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Organizing volumes of right-angled hyperbolic polyhedra

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This article defines a pair of combinatorial operations on the combinatorial structure of compact right-angled hyperbolic polyhedra in dimension three called decomposition and edge surgery. It is shown that these operations simplify the combinatorics of such a polyhedron, while keeping it within the class of right-angled objects, until it is a disjoint union of Löbell polyhedra, a class of polyhedra which generalizes the dodecahedron. Furthermore, these combinatorial operations are shown to have geometric realizations which are volume decreasing. This allows for an organization of the volumes of right-angled hyperbolic polyhedra and allows, in particular, the determination of the polyhedra with smallest and second smallest volumes.

51M10, 57M50; 52B99

1 Introduction

Three-dimensional right-angled hyperbolic compact polyhedra are a well-understood family of hyperbolic objects. A complete classification of them is provided by Andreev's Theorem [3] (see also Roeder, Hubbard and Dunbar [11]), which characterizes when a combinatorial polyhedron admits a geometric realization in $\mathbb{H}^3$ with a given nonobtuse dihedral angle specified at each edge. A simple corollary of this result is a complete characterization of combinatorial types of right-angled hyperbolic polyhedra by means of a small set of conditions on the combinatorics.

Nonetheless, the problem of determining the volume of a given right-angled hyperbolic polyhedra remains difficult. This article will attempt to organize these volumes.

By Mostow rigidity, the hyperbolic structure of a three dimensional right-angled hyperbolic polyhedron is unique and is determined by the combinatorial structure of the polyhedron. In particular, volume is a combinatorial invariant for those polyhedra which admit a right-angled hyperbolic realization.

Following this spirit, volumes of these polyhedra will be organized according to a combinatorial process which, at each step, reduces the complexity of the polyhedron. The two combinatorial operations used in this process are called decomposition and edge surgery. The former is a splitting of the polyhedron under certain conditions,

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while the latter is the deletion of a particular edge. These operations, when applied to sufficiently complicated polyhedra, reduce the complexity of the polyhedron until, after enough applications, the resulting object is a finite disjoint union of polyhedra from an infinite family of exceptional polyhedra, namely the Löbell polyhedra. This family of polyhedra have the property that these operations cannot be applied to them and their geometry, in particular their volume, is very well understood.

It will be shown that for polyhedra not of Löbell type, either the polyhedron is decomposable or there exists an edge for which edge surgery can be performed. Therefore, every such polyhedron will eventually be transformed into a family of Löbell ones via an iteration of this process.

These combinatorial operations will then be studied from a geometric point of view. Decomposition bears a strong resemblance to the decomposition of hyperbolic Haken manifolds along incompressible subsurfaces. In particular, after passing to a manifold cover of a particular sort constructed in Section 5, this is precisely what decomposition is. As such, the result of Agol, Dunfield, Storm and Thurston [1], which gives a description of the effect of Haken decomposition on volumes, can be applied to show that decomposition of right-angled hyperbolic polyhedra is not volume increasing.

Next, it will be shown that the geometric realization of edge surgery is to "unbend" the polyhedron along the edge that is surgered. This means that the polyhedron is deformed so that the dihedral angle measure along this edge increases from $\pi/2$ to π while the dihedral angle measure of every other edge is constant and equal to $\pi/2$. The main sticking point with this is that this deformation passes through obtuse-angled polyhedra which is not the purview of Andreev's theorem. However, results of Rivin and Hodgson [8] generalizing Andreev's theorem will imply that this deformation exists. Then by means of the Schläfli differential formula, it will be shown that the geometric realization of this operation is volume decreasing.

Putting these things together gives chains of inequalities of volumes of right-angled hyperbolic polyhedra determined by the decompositions and edge surgeries used to go from an initial polyhedron to a disjoint union of Löbell polyhedra. Since every right-angled hyperbolic polyhedron which is not of Löbell type can be reduced in this way, a method for organizing volumes of right-angled hyperbolic polyhedra is obtained. This is summarized in the following theorem, which is the main result of this article:

Theorem 9.1 *Let P_0 be a compact right-angled hyperbolic polyhedron. Then there exists a sequence of disjoint unions of right-angled hyperbolic polyhedra $P_1, P_2, \ldots, P_k$ such that for $i = 1, \ldots, k$, P_i is gotten from P_{i-1} by either a decomposition or edge* *surgery, and P_k is a set of Löbell polyhedra. Furthermore,*

$$\text{vol}(P_0) \geq \text{vol}(P_1) \geq \text{vol}(P_2) \geq \cdots \geq \text{vol}(P_k).$$

As the volumes of Löbell polyhedra can be explicitly calculated, the right-angled hyperbolic polyhedra of smallest and second smallest volumes can be identified easily:

Corollary 9.2 *The compact right-angled hyperbolic polyhedron of smallest volume is $L(5)$ (a dodecahedron) and the second smallest is $L(6)$ where $L(n)$ denotes the n–th Löbell polyhedron.*

In fact, if one had an oracle to inform them about the precise volume of a given polyhedron, then this result would provide an algorithm terminating in finite time to determine the ordering of volumes in the sense that the right-angled hyperbolic polyhedron of n–th smallest volume would be identifiable.

It should be noted that this article deals exclusively with compact right-angled hyperbolic polyhedra. Their ideal siblings, those right-angled hyperbolic polyhedra which have vertices lying on the ideal boundary of $\mathbb{H}^3$, are not covered. There are examples of ideal right-angled polyhedra whose volumes are smaller than that of the smallest compact right-angled hyperbolic polyhedron. For example, the right-angled ideal octahedron, all of whose vertices are ideal, has volume strictly smaller than that of the right-angled dodecahedron.

Furthermore, at the present moment, analogous questions about four dimensional and higher dimensional right-angled hyperbolic polyhedra are largely a mystery and cannot be dealt with using the techniques of this article. Indeed, there is no technology which rivals the power of Andreev's theorem to even begin constructing examples of such objects in dimension 4 and higher.

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2 Definitions and notation

The setting for this article is $\mathbb{H}^3$. More generally, $\mathbb{H}^n$ is the unique simply connected Riemannian manifold of dimension n with constant sectional curvatures equal to -1.

If F is a totally geodesic 2–plane in $\mathbb{H}^3$ then a *hyperbolic half-space* H_F is a closed subset of $\mathbb{H}^3$ bounded by H. Similarly, if g is a geodesic in $\mathbb{H}^2$, then a *hyperbolic half-plane* is a closed subset of $\mathbb{H}^2$ bounded by g.

A *hyperbolic polyhedron* is a nonempty compact convex transverse intersection of a finite number of hyperbolic half-spaces. For any hyperbolic polyhedron P, there will be a unique minimal set of hyperbolic half spaces whose intersection is P. It will be assumed that this minimal set will always be the one defining P.

If H_F is a hyperbolic half-space defining P, a *face* of P is the intersection of F with P. The 2–plane F is said to be the plane *supporting* the face. As F is itself isometric to the hyperbolic plane, it is evident that a face is itself isometric to a *hyperbolic polygon*, that is a nonempty convex compact transverse intersection of a finite number of hyperbolic half-planes. By abusing notation, the face supported by F will often be denoted by the same letter F.

If P is a hyperbolic polyhedron, then an *edge* is a nonempty intersection of two distinct faces of P containing more than one point. A *vertex* is a nonempty intersection of three or more distinct faces of P.

A *combinatorial polyhedron* is a 3–ball whose boundary sphere is equipped with a cell structure whose 0–cells, 1–cells and 2–cells will also be called vertices, edges and faces respectively, and which can be realized as a convex polyhedron. By Steinitz's theorem, such objects are exactly those whose 1–skeletons are simple and 3–connected graphs. A hyperbolic polyhedron has a natural description as a combinatorial polyhedron. Passage between the combinatorial perspective and the geometric one will often be done without mention.

Let c be a simple closed curve on ∂P which intersects transversely the interior of exactly k distinct edges. Such a curve is called a *k–circuit*. A k–circuit is a *prismatic k–circuit* if the endpoints of all the edges which c intersects are distinct. Often the distinction between a k–circuit c and the edges it intersects will be blurry, if not completely nonexistent. See Figure 1 for examples of prismatic circuits in combinatorial polyhedra.

Let e be an edge of P given by the intersection of a pair of distinct faces F and G which are supported by planes also denoted F and G. Then the *interior dihedral angle* (often simply the *dihedral angle*) at e is the dihedral angle formed by the planes F and G in the interior of P. Note that because P is a convex set, the interior dihedral angle of an edge always has measure strictly smaller than π. The *exterior dihedral angle* at e is the dihedral angle formed by F and G in the exterior of P which is the supplement of the interior dihedral angle.

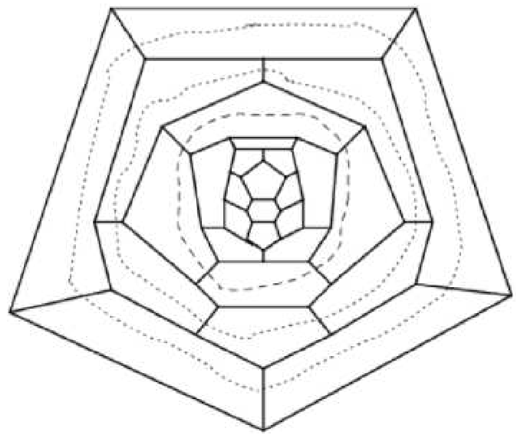

Figure 1: This is the 1–skeleton of a polyhedron with prismatic circuits. The dashed curve is a prismatic 4–circuit and the dotted curves are prismatic 5–circuits.

3 Right-angled hyperbolic polyhedra

The primary objects of study in this article are *right-angled hyperbolic polyhedra* which are hyperbolic polyhedra all of whose dihedral angles have measure equal to $\pi/2$. A hyperbolic polyhedron all of whose dihedral angles have measure less than or equal to $\pi/2$ is said to be *nonobtuse*. When one restricts attention to this class, there are more constraints placed on the combinatorics of the polyhedron than just Steinitz's Theorem. A set of necessary and sufficient conditions for the existence of a nonobtuse hyperbolic polyhedron is described by Andreev's Theorem [3]:

Andreev's Theorem *A combinatorial polyhedron P which is not isomorphic to a tetrahedron or a triangular prism has a geometric realization in $\mathbb{H}^3$ with interior dihedral angle measures $0 < \theta_i \leq \pi/2$ at edge e_i if and only if:*

(1) *The 1–skeleton of P is trivalent.*

(2) *If e_i, e_j, e_k are distinct edges which meet at a vertex, then $\theta_i + \theta_j + \theta_k > \pi$.*

(3) *If e_i, e_j, e_k form a prismatic 3–circuit, then $\theta_i + \theta_j + \theta_k < \pi$.*

(4) *If e_i, e_j, e_k, e_l form a prismatic 4–circuit, then $\theta_i + \theta_j + \theta_k + \theta_l < 2\pi$.*

This geometric realization is unique up to isometry of $\mathbb{H}^3$.

Andreev's original statement and proof of this result in 1970 [3] contains a flaw in one of its combinatorial arguments. A corrected proof is presented in Roeder, Hubbard and Dunbar's paper [11]. Andreev's theorem was also proved independently as a consequence of Rivin and Hodgson's generalization [8]. Details of this can be found in Hodgson's article [7].

The prismatic circuit conditions in Andreev's Theorem can be thought of as a cone manifold translation of the concepts "irreducible" and "atoroidal". For example, a prismatic 3–circuit implies the existence of a topologically embedded triangle in P whose vertices lie on the edges of the circuit. If the dihedral angle measures of these edges add up to more π, then geometrically this triangle is positively curved and so can be thought of as the analogue of an embedded 2–sphere in a three manifold. That the 3–circuit is prismatic is what implies the "incompressibility" of the triangle. If the dihedral angle measures add up to π exactly, then the embedded triangle is flat, and so can be thought of as the analogue of an incompressible torus. A similar discussion works for prismatic 4–circuits. Similarly, the second condition is a cone manifold version of the "spherical link" condition for manifolds.

Note that for prismatic k–circuits with $k \geq 5$, an embedded polygon whose vertices lie on these edges is geometrically negatively curved if the polyhedron is nonobtuse. However, this might fail to be the case without the nonobtuse condition. Thus, the classification of hyperbolic polyhedra without the nonobtuse restriction is necessarily more subtle. I Rivin and C Hodgson have accomplished this generalization which will be discussed later.

By requiring that all dihedral angle measures be $\pi/2$, Andreev's classification of right-angled hyperbolic polyhedra becomes purely combinatorial. In fact, the following classification of right-angled hyperbolic polyhedra was done by A Pogorelov in 1967 [10] before Andreev's work:

Corollary 3.1 *A combinatorial polyhedron P has a geometric realization in $\mathbb{H}^3$ as a right-angled hyperbolic polyhedron if and only if:*

(1) *The 1–skeleton of P is trivalent.*

(2) *There are no prismatic 3 or 4–circuits.*

This geometric realization is unique up to isometry.

