Knot exteriors with additive Heegaard genus and Morimoto's Conjecture

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Given integers $g \geq 2$, $n \geq 1$ we prove that there exist a collection of knots, denoted by $\mathcal{K}_{g,n}$, fulfilling the following two conditions:

(1) For any integer $2 \leq h \leq g$, there exist infinitely many knots $K \in \mathcal{K}_{g,n}$ with $g(E(K)) = h$.
(2) For any $m \leq n$, and for any collection of knots $K_1, \dots, K_m \in \mathcal{K}_{g,n}$, the Heegaard genus is additive:

$$g(E(\#_{i=1}^m K_i)) = \sum_{i=1}^m g(E(K_i)).$$

This implies the existence of counterexamples to Morimoto's Conjecture [17].

57M25; 57M27

1 Introduction and statements of results

Let K_i $(i = 1, 2)$ be knots in the 3–sphere S^3, and let $K_1 \# K_2$ be their connected sum. We use the notation $t(\cdot)$, $E(\cdot)$, and $g(\cdot)$ to denote tunnel number, exterior, and Heegaard genus respectively. It is well known that the union of a tunnel system for K_1, a tunnel system for K_2 and a tunnel on a decomposing annulus for $K_1 \# K_2$ forms a tunnel system for $K_1 \# K_2$. Therefore:

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1.$$

Since $t(K) = g(E(K)) - 1$, this gives:

$$g(E(K_1 \# K_2)) \leq g(E(K_1)) + g(E(K_2)). \tag{1}$$

Given integers $g \geq 0$ and $n \geq 1$, we say that a knot K in a closed orientable manifold M admits a (g, n) position if there exists a genus g Heegaard surface Σ for M, separating M into the handlebodies H_1 and H_2, so that $H_i \cap K$ $(i = 1, 2)$ consists of n arcs that are simultaneously parallel into ∂H_i. We say that K admits a $(g, 0)$ position if $g(E(K)) \leq g$. Note that if K admits a (g, n) position, then K admits both a $(g, n + 1)$ position and a $(g + 1, n)$ position.

Published: 5 July 2008 DOI: 10.2140/agt.2008.8.953

From Morimoto [17, Proposition 1.3], it is known that if K_i ($i = 1$ or 2) admits a $(t(K_i), 1)$ position, then Inequality (1) is strict:

$$g(E(K_1 \# K_2)) < g(E(K_1)) + g(E(K_2)). \tag{2}$$

Morimoto proved that if K_1 and K_2 are m-small knots[1] in S^3, then the converse holds [17, Theorem 1.6]. This result was generalized to arbitrarily many m-small knots in general manifolds by the authors [9]. Morimoto conjectured that the converse holds in general [17, Conjecture 1.5]:

Morimoto's Conjecture *Given knots K_1, $K_2 \subset S^3$,*

$$g(E(K_1 \# K_2)) < g(E(K_1)) + g(E(K_2))$$

if and only if K_i admits a $(t(K_i), 1)$ position (for $i = 1$ or $i = 2$).

Remark 1.1 Morimoto stated the above conjecture in terms of 1–bridge genus $g_1(K)$. It is easy to see that Conjecture 1.5 of [17] is equivalent to the statement above.

In [10] the authors showed that the existence of a knot K satisfying the two conditions below implies the existence of counterexamples to Morimoto's Conjecture:

- K does not admit a $(t(K), 2)$ position.
- K is m-small.

We asked [10, Question 1.9] if there exists a knot K with $g(E(K)) = 2$ that does not admit a $(1, 2)$ position; this question was answered affirmatively by Johnson and Thompson. In fact, in [5, Lemma 4] Johnson showed the existence of knots K with $g(E(K)) = 2$ admitting Heegaard splittings with arbitrarily high distance (see Definition 2.4), and in [6, Corollary 2] Johnson and Thompson showed that (for any n) infinitely many of these knots do not admit a $(1, n)$ position. At about the same time Minsky, Moriah and Schleimer [11, Theorem 3.1] proved a more general result, showing that for any integer $g \geq 2$, there exist infinitely many knots K with $g(E(K)) = g$ admitting a minimal genus Heegaard splitting with arbitrarily high distance. By Proposition 2.6 (for any n) infinitely many of these knots do not admit a $(t(K), n)$ position. However, at the time of writing, the existence of an m-small knot K not admitting a $(t(K), 2)$ position is not known.

[1] A knot K is called *m-small* if its exterior does not admit an essential surface whose boundary consists of a nonempty collection of meridians of K.

Given $n \geq 1$, consider the following conditions:

(1) K does not admit a $(t(K), n)$ position.

(2) $E(K)$ does not admit an essential surface S with $\chi(S) \geq 4 - 2ng(E(K))$.

Our main result is Theorem 1.2 below, which implies the existence of knots fulfilling Conditions (1) and (2) for each $n \geq 1$; specifically, in the proof of Theorem 1.2 we show that some of the knots whose existence was proved in [5] and [11] fulfill these conditions. In Corollary 1.5, we show that this implies the existence of counterexamples to Morimoto's Conjecture.

Theorem 1.2 *Given integers $g \geq 2$ and $n \geq 1$, let $\mathcal{K}_{g,n}$ be the set of all knots $K \subset S^3$ with the following three properties:*

(a) $g(E(K)) \leq g$.

