The Magnus representation and higher-order Alexander invariants for homology cobordisms of surfaces

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The set of homology cobordisms from a surface to itself with markings of their boundaries has a natural monoid structure. To investigate the structure of this monoid, we define and study its Magnus representation and Reidemeister torsion invariants by generalizing Kirk, Livingston and Wang's argument over the Gassner representation of string links. Then, by applying Cochran and Harvey's framework of higher-order (noncommutative) Alexander invariants to them, we extract several information about the monoid and related objects.

1 Introduction

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 1$ with one boundary component. A *homology cylinder over* $\Sigma_{g,1}$ consists of a homology cobordism from $\Sigma_{g,1}$ to itself with markings of its boundary. We denote by $\mathcal{C}_{g,1}$ the set of isomorphisms classes of homology cylinders. Since stacking two homology cylinders gives a new one, we can endow $\mathcal{C}_{g,1}$ with a monoid structure (see Section 2 for the precise definition). The origin of homology cylinders goes back to Goussarov [10], Habiro [11], Garoufalidis–Levine [9] and Levine [21], where the clasper (or clover) surgery theory is effectively used to investigate the structure of $\mathcal{C}_{g,1}$.

As mentioned in [9] and [21], we can construct a homology cylinder from each homology 3–sphere or pure string link. Also, for a given homology cylinder, we can use an element of the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ to give another one by changing its markings. Since these operations preserve each monoid structure, $\mathcal{C}_{g,1}$ can be regarded as a simultaneous generalization of the monoid of homology 3–spheres, that of string links and $\mathcal{M}_{g,1}$, any of which plays an important role in the theory of 3–manifolds. On the other hand, there exist some natural ways (see Section 6.3) to construct closed 3–manifolds from each homology cylinder. Therefore, by using its monoid structure, $\mathcal{C}_{g,1}$ serves as a tool for classifying closed 3–manifolds.

The aim of this paper is to study the structure of $\mathcal{C}_{g,1}$ from an algebraic point of view. The main tools for that are invariants using noncommutative rings arising from

group rings of fundamental groups of homology cylinders. The first half of this paper is occupied by defining the Magnus representation for homology cylinders and observing its fundamental properties. In the latter half, Cochran and Harvey's higher-order Alexander invariants are introduced and used to define numerical invariants for matrices obtained by the Magnus representation or associated Reidemeister torsions, whose entries are in noncommutative rings so that it is difficult to treat directly. This combination of the Magnus representation and higher-order Alexander invariants leads us to derive various facts about the structure of $\mathcal{C}_{g,1}$ and to give applications to the theory of closed 3–manifolds.

The outline of this paper is as follows.

In Section 2, we first recall the definition of homology cylinders over $\Sigma_{g,1}$ by following that of Garoufalidis–Levine [9] and Levine [21]. Then the monoid $\mathcal{C}_{g,1}$ of homology cylinders and its quotient group $\mathcal{H}_{g,1}$ by homology cobordisms are introduced. They are main objects of this paper. An example of homology cylinders comes from the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$. In fact, $\mathcal{M}_{g,1}$ is contained in $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ as a subgroup. We use Stallings' theorem to give filtrations $\{\mathcal{C}_{g,1}[k]\}_{k\geq 1}$ and $\{\mathcal{H}_{g,1}[k]\}_{k\geq 1}$ of $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ respectively, which enable us to apply the Johnson–Morita theory used for $\mathcal{M}_{g,1}$ (see Morita [24] for details) to homology cylinders. Note that, in Goussarov [10] and Habiro [11], the term "homology cylinders" indicates homology cylinders in $\mathcal{C}_{g,1}[2]$.

In Section 3, which is the first main part of this paper, we study the Magnus representation

$$r_k : \mathcal{C}_{g,1} \longrightarrow GL(2g, \mathcal{K}_{N_k}) \quad (k = 2, 3, \ldots)$$

for homology cylinders, where $\mathcal{K}_{N_k}$ is a certain (skew) field. This "representation", which is in fact a crossed homomorphism, extends that for $\mathcal{M}_{g,1}$ defined by Morita [24]. (See also Birman's book [2] for generalities of the ordinary Magnus representation as well as free differentials.) We can easily check that r_k factors through $\mathcal{H}_{g,1}$. One method for defining r_k was already given in our article [28] as an analogue of the extension of the Gassner representation from pure braids to pure string links defined by Le Dimet [17]. More precisely, what we defined in [28] is a noncommutative generalization of the analogue of the Gassner representation. (In Section 4, the definition of a noncommutative generalization of the Gassner representation is described for completeness.) In this paper, we re-define the Magnus representation for $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ (Definition 3.3) by following Kirk–Livingston–Wang [16], where another and simplified way to define the Gassner representation for string links is given by using twisted (co)homology. Kirk–Livingston–Wang's method is also easier to give applications to the theory of 3–manifolds. After observing some fundamental properties of the Magnus representation, we see a method of the computation in Section 3.2.

In Section 4, we apply the computational method given in Section 3.2 to homology cylinders obtained from pure string links by using Levine's construction in [21]. We will show that our Magnus representation contains the Gassner representation for string links. As a corollary, we see that $\mathcal{M}_{g,1}$ is *not* a normal subgroup of $\mathcal{H}_{g,1}$.

The latter half of this paper starts at Section 5. We begin by a brief review of higher-order Alexander invariants due to Cochran [4] and Harvey [12; 13]. Higher-order Alexander invariants are invariants for finitely presentable groups interpreted as degrees of "noncommutative Alexander polynomials", which have some unclear ambiguity except their degrees. Historically, they were first defined for knot groups by Cochran, and then generalized for arbitrary finitely presentable groups by Harvey. Using them, Cochran and Harvey obtained various sharper results than those brought by the ordinary Alexander invariants: lower bounds on the knot genus or the Thurston norm, necessary conditions for realizing a given group as the fundamental group of some compact oriented 3–manifold, and so on. When we apply the theory of higher-order Alexander invariants to our situation, we shall need a slight generalization called *torsion-degree functions* because of the difference of localizations of noncommutative rings used in the Magnus representation and higher-order Alexander invariants (see Remark 3.2). Torsion-degree functions, whose definition occupies the middle of this section, are integer-valued maps

$$d_{\Gamma}^{\psi} \colon M(\mathcal{K}_{\Gamma}) \longrightarrow \mathbb{Z} \qquad (\psi \in H^1(\Gamma), \text{ primitive})$$

from (not necessarily invertible) matrices $M(\mathcal{K}_{\Gamma})$ with entries in the right field of fractions $\mathcal{K}_{\Gamma} = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$ of the group ring of a group Γ in a certain class. The construction of them is given by using the Dieudonné determinant and it is somewhat technical. However, the properties we need are summarized in Example 5.7, where we see that torsion-degree functions give a purely algebraic description of the machinery of higher-order Alexander invariants.

In Section 6, which is the second main part, we give various applications of torsion-degree functions to homology cylinders. We first apply them to the Magnus representation (Section 6.1) and associated Reidemeister torsions (Section 6.2). In some special cases, the results of these subsections are related to higher-order Alexander invariants of closed 3–manifolds associated with homology cylinders by deriving some factorization formulas (Theorems 6.8, 6.11 and 6.12) in Section 6.3. In Section 6.4, we observe the result of Section 6.2 more closely and show that there exist many families of torsion-degree functions which are homomorphisms on $\mathcal{C}_{g,1}[2]$ and linearly independent of each other (Theorem 6.15 and Corollary 6.16) by using an argument similar to Harvey's Realization Theorem [12, Theorem 11.2]. From this, we immediately see that $\mathcal{C}_{g,1}[2], \mathcal{C}_{g,1}[3], \ldots$ are *not* finitely generated.

In Section 7, we apply our machinery of torsion-degree functions to the automorphism group $\mathrm{Aut}\, F_n^{\mathrm{acy}}$ of the acyclic closure of a free group, which can be regarded as an enlargement of the automorphisms group of a free group. We show that the abelianization of certain subgroup of $\mathrm{Aut}\, F_n^{\mathrm{acy}}$ has infinite rank by using the linearly independence of torsion-degree functions.

2 Homology cylinders

Throughout the paper, we work in PL or smooth category. Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus $g \geq 1$ with one boundary component. We take a base point p on the boundary of $\Sigma_{g,1}$, and take $2g$ loops $\gamma_1, \ldots, \gamma_{2g}$ of $\Sigma_{g,1}$ as shown in Figure 1. We consider them to be an embedded bouquet R_{2g} of $2g$ circles tied at the base point $p \in \partial\Sigma_{g,1}$. Then R_{2g} and the boundary loop ζ of $\Sigma_{g,1}$ together with one 2–cell make up a standard cell decomposition of $\Sigma_{g,1}$. The fundamental group $\pi_1\Sigma_{g,1}$ of $\Sigma_{g,1}$ is isomorphic to the free group F_{2g} of rank $2g$ generated by $\gamma_1, \ldots, \gamma_{2g}$, in which $\zeta = \prod_{i=1}^{g}[\gamma_i, \gamma_{g+i}]$.
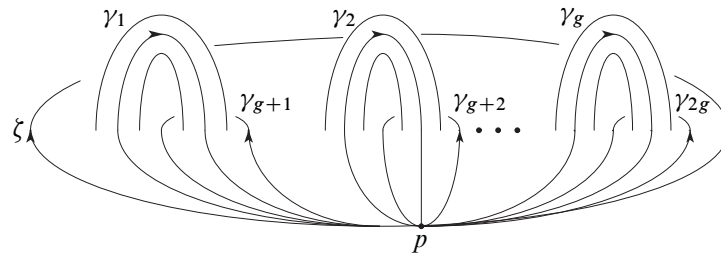

Figure 1: A standard cell decomposition of $\Sigma_{g,1}$

A *homology cylinder* (M, i_+, i_-) (*over* $\Sigma_{g,1}$), which has its origin in Goussarov [10], Habiro [11], Garoufalidis–Levine [9] and Levine [21], consists of a compact oriented 3–manifold M and two embeddings $i_+, i_-\colon \Sigma_{g,1} \to \partial M$ satisfying the following:

(1) i_+ is orientation-preserving and i_- is orientation-reversing,

(2) $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$ and $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$,

(3) $i_+\big|_{\partial\Sigma_{g,1}} = i_-\big|_{\partial\Sigma_{g,1}}$,

(4) $i_+, i_-\colon H_*(\Sigma_{g,1}) \to H_*(M)$ are isomorphisms.

We denote $i_+(p) = i_-(p)$ by $p \in \partial M$ again and consider it to be the base point of M. We write (M, i_+, i_-) or simply M for a homology cylinder.

Two homology cylinders are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism between the underlying 3–manifolds which is compatible with the embeddings of $\Sigma_{g,1}$. We denote the set of isomorphism classes of homology cylinders by $\mathcal{C}_{g,1}$. Given two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$, we can construct a new homology cylinder $M \cdot N$ by

$$M \cdot N = (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-).$$

Then $\mathcal{C}_{g,1}$ becomes a monoid with the unit $1_{\mathcal{C}_{g,1}} := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \mathrm{id} \times 0)$.

From the monoid $\mathcal{C}_{g,1}$, we can construct the *homology cobordism group $\mathcal{H}_{g,1}$ of homology cylinders* as in the following way. Two homology cylinders $M = (M, i_+, i_-)$ and $N = (N, j_+, j_-)$ are *homology cobordant* if there exists a compact oriented 4–manifold W such that

(1) $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\ i_-(x) = j_-(x)) \quad x \in \Sigma_{g,1}$,

(2) the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the homology,

where $-N$ is N with opposite orientation. We denote by $\mathcal{H}_{g,1}$ the quotient set of $\mathcal{C}_{g,1}$ with respect to the equivalence relation of homology cobordism. The monoid structure of $\mathcal{C}_{g,1}$ induces a group structure of $\mathcal{H}_{g,1}$. In the group $\mathcal{H}_{g,1}$, the inverse of (M, i_+, i_-) is given by $(-M, i_-, i_+)$.

Example 2.1 For each element φ of the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$, we can construct a homology cylinder $M_\varphi \in \mathcal{C}_{g,1}$ by setting

$$M_\varphi := (\Sigma_{g,1} \times I, \mathrm{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,1})$ and $i_-(\Sigma_{g,1})$ are stretched half-way along $\partial \Sigma_{g,1} \times I$. As mentioned in Garoufalidis–Levine [9, Section 2.4] and Levine [21, Section 2.1], the existence of the homomorphisms σ_k $(k \geq 2)$ below shows that this construction gives injective homomorphisms $\mathcal{M}_{g,1} \hookrightarrow \mathcal{C}_{g,1}$ and $\mathcal{M}_{g,1} \hookrightarrow \mathcal{H}_{g,1}$ since $\bigcap_{k \geq 2} \mathrm{Ker}(\sigma_k|_{\mathcal{M}_{g,1}}) = \{1\}$. From this, we can regard $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ as enlargements of $\mathcal{M}_{g,1}$.

Let $N_k(G) := G/(\Gamma^k G)$ be the k–th nilpotent quotient of a group G, where we define $\Gamma^1 G = G$ and $\Gamma^l G = [\Gamma^{l-1} G, G]$ for $l \geq 2$. For simplicity, we write $N_k(X)$ for $N_k(\pi_1 X)$ where X is a connected topological space, and write N_k for $N_k(F_{2g}) = N_k(\Sigma_{g,1})$.

Let (M, i_+, i_-) be a homology cylinder. By definition, $i_+, i_-\colon \pi_1 \Sigma_{g,1} \to \pi_1 M$ are both 2–*connected*, namely they induce isomorphisms on the first homology groups and epimorphisms on the second homology groups. Then, by Stallings' theorem [30,

Theorem 3.4], $i_+, i_-\colon N_k \overset{\cong}{\Rightarrow} N_k(M)$ are isomorphisms for each $k \geq 2$. Using them, we obtain a monoid homomorphism

$$\sigma_k\colon \mathcal{C}_{g,1} \longrightarrow \operatorname{Aut} N_k \qquad ((M, i_+, i_-) \mapsto (i_+)^{-1} \circ i_-).$$

It can be easily checked that σ_k induces a group homomorphism $\sigma_k\colon \mathcal{H}_{g,1} \to \operatorname{Aut} N_k$. We define filtrations of $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ by

$$\mathcal{C}_{g,1}[1] := \mathcal{C}_{g,1}, \quad \mathcal{C}_{g,1}[k] := \operatorname{Ker}\left(\mathcal{C}_{g,1} \xrightarrow{\sigma_k} \operatorname{Aut} N_k\right) \text{ for } k \geq 2,$$
$$\mathcal{H}_{g,1}[1] := \mathcal{H}_{g,1}, \ \mathcal{H}_{g,1}[k] := \operatorname{Ker}\left(\mathcal{H}_{g,1} \xrightarrow{\sigma_k} \operatorname{Aut} N_k\right) \text{ for } k \geq 2.$$

(We follow the notation of [28]. Note that $\mathcal{H}_{g,1}[k]$ is denoted by $\mathcal{F}^w_{k-1}(\mathcal{H}_g)$ in Levine [21].)

3   The Magnus representation for homology cylinders

We first summarize our notation. For a matrix A with entries in a ring R, and a ring homomorphism $\varphi\colon R \to R'$, we denote by $^\varphi A$ the matrix obtained from A by applying φ to each entry. A^T denotes the transpose of A. When $R = \mathbb{Z}G$ for a group G or its right field of fractions (if exists), we denote by $\bar{A}$ the matrix obtained from A by applying the involution induced from $(x \mapsto x^{-1}, \ x \in G)$ to each entry.

For a module M, we write M^n and M_n for the modules of column and row vectors with n entries respectively.

For a connected CW-complex X and its regular covering X_Γ with respect to a homomorphism $\pi_1 X \to \Gamma$, we consider the *right* action of Γ on X_Γ as in Harvey [12, Section 5]. Therefore the cellular chain complex $C_*(X_\Gamma)$ of X_Γ becomes a right $\mathbb{Z}\Gamma$–module. For each $(\mathbb{Z}\Gamma, R)$–bimodule A, the twisted chain complex $C_*(X; A)$ is given by the tensor product of the right $\mathbb{Z}\Gamma$–module $C_*(X_\Gamma)$ and the left $\mathbb{Z}\Gamma$–module A, so that $C_*(X; A)$ and $H_*(X; A)$ are right R–modules.

3.1   Definition of the Magnus representation for homology cylinders

In what follows, we fix an integer $k \geq 2$, which corresponds to the class of the nilpotent quotient. The following construction is based on Kirk–Livingston–Wang's work of the Gassner representation for string links in [16].

Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$. By Stallings' theorem, N_k and $N_k(M)$ are isomorphic. Since N_k is a finitely generated torsion-free nilpotent group for each $k \geq 2$, we can embed $\mathbb{Z}N_k$ into the right field of fractions $\mathcal{K}_{N_k} := \mathbb{Z}N_k(\mathbb{Z}N_k - \{0\})^{-1}$. (See

Similarly, we have $\mathbb{Z}N_k(M) \hookrightarrow \mathcal{K}_{N_k(M)} := \mathbb{Z}N_k(M)(\mathbb{Z}N_k(M) - \{0\})^{-1}$.
We consider the fields $\mathcal{K}_{N_k}$ and $\mathcal{K}_{N_k(M)}$ to be local coefficient systems on $\Sigma_{g,1}$ and M respectively.

Lemma 3.1 (Cochran–Orr–Teichner [5, Proposition 2.10])

$$i_\pm \colon H_*(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{N_k(M)}) \to H_*(M, p; \mathcal{K}_{N_k(M)})$$

are isomorphisms as right $\mathcal{K}_{N_k(M)}$–vector spaces.

