On asymptotic dimension of amalgamated products and right-angled Coxeter groups

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We prove the inequality

$$\text{asdim} A *_C B \leq \max\{\text{asdim} A, \text{asdim} B, \text{asdim} C + 1\}.$$

Then we apply this inequality to show that the asymptotic dimension of any right-angled Coxeter group does not exceed the dimension of its Davis complex.

1 Introduction

Asymptotic dimension was introduced by Gromov as an invariant of finitely generated groups [20]. It is defined for metric spaces and applied to finitely generated groups with the word metric. Since by its definition it is quasi-isometry invariant, it does not depend on the choice of a finite generating set. It turns out that the asymptotic dimension is a coarse invariant in the sense of Roe [21]. Since all proper left invariant metrics on any countable group are coarsely equivalent (see Dranishnikov and Smith [16] and Shalom [22]), the notion of asymptotic dimension can be extended to all countable groups.

The interest in the asymptotic dimension was sparked by Gouliang Yu's proof of the Novikov Higher Signature conjecture for manifolds whose fundamental group has finite asymptotic dimension [23]. Similar progress on related conjectures was done under the assumption of finite asymptotic dimension in the works of Bartels [1], Carlsson and Goldfarb [9], the author [12] and the author with Ferry and Weinberger [13].

Finite asymptotic dimensionality is proven for many classes of groups. The exact computation of asymptotic dimension of groups is a more difficult task. To the best of my knowledge it is completed only for polycyclic groups and for hyperbolic groups. For polycyclic groups, the asymptotic dimension equals the Hirsch length: $\text{asdim}\,\Gamma = h(\Gamma)$ (see Bell and Dranishnikov [4] for the inequality in one direction and Dranishnikov and Smith [16] for the other direction). The asymptotic dimension of a finitely generated hyperbolic group equals the covering dimension of its boundary plus one:

asdim $\Gamma = \dim \partial_\infty \Gamma + 1$ by Buyalo [7] and Buyalo and Lebedeva [8]. In view of Bestvina–Mess' formula [6] we have asdim $\Gamma = \mathrm{vcd}(\Gamma)$ for those hyperbolic groups for which the virtual cohomological dimension is defined (eg for residually finite hyperbolic groups).

The Coxeter groups are considered a playground for many problems and conjectures in geometric group theory. All Coxeter groups have finite asymptotic dimension since they are isometrically embeddable in the finite product of trees by the author and Januszkiewicz [14]. The original such embedding is due to Januszkiewicz and it gives an estimate asdim $\Gamma \le |S|$ for a Coxeter system (Γ, S). In [15] we noticed that the Januszkiewicz technique can be pushed to bring a better estimate to asdim $\Gamma \le \mathrm{ch}(N(\Gamma))$ where $N(\Gamma)$ is the nerve of (Γ, S) and $\mathrm{ch}(N(\Gamma))$ is the chromatic number of the 1–skeleton of $N(\Gamma)$. A lower bound for the asymptotic dimension of Coxeter groups is given by Corollary 4.11 in [11]: $\mathrm{vcd}\,\Gamma \le \mathrm{asdim}\,\Gamma$. Since $\dim N + 1 \le \mathrm{ch}(N)$ for every simplicial complex N and $\mathrm{vcd}(\Gamma) \le \dim \Sigma(\Gamma) = \dim N(\Gamma) + 1$, it is natural to assume that asdim $\Gamma \le \dim N(\Gamma) + 1$ where $\Sigma(\Gamma)$ denotes the Davis complex of (Γ, S). In this paper we prove this inequality for right-angled Coxeter groups. Generally, the estimate asdim $\Gamma \le \dim N(\Gamma) + 1$ is not optimal. Perhaps the most natural guess would be that asdim $\Gamma = \mathrm{vcd}\,\Gamma$, but in view of Bestvina's candidate among right-angled Coxeter groups for a counterexample to the Eilenberg–Ganea problem (see Davis [10]), this conjecture seems to be very difficult. The proposed equality has to be checked first for Γ with 2–dimensional acyclic nerves $N(\Gamma)$.

In this paper the estimate asdim $\Gamma \le \dim N(\Gamma) + 1$ is proven by induction on dimension of the nerve. The main ingredient here is the inequality for the asymptotic dimension of the amalgamated product

$$(*) \qquad \mathrm{asdim}\,A *_C B \le \max\{\mathrm{asdim}\,A, \mathrm{asdim}\,B, \mathrm{asdim}\,C + 1\}$$

which is proven in this paper.

This inequality was conjectured as an equality by Bell and the author [3], who proved it when C is a finite group in [4]. Later J-P Caprice found an example among Coxeter groups where the inequality is strict.

As a corollary of $(*)$ this paper gives a new proof of the free product equality

$$\mathrm{asdim}\,A * B = \max\{\mathrm{asdim}\,A, \mathrm{asdim}\,B, 1\}.$$

The existing proof in Bell, Dranishnikov and Keesling [5] is quite long and it appeals to the asymptotic inductive dimension theory developed by the author and Zarichnyi [17].