(b) *K does not admit a $(t(K), n)$ position.*

(c) *$E(K)$ does not admits an essential surface S with $\chi(S) \geq 4 - 2gn$.*

Then $\mathcal{K}_{g,n}$ has the following properties:

(1) *For each h, $2 \leq h \leq g$, there exists infinitely many knots $K \in \mathcal{K}_{g,n}$ with $g(E(K)) = h$.*

(2) *For each $m \leq n$ and for any collection of knots $K_1, \dots, K_m \in \mathcal{K}_{g,n}$ (possibly, $K_i = K_j$ for $i \neq j$) we have:*

$$g(E(\#_{i=1}^m K_i)) = \textstyle\sum_{i=1}^m g(E(K_i)).$$

Moreover, for each g, we have:

$$\bigcap_{n=1}^{\infty} \mathcal{K}_{g,n} = \varnothing.$$

Remark 1.3 The existence of knots K_1, K_2 with $g(E(K_1 \# K_2)) = g(E(K_1)) + g(E(K_2))$ is known from Moriah and Rubinstein [15] and Morimoto, Sakuma and Yokota [18]. Theorem 1.2 is new in the following ways:

(1) It is the first time that the connected sum of more than two knots is shown to have additive Heegaard genus.

(2) The proof in [15] uses minimal surfaces in hyperbolic manifolds and in [18] quantum invariants. Our proof is purely topological.

A knot $K \subset M$ is called *admissible* if $g(E(K)) > g(M)$. Thus any knot $K \subset S^3$ is admissible. We denote the connected sum of m copies of K by mK. By [10, Theorem 1.2] for any admissible knot K, there exists N so that if $m > N$ then $g(E(mK)) < mg(E(K))$. In contrast to this, as an obvious consequence of Theorem 1.2 we have:

Corollary 1.4 *Given integers $g \geq 2$ and $n \geq 1$, there exist infinitely many knots $K \subset S^3$ so that $g(E(K)) = g$ and for any $m \leq n$, $g(E(mK)) = mg$.*

A consequence of Corollary 1.4 is:

Corollary 1.5 *There exists a counterexample to Morimoto's Conjecture. Specifically, there exist knots K_1, $K_2 \subset S^3$ such that the following two conditions hold:*

(1) *K_i does not admit a $(t(K_i), 1)$ position ($i = 1, 2$).*

(2) *There exists an integer $m_0 \geq 4$ such that:*
   (a) $g(E(K_1)) = 4$.
   (b) $g(E(K_2)) = 2(m_0 - 2)$.
   (c) $g(E(K_1 \# K_2)) < 2m_0$.

The argument of the proof of Corollary 1.5 was originally given in [10, Theorem 1.4]. We outline it here for completeness.

Proof of Corollary 1.5 Let K be a knot as in Corollary 1.4, for $g = 2$ and $n = 3$. By [10, Theorem 1.2], for some $m > 1$, $g(E(mK)) < mg(E(K)) = 2m$. Let m_0 be the minimal number with that property. Since we chose K for $n = 3$, $m_0 \geq 4$. Hence $g(E(2K)) = 2g(E(K)) = 4$. By the minimality of m_0, $g(E((m_0 - 2)K)) = (m_0 - 2)g(E(K)) = 2(m_0 - 2)$. Let $K_1 = 2K$ and $K_2 = (m_0 - 2)K$. Note that $K_1 \# K_2 = m_0 K$. Thus:

(a) $g(E(K_1)) = 4$.

(b) $g(E(K_2)) = 2(m_0 - 2)$.

(c) $g(E(K_1 \# K_2)) < 2m_0$.

We claim that K_1 does not admit a $(t(K_1), 1)$ position. Assume for a contradiction it does. By Inequality (2) and the above (a), $g(E(3K)) = g(E(K_1 \# K)) < g(E(K_1)) + g(E(K)) = 6$. Since $m_0 \geq 4$, $g(E(3K)) = 3g(E(K)) = 6$, which is a contradiction.

We claim that K_2 does not admit a $(t(K_2), 1)$ position. Assume for a contradiction it does. By Inequality (2) and the above (b), $g(E((m_0 - 1)K)) < g(E((m_0 - 2)K)) + g(E(K)) = (m_0 - 1)g(E(K))$. By the minimality of m_0, $g(E((m_0 - 1)K)) = (m_0 - 1)g(E(K))$, which is a contradiction. □

We note that K_1 and K_2 are composite knots. This led Moriah to conjecture [13, Conjecture 7.14] that if K_1 and K_2 are prime then Morimoto's Conjecture holds.

Outline Section 2 is devoted to three propositions necessary for the proof of Theorem 1.2: Proposition 2.2 relates strongly irreducible Heegaard splittings and bridge position, Proposition 2.5 relates essential surfaces and the distance of Heegaard splitting (Proposition 2.5 is exactly Theorem 3.1 of Scharlemann [22]), and Proposition 2.6 relates bridge position and distance of Heegaard splittings (Proposition 2.6 is exactly Theorem 1 of Johnson and Thompson [6] except for knots $K \subset M$ that admit a $(t(K), 1)$ position and are isotopic onto a Heegaard surface for M of genus $t(K)$). In Section 3 we calculate the genera of certain manifolds that we denote by $X^{(c)}$ (see Notation 2.1). In Section 4 we prove Theorem 1.2.

Remarks 1.6 (1) Tomova, independently and using different techniques, obtained a stronger result than Proposition 2.6 [28, Theorem 1.3].