Remark 3.2 A statement similar to Lemma 3.1 is true for the homology with coefficients in any left $\mathbb{Z}\pi_1 M$–algebra A satisfying the following property: Every matrix with entries in $\mathbb{Z}\pi_1 M$ sent to an invertible one by the augmentation map $\mathbb{Z}\pi_1 M \to \mathbb{Z}$ is also invertible in A. Note that $\mathcal{K}_{N_k(M)}$ satisfies this property (although $\mathcal{K}_{N_k(M)^\psi}[t^{\pm 1}]$ in Section 5 does not in general).

Since the bouquet R_{2g} is a deformation retract of $\Sigma_{g,1}$ relative to p, we have isomorphisms $\iota_\pm \colon \mathcal{K}_{N_k(M)}^{2g} \xRightarrow{\cong} H_1(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{N_k(M)})$ of right $\mathcal{K}_{N_k(M)}$–vector spaces defined as the inverse map of the composite of

$$
\begin{aligned}
H_1(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{N_k(M)}) \cong H_1(R_{2g}, p; i_\pm^* \mathcal{K}_{N_k(M)}) &= C_1(R_{2g}, p; i_\pm^* \mathcal{K}_{N_k(M)}) \\
&= C_1(\widetilde{R_{2g}}) \otimes_{\pi_1 R_{2g}} i_\pm^* \mathcal{K}_{N_k(M)} \\
&\cong \mathcal{K}_{N_k(M)}^{2g}
\end{aligned}
$$

under a basis

$$\{\widetilde{\gamma_1} \otimes 1, \ldots, \widetilde{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{\pi_1 R_{2g}} i_\pm^* \mathcal{K}_{N_k(M)},$$

where $\widetilde{\gamma_i}$ is a lift of γ_i on the universal covering $\widetilde{R_{2g}}$.

Definition 3.3 (1) For each $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we denote by $r'_k(M) \in GL(2g, \mathcal{K}_{N_k(M)})$ the representation matrix of the right $\mathcal{K}_{N_k(M)}$–isomorphism:

$$
\mathcal{K}_{N_k(M)}^{2g} \xrightarrow[\iota_-]{\cong} H_1(\Sigma_{g,1}, p; i_-^* \mathcal{K}_{N_k(M)}) \xrightarrow[i_-]{\cong} H_1(M, p; \mathcal{K}_{N_k(M)})
$$
$$
\xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; i_+^* \mathcal{K}_{N_k(M)}) \xrightarrow[\iota_+^{-1}]{\cong} \mathcal{K}_{N_k(M)}^{2g}
$$

(2) The *Magnus representation* for $\mathcal{C}_{g,1}$ is the map $r_k \colon \mathcal{C}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ which assigns to $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ the matrix $^{i_+^{-1}} r'_k(M)$. We call $r_k(M)$ the *Magnus matrix* of M.

Remark 3.4 Since the base points of $\Sigma_{g,1}$, R_{2g} and M are the same, $r'_k(M)$ and $r_k(M)$ depend only on the choice of a cell decomposition of $\Sigma_{g,1}$ (or a basis of $\pi_1\Sigma_{g,1}$). See also Proposition 3.10.

While we call $r_k(M)$ the Magnus "representation", it is actually a crossed homomorphism, namely we have the following.

Theorem 3.5 For $M_1, M_2 \in \mathcal{C}_{g,1}$, we have

$$r_k(M_1 \cdot M_2) = r_k(M_1) \cdot {}^{\sigma_k(M_1)} r_k(M_2).$$

Proof We write $M = M_1 \cdot M_2$ for simplicity. Let $i\colon M_1 \to M$ and $j\colon M_2 \to M$ be the natural inclusions. Since $M = (M, i \circ i_+, j \circ j_-)$ and $i \circ i_- = j \circ j_+$, the map

$$H_1(\Sigma_{g,1}, p; j_-^* j^* \mathcal{K}_{N_k(M)}) \xrightarrow{j \circ j_-} H_1(M, p; \mathcal{K}_{N_k(M)})$$

$$\xrightarrow{(i \circ i_+)^{-1}} H_1(\Sigma_{g,1}, p; i_+^* i^* \mathcal{K}_{N_k(M)})$$

is given as the composite of

$$H_1(\Sigma_{g,1}, p; j_-^* j^* \mathcal{K}_{N_k(M)}) \xrightarrow{j_-} H_1(M_2, p; j^* \mathcal{K}_{N_k(M)})$$

$$\xrightarrow{j_+^{-1}} H_1(\Sigma_{g,1}, p; j_+^* j^* \mathcal{K}_{N_k(M)})$$

and $\quad H_1(\Sigma_{g,1}, p; i_-^* i^* \mathcal{K}_{N_k(M)}) \xrightarrow{i_-} H_1(M_1, p; i^* \mathcal{K}_{N_k(M)})$

$$\xrightarrow{i_+^{-1}} H_1(\Sigma_{g,1}, p; i_+^* i^* \mathcal{K}_{N_k(M)}).$$

Hence

$$r'_k(M) = {}^i r'_k(M_1) \cdot {}^j r'_k(M_2)$$

$$\implies {}^{(i\,i_+)^{-1}} r'_k(M) = {}^{(i\,i_+)^{-1} i} r'_k(M_1) \cdot {}^{(i\,i_+)^{-1} j} r'_k(M_2)$$

$$= {}^{i_+^{-1}} r'_k(M_1) \cdot {}^{i_+^{-1} i^{-1} j} r'_k(M_2)$$

$$= {}^{i_+^{-1}} r'_k(M_1) \cdot {}^{i_+^{-1} i_- j_+^{-1}} r'_k(M_2)$$

$$\implies \quad r_k(M) = r_k(M_1) \cdot {}^{\sigma_k(M_1)} r_k(M_2).$$

This completes the proof. □

Theorem 3.6 $r_k \colon \mathcal{C}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$ *factors through* $\mathcal{H}_{g,1}$.

Proof Suppose $M_1 = (M_1, i_+, i_-)$ and $M_2 = (M_2, j_+, j_-) \in \mathcal{C}_{g,1}$ are homology cobordant by a homology cobordism W. Let $i \colon M_1 \to W$, $j \colon M_2 \to W$ be the natural inclusions. We may assume that $M_1 \cup M_2$ is a subcomplex of W and that W has only one 0–cell p. Since $\mathbb{Z}N_k \cong \mathbb{Z}N_k(W)$ by Stallings' theorem, we have $\mathcal{K}_{N_k(W)} := \mathbb{Z}N_k(W)(\mathbb{Z}N_k(W) - \{0\})^{-1}$. We write $I_+ := i \circ i_+ = j \circ j_+$ and $I_- := i \circ i_- = j \circ j_-$. Then we have the following commutative diagram:

$$H_1(\Sigma_{g,1}, p; i_-^* i^* \mathcal{K}_{N_k(W)}) \overset{}{=\!=\!=} H_1(\Sigma_{g,1}, p; I_-^* \mathcal{K}_{N_k(W)}) \overset{}{=\!=\!=} H_1(\Sigma_{g,1}, p; j_-^* j^* \mathcal{K}_{N_k(W)})$$

$$i_- \downarrow \cong \qquad I_- \downarrow \cong \qquad j_- \downarrow \cong$$

$$H_1(M_1, p; i^* \mathcal{K}_{N_k(W)}) \xrightarrow[i]{\cong} H_1(W, p; \mathcal{K}_{N_k(W)}) \xleftarrow[\cong]{j} H_1(M_2, p; j^* \mathcal{K}_{N_k(W)})$$

$$(i_+)^{-1} \downarrow \cong \qquad (I_+)^{-1} \downarrow \cong \qquad (j_+)^{-1} \downarrow \cong$$

$$H_1(\Sigma_{g,1}, p; i_+^* i^* \mathcal{K}_{N_k(W)}) \overset{}{=\!=\!=} H_1(\Sigma_{g,1}, p; I_+^* \mathcal{K}_{N_k(W)}) \overset{}{=\!=\!=} H_1(\Sigma_{g,1}, p; j_+^* j^* \mathcal{K}_{N_k(W)})$$

The left vertical maps give ${}^i r_k'(M_1)$ and the right ones give ${}^j r_k'(M_2)$. Applying I_+^{-1}, we obtain $r_k(M_1) = r_k(M_2)$. $\square$

Consequently, we obtain the Magnus representation $r_k \colon \mathcal{H}_{g,1} \to GL(2g, \mathcal{K}_{N_k})$, which is a crossed homomorphism. If we restrict r_k to $\mathcal{C}_{g,1}[k]$ (resp. $\mathcal{H}_{g,1}[k]$), it becomes a monoid (resp. group) homomorphism.

Example 3.7 For $\varphi \in \mathcal{M}_{g,1} \hookrightarrow \mathrm{Aut}\, F_{2g}$, we can obtain

$$r_k(M_\varphi) = {}^{\rho_k}\overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)}_{1 \le i, j \le 2g}$$

where $\rho_k \colon \mathbb{Z}F_{2g} \to \mathbb{Z}N_k \subset \mathcal{K}_{N_k}$ is the natural map and $\partial / \partial \gamma_i$ are free differentials. From this, we see that r_k generalizes the original Magnus representation for $\mathcal{M}_{g,1}$ in Morita [24].

3.2 Computation of the Magnus matrix

In Kirk–Livingston–Wang [16], the Gassner matrix of a string link is computed from the Wirtinger presentation of the fundamental group of its exterior, which gives a finite presentation whose deficiency coincides with the number of strings. Recall that the deficiency of a finite presentation $\{x_1, \dots, x_n \mid r_1, \dots, r_m\}$ of a finitely presentable group G is $n - m$, and the deficiency of G is the maximum of all over the deficiencies of finite presentations of G. In our context, we do not have such a useful method in general.

Definition 3.8 For $(M, i_+, i_-) \in \mathcal{C}_{g,1}$, a presentation of $\pi_1 M$ is said to be *admissible* if it is of the form

$$\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots, r_{2g+l} \rangle.$$

That is, it is a finite presentation with deficiency $2g$ whose generating set contains $i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), i_+(\gamma_1), \ldots, i_+(\gamma_{2g})$ and is ordered as above.

Note that there does exist an admissible presentation of $\pi_1 M$ for each $(M, i_+, i_-) \in \mathcal{C}_{g,1}$. Indeed, take a relative Morse function with no critical points of indices 0 and 3. Then M can be seen as $\Sigma_{g,1} \times I$ with some 1– and 2–handles. Since the Euler characteristics of $\Sigma_{g,1} \times I$ and M are the same, the numbers of the attached 1– and 2– handles are the same. Therefore any presentation of $\pi_1 M$ obtained from a presentation of $\pi_1(\Sigma_{g,1} \times I) = F_{2g}$ with deficiency $2g$ by adding new generators and relations corresponding to the 1– and 2–handles has deficiency $2g$ again. By adding $i_\pm(\gamma_j)$'s to the presentation with appropriate relations, we obtain an admissible presentation of $\pi_1 M$. (See also Section 6.2.)

Now we see a method for computing Magnus matrices, which is based on Kirk–Livingston–Wang's one for computing Gassner matrices of string links [16, Section 4]. Comparing with the case of Kirk–Livingston–Wang, we need extra care about multiplications since we are dealing with more general situation by using noncommutative rings. However, almost the same argument holds.

Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$. Given an admissible presentation of $\pi_1 M$ as in Definition 3.8, we define $2g \times (2g + l)$, $l \times (2g + l)$ and $2g \times (2g + l)$ matrices A, B, C by

$$A = \overline{\left(\frac{\partial r_j}{\partial i_-(\gamma_i)} \right)}_{\substack{1 \leq i \leq 2g \\ 1 \leq j \leq 2g+l}}, \quad B = \overline{\left(\frac{\partial r_j}{\partial z_i} \right)}_{\substack{1 \leq i \leq l \\ 1 \leq j \leq 2g+l}}, \quad C = \overline{\left(\frac{\partial r_j}{\partial i_+(\gamma_i)} \right)}_{\substack{1 \leq i \leq 2g \\ 1 \leq j \leq 2g+l}}$$

at $\mathbb{Z} N_k(M)$, namely we apply the natural map

$$\mathbb{Z}\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g})\rangle \to \mathbb{Z}\pi_1 M \to \mathbb{Z}N_k(M)$$

to each entry of the matrices obtained by free differentials.

Proposition 3.9 *As matrices with entries in $\mathcal{K}_{N_k(M)}$, we have the following.*

(1) *The square matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is invertible.*

(2) *The following equality holds:*

$$r_k'(M) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(l,2g)} \end{pmatrix}.$$

In particular, the Magnus matrix $r'_k(M)$ can be computed from any admissible presentation of $\pi_1 M$.

Proof (1) Let $\mathfrak{t} \colon \mathbb{Z}N_k(M) \to \mathbb{Z}$ be the augmentation map. ${}^t\!\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ gives a presentation matrix of $H_1(M)/\Phi_+$, where Φ_+ is the subgroup of $H_1(M)$ generated by $i_+(\gamma_1), \ldots, i_+(\gamma_{2g})$. (See Fox [7] for this fact through the concept of *presentations of a pair of groups*.) By definition, $H_1(M)/\Phi_+ = 0$, and we have an exact sequence

$$\mathbb{Z}^{2g+l} \xrightarrow{\;{}^t\!\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)\cdot\;} \mathbb{Z}^{2g+l} \longrightarrow H_1(M)/\Phi_+ = 0.$$

By the Hopfian property of $\mathbb{Z}^{2g+l}$, we see that ${}^t\!\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ is invertible. (1) follows from this. (See Remark 3.2.)

(2) By a standard argument using Eilenberg–MacLane spaces, we can assume that a given admissible presentation is obtained from a cell decomposition of M. We fix isomorphisms $\mathcal{K}_{N_k(M)}^{4g+l} \cong C_1(M, p; \mathcal{K}_{N_k(M)})$ and $\mathcal{K}_{N_k(M)}^{2g+l} \cong C_2(M, p; \mathcal{K}_{N_k(M)})$ by using the ordered bases $\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g})\rangle$ and $\langle r_1, \ldots, r_{2g+l}\rangle$ respectively. The boundary map

$$\partial \colon C_2(M, p; \mathcal{K}_{N_k(M)}) \to C_1(M, p; \mathcal{K}_{N_k(M)})$$

is given by multiplying

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

from the left. Let $e_i = (0_{(1,i-1)}, 1, 0_{(1,2g-i)})^T \in \mathcal{K}_{N_k(M)}^{2g}$, where we write $0_{(k,l)}$ for the zero matrix of size $k \times l$. By the definition of $r'_k(M)$, two 1–cycles

$$\begin{pmatrix} e_i \\ 0_{(l,1)} \\ 0_{(2g,1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0_{(2g,1)} \\ 0_{(l,1)} \\ r'_k(M)e_i \end{pmatrix}$$

in $\mathcal{K}_{N_k(M)}^{4g+l} \cong C_1(M, p; \mathcal{K}_{N_k(M)})$ are homologous, so that there exists $x_i \in \mathcal{K}_{N_k(M)}^{2g+l} \cong C_2(M, p; \mathcal{K}_{N_k(M)})$ satisfying

$$\begin{pmatrix} e_i \\ 0_{(l,1)} \\ 0_{(2g,1)} \end{pmatrix} - \begin{pmatrix} 0_{(2g,1)} \\ 0_{(l,1)} \\ r'_k(M)e_i \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} x_i \in C_1(M, p; \mathcal{K}_{N_k(M)}).$$

We put $X := (x_1, \dots, x_{2g}) \in C_2(M, p; \mathcal{K}_{N_k(M)})_{2g}$. Then we have

$$
\begin{pmatrix} I_{2g} \\ 0_{(l,2g)} \\ 0_{2g} \end{pmatrix} - \begin{pmatrix} 0_{2g} \\ 0_{(l,2g)} \\ r'_k(M) \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} X \in C_1(M, p; \mathcal{K}_{N_k(M)})_{2g},
$$

where $0_k := 0_{(k,k)}$. This equality is split into

$$
\begin{pmatrix} I_{2g} \\ 0_{(l,2g)} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} X
$$

and $-r'_k(M) = CX$. Eliminating X by using (1), we obtain the equality of (2). $\quad\square$

Next, we derive a formula for changing a basis of $\pi_1 \Sigma_{g,1}$. For $(M, i_+, i_-) \in \mathcal{C}_{g,1}$, we take an admissible presentation of $\pi_1 M$ as in Definition 3.8 and construct the matrices A, B, C as before. Let $\gamma'_1, \dots, \gamma'_{2g}$ be another basis of $\pi_1 \Sigma_{g,1}$. We can take $\varphi \in \mathrm{Aut}\, \pi_1 \Sigma_{g,1}$ such that $\gamma'_i = \varphi(\gamma_i)$ for $i = 1, \dots, 2g$. Note that

$$
\left(\frac{\partial \varphi(\gamma_i)}{\partial \gamma_j} \right)_{1 \le i,j \le 2g}
$$

is invertible as a matrix with entries in $\mathbb{Z}\pi_1 \Sigma_{g,1}$. In fact, we have

$$
\left(\frac{\partial \varphi(\gamma_i)}{\partial \gamma_j} \right)^{-1}_{1 \le i,j \le 2g} = {}^{\varphi}\left(\frac{\partial \varphi^{-1}(\gamma_i)}{\partial \gamma_j} \right)_{1 \le i,j \le 2g},
$$

$$
\overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)}^{-1}_{1 \le i,j \le 2g} = {}^{\varphi}\overline{\left(\frac{\partial \varphi^{-1}(\gamma_j)}{\partial \gamma_i} \right)}_{1 \le i,j \le 2g},
$$

which are deduced from the fundamental formula of free calculus (see Birman [2, Proposition 3.4]).