The geometry of a face of a right-angled polyhedron is described by the following theorem:

Theorem 3.2 *If P is a right-angled hyperbolic polyhedron, then all of its faces are right-angled hyperbolic polygons.*

Proof Let F be a face of P and v a vertex of F. Then $\mathrm{link}(v)$ is a right-angled spherical triangle. Such a triangle is unique up to isometry on the round unit sphere and has edge lengths $\pi/2$. These edge lengths are precisely the angles in the faces containing v at v. Therefore, in particular, the angle of F at v is $\pi/2$. □

This fact, combined with the well-known classification of right-angled hyperbolic polygons which states, in particular, that right-angled hyperbolic k–gons exist for $k \geq 5$, (see, for example, Costa and Martínez [6]) implies that each face of a right-angled hyperbolic polyhedron has at least 5 edges. However, it should be noted that requiring all faces to have at least five edges is not a sufficient replacement for the prismatic circuit conditions in Andreev's characterization of right-angled polyhedra (Corollary 3.1). Figure 1 shows an example of a combinatorial polyhedron with trivalent 1–skeleton, all of whose faces have at least five edges, but which possesses a prismatic 4–circuit and therefore cannot be realized as a right-angled hyperbolic polyhedron.

It should also be remarked that the above result generalizes to higher dimensions. That is, if P is a right-angled polytope in $\mathbb{H}^n$, then any lower dimensional face of P is isometric to a right-angled hyperbolic polytope of the appropriate dimension. However, it has been shown that compact right-angled hyperbolic polytopes in $\mathbb{H}^n$ only exist for $n \leq 4$ [2] (although higher dimensional *ideal* right-angled hyperbolic polytopes do exist).

Here are some combinatorial consequences of the above conditions on the structure of right-angled hyperbolic polyhedra which will be of use later:

Lemma 3.3 *Every right-angled hyperbolic polyhedron P has at least 12 pentagonal faces. If a right-angled hyperbolic polyhedron contains only pentagonal faces, then it is a dodecahedron. In particular, the dodecahedron is the right-angled hyperbolic polyhedron with the smallest number of faces.*

Proof Let $F(P)$ denote the set of faces of P and $E(F)$ denote the number of edges which the face F contains. Let v, e and f denote the number of vertices, edges and faces of P. Let

$$c(P) = \sum_{F \in F(P)} E(F) - 5.$$

Note that $c(P)$ is also equal to $2e - 5f$.

By Euler's formula, $v - e + f = 2$. By trivalence of the 1–skeleton of P, $e = 3v/2$. Therefore $f - e/3 = 2$, or $f - 2e + 5f = 12$, so $f - c(P) = 12$.

Let k denote the number of pentagonal faces of P. Then $c(P) \geq f - k$ as each face of P which is not a pentagon contributes at least 1 to $c(P)$. Therefore, $12 = f - c(P) \leq k$ which proves the first claim.

Suppose P contains only pentagonal faces. Then $c(P) = 0$, so $f = 12$. It is easy to see that the only polyhedron which has 12 faces which are all pentagons is the dodecahedron. □

Lemma 3.4 *If A and B are a pair of distinct faces in a right-angled hyperbolic polyhedron P which share a vertex, then they are adjacent in an edge which contains this vertex.*

Proof Suppose A is not adjacent to B in an edge containing the vertex they share. Then there are at least 4 edges emanating from this vertex which contradicts the trivalence of P. □

Lemma 3.5 *If A, B, and C are pairwise adjacent faces of right-angled hyperbolic polyhedron P, then they all share a vertex.*

Proof Let $a = A \cap B$, $b = B \cap C$, $c = C \cap A$. Suppose no two of a, b and c share an endpoint. Then these three edges form a prismatic 3–circuit which is a contradiction.

Suppose then that two of a, b, c share an endpoint v, for example a and b. Then A and C share the vertex v which implies that they are adjacent in an edge which contains this vertex. This edge must be c since a pair of faces in a polyhedron can intersect in at most one edge. Therefore, A, B and C share a vertex. □

Lemma 3.6 *Suppose P is a right-angled hyperbolic polyhedron and A and C are nonadjacent faces both adjacent to a face B. If $D \neq B$ is also adjacent to both A and C, then D is also adjacent to B.*

Proof Suppose for a contradiction that D is not adjacent to B. Consider the cycle of faces A, B, C, D. Let $e_1 = A \cap B$, $e_2 = B \cap C$, $e_3 = C \cap D$, and $e_4 = D \cap A$.

Suppose e_1 and e_2 share an endpoint. Then A and C share a vertex and therefore, by the above proposition, are adjacent which is a contradiction. A similar argument shows e_i and e_j share no endpoints for $i \neq j$.

Therefore, these edges form a prismatic 4–circuit which is a contradiction. □

4 Examples of right-angled hyperbolic polyhedra

This section will describe an important family of right-angled hyperbolic polyhedra, and will describe operations for producing new examples from given ones.

Löbell Polyhedra A *pentagonal flower*, denoted L^n for $n \geq 5$, is a combinatorial 2–complex consisting of an n–gon F surrounded by n pentagons $p_1, \dots, p_n$ with indices ordered cyclically such that p_i is adjacent to F, p_{i-1} and p_{i+1}. In the case when $n = 5$, a pentagonal flower L^5 is called a *dodecahedral flower*.

Let L_1^n and L_2^n be a pair of pentagonal flowers. Let $L(n)$ be the polyhedron whose boundary is obtained by gluing L_1^n to L_2^n along their S^1 boundary in the only way which produces a cellular decomposition of the sphere with trivalent 1–skeleton. This family of combinatorial polyhedra evidently has no prismatic 3 or 4–circuits and so each has a geometric realization as a right-angled hyperbolic polyhedron. For a geometric construction of these polyhedra in $\mathbb{H}^3$; see Vesnin [12]. These polyhedra $L(n)$ are called *Löbell polyhedra*. In particular, $L(5)$ is isomorphic to a dodecahedron.

The first example of a closed orientable hyperbolic 3–manifold was constructed by F Löbell by gluing the faces of eight copies of $L(6)$. This construction is an example of a more general procedure which produces eightfold manifold covers of right-angled hyperbolic polyhedra which will be described in detail below.

Löbell polyhedra are fairly well understood. In particular, their symmetry allows for an explicit computation of their volumes. This was carried out by A Vesnin [12] whose result will be recorded in the following:

Theorem 4.1 *For $n \geq 5$,*

$$\mathrm{vol}(L(n)) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) - \Lambda\left(2\theta_n - \frac{\pi}{2}\right)\right)$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\frac{\pi}{n})}\right)$$

and $\Lambda\colon \mathbb{R} \to \mathbb{R}$ is the Lobachevskii function

$$\Lambda(z) = -\mathrm{int}_0^z \log|2\sin(t)|\, dt.$$

Theorem 4.2 $\mathrm{vol}(L(n))$ *is an increasing function of n.*

Proof Let $v(x)$ denote the function:

$$v(x) = \frac{x}{2}\left(2\Lambda\left(\theta_x\right) + \Lambda\left(\theta_x + \frac{\pi}{x}\right) + \Lambda\left(\theta_x - \frac{\pi}{x}\right) - \Lambda\left(2\theta_x - \frac{\pi}{2}\right)\right)$$

where
$$\theta_x = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos\left(\frac{\pi}{x}\right)}\right).$$

The result will be shown by proving that v is an increasing function for $x \geq 5$.

Here are a smattering of estimates which will be of use and which are stated without their elementary proofs:

Lemma 4.3 *For* $x \geq 5$*,*

(1) $\pi/6 < \theta_x < \pi/4$.

(2) $-\pi/6 < 2\theta_x - \pi/2 < 0$.

(3) $\pi/6 < \theta_x + \pi/x < \pi/2$.

(4) $\theta'_x < 0$.

(5) Λ *is increasing on* $(-\pi/6, \pi/6)$ *and decreasing on* $(\pi/6, 5\pi/6)$.

(6) Λ' *is decreasing on* $(0, \pi/2)$. □

Let
$$g(x) = 2\Lambda(\theta_x) + \Lambda\left(\theta_x + \frac{\pi}{x}\right) + \Lambda\left(\theta_x - \frac{\pi}{x}\right) - \Lambda\left(2\theta_x - \frac{\pi}{2}\right)$$

so that $v(x) = (x/2)(g(x))$. The function v will be shown to be increasing for $x \geq 5$ by showing that g is increasing on this interval.

Now calculate $g'(x)$:

$$\begin{aligned}
g'(x) &= 2\Lambda'(\theta_x)(\theta'_x) + \Lambda'\left(\theta_x + \frac{\pi}{x}\right)\left(\theta'_x - \frac{\pi}{x^2}\right) \\
&\qquad + \Lambda'\left(\theta_x - \frac{\pi}{x}\right)\left(\theta'_x + \frac{\pi}{x^2}\right) - \Lambda'\left(2\theta_x - \frac{\pi}{2}\right)(2\theta'_x) \\
&\geq \Lambda'\left(\theta_x + \frac{\pi}{x}\right)\left(\theta'_x - \frac{\pi}{x^2}\right) + \Lambda'\left(\theta_x - \frac{\pi}{x}\right)\left(\theta'_x + \frac{\pi}{x^2}\right) \\
&\qquad - \Lambda'\left(2\theta_x - \frac{\pi}{2}\right)(2\theta'_x) \\
&> \Lambda'\left(\theta_x - \frac{\pi}{x}\right)\left(\theta'_x - \frac{\pi}{x^2}\right) + \Lambda'\left(\theta_x - \frac{\pi}{x}\right)\left(\theta'_x + \frac{\pi}{x^2}\right) \\
&\qquad - \Lambda'\left(2\theta_x - \frac{\pi}{2}\right)(2\theta'_x) \\
&= 2\Lambda'\left(\theta_x - \frac{\pi}{x}\right)\theta'_x - \Lambda'\left(2\theta_x - \frac{\pi}{2}\right)(2\theta'_x)
\end{aligned}$$

$$= 2\theta_x' \left(\Lambda'\left(\theta_x - \frac{\pi}{x}\right) - \Lambda'\left(2\theta_x - \frac{\pi}{2}\right)\right)$$
$$> 2\theta_x' \left(\Lambda'\left(\theta_x - \frac{\pi}{x}\right)\right)$$
$$> 0.$$

The first inequality follows from statements (1), (4) and (5) of the Lemma which imply $\Lambda'(\theta_x)\theta_x'$ is positive. The second inequality follows from statement (4) and (6) of the Lemma. The third inequality follows from statement (4) which implies that $2\theta_x'$ is negative, and statements (2) and (5) which imply $\Lambda'(2\theta_x - \pi/2)$ is positive. The final inequality follows from (4), (1) and (5).

This computation implies that g, and therefore v, is an increasing function for $x \geq 5$. Therefore, $\mathrm{vol}(L(n)) = v(n)$ is an increasing function for $n \geq 5$. This ends the proof of Theorem 4.2. □

For reference, the first few values of $\mathrm{vol}(L(n))$ as computed by *Mathematica* are recorded in Table 1.

n	$\mathrm{vol}(L(n))$	n	$\mathrm{vol}(L(n))$
5	4.306...	13	15.822...
6	6.023...	14	17.140...
7	7.563...	15	18.452...
8	9.019...	16	19.758...
9	10.426...	17	21.059...
10	11.801...	18	22.356...
11	13.156...	19	23.651...
12	14.494...	20	24.943...

Table 1: The volumes of the first sixteen Löbell polyhedra

Doubling Given a right-angled polyhedron P and a face F of P, a new right angled polyhedron called the *double of P across F* or simply a *double dP* can be constructed as follows. Let r_F be the reflection of $\mathbb{H}^3$ across the plane supporting the face F. Then dP is defined to be $P \cup r_F(P)$. Note that the edges of F disappear in dP, in the sense that the dihedral angle along these geodesic segments in dP has measure π.

Composition Let P_1 and P_2 be a pair of combinatorial polyhedra. Suppose that F_1 is a face of P_1 that is combinatorially isomorphic to a face F_2 of P_2, which just means they have the same number of edges. A new combinatorial polyhedron P can be defined by choosing an isomorphism between F_1 and F_2, identifying P_1 and P_2 along F_i using this isomorphism, deleting the interiors of the edges corresponding to F_i, and demoting the endpoints of these edges to nonvertices. This new polyhedron is called a *composition* of P_1 and P_2. Note that this composition has a distinguished prismatic k–circuit made up of the edges whose interiors were deleted where k is the number of edges of F_i.

Theorem 4.4 *The composition P of a pair of right-angled hyperbolic polyhedra P_1 and P_2 is also a right-angled hyperbolic polyhedron.*

Proof It must be shown that P has trivalent 1–skeleton and contains no prismatic 3 or 4–circuits. That P has trivalent 1–skeleton is clear. Let $F_1 \subset P_1$ and $F_2 \subset P_2$ be the faces being identified in the formation of P and let c denote the distinguished prismatic k–circuit of P. Suppose P contains a prismatic 3 or 4–circuit d. If d can be made to completely miss c by an isotopy that does not change the set of edges that d intersects, then evidently one of P_1 or P_2 contains a prismatic 3 or 4–circuit, a contradiction.