2 Asymptotic dimension

We recall the definition of asymptotic dimension of a metric space [20]: asdim $X \leq n$ *if for every* $r < \infty$ *there exist uniformly bounded,* r*–disjoint families* $\mathcal{U}^0, \ldots, \mathcal{U}^n$ *of subsets of* X *such that* $\bigcup_i \mathcal{U}^i$ *is a cover of* X.

Let $r \in \mathbb{R}_+$ be given and let X be a metric space. We will say that a family $\mathcal{U}$ of subsets of X is r–*disjoint* if $d(U, U') \geq r$ for every $U \neq U'$ in $\mathcal{U}$. Here, $d(U, U') = \inf\{d(x, x') \mid x \in U, x' \in U'\}$.

For a cover $\mathcal{U}$ of a metric space X we denote by $L(\mathcal{U}) = \inf_{U \in \mathcal{U}} \sup_{x \in X} d(x, X \setminus U)$ the Lebesgue number of $\mathcal{U}$. We recall that the order $\mathrm{ord}\,\mathcal{U}$ of a cover $\mathcal{U}$ is the maximal number of elements with nonempty intersection.

We say that (r, d)–dim $X \leq n$ if for every $r > 0$ there exists a d–bounded cover $\mathcal{U}$ of X with $\mathrm{ord}\,\mathcal{U} \leq n + 1$ and with the Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as to an (r, d)–cover of X.

Proposition 2.1 [3] *For a metric space, asdim $X \leq n$ if and only if there is a function $d(r)$ such that $(r, d(r))$–dim $X \leq n$ for all $r > 0$.*

Let $B_R(x)$ denote the closed R–ball centered at x and let $N_R(A) = \{x \in X \mid d(x, A) \leq R\}$ denotes the closed R–neighborhood of A. Thus, $B_R(x) = N_R(\{x\})$.

Proposition 2.2 *Suppose that $X \subset Y$ is given the restricted metric and let $\mathcal{U}$ be an (r, d)–cover of X. Then $N_{r/4}(X)$ admits an $(r/4, d+r)$–cover $\tilde{\mathcal{U}}$ with $\mathrm{ord}\,\tilde{\mathcal{U}} \leq \mathrm{ord}\,\mathcal{U}$.*

Proof For every $U \in \mathcal{U}$ we define

$$\bar{U} = \bigcup \{\mathrm{Int}\, B_{r/2}(x) \mid d(x, X \setminus U) \geq r\}.$$

Clearly, $N_{r/2}(X) \subset \bigcup_{U \in \mathcal{U}} \bar{U}$. We show that $\mathrm{ord}\{\bar{U} \mid U \in \mathcal{U}\} \leq \mathrm{ord}\,\mathcal{U}$. Let $y \in \bar{U}_1 \cap \cdots \cap \bar{U}_k$. Let $x_i \in U_i$ be such that $d(x_i, y) < r/2$ and $d(x_i, X \setminus U_i) \geq r$. Since $d(x_i, x_1) < r$ and $d(x_i, X \setminus U_i) \geq r$, it follows that $x_1 \in U_i$ for all i. Thus, $U_1 \cap \cdots \cap U_k \neq \varnothing$.

Let $\tilde{U} = \bar{U} \cap N_{r/4}(X)$. Then $\tilde{\mathcal{U}} = \{\tilde{U}\}$ is an $(r/4, d+r)$–cover with $\mathrm{ord}\,\tilde{\mathcal{U}} \leq \mathrm{ord}\,\mathcal{U}$. $\square$

Let K be a countable simplicial complex. There is a metric on $|K|$ called *uniform* which comes from the geometric realization of K. It is defined by embedding K into the Hilbert space $\ell^2 = \ell^2(K^{(0)})$ by mapping each vertex $v \in K^{(0)}$ to a corresponding element of an orthonormal basis for ℓ^2 and giving K the metric it inherits as a subspace.

A map $\varphi\colon X \to Y$ between metric spaces is *uniformly cobounded* if for every $R > 0$, $\operatorname{diam}(\varphi^{-1}(B_R(y)))$ is uniformly bounded. We call a map $\varphi\colon X \to |K|$ to a simplicial complex *c–cobounded*, $c \in \mathbb{R}_+$, if $\operatorname{diam}(\varphi^{-1}(\Delta)) < c$ for all simplices $\Delta \subset K$.

The following was proven by Gromov [20] (see also Bell and Dranishnikov [3] and Roe [21]).

Theorem 2.3 *Let X be a metric space. The following conditions are equivalent.*

- $\operatorname{asdim} X \leq n$
- *For every $\epsilon > 0$ there is a uniformly cobounded, ϵ–Lipschitz map $\varphi\colon X \to K$ to a uniform simplicial complex of dimension n.*

This theorem is proved by using projections to the nerves of open covers. The projection $p_{\mathcal{U}}\colon X \to \operatorname{Nerve}(\mathcal{U}) \subset \ell^2(\mathcal{U})$ defined by the formula

$$p_{\mathcal{U}}(x) = (\phi_U)_{U \in \mathcal{U}}, \qquad \phi_U(x) = \frac{d(x, X \setminus U)}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)}$$

is called *canonical*.