Proposition 3.10 *Let $r_k^{\varphi}(M)$ be the Magnus matrix corresponding to the new basis. Then the following holds:*

$$
r_k^{\varphi}(M) = {}^{\rho_k}\overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)}^{-1}_{1 \le i,j \le 2g} \cdot r_k(M) \cdot {}^{\sigma_k(M) \circ \rho_k}\overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)}_{1 \le i,j \le 2g}
$$

Proof We have the following admissible presentation of $\pi_1 M$ with respect to $\gamma'_1, \ldots,$ γ'_{2g}:

$$\pi_1 M \cong \left\langle \begin{array}{c} i_-(\gamma'_1), \ldots, i_-(\gamma'_{2g}) \\ i_-(\gamma_1), \ldots, i_-(\gamma_{2g}) \\ z_1, \ldots, z_l \\ i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \\ i_+(\gamma'_1), \ldots, i_+(\gamma'_{2g}) \end{array} \middle| \begin{array}{c} i_-(\gamma'_1)i_-(\varphi(\gamma_1))^{-1}, \ldots, i_-(\gamma'_{2g})i_-(\varphi(\gamma_{2g}))^{-1} \\ r_1, \ldots r_{2g+l} \\ i_+(\gamma'_1)i_+(\varphi(\gamma_1))^{-1}, \ldots, i_+(\gamma'_{2g})i_+(\varphi(\gamma_{2g}))^{-1} \end{array} \right\rangle.$$

The matrices A', B', C' corresponding to this presentation are given by

$$A' = (I_{2g} \quad 0_{(2g,2g+l)} \quad 0_{2g}),$$

$$B' = \begin{pmatrix} -\overset{i_- \circ \rho_k}{\overline{\left(\dfrac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)}}_{1 \le i,j \le 2g} & A & 0_{2g} \\ 0_{(l,2g)} & B & 0_{(l,2g)} \\ 0_{2g} & C - \overset{i_+ \circ \rho_k}{\overline{\left(\dfrac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)}}_{1 \le i,j \le 2g} \end{pmatrix},$$

$$C' = (0_{2g} \quad 0_{(2g,2g+l)} \quad I_{2g}).$$

Put $J_k := \overset{\rho_k}{\overline{\left(\dfrac{\partial \varphi(\gamma_j)}{\partial \gamma_i}\right)}}_{1 \le i,j \le 2g}$. Then

$$\begin{pmatrix} A' \\ B' \end{pmatrix}^{-1} = \left(\begin{array}{ccc|c} I_{2g} & 0_{(2g,2g+l)} & & 0_{2g} \\ -{}^{i_-}J_k & A & & 0_{2g} \\ 0_{(l,2g)} & B & & 0_{(l,2g)} \\ \hline 0_{2g} & C & & -{}^{i_+}J_k \end{array} \right)^{-1} = \left(\begin{array}{c|c} Z & 0_{(4g+l,2g)} \\ \hline {}^{i_+}J_k^{-1}\left(0_{2g} \ C\right)Z & -{}^{i_+}J_k^{-1} \end{array} \right),$$

where $\quad Z := \left(\begin{array}{c|c} I_{2g} & 0_{(2g,2g+l)} \\ -{}^{i_-}J_k & A \\ 0_{(l,2g)} & B \end{array} \right)^{-1} = \left(\begin{array}{c|c} I_{2g} & 0_{(2g,2g+l)} \\ \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \cdot \begin{pmatrix} {}^{i_-}J_k \\ 0_{(l,2g)} \end{pmatrix} & \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \end{array} \right).$

Using Proposition 3.9 (2), we have

$${}^{i_+}J_k^{-1}\left(0_{2g} \ C\right)Z = {}^{i_+}J_k^{-1}\left(C\begin{pmatrix}A\\B\end{pmatrix}^{-1} \cdot \begin{pmatrix}{}^{i_-}J_k\\0_{(l,2g)}\end{pmatrix} \ \ C\begin{pmatrix}A\\B\end{pmatrix}^{-1}\right)$$

$$= \left(-{}^{i_+}J_k^{-1} r'_k(M) \ {}^{i_-}J_k \quad {}^{i_+}J_k^{-1}C\begin{pmatrix}A\\B\end{pmatrix}^{-1} \right).$$

Therefore

$$
{}^{i_+}r_k^\varphi(M) = -C' \begin{pmatrix} A' \\ B' \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(4g+l,2g)} \end{pmatrix}
$$

$$
= -(0_{(2g,4g+l)} \ \ I_{2g}) \begin{pmatrix} Z & 0_{(4g+l,2g)} \\ {}^{i_+}J_k^{-1} (0_{2g} \ \ C) Z & -{}^{i_+}J_k^{-1} \end{pmatrix} \begin{pmatrix} I_{2g} \\ 0_{(2g+l,2g)} \\ 0_{2g} \end{pmatrix}
$$

$$
= -{}^{i_+}J_k^{-1} (0_{2g} \ \ C) Z \begin{pmatrix} I_{2g} \\ 0_{(2g+l,2g)} \end{pmatrix} = {}^{i_+}J_k^{-1} r_k'(M) \ {}^{i_-}J_k.
$$

Applying i_+^{-1} to each entry, we obtain the formula. $\qquad\qquad\qquad\qquad\square$

4 Example: Relationship to the Gassner representation for string links

In [21, Section 3], Levine gave a method for constructing homology cylinders from pure string links. By this, we can obtain many explicit examples of homology cylinders not belonging to the subgroup $\mathcal{M}_{g,1}$. Here we consider Magnus matrices of such homology cylinders and see a relationship between the Gassner representation for string links and our representation.

First, we recall the definition of pure string links and describe a noncommutative generalization of the Gassner representation, whose original commutative version was defined by Le Dimet [17] and Kirk–Livingston–Wang [16]. Note that our convention differs from theirs.

Let D^2 be a 2–dimensional disk. Given $n \geq 1$, we fix n points $p_1, \ldots, p_n$ in the interior of D^2. An n–component *pure string link* is a smooth embedding of the disjoint union of ordered n unit intervals into $D^2 \times I$ sending 0 and 1 of i–th interval to $p_i \times \{0\}$ and $p_i \times \{1\}$ respectively. The set $\mathcal{L}_n$ of isotopy classes of n–component pure string links has a natural monoid structure. For $L \in \mathcal{L}_n$, its image in $D^2 \times I$ is denoted by L again. Let $D_n^0 := D^2 - \{p_1, \ldots, p_n\}$. $\pi_1 D_n^0$ is the free group generated by n loops $\delta_1, \ldots, \delta_n$ as in Figure 2.

We write i_0 and i_1 for the natural embeddings of D_n^0 into $D^2 \times \{0\}$ and $D^2 \times \{1\}$ respectively. It is easily checked that $i_0, i_1 \colon \pi_1 D_n^0 \to \pi_1(D^2 \times I - L)$ are 2–connected for each $L \in \mathcal{L}_n$. Hence they induce isomorphisms between $N_k^0 := N_k(\pi_1 D_n^0)$ and $N_k(L) := N_k(\pi_1(D^2 \times I - L))$.
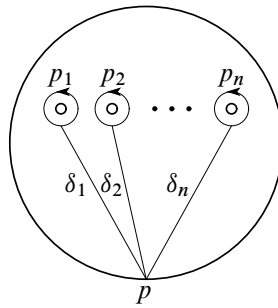
Figure 2: A standard basis of $\pi_1 D_n^0$

Fix an integer $k \geq 2$. Cochran–Orr–Teichner's argument [5, Proposition 2.10] is used to show that $i_j \colon H_1(D_n^0, p; i_j^* \mathcal{K}_{N_k(L)}) \to H_1(D^2 \times I - L, p \times I; \mathcal{K}_{N_k(L)})$ are isomorphisms for $j = 0, 1$. The natural isomorphisms $\iota_j \colon \mathcal{K}_{N_k(L)}^n \overset{\cong}{\Rightarrow} H_1(D_n^0, p; i_j^* \mathcal{K}_{N_k(L)})$ are obtained by using the above basis of $\pi_1 D_n^0$.

Definition 4.1 (1) For each $L \in \mathcal{L}_n$, we denote by $r'_{G,k}(L) \in GL(n, \mathcal{K}_{N_k(L)})$ the representation matrix of the right $\mathcal{K}_{N_k(L)}$–isomorphism

$$\mathcal{K}_{N_k(L)}^n \overset{\cong}{\underset{\iota_0}{\Rightarrow}} H_1(D_n^0, p; i_0^* \mathcal{K}_{N_k(L)}) \overset{\cong}{\underset{i_0}{\rightarrow}} H_1(D^2 \times I - L, p \times I; \mathcal{K}_{N_k(L)})$$

$$\overset{\cong}{\underset{i_1^{-1}}{\rightarrow}} H_1(D_n^0, p; i_1^* \mathcal{K}_{N_k(L)}) \overset{\cong}{\underset{\iota_1^{-1}}{\rightarrow}} \mathcal{K}_{N_k(L)}^n.$$

(2) The *Gassner representation* for $\mathcal{L}_n$ is the map $r_{G,k} \colon \mathcal{L}_n \to GL(n, \mathcal{K}_{N_k^0})$ which assigns to $L \in \mathcal{L}_n$ the matrix $i_1^{-1} r'_{G,k}(L)$. We call $r_{G,k}(L)$ the *Gassner matrix* of L.

$r_{G,k}$ is a crossed homomorphism in a suitable sense. Note that $r_{G,2}$ corresponds to the original version.

As mentioned in Kirk–Livingston–Wang [16, Section 4], for $L \in \mathcal{L}_n$, the Wirtinger presentation gives a presentation of $\pi_1(D^2 \times I - L)$ of the form

$$\langle i_0(\delta_1), \dots, i_0(\delta_n), z_1, \dots, z_l, i_1(\delta_1), \dots, i_1(\delta_n) \mid r_1, \dots, r_{n+l} \rangle.$$

Then the computation of $r_{G,k}(L)$ goes parallel to that of Magnus matrix in Section 3.2 and we omit it.

Next we recall Levine's construction [21, Section 3]. For a g–component pure string link $L \in \mathcal{L}_g$, we now construct a homology cylinder $M_L \in \mathcal{C}_{g,1}$ as follows. Consider
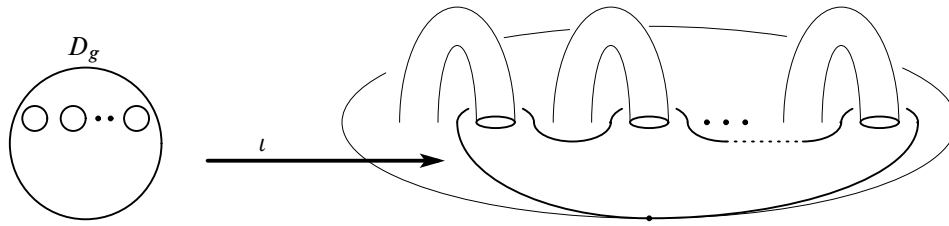
Figure 3

a closed tubular neighborhood of the loops $\gamma_{g+1}, \gamma_{g+2}, \ldots, \gamma_{2g}$ in Figure 1 to be the image of an embedding $\iota\colon D_g \hookrightarrow \Sigma_{g,1}$ of a g–holed disk D_g as in Figure 3.

Let C be the complement of an open tubular neighborhood of L in $D^2 \times I$. For each choice of a framing of L, a homeomorphism $h\colon \partial C \overset{\cong}{\Rightarrow} \partial(\iota(D_g) \times I)$ is fixed. Then the manifold M_L obtained from $\Sigma_{g,1} \times I$ by removing $\iota(D_g) \times I$ and regluing C by h becomes a homology cylinder. Levine showed that this construction gives an injective monoid homomorphism $\mathcal{L}_g \to \mathcal{C}_{g,1}$, and moreover, it induces an injective homomorphism $\mathcal{S}_g \to \mathcal{H}_{g,1}$ from the concordance group $\mathcal{S}_g$ of pure string links to $\mathcal{H}_{g,1}$. In particular, the (smooth) knot concordance group, which coincides with $\mathcal{S}_1$, is embedded in $\mathcal{H}_{g,1}$. If we restrict these embeddings to the pure braid group, which is a subgroup of $\mathcal{L}_g$ and $\mathcal{S}_g$, their images are contained in $\mathcal{M}_{g,1}$. We can easily check that $r_{G,k}$ factors through $\mathcal{S}_g$.

A relationship between the Gassner representation $r_{G,k}$ and the Magnus representation r_k is given as follows.

Theorem 4.2 *For any pure string link $L \in \mathcal{L}_g$, $r_k(M_L) = \begin{pmatrix} * & 0_g \\ * & r_{G,k}(L) \end{pmatrix}$.*

We mention two remarks about this theorem. First we identify $\pi_1 D_g \cong \pi_1 D_g^0 = \langle \delta_1, \ldots, \delta_g \rangle$ with the subgroup $\langle \gamma_{g+1}, \ldots, \gamma_{2g} \rangle$ of $\pi_1 \Sigma_{g,1}$ by the correspondence $(\delta_i \mapsto \gamma_{g+i})$ for $i = 1, \ldots g$. Then the maps $\pi_1 D_g^0 \hookrightarrow \pi_1 \Sigma_{g,1} \twoheadrightarrow \pi_1 D_g^0$, where the second map sends $\gamma_1, \ldots, \gamma_g$ to 1, show that $N_k^0 \subset N_k$ and $\mathcal{K}_{N_k^0} \subset \mathcal{K}_{N_k}$. Second, the embeddings $\mathcal{L}_g \hookrightarrow \mathcal{C}_{g,1}$ and $\mathcal{S}_g \hookrightarrow \mathcal{H}_{g,1}$ have ambiguity with respect to framings. However we can see that the lower right part of $r_k(M_L)$ is independent of the framings.

Proof of Theorem 4.2 To obtain a suitable admissible presentation of $\pi_1 M_L$, we divide M_L into two parts X and C as follows.

We take g points $q_1, \ldots, q_g$ and g paths l_j from the base point p to q_j as in Figure 4.
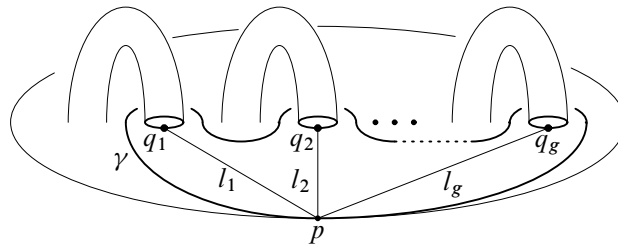
Figure 4

Let X be the union of $\overline{\Sigma_{g,1} \times I - \iota(D_g) \times I}$ and $2g$ paths $i_+(l_j)$ and $i_-(l_j)$ $(j = 1, \ldots, g)$. Then

$$\pi_1 X \cong \left\langle \begin{array}{c} i_-(\widehat{\gamma}_1), \ldots, i_-(\widehat{\gamma}_g) \\ i_+(\widehat{\gamma}_1), \ldots, i_+(\widehat{\gamma}_g) \\ i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \\ \eta_1, \ldots, \eta_g, i_+(\gamma) \end{array} \middle| \begin{array}{c} i_+(\widehat{\gamma}_1) = i_-(\widehat{\gamma}_1)\eta_1 \\ \vdots \\ i_+(\widehat{\gamma}_g) = i_-(\widehat{\gamma}_g)\eta_g \end{array} \right\rangle$$

where the loop $\widehat{\gamma}_j := [\gamma_1, \gamma_{g+1}] \cdots [\gamma_{j-1}, \gamma_{g+j-1}]\gamma_j$ satisfies $\widehat{\gamma}_j \cap \iota(D_g) = l_j$, γ is the loop corresponding to the outer boundary of $\iota(D_g)$ and η_j is the composite of paths $i_-(l_j)$, $\overrightarrow{i_-(q_j)i_+(q_j)}$ and $i_+(l_j^{-1})$. We denote by C the complement of an open tubular neighborhood of L in $D^2 \times I$ as before.

$$\pi_1 C \cong \langle i_-(\gamma_{g+1}), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots r_{g+l} \rangle$$

is given by the Wirtinger presentation of $D^2 \times I - L$. We glue C to X by using a fixed framing. Then it is easily seen that $\pi_1(X \cap C)$ is the free group generated by $\{i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}), \eta_1, \ldots, \eta_g, i_+(\gamma)\}$.

Using the above decomposition, we obtain

$$\pi_1 M_L \cong \left\langle \begin{array}{c} i_-(\widehat{\gamma}_1), \ldots, i_-(\widehat{\gamma}_g) \\ i_-(\gamma_{g+1}), \ldots, i_-(\gamma_{2g}) \\ z_1, \ldots, z_l \\ i_+(\widehat{\gamma}_1), \ldots, i_+(\widehat{\gamma}_g) \\ i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g}) \end{array} \middle| \begin{array}{c} i_+(\widehat{\gamma}_1) = i_-(\widehat{\gamma}_1)\widehat{\eta}_1 \\ \vdots \\ i_+(\widehat{\gamma}_g) = i_-(\widehat{\gamma}_g)\widehat{\eta}_g \\ r_1, \ldots r_{g+l} \end{array} \right\rangle$$

where each $\widehat{\eta}_i$ is a word in $i_-(\gamma_{g+1}), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_{g+1}), \ldots, i_+(\gamma_{2g})$ which depends on the framing. Rewrite the above presentation by using $i_+(\gamma_j)$'s and $i_-(\gamma_j)$'s instead of $i_+(\widehat{\gamma}_j)$'s and $i_-(\widehat{\gamma}_j)$'s. This process does not affect generators $i_-(\gamma_{g+j}), z_j, i_+(\gamma_{g+j})$ and relations r_j. From the resulting admissible presentation, we can compute the Magnus matrix of M_L as seen in Section 3.2. So our claim follows

by comparing it with the method for calculating $r_{G,k}(L)$. (The most important point is that the admissible presentation of $\pi_1 M_L$ obtained contains the Wirtinger presentation of $\pi_1 C = \pi_1(D^2 \times I - L)$.) It is straightforward and we omit the details. □

Corollary 4.3 $\mathcal{M}_{g,1}$ *is not a normal subgroup of* $\mathcal{H}_{g,1}$ *for* $g \geq 3$.