So then d must intersect c, and it must do so in two distinct faces as d and c are simple closed curves on S^2. This curve d in the composition P determines an arc in either P_1 or P_2 whose endpoints lie on c. Furthermore, this arc intersects at most 3 or 4 edges of P_i, two of which are edges of F_i. Closing this arc by adding a line segment in the face $F_i \subset P_i$ joining the endpoints of this arc, produces a prismatic 2–, 3– or 4–circuit in P_i. This is a contradiction. □

Define *decomposition* to be the operation inverse to composition. That is, decomposition splits a polyhedron P along some prismatic k–circuit into a pair of polyhedra P_1 and P_2 each of which admit a right-angled hyperbolic structure. A right-angled hyperbolic polyhedron which admits a decomposition is *decomposable*.

A necessary condition for a polyhedron P to be decomposable is that it have a prismatic k–circuit c with $k \geq 5$ such that if F is a face of P which c intersects, then, in F, the curve c bounds combinatorial polygons on either side which have at least 5 edges. In this case, c will be said to have no *flats*. However, this condition is not sufficient for decomposability as there is no guarantee that the resulting combinatorial polyhedra P_1 and P_2 obtained by splitting P along c admit a right-angled hyperbolic structure. In particular, the polyhedra P_1 and P_2 may contain prismatic 3 or 4–circuits. However, if $k = 5$, then the necessary condition is sufficient:

Theorem 4.5 *Suppose c is a prismatic 5–circuit of a right-angled hyperbolic polyhedron P with no flats. Then the polyhedron P is decomposable along c.*

Proof Denote the combinatorial polyhedra obtained by splitting P along c by P_1 and P_2. It must be shown that these polyhedra admit right-angled hyperbolic structures. It suffices to show this for just one of the two.

It is obvious that P_1 has trivalent 1–skeleton.

For the purposes of establishing a contradiction, suppose d is a prismatic 3 or 4–circuit of P_1. Let F_1 be the pentagonal face of P_1 produced by splitting P along c. If d does not intersect F_1, then the curve d persists in the composition P which is a contradiction as it implies that P has a prismatic 3 or 4–circuit.

So suppose d intersects the pentagon F_1 in a pair of edges d_1 and d_2. These edges must be disjoint in F_1, and since F_1 is a pentagon, there must be an edge e of F_1 adjacent to both d_1 and d_2. For $i = 1, 2$, let e_i denote the edges of P_1 sharing an endpoint with both e and d_i. These edges e_i both belong to some face of P_1 called E.

Via an isotopy, the circuit d can be pushed across the edge e to form a new circuit $\widehat{d}$ so that instead of intersecting F_1, it intersects E, now in the edges e_i. Let D_i denote the face of P_1 which is adjacent to E in the edge e_i. Note that the faces D_i and E and the curve $\widehat{d}$ are disjoint from the face F_1, and so persist in P. By abusing of notation, label all edges and faces and curves in P by their given labels in P_1.

Suppose d is a prismatic 3–circuit in P_1. Then $\widehat{d}$ is a 3–circuit in P which intersects the three faces D_1, E and D_2. Therefore, these faces are pairwise adjacent. Thus by Lemma 3.5, these faces all share a vertex in P. This implies e_1 and e_2 share a vertex, which implies that the edges e, e_1, and e_2 of P_1 form a triangle. This contradicts the hypothesis on c.

Suppose d is a prismatic 4–circuit in P_1. Then $\widehat{d}$ is a 4–circuit in P which intersects the three faces D_1, E and D_2 as well as some fourth face G which is adjacent to each D_i. Therefore, by Lemma 3.6, G must be adjacent to E. This implies that the edges e, e_1, e_2 and $G \cap E$ of P_1 form a quadrilateral. This contradicts the hypothesis on c.

Therefore, P_1 contains no prismatic 3 or 4–circuits and has trivalent 1–skeleton which implies, by Andreev's theorem, that it admits a right-angled hyperbolic structure. □

Because there is a nontrivial moduli space of right-angled polygons [6], this combinatorial composition of polyhedra is not as simple to understand in the geometric setting as the doubling process described above. The difficulty is that F_1 and F_2 may be nonisometric in the geometric realizations of P_1 and P_2 and therefore the composition's geometric realization is not simply P_1 glued to P_2. However, doubles are a special case of this welding operation where the combinatorics and the geometry agree.

5 Manifold covers

In this section, manifold covers of right-angled hyperbolic polyhedra, viewed as hyperbolic Coxeter orbifolds, are constructed.

If P is a right-angled hyperbolic polyhedron, let the *reflection group of P*, denoted Γ_P, be the group generated by reflections in the planes supporting the faces of P. Then evidently Γ_P is a discrete group of isometries, and $\mathbb{H}^3/\Gamma_P = P$, giving P the structure of a Coxeter orbifold.

A group presentation for Γ_P is a simple matter to write down. Let $F_1, \dots, F_k$ be the faces of P listed in no particular order, and let r_i denote the reflection in the plane supporting the face F_i. Then

$$\Gamma_P = \langle r_1, \dots, r_k \mid r_i^2 = 1, (r_i r_j)^2 = 1 \text{ if } F_i \text{ is adjacent to } F_j \rangle.$$

This presentation will be called the *standard presentation* of Γ_P.

The following theorem was originally proved by Mednykh and Vesnin [9]. The proof contained herein is only slightly modified from their original argument.

Theorem 5.1 *Every right-angled hyperbolic polyhedron $P = \mathbb{H}^3/\Gamma_P$ has an eightfold manifold cover.*

Proof Let P be a right-angled hyperbolic polyhedron and Γ_P be its reflection group. Let $g\colon \Gamma_P \to \mathbb{Z}/2\mathbb{Z}$ be the group homomorphism which gives the mod 2 length of a word in Γ_P. Here the group presentation for Γ_P given above is used. Let G_P be the kernel of this homomorphism.

This group G_P determines a double cover of P which is easy to visualize. Take two copies of P and identify each face of one copy to the corresponding face of the other copy. The resulting geometric object is a hyperbolic orbifold which is topologically S^3 with a one dimensional singular set which is isomorphic as a graph to the 1–skeleton of P. As P has right angles, the cone angle around each edge of this singular set is π.

A Wirtinger type presentation then gives a presentation for G_P. A loop around an edge in the singular set corresponding to an edge in P which is contained by the faces F_i and F_j gives the generator of G_P given by $a_{ij} = r_i r_j$. By the relations of Γ_P, $a_{ij} = a_{ji}$. Note that a_{ij} is a composition of reflections in orthogonal planes and so is a rotation of π about the geodesic which supports the edge $F_i \cap F_j$. In particular, there are relations $a_{ij}^2 = 1$. Further relations are given by vertices so that if F_i, F_j and F_k are distinct faces sharing a vertex, then $a_{ij} a_{jk} = a_{ik}$.

To find a torsion free subgroup of G_P that has index four, some facts about colorings of the faces and edges of P will need to be collected. Suppose the faces of P are colored by the four elements of the group $(\mathbb{Z}/2\mathbb{Z})^2$ with the usual condition that if two faces are adjacent, they are colored differently. Such a face coloring is guaranteed by the four color theorem proved by Appel and Haken [4]. Then each edge of P can be colored by the sum of the colors of the faces which contain the edge. Note that each edge is then colored by one of the three nonzero elements of $(\mathbb{Z}/2\mathbb{Z})^2$. Note also that the sum of two distinct nonzero elements of $(\mathbb{Z}/2\mathbb{Z})^2$ is the third nonzero element.

Lemma 5.2 *This is an edge 3–coloring of the 1–skeleton of P. That is, no two edges which are colored the same share an endpoint.*

Proof Suppose that two edges e_1 and e_2 share a vertex v. By trivalence, there is a unique edge d which also has vertex v which is not e_1 or e_2. Let A be the face containing e_1 and e_2, B the face containing e_2 and d, and C the face containing d and e_1. Let L_F denote the color of the face F given by the face coloring. Then the color of the edge e_1 is $L_A + L_C$ while the color of e_2 is $L_A + L_B$. Since $L_B \neq L_C$, these colors are different. □

Let C denote this coloring of P. Define a homomorphism $h\colon G_P \to (\mathbb{Z}/2\mathbb{Z})^2$ which sends the generator a_{ij} to the coloring of the edge shared by the faces F_i and F_j. That this assignment of images for generators of G_P extends to a group homomorphism follows from the comments above. Let $H_{(P,C)}$ denote the kernel of h.

Lemma 5.3 *$H_{(P,C)}$ is torsion free.*

Proof The proof of this result is an induction argument on the length of a freely reduced word in the generators a_{ij}. If w is such a word, let $l(w)$ denote its length. Let e_{ij} denote the edge of P corresponding to the generator a_{ij}. The proof will show that if w is in $H_{(P,C)}$ and is not the identity element, then w is a hyperbolic screw translation, or a *loxodromic* isometry in the vernacular. These isometries have infinite order, and also have the property that the composition of two such is again loxodromic.

First note that if $l(w) = 1$, then w cannot be in $H_{(P,C)}$. Suppose then that $l(w) = 2$, say $w = a_{ij}a_{mn}$. If w is in $H_{(P,C)}$, then $h(a_{ij}) = h(a_{mn})$. Therefore, as w is freely reduced, e_{ij} and e_{mn} do not share any vertices. Let g denote the geodesic which intersects e_{ij} and e_{mn} orthogonally. Then the isometry $w = a_{ij}a_{mn}$ is loxodromic and its translation axis is g. Therefore, in particular, w cannot have finite order.

Suppose $l(w) = 3$ with $h(w) = 0$, say $w = abc$. Note that $h(a)$, $h(b)$ and $h(c)$ must all be different nonzero elements of $(\mathbb{Z}/2\mathbb{Z})^2$. Furthermore, note that if the edges associated to the generators a and b share a vertex, then w can be reduced in the group to an equivalent word of length 2, and thus is loxodromic. So assume these edges are disjoint. Then, ab is a loxodromic isometry and so w is the composition of a loxodromic isometry with the rotation c and is therefore loxodromic.

Now assume, for the purposes of induction, that if $l(w) \leq d$ for $d \geq 3$ with $w \in H_P$, then w is loxodromic and so does not have finite order. Let w be a freely reduced word of length $d+1$ with $h(w) = 0$. Let x be the prefix of w of length $\lfloor l(w)/2 \rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor function. Furthermore, let y be the freely reduced word representing $x^{-1}w$ so that $w = xy$. Note that since $l(w) \geq 4$, the lengths of both x and y are at least 2.

Suppose first that $h(x) = 0$. Then evidently, $h(y) = 0$ and so x and y are words whose lengths are shorter than that of w and live in $H_{(P,C)}$. Therefore, by the induction hypothesis, x and y are loxodromic, and so w, their composition, is also loxodromic.

Suppose then that $h(x) \neq 0$. Let a_{ij} be the first letter of the word y and z the freely reduced word representing $a_{ij}^{-1}y$. If $h(x) = h(a_{ij})$, then $h(xa_{ij}) = 0$ and so $h(z) = 0$. Therefore, induction says that xa_{ij} and z are loxodromic as they are strictly shorter than w, and so $w = xa_{ij}z = xy$, their composition, is also loxodromic.

Finally, suppose $h(x) \neq 0$ but $h(x) \neq h(a_{ij})$. Let e_{ik} and e_{kj} be edges of P which both share the same endpoint with e_{ij} and let a_{ik} and a_{kj} denote the corresponding generators of G_P. Note that one of a_{ik} or a_{kj} must be colored by the color $h(x)$. Suppose without loss of generality that $h(a_{ik}) = h(x)$. Then, in the word w, replace the first letter a_{ij} in y by $a_{ik}a_{kj}$. Note that since $d+1 = l(w) \geq 4$, the words xa_{ik} and $a_{kj}z$ both have length smaller than $d+1$ and at least 2. Then, by construction, $h(xa_{ik}) = 0$ and so $h(a_{kj}z) = 0$ and therefore both these words are loxodromic by the induction hypothesis. Thus their composition $xa_{ik}a_{kj}z = xa_{ij}z = xy = w$ is loxodromic. Note that the words xa_{ik} and $a_{kj}z$ may not be freely reduced. However, since free reduction only reduces length, the induction hypothesis still applies.

This ends the proof of Lemma 5.3. □

Therefore, $H_{(P,C)}$ is a subgroup of index 4 in G_P which has index 2 in Γ_P, and so $H_{(P,C)}$ is a torsion free subgroup of index 8 in Γ_P. This proves Theorem 5.1. □

Suppose F is some face of P. Let $\Gamma_{(P,F)}$ be the group generated by reflections in the planes supporting each face of P except F. This is a subgroup of index 2 in Γ_P. Let $\mathbb{H}_F$ denote the closed connected subset of $\mathbb{H}^3$ which contains P and is bounded by the planes supporting F and its images under the action of $\Gamma_{(P,F)}$. Then the group $\Gamma_{(P,F)}$ acts on $\mathbb{H}_F$ with quotient $\mathbb{H}_F / \Gamma_{(P,F)} = P$. The orbifold structure on P

can be thought of as mirroring every face of P except F. The face F is the totally geodesic boundary of this orbifold structure on P.