(2) We refer the reader to our paper [7], that can be used as an introduction to the ideas in the current paper. In [7] an easy argument is given for a special case of Corollary 1.4, namely, $g = 2$ and $n = 3$. Note that this special case is sufficient for Corollary 1.5.

2 Decomposing $X^{(c)}$

In this and the following sections, we adopt the following notation.

Notation 2.1 Let K be a knot in a closed orientable connected manifold M and X its exterior. For an integer $c \geq 0$ we denote by $X^{(c)}$ the manifold obtained by drilling c curves out of X that are simultaneously parallel to meridians of K. Note that $X^{(0)} = X$.

Proposition 2.2 *Let X, $X^{(c)}$ be as above and $g \geq 0$ an integer. Suppose that for some integer $c > 0$, $X^{(c)}$ admits a strongly irreducible Heegaard surface of genus g. Then one of the following holds:*

(1) *X admits an essential surface S with $\chi(S) \geq 4 - 2g$.*

(2) (a) *$c \leq g$, and*
(b) *for some b, $c \leq b \leq g$, K admits a $(g - b, b)$ position.*

Proof of Proposition 2.2 Assume Conclusion (1) does not hold.

Let $C_1 \cup_\Sigma C_2$ be a genus g strongly irreducible Heegaard splitting of $X^{(c)}$. Since $c > 0$, $X^{(c)}$ admits an essential torus T that gives the decomposition $X^{(c)} = X' \cup_T Q^{(c)}$, where $X' \cong X$ and $Q^{(c)}$ is a c–times punctured annulus cross S^1. Since T is incompressible and Σ is strongly irreducible, we may isotope Σ so that every component of $\Sigma \cap T$ is essential in both surfaces (see, for example, Schultens [26, Lemma 6]). Isotope Σ to minimize $|\Sigma \cap T|$ subject to this constraint. Denote $\Sigma \cap X'$ by Σ_X, and $\Sigma \cap Q^{(c)}$ by Σ_Q. Note that, since T is essential, $\Sigma \cap T \neq \varnothing$. By the minimality of $|\Sigma \cap T|$ no component of Σ_X (resp. Σ_Q) is boundary parallel in X' (resp. $Q^{(c)}$).

We claim that Σ_X is connected and compresses into both sides in X', and that Σ_Q is incompressible in $Q^{(c)}$. We sketch this argument here (see [9, Claim 4.5]). By the minimality of $|\Sigma \cap T|$, for $i = 1, 2$, the components of $T \cap C_i$ are incompressible, non–boundary parallel annuli in C_i. It follows that there is a meridian disk $D_i \subset C_i$ which is disjoint from T. Hence there is some component of Σ cut open along T that compresses into C_1 and some component that compresses into C_2. By strong irreducibility of Σ, the same component compresses into both sides; moreover, all other components are incompressible. As remarked above no component of Σ cut open along T is boundary parallel; hence any incompressible component is essential. If some such component is in X' then Conclusion (1) holds, contradicting our assumption. Hence Σ_X is connected and compresses into both sides, and every component of Σ_Q is essential. This completes the proof of the claim.

Since $Q^{(c)}$ is a punctured annulus cross S^1 and Σ_Q is incompressible and has no boundary parallel or closed component, every component of Σ_Q is a vertical annulus (see, for example, Jaco [4, VI.34]). Hence $\partial\Sigma_X$ consists of meridians of K. For $i = 1, 2$, let Σ_i be the surface obtained by simultaneously compressing Σ_X maximally into $C_i \cap X'$. (By *simultaneous compression*, we mean compressing Σ_X once along a collection of mutually disjoint disks, without iterations.) Then the argument of Claim 6 of [8, page 248] shows that every component of Σ_i is incompressible. Hence, every component of Σ_i is a boundary parallel annulus in X' or a 2–sphere, for otherwise Conclusion (1) holds, contradicting our assumption. Denote the number of boundary parallel annuli by b (note that $b = \frac{1}{2}|\partial\Sigma_X|$ and is the same for Σ_1 and Σ_2). Denote the solid tori that define the boundary parallelism of the annular components of Σ_i by $N_{i,1}, \ldots, N_{i,b}$ ($i = 1, 2$).

Claim 1 *For each i ($i = 1, 2$), $N_{i,1}, \ldots, N_{i,b}$ are mutually disjoint.*

Proof of Claim 1 Assume, for a contradiction, that two components (say $N_{i,1}$ and $N_{i,2}$) intersect, say $N_{i,2} \subset N_{i,1}$. Note that Σ_X is retrieved from Σ_i by tubing. Since

Σ_i is obtained from Σ_X by simultaneously compressing into the C_i side only and Σ_X is connected, all the tubes are contained in $N_{i,1}$. This implies that $N_{i,j} \subset N_{i,1}$ for all j. This shows that Σ is isotopic into $Q^{(c)}$, hence T is isotopic into C_1 or C_2. Since T is essential, this is impossible. This proves Claim 1. □

Remark 2.3 As a part of the proof of Proposition 2.2, we analyze the intersection of Σ with $Q^{(c)}$. When K is a hyperbolic knot, $Q^{(c)}$ is a component of the characteristic subvariety. We point the reader to [23, Theorem 3.8], where Scharlemann and Schultens treat the intersection of a strongly irreducible Heegaard surface with the characteristic subvariety in general. Our setting is more limited, and this allows us to obtain more detailed information, eg Claim 2 below.