A map $f\colon X \to Y$ between metric spaces is a *coarse embedding* if there exist nondecreasing functions ρ_1 and ρ_2, $\rho_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_i \to \infty$ and for every $x, x' \in X$

$$\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')).$$

Such a map is often called a *coarsely uniform embedding* or just a *uniform embedding*. The metric spaces X and Y are *coarsely equivalent* if there is a coarse embedding $f\colon X \to Y$ so that there is some R such that $Y \subset N_R(f(X))$.

Observe that quasi-isometric spaces are coarsely equivalent with linear ρ_i.

Proposition 2.4 *Let $f\colon X \to Y$ be a coarse equivalence. Then $\operatorname{asdim} X = \operatorname{asdim} Y$.*

As a corollary we obtain that $\operatorname{asdim} \Gamma$ is an invariant for finitely generated groups. One can extend this definition of asdim for all countable groups by considering left-invariant proper metrics on Γ. All such metrics are coarsely equivalent [16; 22].

Theorem 2.5 [16] *Let G be a countable group. Then $\operatorname{asdim} G = \sup \operatorname{asdim} F$ where the supremum is taken over all finitely generated subgroups $F \subset G$.*

For a subset $Y \subset X$ of a metric space X when we write $\operatorname{asdim} Y$ we assume that Y is taken with the metric obtained by restriction.

Also we use in this paper the following two theorems from [2]:

Theorem 2.6 (Finite Union Theorem) *For every metric space presented as a finite union $X = \bigcup X_i$ there is the formula*

$$\mathrm{asdim}(\bigcup X_i) = \max\{\mathrm{asdim}\, X_i\}.$$

Theorem 2.7 (Infinite Union Theorem) *Let $X = \bigcup_\alpha X_\alpha$ be a metric space where the family $\{X_\alpha\}$ satisfies the inequality $\mathrm{asdim}\, X_\alpha \le n$ uniformly. Suppose further that for every r there is a $Y_r \subset X$ with $\mathrm{asdim}\, Y_r \le n$ so that $d(X_\alpha \setminus Y_r, X_{\alpha'} \setminus Y_r) \ge r$ whenever $X_\alpha \ne X_{\alpha'}$. Then $\mathrm{asdim}\, X \le n$.*

We recall that the family $\{X_\alpha\}$ of subsets of X satisfies the inequality $\mathrm{asdim}\, X_\alpha \le n$ *uniformly* if for every $r < \infty$ one can find a constant R so that for every α there exist r–disjoint families $\mathcal{U}_\alpha^0, \ldots, \mathcal{U}_\alpha^n$ of R–bounded subsets of X_α covering X_α.

The following Propositions are taken from [3, Proposition 2 and Lemma 1].

Proposition 2.8 *For every simplicial map $g\colon X \to Y$ the mapping cylinder M_g admits a triangulation with the set of vertices equal to the disjoint union of vertices of X and Y.*

We consider the uniform metric on M_g.

For a cover $\mathcal{U}$ of a metric space X we denote by $b(\mathcal{U}) = \sup_{U \in \mathcal{U}} \mathrm{diam}(U)$ the diameter of $\mathcal{U}$. We note that if for two covers $b(\mathcal{V}) < L(\mathcal{U})$ then there is a map $G\colon \mathcal{V} \to \mathcal{U}$ with the property $G(V) \subset U$. Note that any such map $G\colon \mathcal{V} \to \mathcal{U}$ defines a simplicial map $g\colon \mathrm{Nerve}(\mathcal{V}) \to \mathrm{Nerve}(\mathcal{U})$ of the nerves.

We use the notation $\partial N_r(A) = \{x \mid d(x, A) = r\}$ for the boundary of the r–neighborhood and $r-\mathrm{Int}(A) = A \setminus N_r(X \setminus A)$ for the r–*interior* of A.

Lemma 2.9 [3] *For every $n \in N$ there is a monotone function $\mu\colon \mathbb{R}_+ \to \mathbb{R}_+$ tending to infinity with the following property: Given $\epsilon > 0$, let $W \subset X$ be a subset of a geodesic metric space X and let $\lambda \ge 1/\epsilon$. Then for every two covers $\mathcal{V}$ of $N_\lambda(\partial W)$ and $\mathcal{U}$ of W by open subsets of X with order $\le n+1$, and with $L(\mathcal{U}) > b(\mathcal{V}) > L(\mathcal{V}) \ge \mu(\lambda)$, there is a $2b(\mathcal{U})$–cobounded ϵ–Lipschitz map $f\colon W \to M_g$ to the mapping cylinder of a simplicial map $g\colon \mathrm{Nerve}(\mathcal{V}) \to \mathrm{Nerve}(\mathcal{U})$ between the nerves such that $f|_{\partial W} = p_\mathcal{V}|_{\partial W}$ where $p_\mathcal{V}\colon N_\lambda(\partial W) \to \mathrm{Nerve}(\mathcal{V}) \subset M_g$ is the canonical projection.*

We note that the formulation of Lemma 2.9 differs slightly from Lemma 1 in [3]. Namely, Lemma 2.9 becomes Lemma 1 if one considers the case $W = N_r(\partial W)$. Nevertheless the same formula for f and the same proof as in [3] are valid for the general case.