Proof In [16, Section 7], Kirk–Livingston–Wang gave 3–component pure string links denoted by L_5 and L_6 having the condition that L_5 is a pure braid, while the conjugate $L_6 L_5 L_6^{-1} \in \mathcal{S}_3$ is not. To show that $L_6 L_5 L_6^{-1}$ is not a pure braid, they use the fact that $r_{G,2}(L_6 L_5 L_6^{-1})$ has an entry not belonging to $\mathbb{Z} N_2(D_3^0)$. Then our claim follows by applying Theorem 4.2 to this example. □

Example 4.4 Let L be a 2–component pure string link as depicted in the left of Figure 5.
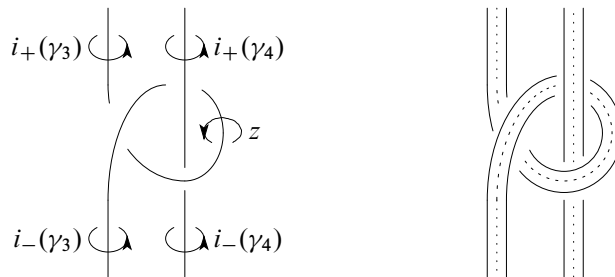

Figure 5: On the left, a 2–component pure string link L and on the right, a tubular neighborhood of L

Then the presentation of $\pi_1 M_L$ is given by

$$\pi_1 M_L \cong \left\langle \begin{array}{c} i_-(\gamma_1), \ldots, i_-(\gamma_4) \\ z \\ i_+(\gamma_1), \ldots, i_+(\gamma_4) \end{array} \middle| \begin{array}{l} i_+(\gamma_1) i_-(\gamma_3)^{-1} i_+(\gamma_4) i_-(\gamma_1)^{-1}, \\ [i_+(\gamma_1), i_+(\gamma_3)] i_+(\gamma_2) z i_-(\gamma_2)^{-1} [i_-(\gamma_3), i_-(\gamma_1)], \\ i_+(\gamma_4) i_-(\gamma_3) i_+(\gamma_4)^{-1} z^{-1}, \\ i_-(\gamma_3) i_+(\gamma_3)^{-1} i_-(\gamma_3)^{-1} z, \\ i_-(\gamma_4) z^{-1} i_+(\gamma_4)^{-1} z \end{array} \right\rangle,$$

where we use the blackboard framing, namely the left dotted line in the right of Figure 5 is glued to $\{q_1\} \times I$ and the right one is glued to $\{q_2\} \times I$ in Figure 4 (with $g = 2$). We now compute the Magnus matrix $r_2(M_L)$. We identify N_2 and $N_2(M_L)$ by using i_+. From the presentation, we have $z = i_-(\gamma_3) = \gamma_3$, $i_-(\gamma_4) = \gamma_4$, $i_-(\gamma_2) = \gamma_2 \gamma_3$

and $i_-(\gamma_1) = \gamma_1 \gamma_3^{-1} \gamma_4$ in N_2. Then

$$
\binom{A}{B} = \begin{pmatrix}
-1 & \gamma_3^{-1} - 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-\gamma_1^{-1}\gamma_3 & 1 - \gamma_1^{-1}\gamma_3\gamma_4^{-1} & \gamma_4^{-1} & 1 - \gamma_3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & \gamma_2^{-1} & -1 & \gamma_3 & \gamma_3 - \gamma_3\gamma_4^{-1}
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
1 & 1 - \gamma_3^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & \gamma_1^{-1} - 1 & 0 & -1 & 0 \\
\gamma_1^{-1}\gamma_3 & 0 & 1 - \gamma_3^{-1} & 0 & -\gamma_3
\end{pmatrix}.
$$

Hence the Magnus matrix

$$
r_2(M_L) = -C \binom{A}{B}^{-1} \binom{I_4}{0_{(1,4)}}
$$

is given by

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\[2mm]
0 & 1 & 0 & 0 \\[2mm]
\dfrac{-\gamma_1^{-1}}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1} - \gamma_4^{-1} + 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{\gamma_3^{-1}}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{\gamma_4^{-1}(\gamma_4^{-1} - 1)}{\gamma_3^{-1} + \gamma_4^{-1} - 1} \\[3mm]
\dfrac{\gamma_1^{-1}\gamma_3\gamma_4^{-1}}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{(1 - \gamma_3^{-1})(\gamma_2^{-1}\gamma_3^{-1} - \gamma_2^{-1} - 1)}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{\gamma_3^{-1} - 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1} & \dfrac{-\gamma_3^{-1}\gamma_4^{-1} + \gamma_3^{-1} + 2\gamma_4^{-1} - 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1}
\end{pmatrix}.
$$

5 Higher-order Alexander invariants and torsion-degree functions

In this section, we first summarize the theory of higher-order Alexander invariants along the lines of Harvey's papers [12; 13]. For our use, we generalize them to functions of matrices called *torsion-degree functions*.

A group Γ is *poly-torsion-free-abelian* (PTFA, for short) if Γ has a normal series of finite length whose successive quotients are all torsion-free abelian. In particular, free nilpotent quotients N_k are PTFA for all $k \geq 2$. Note that every subgroup of a PTFA group is also PTFA. For each PTFA group Γ, the group ring $\mathbb{Z}\Gamma$ is known to be an Ore domain, so that it can be embedded in the *right field of fractions* $\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1}$,

which is a skew field. We refer to Cohn [6] and Passman [26] for localizations of noncommutative rings.

We will also use the following localizations of $\mathbb{Z}\Gamma$ placed between $\mathbb{Z}\Gamma$ and $\mathcal{K}_\Gamma$. Let $\psi\colon \Gamma \twoheadrightarrow \mathbb{Z}$ be an epimorphism. Then we have an exact sequence

$$1 \longrightarrow (\Gamma^\psi := \operatorname{Ker}\psi) \longrightarrow \Gamma \xrightarrow{\psi} \mathbb{Z} \longrightarrow 1.$$

We take a splitting $\xi\colon \mathbb{Z} \to \Gamma$ and put $\tilde{t} := \xi(1) \in \Gamma$. Since Γ^ψ is again a PTFA group, $\mathbb{Z}\Gamma^\psi$ can be embedded in its right field of fractions $\mathcal{K}_{\Gamma^\psi} = \mathbb{Z}\Gamma^\psi(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$. Moreover, we can also construct $\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$. Then the splitting ξ gives an isomorphism between $\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$ and the skew Laurent polynomial ring $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$, in which $at = t(\tilde{t}^{-1}a\tilde{t})$ holds for each $a \in \mathcal{K}_{\Gamma^\psi}$. $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$ is known to be a noncommutative right and left principal ideal domain. By definition, we have inclusions

$$\mathbb{Z}\Gamma \hookrightarrow \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}] \hookrightarrow \mathcal{K}_\Gamma.$$

$\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$ and $\mathcal{K}_\Gamma$ are known to be flat $\mathbb{Z}\Gamma$–modules. On $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$, we have a map $\deg^\psi\colon \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}] \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ assigning to each polynomial its degree. We put $\deg^\psi(0) := \infty$. By this, $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$ becomes a Euclidean domain. The composite

$$\mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1} \xrightarrow{\cong} \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}] \xrightarrow{\deg^\psi} \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

does not depend on the choice of ξ.

Harvey's higher-order Alexander invariants [13] are defined as follows. Let G be a finitely presentable group and let $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. For a PTFA group Γ and an epimorphism $\varphi_\Gamma\colon G \twoheadrightarrow \Gamma$, $(\varphi_\Gamma, \varphi)$ is called an *admissible pair* for G if there exists an epimorphism $\psi\colon \Gamma \twoheadrightarrow \mathbb{Z}$ satisfying $\varphi = \psi \circ \varphi_\Gamma$. For each admissible pair $(\varphi_\Gamma, \varphi)$ for G, we regard $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}] = \mathbb{Z}\Gamma(\mathbb{Z}\Gamma^\psi - \{0\})^{-1}$ as a $(\mathbb{Z}G, \mathcal{K}_{\Gamma^\psi})$– or $(\mathbb{Z}G, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$–module and we define the higher-order Alexander invariant for $(\varphi_\Gamma, \varphi)$ by

$$\overline{\delta}_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}\left(H_1(G; \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])\right) \in \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

$$\delta_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}\left(T_{\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]}H_1(G; \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])\right) \in \mathbb{Z}_{\geq 0},$$

where $T_{\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]}M$ denotes the $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–torsion part of a right $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module M. $\overline{\delta}_\Gamma^\psi(G)$ and $\delta_\Gamma^\psi(G)$ are called the Γ–*degree* and the *refined* Γ–*degree* respectively. (Our definition and notation are slightly different from those of [13], where the Γ–degree is defined to be zero when it is ∞ in our definition.) Note that the right

$\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]$–module $H_1(G; \mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}])$ can be decomposed into

$$H_1(G; \mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]) = (\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}])^r \oplus \left(\bigoplus_{i=1}^{l} \frac{\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]}{p_i(t)\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]} \right)$$

for some $r \in \mathbb{Z}_{\geq 0}$ and $p_i(t) \in \mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]$, then

$$\overline{\delta}_{\Gamma}^{\psi}(G) = \begin{cases} \sum_{i=1}^{l} \deg^{\psi}(p_i(t)) & (r = 0), \\ \infty & (r > 0), \end{cases}$$

$$\delta_{\Gamma}^{\psi}(G) = \sum_{i=1}^{l} \deg^{\psi}(p_i(t)).$$

For a connected space X and an admissible pair $(\varphi_{\Gamma}, \varphi)$ for $\pi_1 X$, we define $\overline{\delta}_{\Gamma}^{\psi}(X) := \overline{\delta}_{\Gamma}^{\psi}(\pi_1 X)$ and $\delta_{\Gamma}^{\psi}(X) := \delta_{\Gamma}^{\psi}(\pi_1 X)$.

For a finitely presentable group G and an admissible pair $(\varphi_{\Gamma}, \varphi)$ for G, the (refined) Γ–degree can be computed from any presentation matrix of the right $\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]$–module $H_1(G; \mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}])$. Therefore we can consider it to be a function on the set $M(\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}])$ of all matrices with entries in $\mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}]$. Here we extend this function to $M(\mathcal{K}_{\Gamma})$ as follows.

First, we extend $\deg^{\psi}: \mathcal{K}_{\Gamma^{\psi}}[t^{\pm 1}] \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to $\deg^{\psi}: \mathcal{K}_{\Gamma} \rightarrow \mathbb{Z} \cup \{\infty\}$ by setting $\deg^{\psi}(fg^{-1}) = \deg^{\psi}(f) - \deg^{\psi}(g)$ for $f \in \mathbb{Z}\Gamma, g \in \mathbb{Z}\Gamma - \{0\}$ (see for instance Cohn [6, Proposition 9.1.1]). It induces a group homomorphism $\deg^{\psi}: (\mathcal{K}_{\Gamma}^{\times})_{ab} \rightarrow \mathbb{Z}$, where $(\mathcal{K}_{\Gamma}^{\times})_{ab}$ is the abelianization of the multiplicative group $\mathcal{K}_{\Gamma}^{\times} = \mathcal{K}_{\Gamma} - \{0\}$.

Since $\mathcal{K}_{\Gamma}$ is a skew field, we have the Dieudonné determinant (see Rosenberg [27, Chapter 2])

$$\det: GL(\mathcal{K}_{\Gamma}) \longrightarrow (\mathcal{K}_{\Gamma}^{\times})_{ab},$$

which is a homomorphism characterized by the following three properties:

(a) $\det I = 1$.

(b) If A' is obtained by multiplying a row of a matrix $A \in GL(\mathcal{K}_{\Gamma})$ by $a \in \mathcal{K}_{\Gamma}^{\times}$ from the left, then $\det A' = a \cdot \det A$.

(c) If A' is obtained by adding to a row of a matrix A a left $\mathcal{K}_{\Gamma}$–linear combination of other rows, then $\det A' = \det A$.

It induces an isomorphism between $K_1(\mathcal{K}_{\Gamma}) \overset{\cong}{\Rightarrow} (\mathcal{K}_{\Gamma}^{\times})_{ab}$.

The following lemma will be used in our generalization of Harvey's invariants. We denote by $M(m, n, \mathcal{K}_{\Gamma})$ the set of all $m \times n$ matrices with entries in $\mathcal{K}_{\Gamma}$.

Lemma 5.1 *For $A \in M(m, n, \mathcal{K}_\Gamma)$ with $\mathrm{rank}_{\mathcal{K}_\Gamma} A = k$, let $U \in M(m-k, m, \mathcal{K}_\Gamma)$, $V \in M(n, n-k, \mathcal{K}_\Gamma)$ be matrices satisfying*

$$\begin{cases} UA = 0, & \mathrm{rank}_{\mathcal{K}_\Gamma} U = m - k, \\ AV = 0, & \mathrm{rank}_{\mathcal{K}_\Gamma} V = n - k. \end{cases}$$

For each $I \subset \{1, 2, \ldots, m\}$, $J \subset \{1, 2, \ldots, n\}$ with $\#I = m - k$, $\#J = n - k$, let U_I denote the square matrix defined by taking i–th columns from U for all $i \in I$, and V_J denote the one defined by taking j–th rows from V for all $j \in J$. We also denote by $A_{I^c J^c}$ the one defined by taking i–th rows from A for all $i \in I^c := \{1, 2, \ldots, m\} - I$ and then taking j–th columns for all $j \in J^c := \{1, 2, \ldots, n\} - J$.

(1)   *If U_I or V_J is not invertible, then $A_{I^c J^c}$ is not invertible.*

(2)   *Otherwise,*

$$\Delta(A; U, V) := \mathrm{sgn}(II^c)\, \mathrm{sgn}(JJ^c) \frac{\det A_{I^c J^c}}{\det U_I \det V_J} \in (\mathcal{K}_\Gamma^\times)_{\mathrm{ab}}$$

is independent of the choice of I and J such that U_I, V_J are invertible, where $\mathrm{sgn}(II^c) \in \{\pm 1\}$ (resp. $\mathrm{sgn}(JJ^c)$) is the signature of the juxtaposition of I and I^c (resp. J and J^c), and we put $\det \varnothing := 1$.

(3)   *For $P_1 \in GL(m, \mathcal{K}_\Gamma)$, $P_2 \in GL(n, \mathcal{K}_\Gamma)$, $Q_1 \in GL(m - k, \mathcal{K}_\Gamma)$ and $Q_2 \in GL(n - k, \mathcal{K}_\Gamma)$,*

$$\Delta(P_1^{-1} A P_2^{-1}; Q_1 U P_1, P_2 V Q_2) = \frac{\Delta(A; U, V)}{\det P_1 \det P_2 \det Q_1 \det Q_2}.$$

Proof   (1) and (2) are deduced from easy observation using noncommutative linear algebra. To prove (3), it suffices to show in the cases where P_1, P_2, Q_1, Q_2 are matrices of elementary transformations, and it can be easily checked. $\square$

Remark 5.2   In the above situation, the sequence

$$0 \longrightarrow \mathcal{K}_\Gamma^{n-k} \xrightarrow{V \cdot} \mathcal{K}_\Gamma^n \xrightarrow{A \cdot} \mathcal{K}_\Gamma^m \xrightarrow{U \cdot} \mathcal{K}_\Gamma^{m-k} \longrightarrow 0$$

is exact. By taking the standard basis for each vector space, we regard the sequence as a based acyclic chain complex. Then we can take its torsion (see Milnor [23] and Turaev [31] for generalities of torsions). This torsion coincides with $\Delta(A; U, V)$ up to sign.

As seen in Lemma 5.1 (3), $\Delta(A; U, V)$ does depend on U and V. The following definition and lemma give special choices of U and V.

Definition 5.3 Let $A \in M(m, n, \mathcal{K}_\Gamma)$ with $\mathrm{rank}_{\mathcal{K}_\Gamma} A = k$. (U, V) is said to be ψ–*primitive* for A if

(1) U, V have entries in $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$,

(2) the row vectors $u_1, \ldots, u_{m-k} \in (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])_m$ of U generate $\mathrm{Ker}(\cdot A) \cap (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])_m$ in $(\mathcal{K}_\Gamma)_m$ as a left $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module,

(3) the column vectors $v_1, \ldots, v_{n-k} \in (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n$ of V generate $\mathrm{Ker}(A \cdot) \cap (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n$ in $(\mathcal{K}_\Gamma)^n$ as a right $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module.

Lemma 5.4 (1) *There exists a pair (U, V) which is ψ–primitive for A.*

(2) *Suppose $U \in M(m - k, m, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ and $V \in M(n, n - k, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ satisfy $UA = 0$ and $AV = 0$. (U, V) is ψ–primitive for A if and only if there exist $\widetilde{P}_1 \in GL(m, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ and $\widetilde{P}_2 \in GL(n, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ satisfying*

$$U \widetilde{P}_1 = (0_{(m-k,k)} \quad I_{m-k}), \quad \widetilde{P}_2 V = (0_{(n-k,k)} \quad I_{n-k})^T.$$

(3) *Suppose (U, V) and (U', V') are ψ–primitive for A. Then there exist $P_1 \in GL(m, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$, $P_2 \in GL(n, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$, $Q_1 \in GL(m - k, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ and $Q_2 \in GL(n - k, \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ such that*

$$UP_1 = U', \quad P_2 V = V', \quad Q_1 U = U', \quad VQ_2 = V'.$$

Proof We prove only for V. The proof for U is similar.

(1) For right $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–homomorphisms

$$(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n \overset{i}{\hookrightarrow} \mathcal{K}_\Gamma^n \overset{A\cdot}{\longrightarrow} \mathcal{K}_\Gamma^m,$$

$\mathrm{Ker}((A \cdot) \circ i) = \mathrm{Ker}(A \cdot) \cap (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n$ is a right free $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module of rank $n - k$. We take a basis $v_1, \ldots, v_{n-k}$ and put $V = (v_1, \ldots, v_{n-k})$. Then V satisfies the desired property.