Claim 2 *K admits a $(g-b,b)$ position.*

Proof of Claim 2 For each i $(i = 1,2)$, let $A_{i,j}$ be the annulus $N_{i,j} \cap T$ $(j = 1,\dots,b)$. Note that $A_{i,j}$ is a longitudinal annulus in $N_{i,j}$. By Claim 1, $C_i \cap X'$ is obtained from $N_{i,1},\dots,N_{i,b}$ and a (possibly empty) collection of 3–balls by attaching 1–handles. Hence $C_i \cap X'$ is a handlebody and $\{A_{i,j}\}_{j=1}^{b}$ is a primitive system of annuli in $\partial(C_i \cap X')$, ie there exists a system of properly embedded disjoint disks $\{\Delta_{i,j}\}_{j=1}^{b}$ such that $\Delta_{i,j} \cap A_{i,k} = \varnothing$ for $j \neq k$, and $\Delta_{i,j} \cap A_{i,j}$ is a spanning arc for $A_{i,j}$.

Since X' is homeomorphic to X, we may perform the trivial Dehn filling on X' to obtain M. In M we cap Σ_X off by attaching $2b$ disks to obtain a genus $g-b$ closed surface, say S. Then S separates M into two parts, denoted H_1 and H_2, so that H_i is obtained from $C_i \cap X'$ by attaching b 2–handles along $A_{i,1},\dots,A_{i,b}$. Since the system $\{A_{i,j}\}_{j=1}^{b}$ is primitive, H_i is a handlebody. Hence $H_1 \cup_S H_2$ is a Heegaard splitting of M.

Up to isotopy, the knot K is the core of the attached solid torus. Thus $K \cap H_i$ $(i = 1,2)$ is the union of the co-cores of the 2–handles, and each co-core is isotopic into ∂H_i via one of the disks $\Delta_{i,j}$. Since the disks $\Delta_{i,j}$ are disjoint, we see that $K \cap H_i$ consists of b simultaneously boundary parallel arcs. Hence $H_1 \cup H_2$ induces a $(g-b,b)$ position of K. This proves Claim 2. □

To complete the proof we need to show that $c \le b \le g$. Since $g-b \ge 0$, it is obvious that $b \le g$ holds. Suppose, for a contradiction, that $b < c$. Note that Σ_Q consists of b vertical annuli that separate $Q^{(c)}$ into $b+1$ components. Note that $\partial X^{(c)}$ consists of $c+1$ tori; thus if $b < c$ then two components of $\partial Q^{(c)}$ are in the same component of $Q^{(c)}$ cut open along Σ_Q. It is easy to see that there is a vertical annulus connecting these tori, which is disjoint from Σ. Hence this annulus is contained in a compression body C_i and connects components of $\partial C_i \setminus \Sigma$. This contradiction completes the proof of Proposition 2.2. $\square$

Definition 2.4 (Hempel [3]) Let $H_1 \cup_\Sigma H_2$ be a Heegaard splitting. The *distance* of Σ, denoted $d(\Sigma)$, is the least integer d so that there exist meridian disks $D_i \subset H_i$ $(i = 1, 2)$ and essential curves $\gamma_0, \dots, \gamma_d \subset \Sigma$ so that $\gamma_0 = \partial D_1$, $\gamma_d = \partial D_2$, and $\gamma_{i-1} \cap \gamma_i = \varnothing$ $(i = 1, \dots, d)$. There are three cases where this definition does not apply: $M \cong S^3$ and $g(\Sigma) = 0$, M is a genus g handlebody and $g(\Sigma) = g$, and M is a lens space and $g(\Sigma) = 1$. In the first two cases on at least one side there are no meridian disks, and in the last case there is no sequence of curves on Σ as required in the definition. In all three cases, we define $d(\Sigma)$ to be zero.

We need two properties of knots whose exteriors admit a Heegaard splittings of high distance. The first is Theorem 3.1 of [22] (for closed surfaces this was shown by Hartshorn [2]):

Proposition 2.5 [22] *Let K be a knot and $d \geq 0$ an integer. Suppose X admits a Heegaard splitting with distance greater than d. Then X does not admit a connected essential surface S with $\chi(S) \geq 2 - d$.*

Proposition 2.6 below was first stated as Theorem 4.1 of [11]. Our proof is a combination of Theorem 1 of [6] and Corollary 4.7 of [24]. The statements of Theorem 1 of [6] and of Proposition 2.6 are very similar; however, the definitions of $(p, 0)$ position used in [6] and here are distinct. In [6] K is said to admit a $(p, 0)$ position[2] if and only if K is isotopic into a genus p Heegaard splitting. Recall that by our definition, K admits a $(p, 0)$ position if and only if $g(X) \leq p$. Thus, if $p < g(X)$ and K is isotopic into a genus p Heegaard surface, then K admits a $(p, 0)$ in the sense of [6], and does not admit a $(p, 0)$ position in our sense; note that in that case K admits a $(p, 1)$ position in our sense. In all other cases, K admits a (p, q) position in the sense of [6] if and only if it admits a (p, q) position in our sense.

Shortly after our paper was posted, Tomova proved a stronger version of Proposition 2.6 using different techniques [28, Theorem 1.3].

Proposition 2.6 *Let $K \subset S^3$ be a knot and p, q integers so that K admits a (p, q) position.*

If $p < g(X)$ then any Heegaard splitting for X has distance at most $2(p + q)$.

[2] The term used in [6] is "K is $(p, 0)$", rather than "K admits a $(p, 0)$ position".