A *partition* of a metric space X is a presentation as a union $X = \bigcup_i W_i$ such that $\mathrm{Int}(W_i) \cap \mathrm{Int}(W_j) = \varnothing$ whenever $i \neq j$.

Theorem 2.10 (Partition Theorem) *Let X be a geodesic metric space. Suppose that for every $R > 0$ there is $d > 0$ and a partition $X = \bigcup_i^\infty W_i$ with $\operatorname{asdim} W_i \leq n$ uniformly in i such that (R, d)–$\dim(\bigcup_i \partial W_i) \leq n - 1$ where $\bigcup_i \partial W_i$ is taken with the metric restricted from X. Then $\operatorname{asdim} X \leq n$.*

Proof We apply Theorem 2.3. Given $\epsilon > 0$ we construct a uniformly cobounded ϵ–Lipschitz map $\phi\colon X \to K$. We apply the assumption with $R = 4\mu(1/\epsilon)$ where μ is taken from Lemma 2.9. Let $\lambda = 1/\epsilon$. Since $\lambda \leq R/4$ and $\mu(t) \geq t$, by Proposition 2.2 there is an $(r/4, 2d)$–cover $\mathcal{V}$ of $N_\lambda(\bigcup_i \partial W_i)$ of order $\leq n$. We may assume that it is a cover by sets that are open in X. Let $\mathcal{V}_i = \mathcal{V}|_{\partial W_i}$ be the restriction, ie, $\mathcal{V}_i$ consists of those elements of $\mathcal{V}$ that have a nonempty intersection with ∂W_i. Let $\mathcal{U}_i$ be a cover of W_i with $L(\mathcal{U}_i) > 2d \geq b(\mathcal{V}_i)$ and with $b(\mathcal{U}_i) < D$ for all i for some fixed D. By Lemma 2.9 there is a $2D$–cobounded ϵ–Lipschitz map $f_i\colon W_i \to M_{g_i}$ to a uniform complex where M_{g_i} is the mapping cylinder of a simplicial map $g_i\colon \mathrm{Nerve}(\mathcal{V}_i) \to \mathrm{Nerve}(\mathcal{U}_i)$ and f_i coincides on ∂W_i with the canonical projection to the nerve $p_\mathcal{V}\colon \bigcup_i \partial W_i \to \mathrm{Nerve}(\mathcal{V})$. We define

$$K = (\mathrm{Nerve}(\mathcal{V}) \bigsqcup \bigsqcup_i M_{g_i})/\sim$$

as the quotient space under identification along the complexes $\mathrm{Nerve}(\mathcal{V}_i)$. Then the union of f_i defines a map $f\colon X \to K$. Clearly, f is $2D$–cobounded. Since X is geodesic and each f_i is ϵ–Lipschitz, f is ϵ–Lipschitz with respect to the uniform metric on K. $\square$

3 Asymptotic dimension of amalgamated product

Let A and B be finitely generated groups and let C be a common subgroup. We fix finite generating symmetric sets S_A and S_B. Let d denote the word metric on $A *_C B$ corresponding to the generating set $S_A \cup S_B$. The group $G = A *_C B$ acts on the Bass–Serre tree whose vertices are the left cosets $G/A \bigsqcup G/B$ and the vertices xA and xB, $x \in G$, and only them are joined by edges. The edges $[xA, xB]$ are labeled by the cosets xC. We consider the action of G on the dual graph K. Thus vertices of K are the left cosets xC. Two vertices xC and $x'C$ are joined by an edge if an only if the edges in the Bass–Serre tree with these labels have a common vertex. Note that K is a tree-graded space in the sense of Drutu–Sapir [18] with pieces $\Delta(A)$ and $\Delta(B)$, the 1–skeletons of the simplices spanned by A/C or B/C. Thus,