Restricting the homomorphism g which computes the mod 2 length of a word gives a map $\Gamma_{(P,F)} \to \mathbb{Z}/2\mathbb{Z}$. The kernel of this map will be denoted $G_{(P,F)}$. This group acts on $\mathbb{H}_F$ and the quotient orbifold can be visualized in the following way. Take two copies of P, and identify each face of one copy of P to the corresponding face in the other copy, but do not perform this identification if the face is F. Then the resulting space is topologically a 3–ball B^3. The orbifold singularities are a graph isomorphic to the 1–skeleton of P with the edges contained in the face F removed. The resulting graph has some number of 1–valent vertices which live on the boundary of the 3–ball. In fact, in the hyperbolic metric, the boundary of the 3–ball is a totally geodesic hyperbolic 2–orbifold. The edges in the singularity graph all have cone angle π as do the cone points on the boundary corresponding to the 1–valent vertices.

As above, a Wirtinger type presentation gives a presentation for $G_{(P,F)}$ where each edge of P but not in F gives a generator of order 2, and additional relations given by the vertices. In fact, it is evident from the presentations that $G_{(P,F)} = G_P \cap \Gamma_{(P,F)}$.

Let the faces of P other than F be colored by elements of $(\mathbb{Z}/2\mathbb{Z})^2$, and the edges of P other than those of F be colored by the sum of the colors which contain the edge as above. Call this coloring C. Let h be the homomorphism $h\colon G_{(P,F)} \to (\mathbb{Z}/2\mathbb{Z})^2$ sending a generator to the color of its corresponding edge. Let $H_{(P,C,F)}$ be the kernel of h, a subgroup of index 8 of $\Gamma_{(P,F)}$.

Theorem 5.4 *$H_{(P,C,F)}$ is torsion free.*

Proof The proof is essentially the same as that of Lemma 5.3. □

Let $M_{(P,C,F)}$ denote the hyperbolic manifold with geodesic boundary $\mathbb{H}_F/H_{(P,C,F)}$ for some choice of coloring C of $P \setminus F$. This is an eightfold cover of the hyperbolic orbifold with geodesic boundary $\mathbb{H}_F/\,\Gamma_{(P,F)}$.

6 Decomposition

Let P_1 and P_2 be a pair of right-angled hyperbolic polyhedra and $F_1 \subset P_1$, $F_2 \subset P_2$ a pair of faces which are isomorphic k–gons for some choice of isomorphism. Let P denote the composition of P_1 and P_2 along these faces using this isomorphism as described in Section 4. The polyhedron P has a distinguished prismatic k–circuit c which, as a simple closed curve, consists of the edges of F_1 and F_2 that were identified and whose interiors were deleted in the formation of the composition P. Let $C_1, \dots, C_k$ denote the faces of P which this prismatic k–circuit c intersects. This curve c bounds in P an embedded topological k–gon which will be denoted F.

For $i = 1, 2$, let $\phi_i\colon P_i \to P$ denote the topological embedding induced by the composition in the obvious way. This map sends faces of P_i which are not adjacent to F_i to faces of P by cellular isomorphisms, sends faces of P_i adjacent to F_i into C_j for some j, and sends the face F_i to the suborbifold F.

In $\Gamma_{(P_i,F_i)}$, the subgroup generated by reflections in the faces adjacent to F_i (or indeed any face of P_i) is itself a reflection group of a right-angled polygon. Let this group be denoted Γ_{F_i}. Then, the isomorphism which identifies F_1 with F_2 in the formation of the composition P induces a group isomorphism between Γ_{F_1} and Γ_{F_2}. Let G denote the isomorphism class of these groups.

Theorem 6.1 *If P is the composition of P_1 and P_2 along the faces $F_1 \subset P_1$ and $F_2 \subset P_2$, then Γ_P is isomorphic to the free product with amalgamation*

$$\Gamma_P \cong \Gamma_{(P_1,F_1)} *_G \Gamma_{(P_2,F_2)}.$$

Proof Suppose that F_1 and F_2 are polygons with k edges. Let $\{s_j\}$ and $\{t_j\}$ denote the generators of $\Gamma_{(P_1,F_1)}$ and $\Gamma_{(P_2,F_2)}$ respectively, which correspond to reflections in the planes supporting each face of P_i except F_i. Index these generators in such a way that $\{s_1, \dots, s_k\}$ and $\{t_1, \dots t_k\}$ correspond to the faces of P_1 and P_2 (resp.) which are adjacent to F_1 and F_2 (resp.). Furthermore, index in such a way that for each $j = 1, \dots k$, s_j and t_j correspond to faces adjacent to F_1 and F_2 (resp.) which are mapped into the face C_j in P under the embeddings ϕ_i of P_i into P.

Then a presentation for the free product with amalgamation $\Gamma_{(P_1,F_1)} *_G \Gamma_{(P_2,F_2)}$ is given by a generating set $\mathcal{S} = \{s_j\} \cup \{t_j\}$ with relations $\mathcal{R}$ given by three types of words:

(1) $(s_j)^2$ and $(t_j)^2$ for all j.

(2) $(s_i s_j)^2$ and $(t_i t_j)^2$ if the corresponding faces in P_1 and P_2 (resp.) are adjacent.

(3) $s_j = t_j$ for $j = 1, \dots k$.

Define a homomorphism ψ from $\Gamma_{(P_1,F_1)} *_G \Gamma_{(P_2,F_2)}$ to Γ_P in the following way. If $u \in \mathcal{S}$ corresponds to a face $U \subset P_i$, then send u to the generator of Γ_P given by the reflection in the face of P containing the image of U under ϕ_i. That this assignment of generators extends to a homomorphism is obvious using the standard presentation for Γ_P.

To show that ψ is an isomorphism, an inverse map will be defined. Let $\{r_i\}$ denote the generators of Γ_P in the standard presentation with $r_1, \dots, r_k$ denoting the reflections in the faces $C_1, \dots, C_k$ respectively. Send a generator r_j with $j > k$ to the generator of the amalgam which was sent to r_j under ψ, and send r_j with $1 < j < k$, to $s_j = t_j$. This map of generators extends to a homomorphism which is inverse to ψ. □

For $i = 1, 2$, let $j_i\colon \Gamma_{(P_i,F_i)} \hookrightarrow \Gamma_P$ denote the natural inclusions. These homomorphisms are induced by the embeddings ϕ_i. Similarly, the natural inclusion $k\colon G \hookrightarrow \Gamma_P$ is induced by the embedding of F into P. Note that F is isomorphic in the category of orbifolds to the quotient of the plane supporting F_i by the action of $\Gamma_{F_i} \cong G$ for either $i = 1$ or $i = 2$. In particular, because k is injective, it is natural to think of this embedded suborbifold as being incompressible in P.

Consider the manifold $M_{(P,C)} = \mathbb{H}^3/H_{(P,C)}$ which is an eightfold cover of the orbifold $P = \mathbb{H}^3/\,\Gamma_P$. Denote the covering projection by ρ. Let Σ denote the closed surface embedded in $M_{(P,C)}$ which covers the embedded suborbifold $F \subset P$. Then the fundamental group of each component of Σ is isomorphic to $G \cap H_{(P,C)}$ and Σ is incompressible in $M_{(P,C)}$. That is, $M_{(P,C)}$ is Haken.

Consider the manifold $M_{(P,C)} - \mathcal{N}(\Sigma)$ obtained by splitting $M_{(P,C)}$ along Σ by removing a tubular neighborhood $\mathcal{N}(\Sigma)$. Since Σ covers the separating suborbifold $F \subset P$, the surface Σ is separating in $M_{(P,C)}$. Therefore, $M_{(P,C)} - \mathcal{N}(\Sigma)$ is a pair of manifolds M_1 and M_2 with boundary homeomorphic to Σ. Furthermore, since F splits P into two parts given by $\phi_i(P_i)$ for $i = 1, 2$, restricting the covering map ρ to M_i is itself a covering map over $\phi_i(P_i)$ and, in particular, is an eightfold covering map. It is the eightfold covering map corresponding to the group $H_{(P_i,C_i,F_i)}$ where C_i is the coloring of $P_i \setminus F_i$ gotten by restricting the coloring C of P to the image of the embedding ϕ_i.

Therefore, by rigidity, M_i is homeomorphic to $M_{(P_i,C_i,F_i)} = \mathbb{H}_{F_i}/H_{(P_i,C_i,F_i)}$. In English, this says that M_i is homeomorphic to an eightfold orbifold cover of the polyhedron P_i. All of this will be recorded in the following theorem:

Theorem 6.2 *The manifold $M_{(P,C)}$ is a Haken manifold with a separating incompressible surface Σ. Splitting $M_{(P,C)}$ along Σ produces a pair of manifolds M_i, $i = 1, 2$ with*

$$M_i \cong M_{(P_i,C_i,F_i)} = \mathbb{H}_{F_i}/H_{(P_i,C_i,F_i)}.$$ □

The following result, proved by Agol, Storm and Thurston [1], gives a description of the effect of decomposition on volumes.

Theorem 6.3 *If M is a closed hyperbolic Haken 3–manifold and $\Sigma \subset M$ is an incompressible surface, then*

$$\mathrm{vol}(M) \geq \frac{1}{2} V_3 ||D(M - \mathcal{N}(\Sigma))||$$

where V_3 denotes the volume of the regular ideal tetrahedron, $||\cdot||$ denotes the Gromov norm, and $D(\cdot)$ denotes manifold doubling.

If $M - \mathcal{N}(\Sigma)$ admits a hyperbolic structure with totally geodesic boundary, then the double $D(M - \Sigma)$ is a disjoint union of closed hyperbolic manifolds and so the simplicial volume $V_3||D(M - \mathcal{N}(\Sigma))||$ coincides with the hyperbolic volume $\mathrm{vol}(D(M - \mathcal{N}(\Sigma)) = 2\,\mathrm{vol}(M - \mathcal{N}(\Sigma))$. Therefore the above theorem implies that $\mathrm{vol}(M) \geq \mathrm{vol}(M - \mathcal{N}(\Sigma))$.

In the particular case of the manifold $M_{(P,C)}$ which when split along its incompressible surface Σ produces the manifolds M_1 and M_2, Theorem 6.3 implies that $\mathrm{vol}(M_{(P,C)}) \geq \mathrm{vol}(M_1) + \mathrm{vol}(M_2)$. Since $M_{(P,C)}$, M_1 and M_2 are eightfold covers of the polyhedra P, P_1 and P_2 respectively, the following result follows immediately:

Theorem 6.4 *If P is a right-angled hyperbolic polyhedron which is the composition of right-angled hyperbolic polyhedra P_1 and P_2, then*

$$\mathrm{vol}(P) \geq \mathrm{vol}(P_1) + \mathrm{vol}(P_2). \qquad \square$$

7 Edge surgery

This section will be devoted to defining and studying a simple combinatorial operation on right-angled polyhedra called *edge surgery*. This operation together with decomposition will simplify any given right-angled hyperbolic polyhedron into a set of Löbell polyhedra. First, some definitions:

Two distinct faces F_1 and F_2 of P are said to be *edge connected* if they are nonadjacent and there exists an edge e of P connecting a vertex of F_1 to a vertex of F_2. Such an edge e is said to *edge connect* F_1 and F_2. Note that since P has trivalent 1–skeleton, every edge of P edge connects a unique pair of faces.

A face F of P is called *large* if it is has 6 or more edges. An edge e is called *good* if it edge connects two large faces. A good edge is called *very good* if it is not a part of any prismatic 5–circuit.

If e is an edge of P, call the combinatorial operation of deleting the interior e and demoting its endpoints to nonvertices *edge surgery along e* (see Figure 2). Call the line

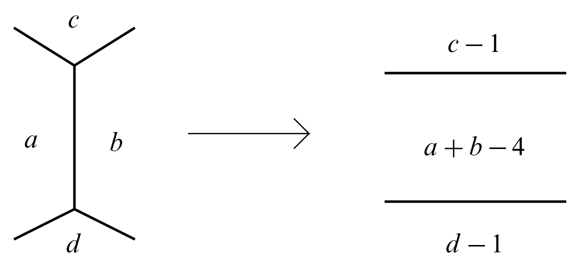

Figure 2: Edge surgery. Here, the faces labelled c and d are edge connected. The labels a, b, c, d also represent the number of edges of the corresponding face, and this figure shows the effect of edge surgery on combinatorics.

segment in P_1 that was removed the *trace* of the edge e and the vertices that were demoted the *traces* of the vertices.

Theorem 7.1 *If a right-angled hyperbolic polyhedron P_0 has a very good edge e and P_1 is the result of edge surgery along e, then P_1 is also a right-angled hyperbolic polyhedron.*

Proof The conditions outlined by Andreev's characterization of right-angled hyperbolic polyhedra must be checked. That is, it must be shown that P_1 has trivalent 1–skeleton and no prismatic 3 or 4–circuits.