Proof Suppose K admits a (p,q) position with $p < g(X)$. By tubing the surface that gives the bridge position r times ($0 \le r \le q$) we obtain a $(p+r, q-r)$ position. We take $r = g(X) - p - 1$; thus $p + r = g(X) - 1 = t(K)$. Let n be the minimal number so that K admits a $(t(K), n)$ position in our sense. We see that $n \le q - r$. Since $t(K) = p + r$, this implies that $t(K) + n \le p + q$. Hence, for the proof of Proposition 2.6, it suffices to show that any Heegaard splitting of X has distance at most $2(t(K)+n)$.

Claim 1 *The knot exterior X admits a minimal genus Heegaard surface with distance at most $2(t(K)+n)$.*

Proof of Claim 1 Let n' be the minimal integer so that K admits a $(t(K), n')$ position according to the definition given in [6]. Assume first that K is not isotopic onto any genus $t(K)$ Heegaard surface of S^3. Then $n = n'$, and the claim then follows directly from [6, Theorem 1].

Thus we may assume that S^3 admits a genus $t(K)$ Heegaard splitting, say $H_1 \cup_\Sigma H_2$, so that $K \subset \Sigma$, ie, $n' = 0$. On the other hand, as explained above $n = 1$. We base our analysis on [19; 20; 21]. We perform a tiny isotopy of K in H_2, pushing it off Σ. Denote the knot obtained by $\widetilde{K} \subset H_2$. The image of the isotopy is an annulus (say A) embedded in H_2 so that one boundary component of A is $\widetilde{K}$ and the other is $K \subset \Sigma$. Let α be a spanning arc for A. Let $\widetilde{H}_1 = H_1 \cup N_{H_2}(\alpha \cup \widetilde{K})$ and let $\widetilde{H}_2 = \mathrm{cl}(M \setminus \widetilde{H}_1)$. It is easy to see that $\widetilde{H}_1$ and $\widetilde{H}_2$ are handlebodies (with $\widetilde{K} \subset \widetilde{H}_1$) and therefore $\partial \widetilde{H}_1 = \partial \widetilde{H}_2$ is a Heegaard surface for S^3, denoted $S_{\widetilde{K}}(\Sigma)$.[3] Denote the exterior of $\widetilde{K}$ by $\widetilde{X}$. Note that $\widetilde{X} \cong X$. In [19] it was shown that $S_{\widetilde{K}}(\Sigma)$ is a Heegaard surface for $\widetilde{X}$. Since $g(S_{\widetilde{K}}(\Sigma)) = g(\Sigma) + 1 = t(K) + 1 = g(\widetilde{X})$, we have that $S_{\widetilde{K}}(\Sigma)$ is a minimal genus Heegaard surface for $\widetilde{X}$.

We claim $d(S_{\widetilde{K}}(\Sigma)) \le 2$. Let $\widetilde{D}_1 \subset \widetilde{H}_1$ be the disk $\mathrm{cl}(\Sigma \setminus S_{\widetilde{K}}(\Sigma))$ and let $\gamma_0 = \partial \widetilde{D}_1$. Since $t(K) > 0$, γ_0 is essential in $S_{\widetilde{K}}(\Sigma)$. Let $\widetilde{D}_2 \subset \widetilde{H}_2$ be the disk $A \cap \widetilde{H}_2$ and let γ_2 be $\partial \widetilde{D}_2$. Since γ_2 is nonseparating it is essential in $S_{\widetilde{K}}(\Sigma)$. Let γ_1 be a longitude of $\partial N_{H_2}(\alpha \cup \widetilde{K})$ chosen so that $\gamma_0 \cap \gamma_1 = \varnothing$ and $\gamma_1 \cap \gamma_2 = \varnothing$. Then γ_1 is essential in $S_{\widetilde{K}}(\Sigma)$. Hence by Definition 2.4, $d(S_{\widetilde{K}}(\Sigma)) \le 2 < 2(t(K)+n)$.[4]

This proves Claim 1. □

Claim 2 *Any Heegaard surface for X has distance at most $2(t(K)+n)$.*

[3] $S_{\widetilde{K}}(\Sigma)$ is called *stabilization of* Σ *along* $\widetilde{K}$ [19, Definition 2.1]. For a detailed description see also Subsection 4.2 of [16].

[4] The referee interprets the proof above as follows: first, we show that $S_{\widetilde{K}}(\Sigma)$ is so-called μ–primitive, and then we show that all μ–primitive Heegaard surfaces have distance at most 2.

Proof of Claim 2 Let Σ be a Heegaard surface as in Claim 1, ie, Σ is minimal genus and $d(\Sigma) \le 2(t(K)+n)$. Let $\tilde{\Sigma}$ be any Heegaard surface for X. By [24, Corollary 4.7] (with Σ corresponding to Q and $\tilde{\Sigma}$ to P) one of the following holds:

(1) Either Σ is isotopic $\tilde{\Sigma}$, or Σ is obtained from $\tilde{\Sigma}$ by stabilizations or boundary stabilizations.

(2) $d(\tilde{\Sigma}) \le 2g(\Sigma)$.

We treat the cases in order:

(1) Since Σ is a minimal genus Heegaard splitting, Σ is isotopic to $\tilde{\Sigma}$. Therefore $d(\tilde{\Sigma}) = d(\Sigma) \le 2(t(K)+n)$.

(2) In this case, $d(\tilde{\Sigma}) \le 2g(\Sigma) = 2(t(K)+1) \le 2(t(K)+n)$.