K is partitioned into these pieces in a way that every two pieces have at most one common vertex and the nerve of the partition is a tree (see Figure 1). The graph K has an additional property that all vertices are the intersection points of exactly two pieces of the different types $\Delta(A)$ and $\Delta(B)$. We consider the simplicial metric on K, ie, every edge has length one and we use the notation $|u, v|$ for the distance between vertices $u, v \in K^{(0)}$. For $u \in K^{(0)}$ by $|u|$ we denote the distance to the vertex with label C. Note K has the unique geodesic property for every pair of vertices. There is a natural projection $\pi\colon G \to K$ defined by the action: $\pi(g) = gC$.
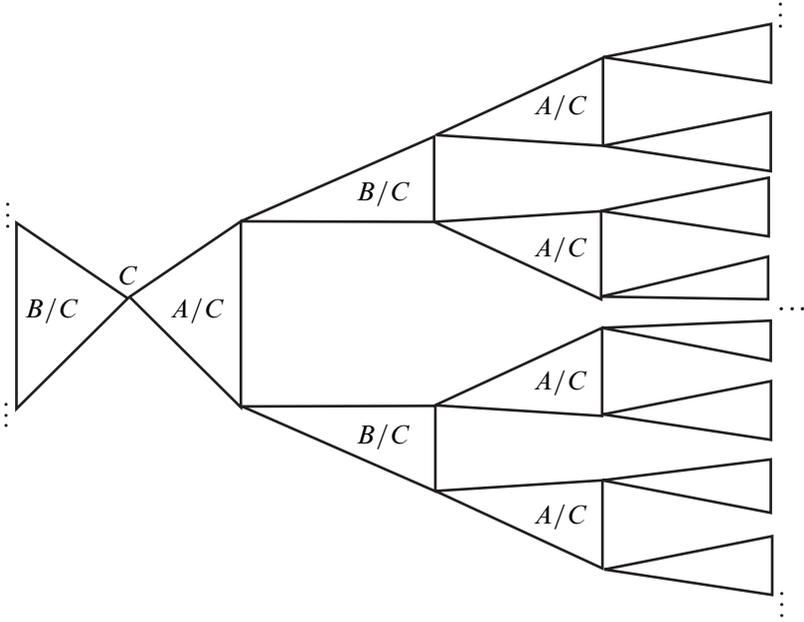

Figure 1

Assertion 3.1 *The map $\pi\colon G \to K$ extends to a simplicial map of the Cayley graph of G, $\pi\colon C(G) \to K$.*

Proof Let $g \in G$ and $s \in S_A \cup S_B$. If $s \in C$, then $\pi(g) = \pi(gs)$ and the edge $[g, gs] \subset C(G)$ is mapped to the vertex $gC = gsC$. With out loss of generality we may assume that $s \in A \setminus C$. We need to show that $\pi(g) = gC$ and $\pi(gs) = gsC$ are joined by an edge in K. Note gA is the common vertex for the edges $[gB, gA]$ and $[gA, gsB]$ in the Bass–Serre tree. So the vertices corresponding these edges are joined by an edge in K. Thus the vertices gC and gsC are joined by an edge in K. $\qquad \square$

As a corollary we obtain that π is 1–Lipschitz.

The base vertex C separates K into two parts $K_A \setminus \{C\}$ and $K_B \setminus \{C\}$. Let $\bar{d}$ denote the graph metric on K. We denote by B_r^A the r–ball, $r \in \mathbb{N}$, in K_A centered at C. There is a partial order on vertices of K defined as follows: $v \leq u$ if and only if v lies in the geodesic segment $[C, u]$ joining the base vertex with u. For $u \in K^{(0)}$ of nonzero level and $r > 0$ we denote by

$$K^u = \{v \in K^{(0)} \mid v \geq u\}, \quad B_r^u = \{v \in K^u \mid |v| \leq |u| + r\}.$$

For every vertex $u \in K^{(0)}$ represented by a coset gC with $|u|$ even we have the equalities $B_r^u = gB_r^A$, $K^u = gK_A$ and hence the equalities $\pi^{-1}(B_r^u) = g\pi^{-1}(B_r^A)$ and $\pi^{-1}(K^u) = g\pi^{-1}(K_A)$.

We say that a set $F \subset G$ *separates* two subsets H_1 and H_2 in G if it separates them in the Cayley graph $C(G)$, that is every path in $C(G)$ with the endpoints in H_1 and H_2 meets F.

Let $D_R = \{x \in G \mid d(x, C) = R\}$ be the boundary of the R–neighborhood of C in $\pi^{-1}(K_A)$, $R \in \mathbb{N}$. For $u \in K^{(0)}$ we denote $D_R^u = g_u(D_R)$, $g_u \in G$, where the coset $g_u C$ represents u. Note that $\pi(D_R^u) \subset B_R^u$.