It is clear that P_1 has trivalent 1–skeleton as the edge that is deleted has its endpoints demoted to nonvertices while all other vertices of P_0 are left unaffected.

Suppose for the purposes of establishing a contradiction that P_1 contains a prismatic 3 or 4–circuit c. If necessary, perturb c slightly so that it does not intersect the traces of the endpoints of e. Let F denote the face of P_1 containing the trace of e. If c does not intersect the face F at all, then it is clear that P_0 contains a prismatic 3 or 4–circuit which is a contradiction. So suppose c intersects F. Then c intersects the boundary of the polygon F in exactly two edges d_1 and d_2.

If d_1 or d_2 is an edge which contains the trace of an endpoint of e, then via an isotopy, c can be made to intersect d_1 and d_2 on the same side of the trace of e and then further pushed by an isotopy to miss the trace of e completely. Therefore the curve c also determines a prismatic 3 or 4–circuit in P_0 which is a contradiction as P_0 is a right angled hyperbolic polyhedron. A similar argument shows that d_1 and d_2 cannot be edges which lie on the same side of the trace of e.

Suppose then that d_1 and d_2 do not contain the trace of a vertex of e and are on opposite sides of the trace of e. Via an isotopy, c can be made to intersect the trace of e in exactly 1 point. Then c, being a prismatic 3 or 4–circuit in P_1, determines a prismatic 4 or 5–circuit (resp.) in P_0 of which e is a member. This is a contradiction as P_0 cannot have a prismatic 4–circuit and e is very good. □

Theorem 7.2 *Let P be a right-angled hyperbolic polyhedron which is not of Löbell type. Then either it has a good edge or it is decomposable. If P does not have a good edge, then P is decomposable into a pair of right-angled polyhedra, one of which is a dodecahedron.*

Proof Let X be a maximal connected subset of ∂P which contains only large faces. Topologically, X is a subsurface of $\partial P \cong S^2$ and therefore is homeomorphic to a sphere with k disks removed. By Lemma 3.3, P must have at least 12 pentagons and if P is not a dodecahedron, then it must have some number of faces which are not pentagons. Thus $X \neq \varnothing$ and $k \geq 1$. Let $D_1, D_2, \dots, D_k$ denote the disks of $\partial P \setminus X$ and label their boundaries $\partial D_i = S_i$.

Note that the S_i are combinatorially polygons, and by maximality in the way X was defined, every edge of S_i is an edge of a pentagon lying in the disk D_i. Let S and D denote some fixed boundary/disk pair. Suppose that Q is a pentagon with at least one edge belonging to S. The proof breaks down into many cases.

Case A Suppose first all 5 edges of Q lie on S. Then evidently $D = Q$. As every face in X is large, any edge of Q is a good edge.

Case B Suppose 4 edges of Q lie on S. If the edges of Q are labelled cyclicly by $e_i, i = 1, 2, \dots 5$, then without loss of generality assume $e_1, \dots, e_4$ lie on S. Then it is evident that e_2 and e_3 are both good edges.

Case C Suppose 3 edges of Q lie on S. Again, label the edges of Q cyclicly by e_i. Suppose that these three edges are adjacent, without loss of generality, say, e_1, e_2, e_3 lie on S. Then it is evident that e_2 is a good edge. But there is another possibility. It could be that, say, e_1, e_2, and e_4 lie on S. In this case, both e_3 and e_5 are good edges.

Case D Suppose then that every pentagon with an edge lying on S meets S in one or two edges. If Q is a pentagon with exactly two nonadjacent edges lying on S, then the edge of Q which is adjacent to both of them must be good. Thus, suppose that every pentagon with exactly two edges lying on S does so in a pair of adjacent edges. Call a pentagon *inward* if it intersects S in one edge and *outward* if in two adjacent edges. This case breaks down into a number of subcases:

Case D.1 Suppose that all pentagons lying on S are inward. Then X is a single large face of P as every edge emanating from a vertex of X passes into the interior of D. Then X along with the faces adjacent to X form a pentagonal flower.

If X is edge connected to some other large face in the interior of D, then a good edge exists. So suppose then X is edge connected only to pentagons. These pentagons can be arranged along the boundary of the pentagonal flower in only one way so that P must be the Löbell polyhedron $L(n)$ where n is the number of edges of X. This is a contradiction.

Case D.2 Suppose that some pentagons lying on S are inward and some are outward. The proof breaks down into even more subcases depending on the number of consecutive outward pentagons incident to S:

Case D.2.1 Suppose there is a set of three consecutive outward pentagons A, B, and C with B adjacent to both A and C. Then there is a face G adjacent to these pentagons lying in D. If G is large, then any edge which is the intersection of two of these pentagons is good.

So suppose instead that G is a pentagon. Then G is adjacent to A, B, C, and two other faces H_1 and H_2 which are adjacent to A and C respectively. Each H_i shares a vertex with A or C lying on S and, therefore, must themselves lie on S and so must be pentagons. Note also that H_1 and H_2 are adjacent as they are each adjacent to G in edges which are adjacent.

Suppose H_1 is an inward pentagon. Label the edge of H_1 lying on S by e_1. Label the edge of H_1 adjacent to e_1 but which is not incident to A by f_1. Then there is a pentagon J which is adjacent to H_1 in f_1 (J cannot be adjacent to A as A is outward). Note that the edge g_1 of J which shares exactly one endpoint with both e_1 and f_1 must lie on S. Note also that J is adjacent to H_2 in some edge f_2 since f_1 contains a vertex of H_2. So H_2 is inward and if e_2 denotes the edge of H_2 lying on S, then e_2 and f_2 are adjacent. Note that the edge g_2 of J which shares exactly one endpoint with both e_2 and f_2 must lie on S. Thus g_1 and g_2 are nonadjacent edges in J which lie on S and so J is neither inward nor outward. This is a contradiction which shows that H_1 is outward. A similar argument applies for H_2.

So H_1 and H_2 must both be outward. Thus every pentagon lying on S is outward, which is a contradiction.

Case D.2.2 Suppose there is a set of exactly two consecutive outward pentagons, call them A_1 and A_2. Let B_1 and B_2 be the inward pentagons lying on S adjacent to A_1 and A_2 respectively. Let C_1 and C_2 be the pentagons lying on S adjacent to B_1 and B_2 respectively.

There is a face G which is adjacent to all four of A_1, A_2, B_1, B_2. If G is large, then the edge $A_1 \cap A_2$ is a good edge (as are $B_1 \cap A_1$ and $A_2 \cap B_2$).

So suppose G is a pentagon. Then there is an edge of G which edge connects B_1 to B_2. Let H be the face adjacent to G through this edge. Then note that H is adjacent to B_1 and B_2 and in particular, is edge connected to a large face in X by the edges $C_1 \cap B_1$ and $B_2 \cap C_2$. Therefore, if H is large, then these edges are good.

So suppose H is a pentagon. Then H is adjacent to G, B_1, B_2, C_1 and C_2. In particular, C_1 and C_2 are adjacent pentagons. An argument almost exactly like the one given in Case D.2.1, which showed that a pair of pentagons were outward, shows, in this case, that C_1 and C_2 are outward pentagons.

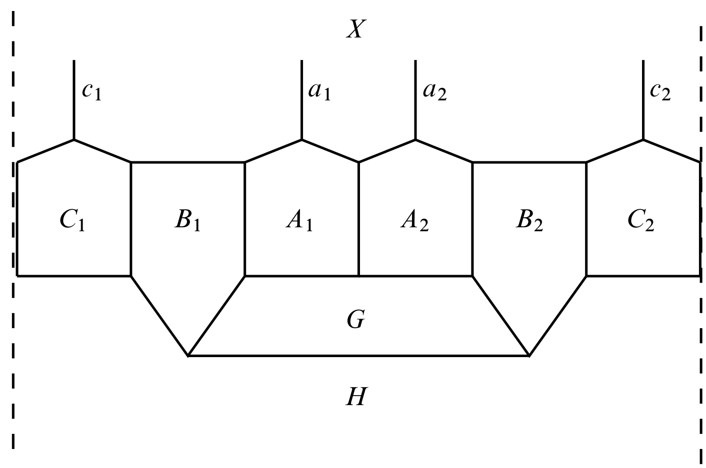

Figure 3: Case D.2.2 of Theorem 7.2. The dashed lines in this figure are identified.

Therefore, there are six pentagons lying on S, four of which are outward. Consider the four edges in X which emanate from vertices of these outward pentagons. The claim is that these edges form a prismatic 4–circuit. Label the edges emanating from A_i by a_i and the edges emanating from C_i by c_i. See Figure 3.

Note that a_1 and a_2 cannot share an endpoint as this would imply that the face in X adjacent to A_1 and A_2 is a quadrilateral. Similarly, c_1 and c_2 do not share an endpoint.

If a_i and c_i share an endpoint, then the face in X of which they are both edges is a pentagon which is a contradiction since X was assumed to consist only of large faces.

Suppose a_1 and c_2 share an endpoint v. By trivalence, there is another edge d emanating from v. Note that this edge d cannot be a_2 as a_1 and a_2 cannot share endpoints by the above argument. Call the face in X for which a_1 and a_2 are edges M and the face for which a_2 and c_2 are edges N. Then M and N are adjacent via the edge a_2, but also intersect in at least v which is disjoint from a_2. This is a contradiction. A similar argument shows that a_2 and c_1 cannot share an endpoint.

Therefore, a_1, a_2, c_1, and c_2 form a prismatic 4–circuit which is a contradiction.

Case D.2.3 Suppose that every outward pentagon lying on S is adjacent to a pair of inward pentagons lying on S, one on each side. Let A be such an outward pentagon, and let B_1 and B_2 be the inward pentagons adjacent to A. Let C_1 and C_2 be the pentagons lying on S adjacent to B_1 and B_2 respectively (but are not A).

There is an edge of A which edge connects B_1 and B_2. Call the face adjacent to A in this edge G. Note that the edges $A \cap B_1$ and $A \cap B_2$ edge connect G to a large face in X. Therefore, if G is large then these edges are good.

So suppose G is a pentagon. Then G is adjacent to A, B_1 and B_2. Let H_1 and H_2 denote the remaining two faces adjacent to G with H_1 adjacent to B_1 and H_2 adjacent to B_2. Note that for $i = 1, 2$ the edge $B_i \cap C_i$ edge connects H_i to some large face in X. Thus if either H_1 or H_2 is large, then P has a good edge.

So suppose both H_1 and H_2 are pentagons. Let D denote the face which is adjacent to each of C_1, C_2, H_1 and H_2. See Figure 4. The remainder of this case breaks down into further subcases depending on the nature of the pentagons C_1 and C_2.

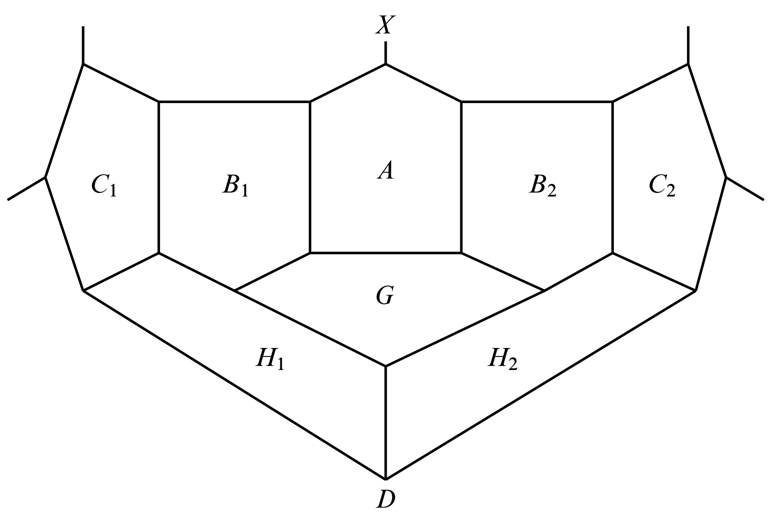

Figure 4: Case D.2.3 of Theorem 7.2

Case D.2.3.1 Suppose C_1 and C_2 are both outward pentagons. Then C_1 and C_2 are nonadjacent since, by assumption, each outward pentagon is adjacent to two inward pentagons. In particular, the face D is in fact an inward pentagon lying on S.

Therefore S has exactly six pentagons lying on it, exactly three of which are outward pointing. Consider the three edges of X emanating from vertices of these outward pointing pentagons. If any two of these edges share an endpoint, then these are edges of a pentagon lying in X which is a contradiction. Therefore, these three edges have distinct endpoints. It is evident then that they form a prismatic 3–circuit in P which is a contradiction.

Case D.2.3.2 Suppose C_1 and C_2 are both inward pentagons. Then the faces C_1 and C_2 cannot be adjacent. For, if they were, then the single face in X which is adjacent to C_1, C_2, B_1, B_2 and A would be adjacent to itself in the edge emanating from the vertex of A lying on S.