This proves Claim 2. □

Claim 2 establishes Proposition 2.6. □

3 Calculating $g(X^{(c)})$

For $X^{(c)}$, recall Notation 2.1. The following lemma is an easy application of the concept of stabilizing along a knot [19, Definition 2.1] that is described in the proof of Proposition 2.6.

Lemma 3.1 *Let $K \subset M$ be a knot, X the exterior of K, and $c \ge 0$ an integer. Denote the genus of X by g. Then*

$$g(X^{(c)}) \le g + c.$$

Proof The proof is an induction on c. For $c = 0$ there is nothing to prove.

Fix $c > 0$. We obtain $X^{(c-1)}$ by Dehn filling a component of $\partial X^{(c)}$ and the core of the attached solid torus (say γ) is isotopic into ∂X. Any Heegaard surface for $X^{(c-1)}$ is obtained from a torus parallel to ∂X and a (possibly empty) collection of tori parallel to other components of $\partial X^{(c-1)}$ by tubing. Hence γ is isotopic onto any Heegaard surface for $X^{(c-1)}$. By stabilizing a minimal genus Heegaard surface for $X^{(c-1)}$ along γ we obtain a Heegaard surface for $X^{(c)}$ of genus $g(X^{(c-1)})+1$. Hence $g(X^{(c)}) \le g(X^{(c-1)})+1$.

By the induction hypothesis, $g(X^{(c-1)}) \le g + (c-1)$; hence we get: $g(X^{(c)}) \le g(X^{(c-1)}) + 1 \le g + (c-1) + 1 = g + c$. □

Proposition 3.2 *Let M be a compact orientable manifold that does not admit a nonseparating surface. Let $K \subset M$ be a knot, and X its exterior. Let $c \geq 0$ be an integer. Denote the genus of X by g. Suppose that X does not admit an essential surface S with $\chi(S) \geq 4-2(g+c)$, and that K does not admit a $(g-1,c)$ position. Then*

$$g(X^{(c)}) = g + c.$$

Proof The proof is an induction on c. For $c=0$ there is nothing to prove.

Fix $c > 0$ and let $\Sigma \subset X^{(c)}$ be a minimal genus Heegaard surface. It follows from the assumptions that X does not admit an essential surface S with $\chi(S) \geq 4-2(g+(c-1))$, and that K does not admit a $(g-1, c-1)$ position; hence the induction hypothesis applies to $X^{(c-1)}$, giving that $g(X^{(c-1)}) = g+c-1$.

The proof is divided into the following two cases:

Case 1 Σ is strongly irreducible.

By Proposition 2.2 one of the following holds:

(1) X admits an essential surface S with $\chi(S) \geq 4-2g(X^{(c)})$.

(2) $c \leq g(X^{(c)})$, and for some b ($c \leq b \leq g(X^{(c)})$), K admits a $(g(X^{(c)})-b, b)$ position.

By Lemma 3.1, we have $4-2g(X^{(c)}) \geq 4-2(g+c)$. By assumption X does not admit an essential surface S with $\chi(S) \geq 4-2(g+c)$, so Case 1 above cannot happen and we may assume that we are in Case 2. Since $b-c \geq 0$, we can tube the Heegaard surface giving the $(g(X^{(c)})-b, b)$ position $b-c$ times to obtain a $(g(X^{(c)})-b+(b-c), b-(b-c)) = (g(X^{(c)})-c, c)$ position.

By assumption K does not admit a $(g-1,c)$ position; this implies that if K admits a (p,c) position for some p, then $p > g-1$. Thus $g(X^{(c)})-c > g-1$. Together with Lemma 3.1, this implies that $g(X^{(c)}) = g+c$.

Case 2 Σ is weakly reducible.

In [27] Sedgwick proved a relative version of Casson and Gordon's seminal theorem [1], proving that an appropriately chosen weak reduction of a minimal genus Heegaard surface yields an essential surface (see the statement and the proof of Theorem 1.1 of [27], cf [14, Theorem 3.1]). Denote by $\widehat{F}$ the essential surface obtained by weakly reducing Σ. Let F be a connected component of $\widehat{F}$. Since $F \subset X^{(c)} \subset M$, it separates. Hence by [9, Proposition 2.13], Σ weakly reduces to F. Note that $\chi(F) \geq \chi(\Sigma) + 4$.

Claim F can be isotoped into $Q^{(c)}$.

Proof of Claim Recall the definitions of T, X' and $Q^{(c)}$ from the proof of Proposition 2.2. Assume, for a contradiction, that F cannot be isotoped into $Q^{(c)}$. Since X does not admit an essential surface S with $\chi(S) \geq 4 - 2(g + c)$, X is irreducible. Minimize $|F \cap T|$. Since F and T are essential and X and $Q^{(c)}$ are irreducible, $F \cap T$ consists of a (possibly empty) collection of curves that are essential in both surfaces. If $F \cap X'$ compresses, then, since the curves of $F \cap T$ are essential in F, so does F, contradiction. Since T is a torus, boundary compression of $F \cap X'$ implies a compression (see, for example, [8, Lemma 2.7]). Finally, minimality of $|F \cap T|$ implies that no component of $F \cap X'$ is boundary parallel. Thus, every component of $F \cap X'$ is essential (including the case $F \subset X'$). Since no component of $F \cap Q^{(c)}$ is a disk or a sphere, $\chi(F \cap X') \geq \chi(F) \geq \chi(\Sigma) + 4$. By Lemma 3.1, $\chi(\Sigma) \geq 2 - 2(g + c)$, thus $\chi(\Sigma) + 4 \geq 6 - 2(g + c)$. Hence $\chi(F \cap X') \geq 6 - 2(g + c)$. Since $X' \cong X$, this contradicts the assumption of Proposition 3.2. This proves the claim. □

Since F is a closed incompressible surface in $Q^{(c)}$, and $Q^{(c)}$ is a punctured annulus cross S^1, F is a vertical torus (see, for example, [4, VI.34]).