Proposition 3.2 *For every vertex $u \in K$ with even $|u|$ and for every $v \in K^{(0)}$ incomparable with u or satisfying $v < u$, the set D_R^u separates $\pi^{-1}(v)$ and $\pi^{-1}(u')$ in $A *_C B$ whenever $u < u'$ and $|u'| - |u| > R$.*

Proof Since G acts by isometries, it suffices to show that D_R separates $\pi^{-1}(K_B)$ and $\pi^{-1}(u')$ with $|u'| > R$ and $u' \in K_A$. In view of the fact that π is 1–Lipschitz, $d(C, \pi^{-1}(u')) > R$. Hence D_R separates C and $\pi^{-1}(u')$. By Assertion 3.1 the image of a path in $C(G)$ is a path in K. Since every path in K from K_A to K_B hits the vertex C, it follows that every path in the Cayley graph from $\pi^{-1}(K_A \setminus C)$ to $\pi^{-1}(K_B \setminus C)$ hits the set $C = \pi^{-1}(\{C\})$. Hence every path from $\pi^{-1}(u')$ to $\pi^{-1}(K_B)$ hits D_R. $\square$

Proposition 3.3 *If $R \leq r/4$, then $d(gD_R, g'D_R) \geq 2R$ for $g, g' \in G$ with $|gC|$, $|g'C| \in nr$, $n \in \mathbb{N}$, and $gC \neq g'C$.*

Proof We use the notation $u = gC$ and $u' = g'C$ for the vertices in K. First consider the case when $|u| \neq |u'|$. Since $\pi(gD_R) \subset B_R^u$, $\bar{d}(B_R^u, B_R^{u'}) \geq r - R \geq 3R$, and π is 1–Lipschitz, we obtain that $d(gD_R, g'D_R) \geq 3R$.

Let $|u| = |u'|$ and let $x \in gD_R$ and $y \in g'D_R$. Since every path in K between $\pi(x)$ and $\pi(y)$ goes through the vertices u and u' and a path in the Cayley graph $C(G)$ is projected to a path in K (see Assertion 3.1), a geodesic from x to y in $C(G)$ passes through gC and $g'C$. Since $d(x, gC) = R$ and $d(y, g'C) = R$ we obtain the inequality $d(x, y) \geq 2R$. $\square$

We fix two set-theoretic sections $s_A\colon A/C \to A$ and $s_B\colon B/C \to B$ of π_A and π_B and denote by $X = \mathrm{im}(s_A) \setminus C$ and $Y = \mathrm{im}(s_B) \setminus C$. These sections give rise the normal presentation of elements in $A *_C B$. Namely, every element $\gamma \in A *_C B$ can be presented uniquely in the following form $\gamma = z_1 \ldots z_k c$ where $c \in C$, $z_i \in X \cup Y$, and z_i are alternating in a sense that if $z_i \in X$ then $z_{i+1} \in Y$ and if $z_i \in Y$ then $z_{i+1} \in X$. Denote by $l(z_1 \ldots z_k c) = k$ the length of the normal presentation. Clearly, $l(\gamma) = |\gamma C|$ where γC is treated as a vertex of K. If X' and Y' is a different choice of representatives and $\gamma = z_1 \ldots z_k c$ and $\gamma = z'_1 \ldots z'_k c'$ are corresponding normal presentations, then $z'_1 = z_1 c_1$, $z'_2 = c_1^{-1} z_2 c_2$, ..., $z'_i = c_{i-1}^{-1} z_i c_i$, ..., $z'_{k-1} = c_{k-2}^{-1} z_{k-1} c_{k-1}$ and $z'_k c' = c_{k-1}^{-1} z_k c$ where $c_i \in C$, $i = 1, \ldots, k$.

Assertion 3.4 *Let $\gamma \in A *_C B$. Then $\|\gamma\| \geq d(\beta_k c, C)$ for the normal presentation $\gamma = \beta_1 \ldots \beta_k c$ for any choice of representatives X and Y.*

Proof Let $\|\gamma\|$ be the word-length of γ with respect to the generating set of $A *_C B$ and let $\gamma = t_1 \ldots t_n$, $n = \|\gamma\|$, be a shortest presentation. Then the word $t_1 \ldots t_n$ can be partitioned into a normal presentation (for some choice of X and Y) $\gamma = \alpha_1 \ldots \alpha_k$, $\alpha_i \in A$ (or $\alpha_i \in B$), with $k = l(\gamma)$. Then $\|\gamma\| \geq \|\alpha_k\| = d(\alpha_k, 1) \geq d(\alpha_k, C)$. If X' and Y' is a different choice of representatives and $\gamma = \beta_1 \ldots \beta_k c$ are corresponding normal presentations, then $\beta_1 = \alpha_1 c_1$, $\beta_2 = c_1^{-1} \alpha_2 c_2$, ..., $\beta_i = c_{i-1}^{-1} \alpha_i c_i$, ..., $\beta_{k-1} = c_{k-2}^{-1} \alpha_{k-1} c_{k-1}$ and $\beta_k c' = c_{k-1}^{-1} \alpha_k$ where $c_i \in C$, $i = 1, \ldots, k$.