(2) Suppose V generates $\mathrm{Ker}(A \cdot) \cap (\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n$. The quotient module $(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n / \mathrm{Ker}((A \cdot) \circ i)$ is $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–torsion free, and hence $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–free. Taking a splitting, we have a direct sum decomposition $(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n \cong ((\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n / \mathrm{Ker}((A \cdot) \circ i)) \oplus \mathrm{Ker}((A \cdot) \circ i)$. We can extend V to obtain a basis $(\widetilde{v}_1, \ldots, \widetilde{v}_k, V)$ for $(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])^n$. Then $\widetilde{P}_2 := (\widetilde{v}_1, \ldots, \widetilde{v}_k, V)^{-1}$ satisfies $\widetilde{P}_2 V = (0_{(n-k,k)} \quad I_{n-k})^T$. The inverse is clear.

(3) The existence of P_2 follows immediately from (2). That of Q_2 is also clear since V and V' are bases of the same right $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module. $\square$

Definition 5.5 Let Γ be a PTFA group and let $\psi\colon \Gamma \twoheadrightarrow \mathbb{Z}$ be an epimorphism.

(1) The *torsion-degree function* $d_\Gamma^\psi\colon M(\mathcal{K}_\Gamma) \to \mathbb{Z}$ is defined by

$$d_\Gamma^\psi(A) := \deg^\psi(\Delta(A; U, V))$$

for a pair (U, V) which is ψ–primitive for A.

(2) The *truncated torsion-degree function* $\overline{d}_\Gamma^\psi\colon M(\mathcal{K}_\Gamma) \to \mathbb{Z} \cup \{\infty\}$ is defined by

$$\overline{d}_\Gamma^\psi(A) := \begin{cases} d_\Gamma^\psi(A) & \text{if } \operatorname{rank} A \geq m - 1, \\ \infty & \text{otherwise} \end{cases}$$

for $A \in M(m, n, \mathcal{K}_\Gamma)$.

That d_Γ^ψ and $\overline{d}_\Gamma^\psi$ are well-defined follows from Lemma 5.1 and Lemma 5.4 together with the following lemma.

Lemma 5.6 *For any $P \in GL(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ and ψ, we have $\deg^\psi(\det P) = 0$.*

Proof Since $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$ is a Euclidean domain, every $P \in GL(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ can be decomposed as products of elementary matrices and diagonal matrices in $GL(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$, each of which, say Q, satisfies $\deg^\psi(\det Q) = 0$. $\qquad\square$

Example 5.7 (1) For $A \in GL(\mathcal{K}_\Gamma)$, we have $d_\Gamma^\psi(A) = \overline{d}_\Gamma^\psi(A) = \deg^\psi(\det A)$. In particular, $d_\Gamma^\psi(P) = \overline{d}_\Gamma^\psi(P) = 0$ holds for any $P \in GL(\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$ (hence, for any $P \in GL(\mathbb{Z}\Gamma)$) and ψ by Lemma 5.6.

(2) Let M be a finitely generated right $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–module, and let A be a presentation matrix of M. Then we have $d_\Gamma^\psi(A) = \dim_{\mathcal{K}_{\Gamma^\psi}}\left(T_{\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]}M\right)$. As for $\overline{d}_\Gamma^\psi(A)$, we can see that $\overline{d}_\Gamma^\psi(A) \in \mathbb{Z}$ if and only if the rank of the $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$–free part of M is less than 2.

(3) Let G be a finitely presentable group. We take a presentation $\langle x_1, \ldots, x_l \mid r_1, \ldots, r_m \rangle$ of G. For an admissible pair $(\varphi_\Gamma, \varphi)$, the matrix

$$A := \overline{{}^{\varphi_\Gamma}\!\left(\frac{\partial r_j}{\partial x_i}\right)}_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}}$$

at $\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]$ gives a presentation matrix of $H_1(G, \{1\}; \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])$. Then Harvey's invariants are given by

$$\delta_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}\left(T_{\mathcal{K}_{\Gamma^\psi}[t^{\pm 1}]}H_1(G; \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])\right) = d_\Gamma^\psi(A),$$

$$\overline{\delta}_\Gamma^\psi(G) = \dim_{\mathcal{K}_{\Gamma^\psi}}\left(H_1(G; \mathcal{K}_{\Gamma^\psi}[t^{\pm 1}])\right) = \overline{d}_\Gamma^\psi(A),$$

where the second equality of each case follows from the direct sum decomposition

$$H_1(G, \{1\}; \mathcal{K}_{\Gamma\psi}[t^{\pm 1}]) \cong H_1(G; \mathcal{K}_{\Gamma\psi}[t^{\pm 1}]) \oplus \mathcal{K}_{\Gamma\psi}[t^{\pm 1}]$$

shown by Harvey in [12, Proposition 5.6].

Remark 5.8 Friedl [8] gave an interpretation of Harvey's invariants by Reidemeister torsions. The definition of our truncated torsion-degree functions has some overlaps with his description. Note also that the restriction of d_Γ^ψ or $\bar{d}_\Gamma^\psi$ to $GL(\mathcal{K}_\Gamma)$ coincides with the degree function in Friedl–Harvey [14, Section 2].

6 Applications of torsion-degree functions to homology cylinders

In this section, we study some invariants of homology cylinders arising from the Magnus representation and Reidemeister torsions by using torsion-degree functions associated with nilpotent quotients N_k of $\pi_1 \Sigma_{g,1}$. N_k is known to be a finitely generated torsion-free nilpotent group. In particular, it is PTFA.

Note that we can take a primitive element of $H^1(\Sigma_{g,1})$ instead of an epimorphism $N_k \twoheadrightarrow \mathbb{Z}$ to define a torsion-degree function since $\mathrm{Hom}(N_k, \mathbb{Z}) = H^1(N_k) = H^1(N_2) = H^1(\Sigma_{g,1})$. We denote by $PH_1(\Sigma_{g,1})$ the set of primitive elements of $H^1(\Sigma_{g,1})$.

6.1 The Magnus representation and torsion-degree functions

First, we apply torsion-degree functions to the Magnus matrix. However, it turns out that the result is trivial.

Theorem 6.1 Let M be a homology cylinder. For any $\psi \in PH^1(\Sigma_{g,1})$, the torsion-degree $d_{N_k}^\psi(r_k(M))$ is always zero.

Proof By definition, $d_{N_k}^\psi$ is additive for products of invertible matrices, and invariant under taking the transpose and operating the involution. Moreover, it vanishes for matrices in $GL(\mathbb{Z}N_k)$ as seen in Example 5.7 (1). In [29], we show that there exists a matrix $\tilde{J} \in GL(2g, \mathbb{Z}N_k)$ satisfying the equality

$$\overline{r_k(M)^T} \, \tilde{J} \, r_k(M) = {}^{\sigma_k(M)}\tilde{J}.$$

By applying $d_{N_k}^\psi$ to it, we obtain $2d_{N_k}^\psi(r_k(M)) = 0$. This completes the proof. $\qquad\square$

Example 6.2 Consider the homology cylinder M_L in Example 4.4. $d_{N_2}^\psi(r_2(M_L))$ is given by the degree of

$$\det r_2(M_L) = \frac{\gamma_3 + \gamma_4 - 1}{\gamma_3 \gamma_4 (\gamma_3^{-1} + \gamma_4^{-1} - 1)}$$

with respect to ψ. It is zero.

To extract some information from $r_k(M)$, we next apply torsion-degree functions to $I_{2g} - r_k(M)$. The functions $d_{N_k}^\psi(I_{2g} - r_k(\cdot)): \mathcal{C}_{g,1} \to \mathbb{Z}$ and $\bar{d}_{N_k}^\psi(I_{2g} - r_k(\cdot)): \mathcal{C}_{g,1} \to \mathbb{Z} \cup \{\infty\}$ factor through $\mathcal{H}_{g,1}$ since r_k does. Hence we can regard them as numerical invariants for $\mathcal{H}_{g,1}$. In the rest of this subsection, we derive an explicit formula of $\bar{d}_{N_k}^\psi(I_{2g} - r_k(\cdot))$ under the assumption that $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$. Note that for every $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$, two inclusions i_+ and i_- induce the same isomorphism $i_+ = i_-: N_k \overset{\cong}{\Rightarrow} N_k(M)$, so that we can naturally identify them. Under this identification, we have the following.

Lemma 6.3 Let M be a homology cylinder belonging to $\mathcal{C}_{g,1}[k]$.

(1) $(1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1})(I_{2g} - r_k(M)) = 0$.

(2) $(I_{2g} - r_k(M))\overline{\left(\frac{\partial \zeta}{\partial \gamma_1}, \ldots, \frac{\partial \zeta}{\partial \gamma_{2g}}\right)}^T = 0$.

Proof We take an admissible presentation of $\pi_1 M$ as in Definition 3.8. We also take the matrices $A, B, C \in M(\mathbb{Z}N_k)$ corresponding to it. For simplicity, we put

$$\overrightarrow{1 - \vec{\gamma}} := (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}),$$

$$\overrightarrow{1 - \vec{z}} := (1 - z_1^{-1}, \ldots, 1 - z_l^{-1}),$$

$$\frac{\partial \zeta}{\partial \vec{\gamma}} := \left(\frac{\partial \zeta}{\partial \gamma_1}, \ldots, \frac{\partial \zeta}{\partial \gamma_{2g}}\right).$$

(1) Using the fundamental formula of free calculus, we have

$$(\overrightarrow{1 - \vec{\gamma}} \quad \overrightarrow{1 - \vec{z}} \quad \overrightarrow{1 - \vec{\gamma}})\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0.$$

Then, by Proposition 3.9,

$$(\overrightarrow{1 - \vec{\gamma}} \quad \overrightarrow{1 - \vec{z}}) = -(\overrightarrow{1 - \vec{\gamma}})C\begin{pmatrix} A \\ B \end{pmatrix}^{-1} = (\overrightarrow{1 - \vec{\gamma}})(r_k(M) \quad Z)$$

for some $Z \in M(2g, l, \mathcal{K}_{N_k})$. Our claim follows by taking their first 2g columns.

(2)   Let $\tau_\zeta \in \mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$ be the Dehn twist along ζ. It belongs to the center of $\mathcal{C}_{g,1}$ and acts on N_k by conjugation $x \mapsto \zeta^{-1} x \zeta$. Then

$$
\begin{aligned}
r_k(M) &= r_k(\tau_\zeta^{-1} M \tau_\zeta) \\
&= r_k(\tau_\zeta^{-1}) \cdot {}^{\sigma_k(\tau_\zeta^{-1})} r_k(M \tau_\zeta) \\
&= {}^{\sigma_k(\tau_\zeta^{-1})} r_k(\tau_\zeta)^{-1} \cdot {}^{\sigma_k(\tau_\zeta^{-1})}\left(r_k(M) \cdot {}^{\sigma_k(M)} r_k(\tau_\zeta) \right) \\
&= {}^{\sigma_k(\tau_\zeta^{-1})}\left(r_k(\tau_\zeta)^{-1} \cdot r_k(M) \cdot r_k(\tau_\zeta) \right) \\
&= (\zeta I_{2g}) \cdot r_k(\tau_\zeta)^{-1} \cdot r_k(M) \cdot r_k(\tau_\zeta) \cdot (\zeta^{-1} I_{2g})
\end{aligned}
$$

where the fourth equality follows from the fact that M acts on N_k trivially. On the other hand, it is easily checked that

$$
r_k(\tau_\zeta) = \left(I_{2g} - \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) \right) (\zeta I_{2g})
$$

by using free differentials. Then

$$
\begin{aligned}
r_k(M) &= \left(I_{2g} - \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) \right)^{-1} r_k(M) \left(I_{2g} - \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) \right) \\
\implies & \left(I_{2g} - \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) \right) r_k(M) = r_k(M) \left(I_{2g} - \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) \right) \\
\implies & \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) r_k(M) = r_k(M) \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}).
\end{aligned}
$$

From (1), we see

$$
\overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}) r_k(M) = \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (\overrightarrow{1-\overrightarrow{\gamma}}).
$$

Comparing first columns, we have

$$
\overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (1 - \gamma_1^{-1}) = r_k(M) \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \, (1 - \gamma_1^{-1}).
$$

(2) follows from this. $\qquad\qquad\square$

Proposition 6.4   If $M \in \mathcal{C}_{g,1}[k]$ satisfies $\mathrm{rank}_{\mathcal{K}_{N_k}}(I_{2g} - r_k(M)) = 2g - 1$, then

$$
\overline{d}_{N_k}^{\psi}(I_{2g} - r_k(M)) = \deg^{\psi}\left(\Delta\left(I_{2g} - r_k(M); \overrightarrow{1-\overrightarrow{\gamma}}, \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} \, T \right) \right).
$$

Otherwise $\overline{d}_{N_k}^{\psi}(I_{2g} - r_k(M)) = \infty$.

Proof   By Lemma 6.3, the rank of $I_{2g} - r_k(M)$ is at most $2g - 1$. The case where $\mathrm{rank}_{\mathcal{K}_{N_k}}(I_{2g} - r_k(M)) < 2g - 1$ is clear from Definition 5.5 (2). Suppose that

$\mathrm{rank}_{\mathcal{K}_{N_k}}(I_{2g} - r_k(M)) = 2g - 1$. The task is to show that

$$(\overrightarrow{1 - \overrightarrow{\gamma}}) \quad \text{and} \quad \overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} T$$

satisfy the conditions (2) and (3) of Definition 5.3 respectively. Suppose $(\overrightarrow{1 - \overrightarrow{\gamma}})$ does not satisfy (2). Then $(\overrightarrow{1 - \overrightarrow{\gamma}})$ can be written as $(\overrightarrow{1 - \overrightarrow{\gamma}}) = fv$, where $v \in (\mathcal{K}_{N_k^\psi}[t^{\pm 1}])_{2g}$ and $f \in \mathcal{K}_{N_k^\psi}[t^{\pm 1}]$ with $\deg^\psi(f) \geq 1$. Hence each entry of $(\overrightarrow{1 - \overrightarrow{\gamma}})$ has degree greater than 0. When $A \in GL(\mathbb{Z} N_k)$, the same holds for each entry of $(\overrightarrow{1 - \overrightarrow{\gamma}})A$ which is nonzero. However, if we choose $f \in \mathrm{Aut}\, F_{2g}$ such that $\psi(f(\gamma_1)) = 0$, which does exist,

$$(1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}) \overline{\left(\frac{\partial f(\gamma_j)}{\partial \gamma_i} \right)}_{i,j} = (1 - f(\gamma_1)^{-1}, \ldots, 1 - f(\gamma_{2g})^{-1}),$$

a contradiction. Hence $(\overrightarrow{1 - \overrightarrow{\gamma}})$ satisfies (2). By a similar argument, we can show that

$$\overline{\frac{\partial \zeta}{\partial \overrightarrow{\gamma}}} T$$

satisfies (3), where we use $f \in \mathrm{Aut}\, F_{2g}$ preserving ζ and the chain rule for free differentials (see Birman [2, Proposition 3.3]). $\square$

Note that $\overline{d}_{N_k}^\psi(I_{2g} - r_k(M))$ does not depend on the choice of a basis of $\pi_1 \Sigma_{g,1}$. This follows from the formulas in Proposition 3.10 and Lemma 5.1 (3). In Section 6.3.3, we will see some examples showing that this invariant is nontrivial at least in the case of $k = 2$ for any $g \geq 1$.

6.2 N_k–torsion and torsion-degree functions

Let $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$. Since the relative complex $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k(M)})$ obtained from any smooth triangulation of $(M, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.1, we can consider its Reidemeister torsion $\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k(M)}))$.

Definition 6.5   The N_k–torsion of a homology cylinder $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ is given by

$$\tau_{N_k}(M) := {}^{i_+^{-1}}\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k(M)})) \in K_1(\mathcal{K}_{N_k})/(\pm N_k).$$

Recall that Reidemeister torsions are invariant under subdivision of the cell complex $(M, i_+(\Sigma_{g,1}))$ and simple homotopy equivalence.

Now we consider $\tau_{N_k}(M)$ more closely. First we give a cell decomposition of $\partial M \cong \Sigma_{g,1} \cup (-\Sigma_{g,1})$ by pasting two copies of that of $\Sigma_{g,1}$ in Figure 1. We denote by R_{2g}^+ the

subcomplex $i_+(R_{2g})$. Take a triangulation of ∂M by refining the cell decomposition, and extend it to one of M. Then

$$\tau(C_*(M, R_{2g}^+; \mathcal{K}_{N_k(M)})) = \tau(C_*(i_+(\Sigma_{g,1}), R_{2g}^+; \mathcal{K}_{N_k(M)}))$$
$$\cdot \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k(M)}))$$
$$= \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k(M)}))$$

by the multiplicativity of torsions (see Milnor [23, Theorem 3.1], for example) and the fact that $i_+(\Sigma_{g,1})$ is simple homotopy equivalent to R_{2g}^+.

