so that each resolution is replaced, via an isomorphism, by a disjoint union of basic closed webs with bivalent vertices (as ) and oriented loops. Moreover, applying the isomorphisms from Corollary 2.8, we can replace each basic closed web by an oriented loop, in such a way that starting from outside, the orientation of the loop is (say) clockwise, and as we go inside of a nesting set of loops, orientations alternate. After this operation we are left with a formal complex whose objects are column matrices of nested oriented loops, so that the outermost loop is oriented (say) clockwise and then the orientations alternate.

We consider now the Khovanov formal chain complex associated to L with its unoriented objects, and orient them such that we end up with the same chain complex described above. Notice that this way of orienting the circles yields well-defined oriented cobordisms between oriented loops. Finally, recalling how our TQFT is defined for $a = 0$ and $a = 1$, and that is the same as the functor $\mathcal{F}_{\varnothing}$, we reach our goal.

References

[1] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499 MR2174270

[2] **D Bar-Natan**, *Fast Khovanov homology computations*, J. Knot Theory Ramifications 16 (2007) 243–255 MR2320156

[3] **C Caprau**, *An* $\mathfrak{sl}(2)$ *tangle homology and seamed cobordisms* (2007) `arXiv:math.GT/07073051`

[4] **J S Carter**, **M Saito**, *Reidemeister moves for surface isotopies and their interpretation as moves to movies*, J. Knot Theory Ramifications 2 (1993) 251–284 MR1238875

[5] **D Clark**, **S Morrison**, **K Walker**, *Fixing the functoriality of Khovanov homology* (2007) `arXiv:math.GT/0701339`

[6] **M Jacobsson**, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. 4 (2004) 1211–1251 MR2113903

[7] **L Kadison**, *New examples of Frobenius extensions*, University Lecture Series 14, American Mathematical Society, Providence, RI (1999) MR1690111

[8] **L H Kauffman**, *Knots and physics*, third edition, Series on Knots and Everything 1, World Scientific Publishing Co., River Edge, NJ (2001) MR1858113

[9] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR1740682

[10] **M Khovanov**, *sl(3) link homology*, Algebr. Geom. Topol. 4 (2004) 1045–1081 MR2100691

[11] **M Khovanov**, *An invariant of tangle cobordisms*, Trans. Amer. Math. Soc. 358 (2006) 315–327 MR2171235

[12] **M Khovanov**, *Link homology and Frobenius extensions*, Fund. Math. 190 (2006) 179–190 MR2232858

[13] **E S Lee**, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005) 554–586 MR2173845

[14] **M Mackaay**, **P Vaz**, *The universal* sl_3*–link homology*, Algebr. Geom. Topol. 7 (2007) 1135–1169 MR2336253

Department of Mathematics, California State University
5245 North Backer Avenue M/S PB 108, Fresno, CA 93740-8001, USA

ccaprau@csufresno.edu

http://zimmer.csufresno.edu/~ccaprau

Received: 9 January 2008