Thus, $\beta_k c' = c \alpha_k$. This implies that $\|\gamma\| \geq d(\beta_k c, C)$. $\qquad \square$

Lemma 3.5 *Let $\mathrm{asdim}\, A, \mathrm{asdim}\, B \leq n$. Then $\mathrm{asdim}(AB)^m \leq n$ for all m where $(AB)^m = AB \ldots AB \subset A *_C B$.*

Proof We prove that $\mathrm{asdim}\, AB \ldots A(B) \leq n$ by induction on the length of the product k. The inequality is a true statement for $k = 1$. Assume that it holds for k. For the sake of concreteness assume that k is odd. Thus, $\mathrm{asdim}\, F_1 \ldots F_k \leq n$ where $F_{2i-1} = A$ and $F_{2i} = B$. We show that $\mathrm{asdim}\, F_1 \ldots F_k B \leq n$. Consider the family $\{wB \mid l(w) = k\}$. Since all sets wB are isometric to B, $\mathrm{asdim}\, wB \leq n$ uniformly.

Given r we define $Y_r = ABAB \ldots ABACB_r \subset (AB)^{(k+1)/2} = F_1 \ldots F_{k+1}$ where B_r is the r–ball in B. Note that Y_r is quasi-isometric to the k–alternating product $AB \ldots ABA$ and hence $\mathrm{asdim}\, Y_r \leq n$ by the induction assumption. We show that $d(wB \setminus Y_r, w'B \setminus Y_r) \geq r$ whenever wB is not equal to $w'B$. Let $b, b' \in B \setminus CB_r$. Then $d(b, C) \geq r$ and $d(b', C) \geq r$. Since $w^{-1}w' \notin B$, the normal presentation of $b^{-1} w^{-1} w' b'$ ends with $c'b'$ for $c' \in C$. Then by [Assertion 3.4](#), $\|b^{-1} w^{-1} w' b'\| \geq d(c'b', C) \geq r$. Then by the Infinite Union Theorem ([Theorem 2.7](#)) we obtain that

asdim$(F_1 \ldots F_k \cap L_k)B \leq n$ where L_k is the set of all elements $w \in A *_C B$ with $l(w) = k$, $F_i = A(B)$ are alternating, and $F_1 = A$. Let $L_{<k}$ be the set of all elements $w \in A *_C B$ with $l(w) < k$. The inequality asdim$(F_1 \ldots F_m \cap L_{<m})B \leq n$ follows from induction assumption and the Finite Union Theorem (Theorem 2.6). □

This Lemma first appeared in [2]. We present it here with a proof since the argument in [2] contains a gap in the proof of Proposition 3.

Theorem 3.6 *For any finitely generated groups A and B and a common subgroup C there is the inequality*

$$\text{asdim } A *_C B \leq \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\}.$$

Proof Let $n = \max\{\text{asdim } A, \text{asdim } B, \text{asdim } C + 1\}$ and let $\pi\colon A *_C B \to K = K_A \cup K_B$ be the projection to the graph dual to the Bass–Serre tree. In view of the Finite Union Theorem (Theorem 2.6) it suffices to show that asdim $\pi^{-1}(K_A) \leq n$ and asdim $\pi^{-1}(K_B) \leq n$. We prove the first.

We apply the Partition Theorem (Theorem 2.10). Let $R > 0$ be given. Take $r > 4R$.

In view of Proposition 3.2 $G = X_+ \cup X_-$ with $X_+ \cap X_- = D_R$ such that $X_+ \subset \pi^{-1}(K_A)$, $\pi^{-1}(K_B) \subset X_-$ and D_R separates $X_+ \backslash D_R$ and $X_- \backslash D_R$. For every vertex $u \in K_A$ we fix an element $g_u \in G$ such that the coset $g_u C$ represents u. We denote by $X_\pm^u = g_u(X_\pm)$ and define $V_r = X_+ \cap (\bigcap_{|u|=r} X_-^u)$. Note that $\pi(V_r) \subset B_{r+R}$. Let V_r^u denote $g_u(V_r)$. Consider the partition

$$\pi^{-1}(K_A) = \bigcup_{|u|=nr, n \in \mathbb{N}_+} V_r^u \cup N_R^A(C)$$

where $N_R^A(C) = N_R(C) \cap \pi^{-1}(K_A)$.

Clearly, if $V_r^u \cap V_r^w \neq \varnothing$, then either $u < w$ and $|w| = |u| + r$ or $w < u$ and $|u| = |w| + r$. If $V_r^u \cap V_r^w \neq \varnothing$ and $u < w$ then $V_r^u \cap V_r^w = D_R^w$ where $D_R^w = g_w D_R$.

Thus
$$Z = \bigcup_{|u|=nr, n \in \mathbb{N}_+} \partial V_r^u = \bigcup_{|u|=nr, n \in \mathbb{N}_+} D_R^u.$$

We show that (R, d)–asdim $Z \leq n-1$ for some $d > 0$. Since D_R is coarsely equivalent to C, we have asdim $D_R \leq n-1$. Hence there is $d > 0$ and an (R, d)–cover $\mathcal{U}$ of D_R with ord$\mathcal{U} \leq n$. In view of Proposition 3.3, $\widetilde{\mathcal{U}} = \bigcup_{|u|=nr, n \in \mathbb{N}_+} g_u(\mathcal{U})$ is an (R, d)–cover of Z.