Starting from a 3–simplex of M facing the boundary, we can deform M onto a 2–dimensional subcomplex M' by a simple homotopy equivalence keeping the 1–skeleton of M invariant. Then $\tau(C_*(M, R_{2g}^+; \mathcal{K}_{N_k(M)})) = \tau(C_*(M', R_{2g}^+; \mathcal{K}_{N_k(M)}))$. Next, we take a maximal tree T of the 1–skeleton of M' and collapse it to a point. This process also preserves the simple homotopy type of (M', R_{2g}^+), so that

$$\tau(C_*(M', R_{2g}^+; \mathcal{K}_{N_k(M)})) = \tau(C_*(M'/T, R_{2g}^+/(T \cap R_{2g}^+); \mathcal{K}_{N_k(M)})).$$

Consequently, $\tau_{N_k}(M) = \overset{i_+^{-1}}{} \tau(C_*(M'/T, R_{2g}^+/(T \cap R_{2g}^+); \mathcal{K}_{N_k(M)}))$. M'/T is a 2–dimensional cell complex with only one 0–cell, and $R_{2g}^+/(T \cap R_{2g}^+)$ is a subcomplex consisting of one 0–cell and $2g$ 1–cells. The pair $(M'/T, R_{2g}^+/(T \cap R_{2g}^+))$ gives an admissible presentation

$$\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots r_{2g+l} \rangle$$

of $\pi_1 M$. For this presentation, we take the matrices $A, B, C \in M(\mathbb{Z}N_k)$ as in Section 3.2. Then

$$\tau_{N_k}(M) = \overset{i_+^{-1}}{} \tau(C_*(M'/T, R_{2g}^+/(T \cap R_{2g}^+); \mathcal{K}_{N_k(M)})) = \overset{i_+^{-1}}{} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Note that the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is a presentation matrix of

$$H_1(M'/T, R_{2g}^+/(T \cap R_{2g}^+); \mathbb{Z}N_k(M))$$

$$\cong H_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k(M)).$$

Since multiplying an element of $\pm N_k$ does not contribute to the degree, we have

$$d_{N_k}^\psi(\tau_{N_k}(M)) = d_{N_k}^\psi\left(\overset{i_+^{-1}}{} \begin{pmatrix} A \\ B \end{pmatrix} \right) = \dim_{\mathcal{K}N_k^\psi} H_1(M, i_+(\Sigma_{g,1}); (i_+^{-1})^* \mathcal{K}_{N_k}^\psi[t^{\pm 1}])$$

for each $\psi \in PH^1(\Sigma_{g,1})$. From this, we see that $d_{N_k}^\psi(\tau_{N_k}(M))$ can be computed from any admissible presentation of $\pi_1 M$ and that it does not depend on choices of cell decompositions of $\Sigma_{g,1}$ and M.

Proposition 6.6 Let $M_1 = (M_1, i_+, i_-)$, $M_2 = (M_2, j_+, j_-) \in \mathcal{C}_{g,1}$. Then

$$d_{N_k}^{\psi}(\tau_{N_k}(M_1 \cdot M_2)) = d_{N_k}^{\psi}(\tau_{N_k}(M_1)) + d_{N_k}^{\psi \circ \sigma_2(M_1)}(\tau_{N_k}(M_2))$$

holds for every $\psi \in PH^1(\Sigma_{g,1})$.

Proof We take an admissible presentation of $\pi_1 M_1$ and the matrices $A_{M_1}, B_{M_1}, C_{M_1}$ corresponding to it. We denote this presentation by

$$\pi_1 M_1 \cong \langle i_-(\overrightarrow{\gamma}), \overrightarrow{z}, i_+(\overrightarrow{\gamma}) \mid \overrightarrow{r} \rangle,$$

for short. Similarly, we take an admissible presentation

$$\pi_1 M_2 \cong \langle j_-(\overrightarrow{\gamma}), \overrightarrow{w}, j_+(\overrightarrow{\gamma}) \mid \overrightarrow{s} \rangle$$

of $\pi_1 M_2$ and the matrices $A_{M_2}, B_{M_2}, C_{M_2}$. Then we obtain an admissible presentation

$$\pi_1(M_1 \cdot M_2) \cong \langle j_-(\overrightarrow{\gamma}), \overrightarrow{w}, j_+(\overrightarrow{\gamma}), i_-(\overrightarrow{\gamma}), \overrightarrow{z}, i_+(\overrightarrow{\gamma}) \mid \overrightarrow{s}, j_+(\overrightarrow{\gamma})i_-(\overrightarrow{\gamma})^{-1}, \overrightarrow{r} \rangle$$

of $\pi_1(M_1 \cdot M_2)$. The corresponding partial matrix $\begin{pmatrix} A_{M_1 \cdot M_2} \\ B_{M_1 \cdot M_2} \end{pmatrix}$ at $\mathbb{Z}N_k(M_1 \cdot M_2)$ is given by

$$\begin{pmatrix} {}^j A_{M_2} & 0 & 0 \\ {}^j B_{M_2} & 0 & 0 \\ {}^j C_{M_2} & I_{2g} & 0 \\ 0 & -I_{2g} & {}^i A_{M_1} \\ 0 & 0 & {}^i B_{M_1} \end{pmatrix},$$

where $i \colon M_1 \to M_1 \cdot M_2$ and $j \colon M_2 \to M_1 \cdot M_2$ are the natural inclusions. From this, we have

$$d_{N_k}^{\psi}(\tau_{N_k}(M_1 \cdot M_2)) = d_{N_k}^{\psi}\left({}^{i \circ i_+^{-1}}\begin{pmatrix} A_{M_1 \cdot M_2} \\ B_{M_1 \cdot M_2} \end{pmatrix} \right)$$

$$= d_{N_k}^{\psi}\left({}^{i_+^{-1}}\begin{pmatrix} A_{M_1} \\ B_{M_1} \end{pmatrix} \right) + d_{N_k}^{\psi}\left({}^{i_+^{-1}i^{-1}j}\begin{pmatrix} A_{M_2} \\ B_{M_2} \end{pmatrix} \right)$$

$$= d_{N_k}^{\psi}\left({}^{i_+^{-1}}\begin{pmatrix} A_{M_1} \\ B_{M_1} \end{pmatrix} \right) + d_{N_k}^{\psi}\left({}^{\sigma_k(M_1)j_+^{-1}}\begin{pmatrix} A_{M_2} \\ B_{M_2} \end{pmatrix} \right)$$

$$= d_{N_k}^{\psi}(\tau_{N_k}(M_1)) + d_{N_k}^{\psi \circ \sigma_2(M_1)}(\tau_{N_k}(M_2)).$$

This completes the proof. □

Remark 6.7 Proposition 6.6 can be seen as a generalization of Le Dimet [18, Proposition 1.11].

6.3 Factorization formulas

6.3.1 The N_k–degree for the closing of a homology cylinder For each $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we can construct a closed 3–manifold defined by

$$C_M := M/(i_+(x) = i_-(x)), \quad x \in \Sigma_{g,1}.$$

We call it the *closing* of M. It is easily seen that if $M \in \mathcal{C}_{g,1}[k]$, we have the natural isomorphisms $N_k = N_k(\Sigma_{g,1}) \cong N_k(M) \cong N_k(C_M)$. Here we identify these groups.

Theorem 6.8 Let $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$. For each $\psi \in PH^1(N_k)$, we have

$$\overline{\delta}^{\psi}_{N_k}(C_M) = d^{\psi}_{N_k}(\tau_{N_k}(M)) + \overline{d}^{\psi}_{N_k}(I_{2g} - r_k(M)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Proof Take an admissible presentation of $\pi_1 M$ as in Definition 3.8, and construct the corresponding matrices $A, B, C \in M(\mathbb{Z}N_k)$.

Adding $2g$ relations $i_+(\gamma_j) = i_-(\gamma_j)$ ($j = 1, \ldots, 2g$) and deleting the generators $i_+(\gamma_j)$ by using them, we obtain a presentation of $\pi_1 C_M$. From this presentation, we have a presentation matrix J_{C_M} of $H_1(C_M, p; \mathbb{Z}N_k)$ given by

$$J_{C_M} = \begin{pmatrix} A + C \\ B \end{pmatrix} = \begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(l,2g)} & I_l \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix},$$

where $Z \in M(2g, l, \mathcal{K}_{N_k})$ is defined by the equality $(r_k(M) \quad Z) = -C \left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)^{-1}$ (see Proposition 3.9) so that $r_k(M)A + ZB = -C$ holds. Since $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ is invertible in $\mathcal{K}_{N_k}$,

$$\mathrm{rank}_{\mathcal{K}_{N_k}} J_{C_M} = \mathrm{rank}_{\mathcal{K}_{N_k}} \begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(l,2g)} & I_l \end{pmatrix}$$
$$= \mathrm{rank}_{\mathcal{K}_{N_k}}(I_{2g} - r_k(M)) + l \leq 2g + l - 1.$$

Hence to show our claim, it suffices to prove the case where this value is just $2g + l - 1$ (see Definition 5.5 (2)).

By the fundamental formula of free calculus, we have

$$(\overrightarrow{1-\gamma} \quad \overrightarrow{1-z}) \, J_{C_M} = (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}, 1 - z_1^{-1}, \ldots, 1 - z_l^{-1}) \, J_{C_M} = 0.$$

On the other hand, we have

$$J_{C_M} \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \overline{\left(\frac{\partial \zeta}{\partial \overrightarrow{\gamma}} \quad 0_{(1,l)}\right)}^T = \begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(l,2g)} & I_l \end{pmatrix} \overline{\left(\frac{\partial \zeta}{\partial \overrightarrow{\gamma}} \quad 0_{(1,l)}\right)}^T = 0$$

by Lemma 6.3 (2). Then we can define $\Delta\left(J_{C_M}; \xi, \mu\right)$, where we put

$$\xi := (1 - \gamma_1^{-1}, \ldots, 1 - \gamma_{2g}^{-1}, 1 - z_1^{-1}, \ldots, 1 - z_l^{-1}),$$

$$\mu := \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \overline{\left(\frac{\partial \zeta}{\partial \vec{\gamma}} \quad 0_{(1,l)}\right)}^T.$$

Lemma 6.9 μ belongs to $(\mathbb{Z}N_k)^{2g+l}$.

Proof Recall that $\begin{pmatrix} A \\ B \end{pmatrix}$ is a presentation matrix of $H_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k)$, so that we have an exact sequence

$$0 \longrightarrow (\mathbb{Z}N_k)^{2g+l} \xrightarrow{\begin{pmatrix} A \\ B \end{pmatrix}\cdot} (\mathbb{Z}N_k)^{2g+l} \longrightarrow H_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k) \longrightarrow 0,$$

where the injectivity of the second map follows since $H_1(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_k}) = 0$. Hence to prove the lemma, it suffices to show that in the third term $(\mathbb{Z}N_k)^{2g+l} = C_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k)$,

$$\overline{\left(\frac{\partial \zeta}{\partial \vec{\gamma}} \quad 0_{(1,l)}\right)}^T$$

is mapped to $0 \in H_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k)$. In the exact sequence

$$0 \to C_1(i_+(\Sigma_{g,1}), p; \mathbb{Z}N_k) \to C_1(M, p; \mathbb{Z}N_k) \to C_1(M, i_+(\Sigma_{g,1}); \mathbb{Z}N_k) \to 0,$$

the cycle

$$\overline{\left(\frac{\partial \zeta}{\partial \vec{\gamma}} \quad 0_{(1,l)}\right)}^T$$

is attained by

$$\overline{\left(\frac{\partial \zeta}{\partial \vec{\gamma}} \quad 0_{(1,l)} \quad 0_{(1,2g)}\right)}^T \in C_1(M, p; \mathbb{Z}N_k) = (\mathbb{Z}N_k)^{2g+l} \oplus (\mathbb{Z}N_k)^{2g}.$$

Then by observing the boundary corresponding to the relation

$$\prod_{j=1}^{g}[i_+(\gamma_j), i_+(\gamma_{g+j})] \left(\prod_{j=1}^{g}[i_-(\gamma_j), i_-(\gamma_{g+j})]\right)^{-1},$$

we see that

$$\overline{\left(\frac{\partial \zeta}{\partial \vec{\gamma}} \quad 0_{(1,l)} \quad 0_{(1,2g)}\right)}^T \quad \text{is homologous to} \quad \overline{\left(0_{(1,2g)} \quad 0_{(1,l)} \quad \frac{\partial \zeta}{\partial \vec{\gamma}}\right)}^T,$$

which comes from $C_1(i_+(\Sigma_{g,1}), p; \mathbb{Z}N_k)$. Our claim follows from this. $\qquad \square$

Now we continue the proof of Theorem 6.8. We can show that (ξ, μ) is ψ–primitive for J_{C_M} as in the proof of Proposition 6.4. Then we have

$$\Delta\left(J_{C_M}; \xi, \mu\right) = \Delta\left(J_{C_M}; (\overrightarrow{1-\gamma} \quad \overrightarrow{1-z}), \left(\frac{A}{B}\right)^{-1} \overline{\left(\frac{\overrightarrow{\partial\zeta}}{\partial\overrightarrow{\gamma}} \quad 0_{(1,l)}\right)}^T\right)$$

$$= \Delta\left(J_{C_M}\left(\frac{A}{B}\right)^{-1}; (\overrightarrow{1-\gamma} \quad \overrightarrow{1-z}), \overline{\left(\frac{\overrightarrow{\partial\zeta}}{\partial\overrightarrow{\gamma}} \quad 0_{(1,l)}\right)}^T\right) \cdot \det\left(\frac{A}{B}\right)$$

$$= \Delta\left(\begin{pmatrix} I_{2g} - r_k(M) & -Z \\ 0_{(l,2g)} & I_l \end{pmatrix}; (\overrightarrow{1-\gamma} \quad \overrightarrow{1-z}), \overline{\left(\frac{\overrightarrow{\partial\zeta}}{\partial\overrightarrow{\gamma}} \quad 0_{(1,l)}\right)}^T\right) \cdot \det\left(\frac{A}{B}\right)$$

$$= \Delta\left(I_{2g} - r_k(M); \overrightarrow{1-\gamma}, \overline{\frac{\overrightarrow{\partial\zeta}}{\partial\overrightarrow{\gamma}}}^T\right) \cdot \det\left(\frac{A}{B}\right).$$

From the above argument, we obtain

$$\overline{\delta}^{\psi}_{N_k}(C_M) = \overline{d}^{\psi}_{N_k}\left(J_{C_M}\right) = \deg^{\psi}\left(\Delta\left(J_{C_M}; \xi, \mu\right)\right)$$

$$= \deg^{\psi}\left(\Delta\left(I_{2g} - r_k(M); \overrightarrow{1-\gamma}, \overline{\frac{\overrightarrow{\partial\zeta}}{\partial\overrightarrow{\gamma}}}^T\right)\right) + \deg^{\psi}\left(\det\left(\frac{A}{B}\right)\right)$$

$$= \overline{d}^{\psi}_{N_k}(I_{2g} - r_k(M)) + d^{\psi}_{N_k}(\tau_{N_k}(M)).$$

This completes the proof of Theorem 6.8. $\hfill\square$

Remark 6.10 When $M \in \mathcal{C}_{g,1}[k] \cap \mathcal{M}_{g,1}$, $I_{2g} - r_k(M)$ itself gives a presentation matrix of $H_1(C_M, p; \mathbb{Z}N_k)$. Hence we have $\overline{\delta}^{\psi}_{N_k}(C_M) = \overline{d}^{\psi}_{N_k}(I_{2g} - r_k(M))$, and moreover $\delta^{\psi}_{N_k}(C_M) = d^{\psi}_{N_k}(I_{2g} - r_k(M))$ for this case.

6.3.2 The $N_{k,T}$–degree for the mapping torus of a homology cylinder Given $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we have another method for obtaining a closed 3–manifold T_M as follows. First we attach a 2–handle $I \times D^2$ to M along $I \times i_{\pm}(\partial\Sigma_{g,1})$ as in Figure 6. Let M' be the resulting 3–manifold. The markings i_+ and i_- are naturally extended to embeddings $i'_+, i'_- : \Sigma_g \hookrightarrow \partial M'$, where $\Sigma_g := \Sigma_{g,1} \cup D^2$ is an oriented closed surface of genus g.

Then we put

$$T_M := M'/(i'_+(x) = i'_-(x)), \quad x \in \Sigma_g$$

and call T_M the *mapping torus* of M. Indeed, for $M_\varphi \in \mathcal{M}_{g,1} \subset \mathcal{C}_{g,1}$, the resulting manifold T_{M_φ} is the usual mapping torus of φ extended naturally to a mapping class of Σ_g. If we take an admissible presentation of $\pi_1 M$ denoted by $\langle i_-(\overrightarrow{\gamma}), \overrightarrow{z}, i_+(\overrightarrow{\gamma}) \mid \overrightarrow{r}\rangle$
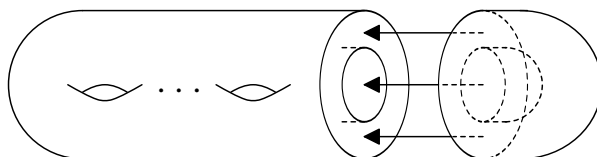
Figure 6: Attaching a 2–handle to M

briefly, then a presentation of $\pi_1 T_M$ is given by

$$\pi_1 T_M \cong \langle i_-(\overrightarrow{\gamma}), \overrightarrow{z}, \lambda, i_+(\overrightarrow{\gamma}) \mid \overrightarrow{r}, \prod_{j=1}^g [i_-(\gamma_j), i_-(\gamma_{g+j})], i_-(\overrightarrow{\gamma})\lambda i_+(\overrightarrow{\gamma})^{-1}\lambda^{-1}\rangle,$$

where λ is the loop $I \times \{p\} \subset I \times D^2 \subset T_M$. If $M \in \mathcal{C}_{g,1}[k]$, we have natural isomorphisms $N_k(\Sigma_g) \cong N_k(M')$ and $N_k(T_M) \cong N_k(\Sigma_g) \times \langle \lambda \rangle$. Note that these groups are torsion-free nilpotent. We consider $N_k(\Sigma_g)$ to be a subgroup of $N_k(T_M)$. For simplicity, we denote $N_k(\Sigma_g)$ by $N_{k,0}$ and $N_k(T_M)$ by $N_{k,T}$.

We can show that $H_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}}) = 0$ (see Remark 3.2). Hence by a similar argument, the Magnus representation $r_{k,T} \colon \mathcal{C}_{g,1} \to GL(2g, \mathcal{K}_{N_{k,T}})$ and the $N_{k,T}$–torsion

$$\tau_{N_{k,T}}(M) := \tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_{k,T}})) \in K_1(\mathcal{K}_{N_{k,T}})/(\pm N_{k,T})$$

are defined. Then we obtain the following factorization formula of the $N_{k,T}$–degree for the mapping torus of a homology cylinder.