First, suppose that F is not boundary parallel in $Q^{(c)}$. Then F decomposes $X^{(c)}$ as $X^{(p+1)} \cup_F D(c-p)$, where $0 \leq p \leq c$ is an integer and $D(c-p)$ is a disk with $c-p$ holes cross S^1. Note that since F is not parallel to a component of $\partial Q^{(c)}$, $c - p \geq 2$. Therefore $p + 1 < c$. This, together with the assumption of the proposition, implies that X does not admit an essential surface S with $\chi(S) \geq 4 - 2(g + (p + 1))$, and that K does not admit a $(g-1, p)$ position; hence the induction hypothesis applies to $X^{(p+1)}$, giving that $g(X^{(p+1)}) = g + p + 1$. By Schultens [25], $g(D(c-p)) = c - p$. Since F was obtained by weakly reducing a minimal genus Heegaard surface [9, Proposition 2.9] (see also [25, Remark 2.7]) gives:

$$\begin{aligned} g(X^{(c)}) &= g(X^{(p+1)}) + g(D(c-p)) - g(F) \\ &= (g + p + 1) + (c - p) - 1 \\ &= g + c. \end{aligned}$$

Next, suppose that F is boundary parallel in $Q^{(c)}$. Since F is essential in $X^{(c)}$, it cannot be isotopic to a component of $\partial X^{(c)}$ and must therefore be isotopic to $\partial Q^{(c)} \setminus \partial X^{(c)} = T$. This gives the decomposition $X^{(c)} = X' \cup_F Q^{(c)}$. Since $X' \cong X$,

$g(X') = g$. By [25] $g(Q^{(c)}) = c + 1$. We get, as above:

$$\begin{aligned} g(X^{(c)}) &= g(X') + g(Q^{(c)}) - g(F) \\ &= g + (c+1) - 1 \\ &= g + c. \end{aligned}$$

This completes the proof of Proposition 3.2. $\square$

Proposition 3.3 *Let $m \geq 1$ and $c \geq 0$ be integers, and let $\{K_i \subset M_i\}_{i=1}^m$ be knots in closed orientable manifolds. Suppose that M_i does not admit a nonseparating surface ($1 \leq i \leq m$). Denote the exterior of K_i by X_i, and the exterior of $\#_{i=1}^m K_i$ by X. Let g be an integer so that $g(X_i) \leq g$ ($1 \leq i \leq m$).*

Suppose that no X_i admits an essential surface S with $\chi(S) \geq 4 - 2g(m+c)$, and that no K_i admit a $(g(X_i) - 1, m + c - 1)$ position. Then we have:

$$g(X^{(c)}) = \textstyle\sum_{i=1}^m g(X_i) + c.$$

Proof Suppose first that $m = 1$. Note that $4 - 2g(1+c) \leq 4 - 2(c+g)$; therefore Proposition 3.3 follows from Proposition 3.2 in this case. Assume from now on $m \geq 2$.

We induct on (m, c) ordered lexicographically, where m is the number of summands and c is the number of curves drilled. Note that by Miyazaki [12], m is well defined (see [9, Claim 1]).

By Lemma 3.1, Inequality (1) in Section 1, and the assumption that $g(X_i) \leq g$ for all i, we get: $g(X^{(c)}) \leq g(X) + c \leq \sum_{i=1}^m g(X_i) + c \leq mg + c$. Since $g \geq 2$, we have that $g(X^{(c)}) \leq g(m+c)$.

By assumption, for all i, X_i does not admit an essential surface S with $\chi(S) \geq 4 - 2g(m+c)$. Hence by the Swallow Follow Torus Theorem [9, Theorem 4.1], any minimal genus Heegaard surface for $X^{(c)}$ weakly reduces to a swallow follow torus F giving the decomposition $X^{(c)} = X_I^{(c_1)} \cup_F X_J^{(c_2)}$, where $I \subset \{1, \dots, m\}$, $K_I = \#_{i \in I} K_i$, $K_J = \#_{i \notin I} K_i$, $X_I = E(K_I)$, $X_J = E(K_J)$, and $c_1 + c_2 = c + 1$ (for details see the first paragraph of Section 4 of [9]). Denote the number of factors of K_I, $|I|$, by m_1, and the number of factors of K_J, $m - |I|$, by m_2. Note that $m_1 = 0$ or $m_2 = 0$ are possible. However, at least one of m_1 or m_2 is not zero so by symmetry we may assume $m_1 \neq 0$.