Since $\pi^{-1}(B_s) \subset (AB)^{s+1}$, by Lemma 3.5 we have asdim $\pi^{-1}(B_s) \leq n$. Hence asdim $\pi^{-1}(B_{r+R}) \leq n$ and so, asdim $V_r^u \leq n$ uniformly. Note that asdim $N_R^A(C) \leq n$ (in fact, it is $\leq n-1$).

By the Partition Theorem (Theorem 2.10), asdim $\pi^{-1}(K_A) \leq n$. $\qquad\qquad\qquad\square$

4   Asymptotic dimension of right-angled Coxeter groups

A symmetric matrix $M = (m_{ss'})_{s,s' \in S}$ is called a Coxeter matrix if $m_{ss'} \in \mathbb{N} \cup \{\infty\}$, and $m_{ss} = 1$ for all $s \in S$. A group Γ with a generating set S is called a Coxeter group if there is a Coxeter matrix $M = (m_{ss'})_{s,s' \in S}$ such that Γ admits a presentation

$$\langle S \mid (ss')^{m_{ss'}}, s, s' \in S \rangle.$$

A Coxeter group Γ is called *even* if all finite nondiagonal entries of M are even. It is called *right-angled* if all finite nondiagonal entries equal 2.

Every subset $W \subset S$ defines a subgroup $\Gamma_W \subset \Gamma$ which will be called *parabolic*. Let (Γ, S) be a Coxeter group with a generating set S and with a presentation given by means of a Coxeter $S \times S$ matrix M. The *nerve* $N(\Gamma)$ is a simplicial complex with the set of vertices S where a subset $W \subset S$ spans a simplex if and only if the group Γ_W is finite. Thus, $s, s' \in S$, $s \neq s'$, form an edge if and only if $m_{ss'} \neq \infty$. We call the number $m_{ss'}$ a *label* of the edge $[ss']$. By N' we denote the barycentric subdivision of N. The cone $C = \text{Cone } N'$ over N' is called a *chamber* for Γ. *The Davis complex* $\Sigma = \Sigma(\Gamma, S)$ is the image of a simplicial map $q: \Gamma \times C \to \Sigma$ defined by the following equivalence relation on the vertices: $a \times v_\sigma \sim b \times v_\sigma$ provided $a^{-1}b \in \Gamma_\sigma$ where σ is a simplex in N and v_σ is the barycenter of σ. We identify C with the image $q(e \times C)$. The group Γ acts properly and simplicially on Σ with the orbit space equivalent to the chamber. Thus, the Davis complex is obtained by gluing the chambers γC, $\gamma \in \Gamma$ along their boundaries. The main feature of Σ is that it is contractible (see Davis [10]).

Theorem 4.1 *For every right-angled Coxeter group* asdim $\Gamma \leq \dim N(\Gamma) + 1$.

Proof   We prove this inequality by induction on the dimension of the nerve $N = N(\Gamma)$. If $\dim N = 0$, then Γ is virtually free group (possibly of zero rank) and hence asdim $\Gamma \leq 1 = \dim N + 1$.

Let $\dim N = n$ and let N be finite. We prove the inequality asdim $\Gamma \leq n + 1$ by induction on the number of vertices in N. If this number is minimal, ie, $n + 1$, the inequality holds since the group Γ in this case is finite. We assume that there is a vertex $v \in N$ such that the star $\text{st}(v, N)$ does not contain all other vertices of N. If there is no such v, then the 1–dimensional skeleton $N^{(1)}$ coincides with the 1–skeleton of a simplex Δ. Since the group is right-angled, $N = \Delta$, and the group is finite, so this case has been already considered. We take K to be the link $\text{Lk}(v, N)$ of such vertex

v and take N_1 to be the star $\mathrm{st}(v, N)$ of this vertex. We define $N_2 = N \setminus \mathrm{Ost}(v, N)$ where $\mathrm{Ost}(v, N)$ is the open star of v.

Then $\Gamma = \Gamma_{N_1} *_{\Gamma_K} \Gamma_{N_2}$. By the induction assumption, $\mathrm{asdim}\, \Gamma_K \le n$. By the internal induction, $\mathrm{asdim}\, \Gamma_{N_i} \le n+1$, $i = 1, 2$. Then Theorem 3.6 implies $\mathrm{asdim}\, \Gamma \le n+1$. □

Corollary 4.2 *For every right-angled Coxeter group* $\mathrm{asdim}\, \Gamma \le \dim \Sigma(\Gamma)$ *where* $\Sigma(\Gamma)$ *is the Davis complex.*

In view of recent result of Dymara and Schick [19] we obtain:

Corollary 4.3 *For a right-angled building* X, $\mathrm{asdim}\, X \le \dim X$.

Remark In order to extend Theorem 4.1 to all Coxeter groups one needs to show the inequality $\mathrm{asdim}\, \Gamma \le \dim N + 1$ in the case when $N^{(1)}$ is the 1–skeleton of a simplex.

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