Theorem 6.11 *Let $M \in \mathcal{C}_{g,1}[k]$. For each primitive element $\psi \in H^1(N_{k,T}) = H^1(T_M)$, the $N_{k,T}$–degree $\overline{\delta}_{N_{k,T}}^{\psi}(T_M)$ is finite, and we have*

$$\overline{\delta}_{N_{k,T}}^{\psi}(T_M) = \delta_{N_{k,T}}^{\psi}(T_M)$$
$$= d_{N_{k,T}}^{\psi}(\tau_{N_{k,T}}(M)) + d_{N_{k,T}}^{\psi}(I_{2g} - \lambda r_{k,T}(M)) - 2|\psi(\lambda)|.$$

Proof The first assertion is a slight generalization of Harvey [12, Proposition 8.4], and we now follow the proof. Let $\psi \in H^1(T_M)$ be the Poincaré dual of the surface $i'_+(\Sigma_g) = i'_-(\Sigma_g)$. This gives an exact sequence $1 \to N_{k,0} \to N_{k,T} \xrightarrow{\psi} \mathbb{Z} \to 1$. Then our claim is proved by showing that $\overline{\delta}_{N_{k,T}}^{\psi}(T_M)$ is finite for this ψ.

Let $(T_M)_{N_{k,T}}$ be the $N_{k,T}$–cover of T_M, and let $(T_M)_\psi$ be the $\mathbb{Z}$–cover of T_M with respect to ψ. $(T_M)_\psi$ is the product $\cdots \cdot M' \cdot M' \cdot M' \cdots$ of countably many

copies of M', and $(T_M)_{N_{k,T}}$ can be regarded as the $N_{k,0}$–cover of $(T_M)_\psi$. Then

$$H_*(T_M; \mathcal{K}_{N_{k,T}^\psi}[t^{\pm 1}]) = H_*(C_*((T_M)_{N_{k,T}}) \otimes_{N_{k,T}} \mathbb{Z}N_{k,T}(\mathbb{Z}N_{k,0} - \{0\})^{-1})$$

$$\cong H_*(C_*((T_M)_{N_{k,T}}) \otimes_{N_{k,0}} \mathbb{Z}N_{k,0}(\mathbb{Z}N_{k,0} - \{0\})^{-1})$$

$$= H_*(C_*(((T_M)_\psi)_{N_{k,0}}) \otimes_{N_{k,0}} \mathcal{K}_{N_{k,T}^\psi})$$

$$= H_*((T_M)_\psi; \mathcal{K}_{N_{k,T}^\psi}).$$

Here we remark that the image of the composite $\pi_1((T_M)_\psi) \to \pi_1 T_M \to N_{k,T}$ is contained in $N_{k,0}$. The same holds for the composite $\pi_1 M' \to \pi_1 T_M \to N_{k,T}$. We also remark that $\mathcal{K}_{N_{k,T}^\psi} = \mathcal{K}_{N_{k,0}}$.

We denote by Σ again for a lift of $\Sigma \subset T_M$ on $(T_M)_\psi$. We divide $(T_M)_\psi$ at Σ, and obtain two parts $(T_M)_\psi^+$ and $(T_M)_\psi^-$. Then $(T_M)_\psi^\pm = \varinjlim_l (M')^l$, and the inclusion $\Sigma \hookrightarrow (M')^l$ induces an isomorphism on homology. We can show that $H_*((M')^l, \Sigma; \mathcal{K}_{N_{k,T}\psi}) = 0$ by the same way as mentioned in [Lemma 3.1](). Thus $H_*((T_M)_\psi^\pm, \Sigma; \mathcal{K}_{N_{k,T}\psi}) = \varinjlim_l H_*((M')^l, \Sigma; \mathcal{K}_{N_{k,T}\psi}) = 0$, and therefore $H_*((T_M)_\psi, \Sigma; \mathcal{K}_{N_{k,T}\psi}) = 0$. This shows that

$$H_*(T_M; \mathcal{K}_{N_{k,T}^\psi}[t^{\pm 1}]) \cong H_*((T_M)_\psi; \mathcal{K}_{N_{k,T}^\psi}) \cong H_*(\Sigma; \mathcal{K}_{N_{k,T}^\psi})$$

is a finite dimensional $\mathcal{K}_{N_{k,T}^\psi}$–vector space, so that $\overline{\delta}_{N_{k,T}}^\psi(T_M)$ is finite.

To show the second assertion, we take an admissible presentation of $\pi_1 M$ and construct the matrices A, B, C as before. We regard them as matrices with entries in $\mathbb{Z}N_{k,T}$. From the presentation, we have a presentation matrix J_{T_M} of $H_1(T_M, p; \mathbb{Z}N_{k,T})$ given by

$$J_{T_M} = \begin{pmatrix} A & \overrightarrow{\frac{\partial \xi}{\partial \gamma}}T & I_{2g} \\ B & 0_{(l,1)} & 0_{(l,2g)} \\ 0_{(1,2g+l)} & 0 & -\overrightarrow{(1-\gamma)} \\ C & 0_{(2g,1)} & -\lambda^{-1}I_{2g} \end{pmatrix},$$

where $\overrightarrow{\gamma} := i_+(\overrightarrow{\gamma}) = i_-(\overrightarrow{\gamma})$. We remark that λ belongs to the center in $N_{k,T}$. As presentation matrices of $H_1(T_M, p; \mathbb{Z}N_{k,T})$, this matrix is equivalent to the square matrix

$$J'_{T_M} = \begin{pmatrix} A + \lambda C & \overrightarrow{\frac{\partial \xi}{\partial \gamma}}T \\ B & 0_{(l,1)} \\ -\overrightarrow{(1-\gamma)}\lambda C & 0 \end{pmatrix}.$$

By Proposition 3.9, Lemma 6.3 and the fundamental formula of free calculus, we have

$$
J'_{T_M} = \begin{pmatrix} A + \lambda C & \overline{\dfrac{\partial \zeta}{\partial \vec{\gamma}}}{}^T \\ B & 0_{(l,1)} \\ \lambda(\overrightarrow{1-\vec{\gamma}} \quad \overrightarrow{1-\vec{z}}) \begin{pmatrix} A \\ B \end{pmatrix} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} I_{2g} - \lambda r_{k,T}(M) & -\lambda Z & \overline{\dfrac{\partial \zeta}{\partial \vec{\gamma}}}{}^T \\ 0_{(l,2g)} & I_l & 0_{(l,1)} \\ \lambda(\overrightarrow{1-\vec{\gamma}}) & \lambda(\overrightarrow{1-\vec{z}}) & 0 \end{pmatrix} \begin{pmatrix} A & 0_{(2g,1)} \\ B & 0_{(l,1)} \\ 0_{(1,2g+l)} & 1 \end{pmatrix}.
$$

Note that
$$
\widetilde{\begin{pmatrix} A \\ B \end{pmatrix}} := \begin{pmatrix} A & 0_{(2g,1)} \\ B & 0_{(l,1)} \\ 0_{(1,2g+l)} & 1 \end{pmatrix}
$$

is invertible in $\mathcal{K}_{N_{k,T}}$. Then it is easily checked that

$$
(\overrightarrow{1-\vec{\gamma}} \quad \overrightarrow{1-\vec{z}} \quad 1-\lambda^{-1})\, J'_{T_M} = 0,
$$

$$
J'_{T_M} \widetilde{\begin{pmatrix} A \\ B \end{pmatrix}}^{-1} \overline{(\overrightarrow{\dfrac{\partial \zeta}{\partial \vec{\gamma}}} \quad 0_{(1,l)} \quad \lambda^{-1}-1)}{}^T = 0.
$$

We put
$$
\widetilde{\xi} := (\overrightarrow{1-\vec{\gamma}} \quad \overrightarrow{1-\vec{z}} \quad 1-\lambda^{-1})
$$

$$
\widetilde{\mu} := \widetilde{\begin{pmatrix} A \\ B \end{pmatrix}}^{-1} \overline{(\overrightarrow{\dfrac{\partial \zeta}{\partial \vec{\gamma}}} \quad 0_{(1,l)} \quad \lambda^{-1}-1)}{}^T.
$$

As in Proposition 6.4 and Lemma 6.9, we can show that $(\widetilde{\xi}, \widetilde{\mu})$ is ψ–primitive for J'_{T_M}. Then

$$
\delta^{\psi}_{N_{k,T}}(T_M) = d^{\psi}_{N_{k,T}}\left(J'_{T_M}\right) = \deg^{\psi}\left(\Delta\left(J'_{T_M}; \widetilde{\xi}, \widetilde{\mu}\right)\right)
$$

$$
= \deg^{\psi}\left(\Delta\left(J'_{T_M} \widetilde{\begin{pmatrix} A \\ B \end{pmatrix}}^{-1}; \widetilde{\xi}, \widetilde{\begin{pmatrix} A \\ B \end{pmatrix}}\widetilde{\mu}\right) \cdot \det\left(\widetilde{\begin{pmatrix} A \\ B \end{pmatrix}}\right)\right)
$$

$$= \deg^{\psi}\left(\Delta\left(\left(\begin{array}{ccc} I_{2g} - \lambda r_{k,T}(M) & -\lambda Z & \overline{\frac{\partial \zeta}{\partial \vec{\gamma}}}^{T} \\ 0_{(l,2g)} & I_l & 0_{(l,1)} \\ \lambda(\overrightarrow{1-\vec{\gamma}}) & \lambda(\overrightarrow{1-\vec{z}}) & 0 \end{array} \right); \widetilde{\xi}, \widetilde{\left(\begin{array}{c} A \\ B \end{array} \right)} \widetilde{\mu} \right) \cdot \det \widetilde{\left(\begin{array}{c} A \\ B \end{array} \right)} \right)$$

$$= \deg^{\psi}\left(\det\left(\begin{array}{cc} I_{2g} - \lambda r_{k,T}(M) & -\lambda Z \\ 0_{(l,2g)} & I_l \end{array} \right) \cdot \lambda^{-1}(1-\lambda^{-1})^{-2} \cdot \det \widetilde{\left(\begin{array}{c} A \\ B \end{array} \right)} \right)$$

$$= \deg^{\psi}\left(\det(I_{2g} - \lambda r_{k,T}(M)) \cdot \lambda^{-1}(1-\lambda^{-1})^{-2} \cdot \det \widetilde{\left(\begin{array}{c} A \\ B \end{array} \right)} \right)$$

$$= \deg^{\psi}\left(\det(I_{2g} - \lambda r_{k,T}(M)) \right) + \deg^{\psi}\left(\det(\tau_{N_k,T}(M)) \right) - 2|\psi(\lambda)|.$$

This completes the proof. □

6.3.3 The case of $k = 2$ (commutative case) Since $\mathbb{Z}N_2 = \mathbb{Z}N_2(\Sigma_g)$ and $\mathcal{K}_{N_2} = \mathcal{K}_{N_2(\Sigma_g)}$ are commutative, we can use the ordinary determinant for computations. Moreover, we can obtain some invariants before taking degrees. For example, define

$$\Delta(M) := (-1)^{i+j} \frac{\det\left((I_{2g} - r_2(M))_{(i,j)} \right)}{(1 - \gamma_i^{-1})(\overline{\frac{\partial \zeta}{\partial \gamma_j}})} \in \mathcal{K}_{N_2},$$

where $A_{(i,j)}$ is the matrix obtained from a matrix A by removing its i–th row and j–th column. $\Delta(M)$ is well-defined by Lemma 5.1. Note that this invariant is based on that for string links given in Kirk–Livingston–Wang [16, Definition 6.1], and we call it the *Alexander rational function* of M.

Theorem 6.12 *Let $M \in \mathcal{C}_{g,1}[2]$, and let Δ_{C_M}, Δ_{T_M} be the Alexander polynomials of C_M, T_M, respectively. Then*

$$\Delta_{C_M} \doteq \overline{\tau_{N_2}(M) \cdot \Delta(M)},$$
$$\Delta_{T_M} \doteq \overline{\tau_{N_2}(M) \cdot \det\left(I_{2g} - \lambda r_{2,T}(M) \right) \cdot (1-\lambda^{-1})^{-2}},$$

where $\doteq$ means that the left hand side coincides with the right hand side in $\mathcal{K}_{N_2}$ up to $\pm N_2$ or in $\mathcal{K}_{N_2(T_M)}$ up to $\pm N_2(T_M)$.

Proof We prove only the first assertion. The proof is almost the same as that for Theorem 6.8. We follow the notation used there. We may assume that $\mathrm{rank}_{\mathcal{K}_{N_2}} J_{C_M} = \mathrm{rank}_{\mathcal{K}_{N_2}} r_2(M) + l = 2g + l - 1$.

By definition, Δ_{C_M} is the greatest common divisor of $\{\det \overline{J_{C_M\,(i,j)}}^T\}_{1\le i,j\le 2g+l}$. We show that it is nothing other than

$$\Delta := \overline{\Delta\left(J_{C_M};(\overrightarrow{1-\gamma}\quad\overrightarrow{1-z}),\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)^{-1}\overline{\left(\begin{smallmatrix}\frac{\partial\zeta}{\partial\overrightarrow{\gamma}}&0_{(1,l)}\end{smallmatrix}\right)}^T\right)} \doteq \overline{\det\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)}\cdot\Delta(M).$$

As seen in Lemma 6.9,

$$\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)^{-1}\overline{\left(\begin{smallmatrix}\frac{\partial\zeta}{\partial\overrightarrow{\gamma}}&0_{(1,l)}\end{smallmatrix}\right)}^T$$

is a vector in $(\mathbb{Z}N_2)^{2g+l}$. If Δ is in $\mathbb{Z}N_2$, it attains the greatest common divisor. To show it, suppose $\Delta = h_1/h_2$ where $h_1\in\mathbb{Z}N_2$ and $h_2\in\mathbb{Z}N_2-\{0\}$ are relatively prime. From the definition of Δ, we have

$$\frac{(1-\gamma_i^{-1})\left(\overline{\frac{\partial\zeta}{\partial\gamma_j}}\right)h_1}{h_2} = (-1)^{i+j}\det\overline{J_{C_M\,(i,j)}}^T \in \mathbb{Z}N_2.$$

Hence h_2 is a common divisor of each

$$\left\{(1-\gamma_i^{-1})\left(\overline{\frac{\partial\zeta}{\partial\gamma_j}}\right)\right\}_{i,j},$$

and it is 1. That is, h_2 is a unit in $\mathbb{Z}N_2$.

The determinant $\det\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)\in\mathcal{K}_{N_2}$ (up to $\pm N_2$) does not depend on the choice of an admissible presentation, and it gives $\tau_{N_2}(M)$. Indeed the matrix $\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right)$ is a presentation matrix of $H_1(M,i_+(\Sigma_{g,1});\mathbb{Z}N_2)$ and its determinant gives a generator of the 0–th elementary ideal, which is principal and invariant under Tietze transformations. This completes the proof. $\qquad\square$

Recall that the equality in Theorem 6.8 holds as elements of $\mathbb{Z}\cup\{\infty\}$. It is easily checked that $\overline{\delta}^\psi_{N_k}(C_M) = \infty$ if and only if $\overline{d}^\psi_{N_k}(I_{2g}-r_k(M)) = \infty$, and this occurs when $H_1(C_M;\mathcal{K}^\psi_{N_k}[t^{\pm 1}])$ has a nontrivial free part. The following are some examples of homology cylinders which have nontrivial (nonzero) Alexander rational functions. For these examples, the equality holds as elements of $\mathbb{Z}$. When $k\ge 3$, the computation becomes quite difficult in general.

Example 6.13 Assume that $g=1$. The Dehn twist $\tau_\zeta\in\mathcal{M}_{1,1}$ belongs to $\mathcal{C}_{1,1}[3]$. A straightforward calculation shows that

$$r_2(\tau_\zeta) = \begin{pmatrix} \gamma_1^{-1}+\gamma_2^{-1}-\gamma_1^{-1}\gamma_2^{-1} & -1+2\gamma_2^{-1}-\gamma_2^{-2} \\ 1-2\gamma_1^{-1}+\gamma_1^{-2} & 2-\gamma_1^{-1}-\gamma_2^{-1}+\gamma_1^{-1}\gamma_2^{-1} \end{pmatrix}.$$

Then $\Delta(\tau_\zeta)\doteq 1\in\mathbb{Z}N_2$, which is nontrivial.

Example 6.14 Assume that $g \geq 2$. Let τ_1, τ_2 and τ_3 be Dehn twists along simple closed curves c_1, c_2 and c_3 as in Figure 7.
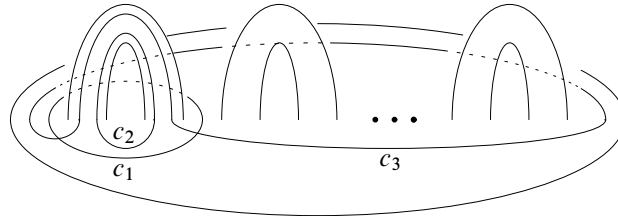

Figure 7

Then $\tau_1 \tau_2^{-1}$, $\tau_3 \in \mathcal{C}_{g,1}[2]$. By a direct computation, we can check that $\Delta(\tau_1 \tau_2^{-1} \cdot \tau_3) \doteq -(\gamma_1 - 1)^{2g-2}$, although $\Delta(\tau_1 \tau_2^{-1}) = \Delta(\tau_3) = 0$.

6.4 N_k–torsions and Harvey's Realization Theorem

By Proposition 6.6, the degree of the N_k–torsion gives a monoid homomorphism

$$d_{N_k}^{\psi}(\tau_{N_k}(\cdot))\colon \mathcal{C}_{g,1}[2] \longrightarrow \mathbb{Z}_{\geq 0}$$

for each $\psi \in PH^1(\Sigma_{g,1})$ and an integer $k \geq 2$. To see some properties of these homomorphisms, including their nontriviality, we use a variant of Harvey's Realization Theorem in [12, Theorem 11.2] which gives a method for performing surgery on a compact orientable 3–manifold to obtain a homology cobordant one having distinct higher-order degrees.