Therefore there is at least one more pentagon E adjacent to, say, C_1. Suppose E is outward and adjacent to C_2. Let e denote the edge emanating from E into X. Let a denote the edge emanating from A into X. If $a = e$, then X contains a pentagon which is a contradiction. If a and e share an endpoint, then there is a pair of faces in X which intersect in both a and e which is again a contradiction.

Thus either E is inward or there are more pentagons lying on S. Thus the face D which is adjacent to E, C_1, C_2, H_1 and H_2 must have at least 6 edges and so is large. Since the edge $C_1 \cap E$ edge connects F to a face in X, this edge is good.

Case D.2.3.3 Suppose exactly one of C_1 and C_2 is outward. Without loss of generality, suppose C_1 is outward and C_2 is inward. Then C_1 and C_2 cannot be adjacent since if they were, the faces C_1, C_2, H_1 and D would all share a vertex which contradicts trivalence. So there is another inward pentagon adjacent to C_1. This pentagon must be D by trivalence. The edge of D which lies on S edge connects C_1 and C_2 and is incident to the edge shared by D and C_2. This contradicts the assumption that C_2 is inward.

Case D.3 Suppose all pentagons lying on S are outward. Then D is a pentagonal flower L^n. Recall this means that D looks like an n–gon surrounded on all sides by a pentagon. If $n \geq 6$, then the edges which are intersections of adjacent pentagonal petals of L^n are good edges as they edge connect the n–gon in D to a face in X.

So suppose that $n = 5$ so that D is a dodecahedral flower. Let $G_1, \ldots, G_5$ denote the ring of large faces in X indexed cyclically, each of which are adjacent to a pair of pentagons in D. Let c_i denote the edge $G_i \cap G_{i+1}$. Then there is a simple closed curve c which lies entirely in the union of the G_i which intersects each c_i transversely. This curve c is a prismatic 5–circuit.

On the side of c which contains D, c bounds a pentagon in each G_i. Since each G_i is large, in each G_i, c bounds a polygon on the other side which has at least 5 edges. Therefore, by Theorem 4.5, P is decomposable along c. The component of the decomposition which contains D is a dodecahedron.

This concludes the proof of Theorem 7.2. □

Theorem 7.3 *If P a right-angled hyperbolic polyhedron not of Löbell type, then either it has a very good edge or it is decomposable.*

Proof Suppose P is not decomposable. By Theorem 7.2, P has a good edge. The only way for this edge to fail to be a very good edge is if it is a member of some prismatic 5–circuit in P. Some facts about prismatic 5–circuits will need to be collected.

Suppose $c = \{e_1, \dots, e_5\}$ is a prismatic 5–circuit with indices taken cyclicly. Recall that this means that there is a simple closed curve c in ∂P which intersects the 1–skeleton of P exactly in e_i and does so transversely, and furthermore all vertices of these edges are distinct. Then e_i and e_{i+1} are edges contained in some face which will be denoted F_i. This ring of faces $\{F_i\}$ of P is called the set of faces the prismatic 5–circuit c intersects. See Figure 5.

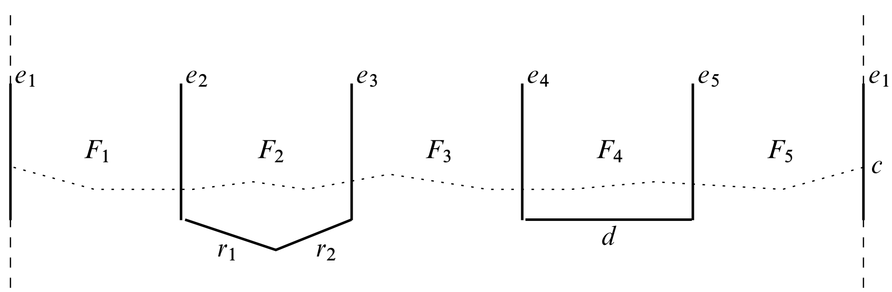

Figure 5: A prismatic 5–circuit and the faces it intersects. As usual, identify the dashed lines. The edge d is a flat of the prismatic circuit c, while edges r_1 and r_2 form a roof of c.

Lemma 7.4 *$F_i \neq F_j$ for $i \neq j$.*

Proof Suppose for contradiction that $F_i = F_j$ and without loss of generality suppose $i = 1$. By transversality when c intersects an edge of F_1, it must immediately leave F_1, therefore $j \neq 2, 5$. Suppose then $j = 3$ or $j = 4$. By turning the labelling around only the case when $j = 3$ needs to be checked. In this case, F_2 intersects $F_1 = F_3$ in edges e_1 and e_2 which are distinct. This is a contradiction. □

Call a prismatic 5–circuit *trivial* if every edge has exactly an endpoint belonging to a fixed pentagonal face and *nontrivial* if otherwise. Note that none of the edges belonging to a trivial prismatic 5–circuit can possibly be good.

Call an edge d of F_i a *flat* of the prismatic 5–circuit c if d and $c \cap F_i$ are opposite edges of some combinatorial quadrilateral in F_i. For example, the edge d in Figure 5 is a flat of the prismatic circuit c.

Lemma 7.5 *A nontrivial prismatic 5–circuit c in a right-angled hyperbolic polyhedron cannot have five adjacent flats on the same side of c.*

Proof By trivalence, there must be a single face adjacent to all these flats. Evidently, this face must be a pentagon and the prismatic 5–circuit must be trivial. □

Lemma 7.6 *A nontrivial prismatic 5–circuit c in a right-angled hyperbolic polyhedron cannot have four adjacent flats on the same side of c.*

Proof Without loss of generality, suppose for a contradiction that $F_1, \dots F_4$ have flat edges on the same side of c. There is a face G which is adjacent to each of $F_1, \dots F_4$. Note that F_5 cannot have a flat on the same side of c as the other flats by the nontriviality of c. Then G intersects F_5 in at least two edges, which contradicts the combinatorics of a polyhedron. □

Lemma 7.7 *A nontrivial prismatic 5–circuit c in a right-angled hyperbolic polyhedron cannot have three adjacent flats on the same side of c.*

Proof Without loss of generality, suppose for a contradiction that F_1, F_2, F_3 have flat edges on the same side of c. There is a face G which is adjacent to each of these. By trivalence, G also intersects each of F_4 and F_5. Consider the sequence of three edges d_i given by

$$d_1 = G \cap F_5, \quad d_2 = G \cap F_4, \quad d_3 = F_4 \cap F_5.$$

Note that neither d_1 nor d_2 can share an endpoint with d_3 since this would imply four adjacent flats.

Suppose d_1 and d_2 share an endpoint so that G is a pentagon. Let q be the edge which shares an endpoint with d_1 and d_2 but is not d_1 nor d_2. By the above, q cannot be d_3 either. However, q is an edge of both F_4 and F_5 which implies these faces intersect in at least two edges which is a contradiction.

Therefore d_1, d_2, and d_3 form a prismatic 3–circuit which is a contradiction. □

Lemma 7.8 *A nontrivial prismatic 5–circuit c in a right-angled hyperbolic polyhedron cannot have adjacent flats on the same side of c.*

Proof Without loss of generality, suppose F_1 and F_2 have flat edges of the same side of c. Then there is a face G which is adjacent to both F_1 and F_2. By trivalence, G is also adjacent to F_5 and F_3. Consider the sequence of four edges d_i:

$$d_1 = F_5 \cap G, \quad d_2 = G \cap F_3, \quad d_3 = F_3 \cap F_4, \quad d_4 = F_4 \cap F_5.$$

See Figure 6.

Note that d_3 and d_4 are contained in the prismatic 5–circuit c and therefore must have distinct endpoints. Note also that d_1 and d_2 have distinct endpoints since the face G must have at least 5 vertices.

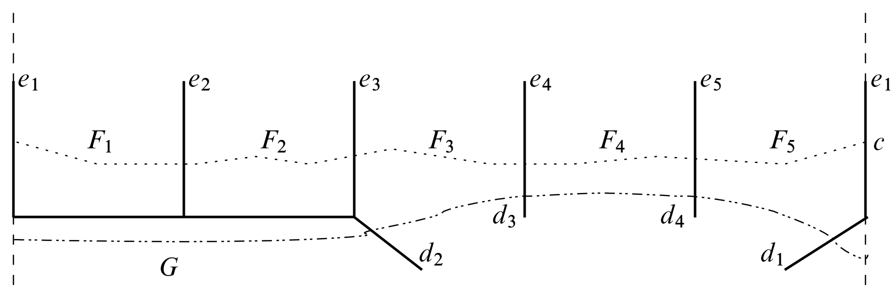

Figure 6: Lemma 7.8

Suppose d_1 and d_3 share an endpoint. Since d_3 is an edge of F_3 and F_4, by trivalence and the property of polyhedra that every edge belong to exactly two faces, d_1 must also be an edge of F_3 or F_4. But d_1 is an edge of the faces G and F_5. The latter, by the above lemma, cannot be F_3 or F_4. Evidently, G is not F_3 and cannot be F_4 since this would imply that $d_2 = d_3 = e_3$ and therefore e_2 and e_3 share an endpoint. This is a contradiction and so d_1 and d_3 are disjoint. A similar argument by a relabelling of edges shows d_2 and d_4 share no endpoints.

Suppose d_1 and d_4 share an endpoint. Then d_1 is a flat of the face F_5 and so there are three adjacent flats which, by the above lemma, is a contradiction. A similar argument shows d_2 and d_3 cannot share an endpoint.

Therefore, d_1, d_2, d_3, d_4 all have distinct endpoints and therefore form a prismatic 4–circuit in P, which is a contradiction. □

For a prismatic k–circuit c, define a *roof* of c to be a pair of edges r_1 and r_2 in a particular face F which c intersects such that the three segments r_1, r_2, and $F \cap c$

together bound a combinatorial pentagon in F. For example, the edges r_1 and r_2 in Figure 5 form a roof of the prismatic circuit c.

Lemma 7.9 *A prismatic 5–circuit c in a right-angled hyperbolic polyhedron cannot have a pair of flats on the same side of c separated by a single roof.*

Proof For the purposes of establishing a contradiction, and without loss of generality, suppose F_1 and F_3 contain flat edges on the same side of c, and F_2 contains a roof of c separating these flats. Then adjacent to each of F_1 and F_3 along their flats are faces G_1 and G_3 which are themselves both adjacent to F_2 by Lemma 3.4 as they evidently share a vertex with it. Note also that G_1 is adjacent to F_5 and G_3 is adjacent to F_4 for similar reasons.

Let $g_1 = G_1 \cap F_2$ and $g_3 = G_3 \cap F_2$. These edges g_1 and g_3 evidently form the roof of c contained in F_2, and therefore are adjacent. Therefore, G_1 and G_3 are themselves adjacent, again, by Lemma 3.4.

Consider then a curve which passes through the edges

$$d_1 = G_1 \cap F_5, \quad d_2 = G_1 \cap G_3, \quad d_3 = G_3 \cap F_4, \quad d_4 = F_4 \cap F_5.$$

If d_1 shares an endpoint with d_4, then d_1 is a flat of the face F_5. This implies c has a pair of adjacent flats on the same side which is a contradiction to Lemma 7.8. A similar argument shows d_3 and d_4 do not share an endpoint.

Furthermore, d_1 and d_2 cannot share endpoints as the face G_1 must have at least five edges. Similarly, d_2 and d_3 cannot share endpoints.

If d_1 and d_3 share an endpoint, then F_4 and F_5 are adjacent in some edge other than $e_5 = d_4$. This is a contradiction. Similarly, if d_2 and d_4 share an endpoint, then G_1 and F_5 intersect in a vertex not in d_1 as d_2 and d_1 were shown to be disjoint. This is also a contradiction.

Therefore, d_1, d_2, d_3, d_4 all have distinct endpoints and so form a prismatic 4–circuit which is a contradiction. This proves Lemma 7.9. □

Lemma 7.10 *If a flat e of a prismatic 5–circuit c is a good edge, then it cannot be a part of a prismatic 5–circuit and so is very good.*

Proof For the purposes of establishing a contradiction, suppose that e is an edge of a prismatic 5–circuit d. Label the edges of c by $\{c_i\}$, the edges of d by $\{d_i\}$, the faces that c intersects $\{F_i\}$ and the faces that d intersects $\{G_i\}$. Without loss of generality, suppose e is an edge of F_1, suppose $G_1 = F_1$ and suppose G_2 is the face adjacent to G_1 in e.

Note that since e is a flat of c, it edge connects F_5 and F_2. Since e is a good edge, F_5 and F_2 are large faces.

Sublemma 7.11 *Either G_3 or G_4 is either F_3 or F_4.*

Proof Consider the simple closed curves c and d. These curves intersect in the face $F_1 = G_1$ since the edge e is a flat of c, which means d must intersect a pair of edges which lie on opposite sides of c, one of which is e. Note that an isotopy can be performed so that all intersections between c and d occur in the interiors of faces of P and within any face, c and d intersect at most once. Since ∂P is topologically S^2, these curves must intersect at least twice. This implies that some G_i is equal to some F_j for $i, j \neq 1$.