First assume that $m_1 = m$. Then $m_2 = 0$ and $X_J^{(c_2)}$ is a disk with c_2 holes cross S^1. Since F is essential [27, Theorem 1.1], $c_2 \geq 2$. Then $c_1 = c - c_2 + 1 \leq c - 1$. Since $m_1 = m$, we see that $m_1 + c_1 \leq c + m - 1$. By assumption, no X_i ($1 \leq i \leq m$) admits an essential surface S with $\chi(S) \geq 4-2g(m_1+c_1) > 4-2g(m+c)$. Hence, the induction hypotheses applies to $X_I^{(c_1)} \cong X^{(c_1)}$, showing that

$$g(X_I^{(c_1)}) = \sum_{i=1}^m g(X_i) + c_1.$$

Since $X_J^{(c_2)}$ is homeomorphic to a disk with c_2 holes cross S^1, $g(X_J^{(c_2)}) = c_2$ by [25]. Since F was obtained by weakly reducing a minimal genus Heegaard surface, Proposition 2.9 of [9] and the fact that $c_1 + c_2 = c+1$, we get:

$$\begin{aligned} g(X^{(c)}) &= g(X_I^{(c_1)}) + g(X_J^{(c_2)}) - g(F) \\ &= \big(\textstyle\sum_{i=1}^m g(X_i) + c_1\big) + c_2 - 1 \\ &= \textstyle\sum_{i=1}^m g(X_i) + c. \end{aligned}$$

This proves Proposition 3.3 when $m_1 = m$.

Next assume that $m_1 < m$. By assumption $m_1 > 0$, hence $m_2 < m$. By construction $c_1 \leq c+1$, and $c_2 \leq c+1$. Hence $m_1 + c_1 \leq m+c$, and $m_2 + c_2 \leq m+c$. By assumption, no X_i $(1 \leq i \leq m)$ admits an essential surface S with $\chi(S) \geq 4-2(m_j+c_j)g \geq 4-2(m+c)g$ $(j=1,2)$. Hence the induction hypothesis applies to $X_I^{(c_1)}$ and $X_J^{(c_2)}$, giving $g(X_I^{(c_1)}) = \sum_{i \in I} g(X_i) + c_1$, and $g(X_J^{(c_2)}) = \sum_{i \notin I} g(X_i) + c_2$. We get, as above:

$$\begin{aligned} g(X^{(c)}) &= g(X_I^{(c_1)}) + g(X_J^{(c_2)}) - g(F) \\ &= \big(\textstyle\sum_{i \in I} g(X_i) + c_1\big) + \big(\textstyle\sum_{i \notin I} g(X_i) + c_2\big) - 1 \\ &= \textstyle\sum_{i=1}^m g(X_i) + c_1 + c_2 - 1 \\ &= \textstyle\sum_{i=1}^m g(X_i) + c. \end{aligned}$$

This completes the proof of Proposition 3.3. □

Remark 3.4 For $m \geq 2$, the proof is an application of the Swallow Follow Torus Theorem [9, Theorem 4.1]. In [9, Remark 4.2] it was shown by means of a counterexample that the Swallow Follow Torus Theorem does not apply to $X^{(c)}$ when $m=1$. Hence the argument of the proof of Proposition 3.3 cannot be used to simplify the proof of Proposition 3.2.

4 Proof of Theorem 1.2

Fix $g \geq 2$ and $n \geq 1$. Let $\mathcal{K}_{g,n}$ be the set of all knots $K \subset S^3$ with the following three properties:

(a) $g(E(K)) \leq g$.

(b) K does not admit a $(t(K), n)$ position.

(c) $E(K)$ does not admits an essential surface S with $\chi(S) \geq 4 - 2gn$.

Fix h satisfying $2 \leq h \leq g$. There exist infinitely many knots in S^3, each admitting a genus h Heegaard splitting of distance greater than $\max\{2gn-2, 2(h+n-1)\}$, by [11, Theorem 3.1]. Let K_h be such a knot, and X_h its exterior.

Since X_h admits a genus h Heegaard splitting with distance greater than $2(h+n-1) \geq 2h$ (as $n \geq 1$), by [24, Corollary 4.7] this splitting must be minimal genus; in particular, $g(E(K_h)) = h$. Since X_h admits a Heegaard splitting with distance greater than $2(h+n-1)$, by Proposition 2.6, K_h does not admit a $(h-1, n) = (t(K), n)$ position. Since X_h admits a Heegaard splitting with distance greater than $2gn-2$, by Proposition 2.5, X_h does not admits an essential surface S with $\chi(S) \geq 4 - 2gn$. We see that $K_h \in \mathcal{K}_{g,n}$ and hence $\mathcal{K}_{g,n}$ contains infinitely many knots K with $g(X) = h$. This proves that $\mathcal{K}_{g,n}$ fulfills Conclusion (1) of Theorem 1.2.

Since (for any $K \in \mathcal{K}_{g,n}$) X does not admit an essential surface S with $\chi(S) \geq 4 - 2gn$, and K does not admit a $(t(K), n)$ position, applying Proposition 3.3 with $m \leq n$ and $c = 0$, we see that the knots in $\mathcal{K}_{g,n}$ fulfill Conclusion (2) of Theorem 1.2.

By [10, Theorem 1.2] for any knot $K' \subset S^3$, there exists N so that if $n > N$, then $g(E(nK')) < ng(E(K'))$. This shows that $K' \notin \mathcal{K}_{g,n}$ for $n > N$. Hence $K' \notin \bigcap_{n=1}^{\infty} \mathcal{K}_{g,n}$. As K' was arbitrary, $\bigcap_{n=1}^{\infty} \mathcal{K}_{g,n} = \varnothing$.

This completes the proof of Theorem 1.2.

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Received: 1 May 2007 Revised: 24 April 2008