Theorem 6.15 *Let $M \in \mathcal{C}_{g,1}$ be a homology cylinder. For each primitive element x of $H_1(\Sigma_{g,1})$ and any integers $n \geq 2$ and $k \geq 1$, there exists a homology cylinder M' such that*

(1)   *M' is homology cobordant to M,*

(2) $d_{N_l}^{\psi}(\tau_{N_l}(M')) = d_{N_l}^{\psi}(\tau_{N_l}(M))$ *for $2 \leq l \leq n-1$,*

(3) $d_{N_n}^{\psi}(\tau_{N_n}(M')) \geq d_{N_n}^{\psi}(\tau_{N_n}(M)) + k|p|$

for any $\psi \in PH^1(\Sigma_{g,1})$ satisfying $\psi(x) = p$.

Proof The proof is based on Harvey's proof of Realization Theorem in [12, Theorem 11.2]. However, since we now use the lower central series instead of the rational derived series, we can shorten the argument.

We take a loop representing $x \in H_1(\Sigma_{g,1})$, and denote it by x again. We also take a loop γ whose homology class in $H_1(\Sigma_{g,1})$ is independent of x.

We attach a 1–handle to $M \times \{1\} \subset M \times I$, and then attach a 2–handle to obtain a 4–manifold W. Here the 2–handle are attached along the loop $\alpha[X_{n-1}, A_{k+1}]$, where $\alpha \in \pi_1 M$ is a loop corresponding to the added 1–handle, and $X_{n-1}, A_{k+1} \in \pi_1 M$ are inductively defined by

$$X_1 = i_+(x), \ \ X_l = [i_+(\gamma), X_{l-1}] \ \text{ for } l \geq 2,$$
$$A_1 = \alpha, \qquad A_l = [i_+(x), A_{l-1}] \ \text{ for } l \geq 2.$$

It is easily seen that $X_l \in \Gamma^l(\pi_1 M) - \Gamma^{l+1}(\pi_1 M)$. M' is defined as another part of ∂W, namely $\partial W = M \cup M'$ and $M \cap M' = \partial M = \partial M'$. From the construction, we have $H_*(W, M) = 0$. We also have $H_*(W, M') = 0$ by using the Poincaré–Lefschetz duality and the universal coefficient theorem. Hence $(M', i_+, i_-) \in \mathcal{C}_{g,1}$, and it is homology cobordant to M. Stallings' theorem shows that

$$N_l \xrightarrow{i_+} N_l(M) \to N_l(W) \leftarrow N_l(M') \xleftarrow{i_+} N_l$$

are all isomorphisms. Using them, we identify $N_l, N_l(M), N_l(M')$ and $N_l(W)$.

For simplicity, we put $K_l := \mathcal{K}_{N_l^\psi}[t^{\pm 1}] = \mathbb{Z}N_l(\mathbb{Z}N_l^\psi - \{0\})^{-1}$. Recall that

$$H_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{N_l}) = H_*(M', i_+(\Sigma_{g,1}); \mathcal{K}_{N_l}) = 0$$

as in Lemma 3.1. By the same proof, we have $H_*(W, i_+(\Sigma_{g,1}); \mathcal{K}_{N_l}) = 0$. Hence $H_*(M, i_+(\Sigma_{g,1}); K_l), H_*(M', i_+(\Sigma_{g,1}); K_l)$ and $H_*(W, i_+(\Sigma_{g,1}); K_l)$ are all finite dimensional $\mathcal{K}_{N_l^\psi}$–vector spaces. As seen in Section 6.2,

$$d_{N_l}^\psi(\tau_{N_l}(M)) = \dim_{\mathcal{K}_{N_l^\psi}} H_1(M, i_+(\Sigma_{g,1}); K_l).$$

If we take an admissible presentation of $\pi_1 M$ and the matrices $A, B \in \mathbb{Z}N_k$ as before, $\binom{A}{B}$ gives a presentation matrix of $H_1(M, i_+(\Sigma_{g,1}); K_l)$. Then one of $H_1(W, i_+(\Sigma_{g,1}); K_l)$ is given by

$$\begin{pmatrix} \begin{matrix} A \\ B \end{matrix} & * \\ 0_{(1,2g+l)} & \frac{\partial \alpha[X_{n-1},A_{k+1}]}{\partial \alpha} \end{pmatrix},$$

so that

$$\dim_{\mathcal{K}_{N_l^\psi}} H_1(W, i_+(\Sigma_{g,1}); K_l) = d_{N_l}^\psi(\tau_{N_l}(M)) + \deg^\psi\left(\frac{\partial \alpha[X_{n-1},A_{k+1}]}{\partial \alpha}\right).$$

By a direct computation,

$$\frac{\partial\alpha[X_{n-1},A_{k+1}]}{\partial\alpha} = 1 + \alpha\left\{(1 - X_{n-1}A_{k+1}X_{n-1}^{-1})\frac{\partial X_{n-1}}{\partial\alpha} + (X_{n-1} - [X_{n-1},A_{k+1}])\frac{\partial A_{k+1}}{\partial\alpha}\right\}.$$

When $2 \le l \le n-1$, we have $X_{n-1} = A_{k+1} = 1 \in N_l$, so that $\frac{\partial\alpha[X_{n-1},A_{k+1}]}{\partial\alpha} = 1$, and $H_1(M, i_+(\Sigma_{g,1}); K_l) \cong H_1(W, i_+(\Sigma_{g,1}); K_l)$. When $l = n$, we have $X_{n-1} \ne A_{k+1} = 1 \in N_l$, so that

$$\frac{\partial\alpha[X_{n-1},A_{k+1}]}{\partial\alpha} = 1 + (X_{n-1} - 1)\frac{\partial A_{k+1}}{\partial\alpha} = 1 + (X_{n-1} - 1)(x - A_{k+1})\frac{\partial A_k}{\partial\alpha}$$

$$= \cdots = 1 + (X_{n-1} - 1)(x - A_{k+1})(x - A_k)\cdots(x - A_2),$$

and $\quad \deg^\psi\left(\overline{\frac{\partial\alpha[X_{n-1},A_{k+1}]}{\partial\alpha}}\right) = \deg^\psi\left(\frac{\partial\alpha[X_{n-1},A_{k+1}]}{\partial\alpha}\right) = \begin{cases} k|p| & (n \ge 3) \\ (k+1)|p| & (n = 2). \end{cases}$

In each case, $\dim_{\mathcal{K}_{N_n^\psi}} H_1(W, i_+(\Sigma_{g,1}); K_n) \ge d_{N_n}^\psi(\tau_{N_n}(M)) + k|p|$.

By considering the dual handle decomposition, we see that W is obtained from $M' \times I$ by attaching a 2–handle and a 3–handle. Hence $H_1(M', i_+(\Sigma_{g,1}); K_l) \to H_1(W, i_+(\Sigma_{g,1}); K_l)$ is an epimorphism. In particular, when $l = n$,

$$d_{N_n}^\psi(\tau_{N_n}(M')) \ge \dim_{\mathcal{K}_{N_n^\psi}} H_1(W, i_+(\Sigma_{g,1}); K_n) \ge d_{N_n}^\psi(\tau_{N_n}(M)) + k|p|.$$

It remains to proof that the map $H_1(M', i_+(\Sigma_{g,1}); K_l) \to H_1(W, i_+(\Sigma_{g,1}); K_l)$ is injective when $2 \le l \le n-1$. We now show that $H_2(W, M'; K_l) = 0$. By the Poincaré–Lefschetz duality, $H_2(W, M'; K_l) \cong H^2(W, M; K_l)$. On the other hand, it is easily checked that $H_0(W, M; K_l) = H_1(W, M; K_l) = H_2(W, M; K_l) = 0$. Then the universal coefficient spectral sequence (see Levine [19, Theorem 2.3]) shows our claim. Consequently, $H_1(M, i_+(\Sigma_{g,1}); K_l) \cong H_1(W, i_+(\Sigma_{g,1}); K_l) \cong H_1(M', i_+(\Sigma_{g,1}); K_l)$ and $d_{N_l}^\psi(\tau_{N_l}(M')) = d_{N_l}^\psi(\tau_{N_l}(M))$. This completes the proof.
$\square$

Corollary 6.16 *For any $\psi \in PH^1(\Sigma_{g,1})$, the maps $\{d_{N_k}^\psi(\tau_{N_k}(\cdot))\colon \mathcal{C}_{g,1}[2] \to \mathbb{Z}_{\ge 0}\}_{k \ge 2}$ are all nontrivial homomorphisms, and independent of each other. In particular, $\mathcal{C}_{g,1}[2], \mathcal{C}_{g,1}[3], \dots, \mathrm{Ker}(\mathcal{C}_{g,1} \to \mathcal{H}_{g,1})$ are not finitely generated as monoids.*

In fact, we can show them by constructing homology cylinders that are homology cobordant to the unit $1_{\mathcal{C}_{g,1}}$. Note that $d_{N_k}^\psi(\tau_{N_k}(M)) = 0$ if $M \in \mathcal{M}_{g,1}$ since $\Sigma_{g,1} \times I$ is simple homotopy equivalent to $\Sigma_{g,1}$ and hence $\tau_{N_k}(M)$ is trivial.

7 Application of torsion-degree functions to Aut F_n^{acy}

In [28, Section 5], we defined the Magnus representation

$$r_k \colon \operatorname{Aut} F_n^{\mathrm{acy}} \to GL(n, \mathcal{K}_{N_k(F_n)})$$

for the automorphism group of the *acyclic closure (or HE-closure)* F_n^{acy} of F_n, where F_n^{acy} is a completion of F_n in a certain sense defined by Levine [20]. We also refer to Cha [3], where acyclic closures are called "$\mathbb{Z}$–closures".

The natural map $F_n \to F_n^{\mathrm{acy}}$ is known to be injective and 2–connected. In particular, $N_k(F_n) = N_k(F_n^{\mathrm{acy}})$ by Stallings' theorem, and we denote it briefly by N_k in this subsection. Aut F_n^{acy} can be regarded as an enlargement of Aut F_n. Indeed we have the enlarged Dehn–Nielsen homomorphism $\sigma^{\mathrm{acy}} \colon \mathcal{H}_{g,1} \to \operatorname{Aut} F_{2g}^{\mathrm{acy}}$ extending the classical one $\sigma \colon \mathcal{M}_{g,1} \hookrightarrow \operatorname{Aut} F_{2g}$ [28, Section 6]. That is, we have the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Aut} F_{2g} & \hookrightarrow & \operatorname{Aut} F_{2g}^{\mathrm{acy}} \\
\Big\uparrow{\scriptstyle\sigma} & & \Big\uparrow{\scriptstyle\sigma^{\mathrm{acy}}} \\
\mathcal{M}_{g,1} & \hookrightarrow & \mathcal{H}_{g,1}.
\end{array}
$$

Note that σ^{acy} is not injective. The Magnus representation for homology cylinders is nothing other than the composite

$$\mathcal{H}_{g,1} \xrightarrow{\sigma^{\mathrm{acy}}} \operatorname{Aut} F_{2g}^{\mathrm{acy}} \xrightarrow{r_k} GL(2g, \mathcal{K}_{N_k}).$$

We now consider the map $d_{N_k}^{\psi} \circ r_k \colon \operatorname{Aut} F_n^{\mathrm{acy}} \to \mathbb{Z}$ for $\psi \in PH^1(F_n)$, where $PH^1(F_n)$ denotes the set of primitive elements of $H^1(F_n)$. Since $d_{N_k}^{\psi}(A) = 0$ for $A \in GL(\mathbb{Z}N_k)$ by Example 5.7 (1), it follows that $d_{N_k}^{\psi} \circ r_k \big|_{\operatorname{Aut} F_n}$ is trivial. When $n = 2g$, $d_{N_k}^{\psi} \circ r_k \big|_{\operatorname{Im} \sigma^{\mathrm{acy}}}$ is also trivial as seen in Theorem 6.1. On the other hand, $d_{N_k}^{\psi} \circ r_k$ is actually nontrivial on Aut F_n^{acy} as we will see below. Since r_k is a crossed homomorphism, we have the following.

Proposition 7.1 For $f, g \in \operatorname{Aut} F_n^{\mathrm{acy}}$ and $\psi \in PH^1(F_n)$, we have

$$d_{N_k}^{\psi}(r_k(fg)) = d_{N_k}^{\psi}(r_k(f)) + d_{N_k}^{\psi \circ f}(r_k(g)).$$

In particular, if we restrict $d_{N_k}^{\psi} \circ r_k$ to

$$\operatorname{IAut} F_n^{\mathrm{acy}} := \operatorname{Ker}(\operatorname{Aut} F_n^{\mathrm{acy}} \to \operatorname{Aut} N_2 \cong GL(n, \mathbb{Z})),$$

it becomes a homomorphism.

Remark 7.2 As Aut F_n^{acy} acts on $PH^1(F_n)$ from the right, it acts on $\text{Map}(PH^1(F_n), \mathbb{Z})$ from the left. Regard $\dot{d}_{N_k}(r_k(\cdot))$ as a map Aut $F_n^{\text{acy}} \to \text{Map}(PH^1(F_n), \mathbb{Z})$. Proposition 7.1 shows that $\dot{d}_{N_k}(r_k(\cdot))$ is a 1–cocycle in $C^1(\text{Aut } F_n^{\text{acy}}, \text{Map}(PH^1(F_n), \mathbb{Z}))$. We can see that it is nontrivial in $H^1(\text{Aut } F_n^{\text{acy}}, \text{Map}(PH^1(F_n), \mathbb{Z}))$ from the proof of Theorem 7.3 below.

IAut F_n^{acy} can be regarded as an enlargement of IAut $F_n := \text{Ker}(\text{Aut } F_n \to \text{Aut } N_2)$. Nielsen [25] showed that IAut $F_2 \cong F_2$. Magnus [22] gave a finite generating set of IAut F_n for all $n \geq 3$. Andreadakis [1] (for $n = 3$) and Cohen–Pakianathan, Farb and Kawazumi [15] (for all $n \geq 3$, independently), showed that the abelianization $H_1(\text{IAut } F_n)$ of IAut F_n is isomorphic to $\text{Hom}(F_n, \mathbb{Z}) \otimes \wedge^2(H_1(F_n))$.

As for IAut F_n^{acy}, we have the following.

Theorem 7.3 *For every $n \geq 2$, IAut F_n^{acy} is not finitely generated. Moreover, the abelianization $H_1(\text{IAut } F_n^{\text{acy}})$ of IAut F_n^{acy} has infinite rank.*

Proof Let $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ and $\psi := x_1^* \in PH^1(F_n)$. We show that the homomorphisms $\{d_{N_k}^{\psi}(r_k(\cdot)): \text{IAut } F_n^{\text{acy}} \to \mathbb{Z}\}_{k \geq 2}$ are all nontrivial, and independent of each other.

Consider the endomorphism f_k of F_n given by

$$f_k(x_1) = x_1[Y_{k-1}, Y_k], \quad f_k(x_i) = x_i \text{ for } i \geq 2,$$

where we define $Y_1 = x_1$ and $Y_l = [x_2, Y_{l-1}]$ for $l \geq 2$. Since f_k is 2–connected, it induces an automorphism of F_n^{acy} [28, Section 4]. We denote it by f_k again. It belongs to IAut F_n^{acy}. For such an automorphism, the Magnus matrix $r_l(f_k)$ can be computed by using free differentials. That is, we have

$$r_l(f_k) = \begin{pmatrix} \overline{\frac{\partial f(x_1)}{\partial x_1}} & 0_{(1,n-1)} \\ \overline{\frac{\partial f(x_1)}{\partial x_2}} & \\ \vdots & I_{n-1} \\ \overline{\frac{\partial f(x_1)}{\partial x_n}} & \end{pmatrix}$$

at $\mathbb{Z}N_k$. Then $d_{N_l}^{\psi}(r_l(f_k)) = \deg^{\psi}(\det(r_l(f_k))) = \deg^{\psi}\left(\overline{\frac{\partial f(x_1)}{\partial x_1}}\right) = \deg^{\psi}\left(\frac{\partial f(x_1)}{\partial x_1}\right)$. By a direct computation, we have

$$\frac{\partial f(x_1)}{\partial x_1} = 1 + x_1 \left\{ (1 - Y_{k-1}Y_k Y_{k-1}^{-1})\frac{\partial Y_{k-1}}{\partial x_1} + (Y_{k-1} - [Y_{k-1}, Y_k])\frac{\partial Y_k}{\partial x_1} \right\}.$$

When $2 \leq l \leq k-1$, we have $Y_{k-1} = Y_k = 1 \in N_l$, so that

$$d_{N_l}^{\psi}(r_l(f_k)) = \deg^{\psi}\left(\frac{\partial f(x_1)}{\partial x_1}\right) = \deg^{\psi}(1) = 0.$$

When $l = k$, we have $Y_{k-1} \neq 1 \in N_k$, so that

$$\frac{\partial f(x_1)}{\partial x_1} = 1 + x_1(Y_{k-1} - 1)\frac{\partial Y_k}{\partial x_1} = 1 + x_1(Y_{k-1} - 1)(x_2 - Y_k)\frac{\partial Y_{k-1}}{\partial x_1}$$
$$= \cdots = 1 + x_1(Y_{k-1} - 1)(x_2 - Y_k)(x_2 - Y_{k-1})\cdots(x_2 - Y_2),$$

and
$$d_{N_k}^{\psi}(r_k(f_k)) = \deg^{\psi}\left(\frac{\partial f(x_1)}{\partial x_1}\right) = \begin{cases} 1 & (k \geq 3) \\ 2 & (k = 2). \end{cases}$$

Our claim follows from this. □

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