G_5 cannot be F_2 or F_5 since $G_1 \cap F_2 = c_1$ and $G_1 \cap F_5 = c_5$ both share an endpoint with e. Also G_5 cannot be F_3 or F_4. For example, if G_5 is F_3, then $F_1 = G_1$ is adjacent to $F_3 = G_5$ in an edge which either shares a vertex with c_2 or does not. In the former case, F_1, F_2 and F_3 for pairwise adjacent faces and so must share a vertex which contradicts c being a prismatic 5–circuit. In the latter case, c_1, c_2 and $F_1 \cap F_3$ form a prismatic 3–circuit which is also a contradiction. A similar proof shows G_5 cannot be F_4. Therefore, G_5 is not F_i for any i.

As $e = G_2 \cap F_1$ is a flat of c, G_2 cannot be F_2 or F_5. G_2 also cannot be F_3 or F_4. For if, for example, G_2 was F_3, then F_1, F_2, and F_3 would be pairwise adjacent faces and so must share a vertex which contradicts c being a prismatic 5–circuit. Therefore, G_5 is not F_i for any i.

Note also G_3 cannot be F_5 or F_2 as this would imply that G_1, G_2 and G_3 are pairwise adjacent faces and thus must share a vertex which contradicts d being a prismatic 5–circuit. Similarly G_4 cannot be F_5 or F_2.

Therefore, G_3 or G_4 is F_3 or F_4. This concludes the proof of the sublemma. □

Suppose first that G_3 is F_3. Note that as G_2 and F_2 are distinct faces which share a vertex, namely $e \cap c_1$, by Lemma 3.4 G_2 and F_2 are adjacent. Furthermore, by definition, $G_3 = F_3$ is adjacent to F_2 and G_2, and so these three faces all share a vertex. Therefore the edge $G_2 \cap F_2$ has endpoints $e \cap c_1$ and an endpoint of c_2 and is thus a flat of c. But this implies c has a two adjacent flats on the same side which contradicts Lemma 7.8.

A similar argument shows G_3 cannot be F_4.

Suppose next that G_4 is F_3. Then G_5 is adjacent to both $G_1 = F_1$ and $G_4 = F_3$. Furthermore F_2 is adjacent to F_1 and F_3, and, as the proof of the above sublemma shows, G_5 is not F_2. Therefore, by Lemma 3.6, G_5 and F_2 are adjacent.

Now consider G_3. This face is adjacent to G_2 and $G_4 = F_3$. Furthermore F_2 is also adjacent to G_2 and G_4, and, as the proof of the above sublemma shows, G_3 is not F_2. Therefore, again by Lemma 3.6, G_3 and F_2 are adjacent.

Therefore, for each i, G_i, F_2, and G_{i+1} and pairwise adjacent and thus share a vertex. This evidently implies F_2 must be a pentagon. This is a contradiction as e was assumed to be a good edge. Therefore G_4 cannot be F_3.

A similar argument shows G_4 cannot be F_4.

Therefore, the good flat e must be a very good edge. This concludes the proof of Lemma 7.10. □

An obvious result which will be used often is the following:

Lemma 7.12 *Suppose F_i is a pentagon. Then c has a flat which is an edge of this pentagon. Furthermore, c has a roof formed by a pair of adjacent edges of this pentagon.*

Proof The curve c intersects two nonadjacent edges of F_i. The remaining three edges must be distributed so that exactly two are on one side of c and one on the other side. The former two edges are a roof and the latter edge is a flat. □

At last, returning to the proof of Theorem 7.3, suppose that P has a good edge e which is a member of a prismatic 5–circuit c. Then c cannot be a trivial prismatic circuit.

The proof of Theorem 7.3 now breaks up into six cases with possibly some subcases depending on the number of large faces the prismatic 5–circuit c intersects. All but one of these cases either produces a contradiction to the combinatorial facts about prismatic 5–circuits collected above, or produces a very good edge via Lemma 7.10. The remaining case implies that P is decomposable.

As usual, let F_i be the face that contains both the edges c_i and c_{i+1}.

Case A Suppose no F_i is large. Then, by Lemma 7.12 c has five flats, one for each pentagon F_i. There is no possible arrangement of these pentagons for which there are no adjacent pairs of flats on the same side of c. This is a contradiction to Lemma 7.8.

Case B Suppose exactly one of F_i is large. Then by Lemma 7.12, each of these pentagons have an edge which is a flat of c on one side, and a roof of c on the other. Because there cannot be adjacent flats on the same side of c, there is only way for the pentagons to be arranged up to isomorphism. That is with flats "alternating" on either side of c. Then on either side of c, there is a flat, then roof, then flat, then roof coming from the edges of the string of four pentagons. This contradicts Lemma 7.9.

Case C Suppose exactly two of the F_i are large. Then either the three pentagons are arranged in a row, or they are not. If they are not, then there is a pentagon whose neighbors are the large faces in $\{F_i\}$. These faces are edge connected by the flat of the pentagon described by Lemma 7.12 and therefore is a very good edge by Lemma 7.10.

Suppose then that the three pentagons are arranged in a row. Then since each pentagon has a flat on one side of c and a roof on the other by Lemma 7.12, either there are two adjacent flats on the same side of c which contradicts Lemma 7.8, or there is a flat, then roof, then flat on the same side of c which contradicts Lemma 7.9.

Case D Suppose exactly three of the F_i are large. Then either the two pentagons are adjacent or they are not. Each case will be treated separately:

Case D.1 If they are not adjacent, then the flat edge of a pentagon described by Lemma 7.12 edge connects two large faces and so, by Lemma 7.10, is very good.

Case D.2 Suppose the two pentagons are adjacent. Without loss of generality, suppose F_2 and F_3 are the pentagons. Then F_1 and F_4 are large faces. Make a choice of side of c by choosing the side which contains the flat of the pentagon F_2. Let n_c denote the number of edges of F_1 and F_4 contained entirely in this chosen side of c. Note that this number n_c must be at least 2 as c is prismatic.

Suppose $n_c = 2, 3$. Then one of F_1 or F_4 contains a flat of c on the chosen side. If F_1 is the culprit, then c has a pair of adjacent flats on the chosen side which contradicts Lemma 7.8. If F_4 is the culprit, then since F_3 must contain a roof of c on the chosen side, there is a contradiction to Lemma 7.9. Thus $n_c \geq 4$ and, in particular, the contribution from each of F_1 and F_4 to n_c must be at least two edges.

Label the face of P adjacent to F_2 through the edge which is a flat of c by G. Then G is adjacent to both F_1 and F_3. Note also that G is edge connected to the face F_4 by an edge of F_3. Call this edge g. Consider the curve d which passes through the edges:

$$d_1 = F_1 \cap F_5, \quad d_2 = F_1 \cap G, \quad d_3 = G \cap F_3, \quad d_4 = F_3 \cap F_4, \quad d_5 = F_4 \cap F_5.$$

This curve d forms a prismatic 5–circuit. Furthermore, the edge g is a flat of d. Thus, if G is a large face, then g is a good edge which is also a flat of a prismatic 5–circuit and so is very good by Lemma 7.10.

So suppose G is a pentagon. Then the prismatic 5–circuit d satisfies the conditions of Case D.2 and furthermore, by choosing the side of d which contains the flat g one has $n_d = n_c - 1$ with the contribution of F_1 to n_d being one less than that to n_c. See Figure 7.

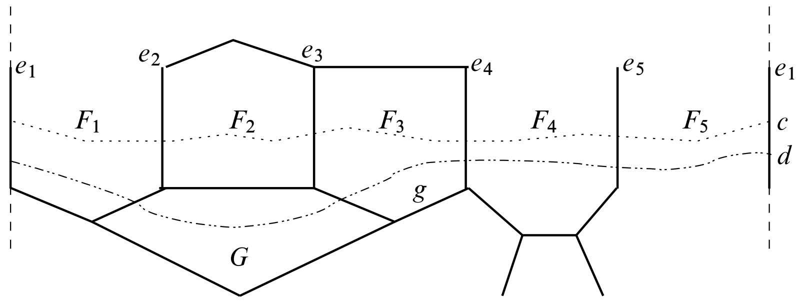

Figure 7: Case D.2 of Theorem 7.3. The prismatic 5–circuit c has $n_c = 5$. As G is a pentagon here, the prismatic circuit d contradicts Lemma 7.9.

Now an argument by induction on these numbers n can proceed. At each step, either a very good edge is exhibited, or a new prismatic 5–circuit is produced with smaller n. If n gets too small, that is $n = 3$, or the contribution from the faces F_1 or F_4 to n is too small, that is a single edge, then the above argument gives a contradiction.

Case E Suppose exactly four of the F_i are large. Without loss of generality, suppose F_1 is a pentagon. Then by Lemma 7.12, F_1 has an edge f which is a flat of c. As it edge connects F_5 with F_2 which are large faces, this edge f is good. By Lemma 7.10, this edge is very good.

Case F Suppose all F_i are large. Then either c has a flat or it doesn't. If c has a flat, then this flat evidently edge connects two large faces F_j and F_{j+2}, and thus by Lemma 7.10, is very good. If, on the other hand, c does not have a flat, then P is decomposable by Theorem 4.5, which is a contradiction.

This concludes the proof of Theorem 7.3. □

8 Geometric realization of edge surgery

Edge surgery as defined is a purely combinatorial operation. It does, however, have a geometric realization as a cone manifold deformation. Intuitively, an edge surgery looks like "unbending" along the edge until the supporting planes of the faces adjacent to the edge coincide. That is, the dihedral angle of the edge being surgered will increase from $\pi/2$ to π while all other edges retain their right-angles.

This section will be devoted to showing that this deformation is a path through the space of hyperbolic polyhedra which begins and ends at right-angled polyhedra. By use of the Schläfli differential formula, it will be shown that the volume of the initial right-angled polyhedron is greater than that of the final right-angled polyhedron.

To prove this deformation is through hyperbolic polyhedra, a generalization of Andreev's Theorem that accounts for obtuse angled hyperbolic polyhedra will be required. I Rivin in his thesis accomplished such a generalization. This result is communicated by a paper of Rivin and Hodgson [8].

Let $P \subset \mathbb{H}^3$ be a hyperbolic polyhedron, not necessarily nonobtuse angled. Define the *spherical polar* v^* of a vertex v of P by associating to each vertex v the set of outward unit normal vectors to planes which are incident to v but are disjoint from the interior of P (the *support planes* to P at v). Then v^* is a spherical polygon whose edge lengths are the exterior dihedral angle measures of edges incident to v in P. It follows that the interior angle measures of v^* are the complementary angle measures of the face angles of P at v.

Define the *spherical polar* P^* of P to be the piecewise spherical metric space constructed by gluing spherical polygons v^* and w^* associated to a pair of vertices v and w of P exactly when v and w are connected by an edge. Note that this metric space P^* is topologically $\mathbb{S}^2$ and is combinatorially dual to the cell structure of P (or, more precisely, the boundary of P). However, P^* will often not be isometric to $\mathbb{S}^2$.

To illustrate this point by way of example, suppose P is a right-angled polyhedron. Then the spherical polar of each vertex of P is a right-angled triangle in $\mathbb{S}^2$. Since every face of P has at least 5 edges, around every vertex in P^* are 5 or more of these right-angled triangles, and in particular, the sum of the angles around such a vertex is greater than 2π. Thus the vertices of P^* are singularities in the spherical metric. They represent accumulations of negative curvature.

Such singular points where the curvature is not 1 are called *cone singularities*. The *cone angle* of such a singularity is the sum of the angles around the singularity, or more intrinsically, the length of the link of the singularity viewed as a piecewise circular metric space. A cone angle of 2π corresponds to a nonsingular point.

I Rivin's generalization of Andreev's Theorem [8] characterizes those metric spaces homeomorphic to $\mathbb{S}^2$ which arise as polars of hyperbolic polyhedra:

Theorem 8.1 *A metric space Q homeomorphic to $\mathbb{S}^2$ is the spherical polar of a compact convex polyhedron P in $\mathbb{H}^3$ if and only if each of the following conditions hold:*

(1) *Q is piecewise spherical with constant curvature 1 away from a finite collection of cone points c_i.*

(2) *The cone angles at the points c_i are greater than 2π.*

(3) *The lengths of closed geodesics are all greater than 2π.*

The metric space Q determines the polyhedron P completely up to isometry.

The word "geodesic" in the statement of condition (3) will be taken to mean a locally distance minimizing path/loop. A geodesic in a piecewise spherical metric space is made up of arcs of great circles in cells such that at any point along the path of the geodesic, the angle subtended has measure greater than or equal to π on either side.

Suppose that P_0 is a right-angled hyperbolic polyhedron with a very good edge e. Think of P_0 combinatorially as the underlying combinatorial polyhedron, together with a labelling of each edge by the dihedral angle measure in its geometric realization. In this case, each edge of P_0 is labelled by $\pi/2$.

For each $t \in [0,1)$, let P_t denote the combinatorial polyhedron isomorphic to the combinatorial polyhedron underlying P_0, every edge other than e labelled by $\pi/2$, and the edge e labelled by $\theta_t = (1-t)(\pi/2) + t\pi$. Let P_1 denote the combinatorial polyhedron obtained by edge surgery of P_0 along the edge e with each edge labelled by $\pi/2$.

Theorem 8.2 *Each P_t has a geometric realization as a hyperbolic polyhedron.*

Proof By Theorem 7.1 and assumption, P_0 and P_1 have geometric realizations as hyperbolic polyhedra (in fact, they are right-angled). So assume t lies in the open interval $(0,1)$, so that P_t is combinatorially isomorphic to P_0.

Let Q_t denote the piecewise spherical metric space constructed in the following way. For each vertex v of P_t, construct a spherical triangle whose edge lengths are the complementary angle measures of the dihedral angle measures of the edges of P_t incident to v. Identify the edges of two such triangles if and only if their associated vertices are connected by an edge of P_t. Then Q_t is evidently a metric space which is homeomorphic to $\mathbb{S}^2$. It will be shown that Q_t satisfies the conditions of Theorem 8.1 implying that Q_t is the spherical polar of a hyperbolic polyhedron. The theorem will be proved when it is shown that the hyperbolic polyhedron in question is in fact the geometric realization of P_t by controlling for the combinatorics of the spherical cell division of Q_t.

To show condition (1) of Theorem 8.1 holds for Q_t, note that the singular points of Q_t correspond to the faces of P_t since every other point lies in the interior of a face or the interior of an edge where two triangles meet. These points have curvature 1. Since P_t has only finitely many faces, there are only finitely many cone points, thus demonstrating the first condition.

The cone angle of a singular point c_i is the sum of the angle measures at c_i of the spherical triangles incident to c_i. Suppose F_i is the face of P_t corresponding to c_i. If F_i is disjoint from the edge $e \subset P_t$, then each triangle incident to c_i is isometric to a right-angled spherical triangle. As F_i has at least $k \geq 5$ edges, the cone angle at c_i is $k(\pi/2) > 2\pi$.

Suppose that F_i contains the edge e and let $k \geq 5$ be the number of edges of F_i. Then c_i is incident to $k-2$ right-angled triangles, and two triangles whose lengths are $\pi/2$, $\pi/2$, and $\pi - \theta_t$. A bit of elementary spherical geometry reveals that such a triangle has interior angle measures equal to the length of the edge opposite the angle. Therefore, these two triangles are incident to c_i in right angles, and so the cone angle around c_i is $k(\pi/2) > 2\pi$.

Suppose that F_i contains a single vertex of e. Then e edge connects F_i to some other face of P_t, and thus, by assumption, is very good. This implies that the face F_i is large and so has $k \geq 6$ edges. Thus c_i, has $k \geq 6$ triangles incident to it. All of the triangles incident to c_i except one are right-angled. The exceptional triangle is isometric to the sort of triangle described above whose lengths are $\pi/2$, $\pi/2$, and $\pi - \theta_t$. This triangle is incident to c_i in the non–right angle, and therefore the cone angle of this point is given by $(k-1)(\pi/2) + \pi - \theta_t > 2\pi$. This shows condition (2).

Denote the edge of Q_t which is dual to $e \subset P_t$ by e^*. This edge e^* is contained in a pair of triangles T_1 and T_2 isometric to the ones described above with edge lengths $\pi/2$, $\pi/2$, and $\pi - \theta_t$. Let B_t denote the spherical bigon which is the union of these two triangles. Note that the two edges of B_t both have length π which meet at two points at an angle whose measure is $\pi - \theta_t$. Denote the endpoints of e^* by w_1 and w_2 and denote the vertices of B_t which are not endpoints of e^* by v_1 and v_2. See Figure 8.

Note that if a point in B_t is distance d from v_1, then it is distance $\pi - d$ from v_2. Therefore, B_t is foliated by arcs of circles of radius d from the point v_1 where $d \in [0, \pi]$, and the leaf space of this foliation is isometric to the interval $[0, \pi]$.

Note also that as t approaches 1, the bigon $B_t \subset Q_t$ degenerates into a line segment B_1 isometric to $[0, \pi]$.

Fix $t \in (0, 1)$ and let $\overline{f}\colon B_t \to [0, \pi] \cong B_1$ denote the projection onto the leaf space. Let $f\colon Q_t \to Q_1$ be the continuous map which when restricted to any cell of Q_t which does not belong to B_t realizes the natural correspondence of cells by an isometry, and which when restricted to B_t, is the map $\overline{f}$.

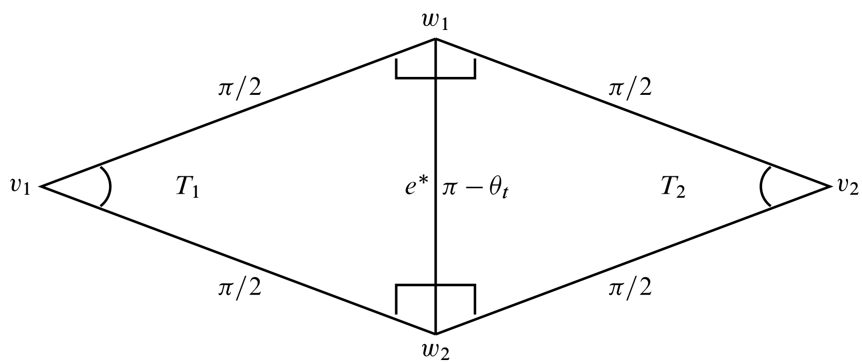

Figure 8: The bigon B_t in the spherical dual of P_t

Lemma 8.3 *This map f is distance nonincreasing and is a local isometry when restricted to $Q_t \setminus B_t$. In particular, it is an isometry when restricted to the star of any vertex of Q_t not contained in the bigon B_t.*

Proof Let x and y be a pair of points of Q_t and let γ be a geodesic segment connecting them whose length is the distance between them. If γ does not intersect the bigon B_t, then $f(\gamma)$ is a geodesic segment in Q_1 and so $d_t(x,y) = d_1(f(x), f(y))$ where d_t and d_1 denote the metrics in Q_t and Q_1 respectively.

So suppose the geodesic segment γ intersects the bigon B_t, and let $\tau = \gamma \cap B_t$. Denote the endpoints of τ in B_t by a and b. Then $f(\gamma)$ is a broken geodesic with a segment given by $f(\tau)$ lying on the interval $B_1 = [0,\pi]$ with endpoints given by $f(a), f(b) \in [0,\pi]$.

Without loss of generality, assume that $f(a) \geq f(b)$. Then the length of $f(\tau)$ is $f(a) - f(b)$. Since f is a local isometry outside of B_t, to prove the lemma it will be shown that the length of τ in Q_t, denoted $l(\tau)$, is greater than or equal to $f(a) - f(b)$.

If $f(a) - f(b) > l(\tau)$, then in B_t, the path τ concatenated with the geodesic segment connecting b to v_1, running along the boundary, of B_t has endpoints a and v_1 and has length $l(\tau) + f(b)$ which is strictly less than $f(a)$. However, the distance between a and v_1 is given by $f(a)$. This is a contradiction. □

If v is a vertex of Q_t, let $\mathrm{st}(v)$ denote the *star* of v. Let $\mathrm{ost}(v)$ denote the *open star* of v, which is the interior of $\mathrm{st}(v)$. The following well-known and important lemma gives an estimate on the length of a geodesic arc contained in $\mathrm{st}(v)$. For a proof, see Charney and Davis [5].

Lemma 8.4 *Suppose Q is a piecewise spherical metric space which is the spherical polar of a nonobtuse hyperbolic polyhedron. Let v be a vertex of Q and γ a geodesic segment in $\mathrm{st}(v)$ joining two points of $\partial\,\mathrm{st}(v)$ such that $\gamma \cap \mathrm{ost}(v) \neq \varnothing$. Then the length of γ is at least π.*

Let γ now denote a closed geodesic in Q_t. If γ does not intersect $\mathrm{int}(B_t)$, then $f(\gamma)$ is a closed geodesic in Q_1 and so therefore has length greater than 2π by Rivin's Theorem 8.1 as Q_1 is the spherical polar of the right-angled hyperbolic polyhedron P_1. Therefore, the length of γ is also greater than 2π by Lemma 8.3.

So suppose γ intersects $\mathrm{int}(B_t)$. It is clear that γ cannot be completely contained in B_t. Let $\gamma_i \subset \gamma$ denote the closure of a component of $\gamma \cap \mathrm{int}(B_t)$.

Fixing notation, let T_1 be the triangle in B_t with vertices v_1, w_1, and w_2, and T_2 the triangle with vertices v_2, w_1 and w_2.

Suppose first that γ_i contains v_1 or v_2. Then a bit of elementary spherical geometry implies that γ_i contains the other vertex v_2 or v_1 and that the length of γ_i is π. Let X and Y denote the right-angled triangles γ enters after leaving B_t through v_1 and v_2 respectively. Then the lengths of both $\gamma \cap X$ and $\gamma \cap Y$ are $\pi/2$. Note that if X and Y are adjacent in Q_t, then the polyhedron P_t contains a prismatic 4–circuit which is a contradiction. Therefore, X and Y are not adjacent, and so the length of γ is larger than 2π.

Suppose once and for all that γ_i misses v_1 and v_2. If γ_i misses e^*, then it is completely contained in one of the triangles T_1 or T_2. Without loss of generality, suppose $\gamma_i \subset T_1$. Let X and Y denote the right angled triangles adjacent to T_1. let x and y denote the vertices of X and Y respectively which are not contained in T_1. Then γ evidently intersects $\mathrm{ost}(x)$ and $\mathrm{ost}(y)$. Denote the closure of the component of $\mathrm{ost}(x) \cap \gamma$ which is adjacent to γ_i by γ_x and define γ_y similarly. Then $f(\gamma_x) \subset Q_1$ is a geodesic segment contained in $\mathrm{st}(f(x))$ by Lemma 8.3 which intersects $\mathrm{ost}(f(x))$ and therefore, by Lemma 8.4 has length at least π. Therefore γ_x also has length at least π. A similar argument shows γ_y has length at least π as well. Note that $\mathrm{ost}(x) \cap \mathrm{ost}(y) = \varnothing$ since if not, then P_t would contain a prismatic 3 or 4–circuit. Since γ_i has nonzero length, γ must have length larger than 2π.

Suppose that γ_i intersects $\mathrm{int}(e^*)$ transversely. Let X and Y denote the right-angled triangles adjacent to each of T_1 and T_2 respectively which contain the endpoints of γ_i. A bit of simple spherical geometry shows that X and Y must lie on opposite sides of B_t in the sense that if X contains w_1 say, then Y contains w_2. Let x and y denote the vertices of X and Y respectively which are not contained in T_1 and T_2

respectively. Then γ evidently intersects $\mathrm{ost}(x)$ and $\mathrm{ost}(y)$. By an argument similar to the one above, this implies that the length of γ is larger than 2π.

Suppose γ_i contains the edge e^*. Then $f(\gamma)$ is a closed geodesic in Q_1 and therefore must have length larger than 2π. Thus by Lemma 8.3, γ must also have length larger than 2π.

Finally, suppose that γ_i intersects e^* exactly in one of its endpoints w_1 or w_2, and without loss of generality, suppose $w_1 \in \gamma_i$. Then γ_i is contained in either T_1 or T_2, so suppose also without loss of generality that $\gamma_i \subset T_1$. Let X denote the right-angled triangle adjacent to T_1 which contains the vertices w_2 and v_1 and let x denote the remaining vertex of X. Then γ evidently intersects $\mathrm{ost}(x)$. Let Y denote the right-angled triangle other than T_1 that contains the vertex w_1 and intersects γ. Then Y is contained in the star of some vertex which is not v_i or w_i. Denote this vertex by y. So γ intersects both $\mathrm{ost}(x)$ and $\mathrm{ost}(y)$ and by an argument similar to the one above, γ has length larger than 2π.

This shows that Q_t satisfies condition (3) of Theorem 8.1 and thus is the spherical polar of some hyperbolic polyhedron. However, it is too hasty to conclude that this polyhedron is the geometric realization of P_t as it is possible that there is more than one cell decomposition of Q_t which satisfy the conditions of being a spherical polar of a hyperbolic polyhedron.

To prove that this cannot be the case, denote by $\mathcal{C}$ the cell decomposition used to construct Q_t , and denote by $\mathcal{D}$ the cell decomposition of Q_t as the spherical polar of the hyperbolic polyhedron coming from Theorem 8.1. It will be shown that $\mathcal{C} = \mathcal{D}$.

$\mathcal{D}$ must satisfy the following three conditions:

(1) The vertex set is exactly the set of cone points.

(2) The edges must be geodesic arcs of length less than π.

(3) The interior face angles at each cone point must have measure less than π.

The first condition is clear from the construction of spherical polars. The second condition is implied by the convexity of the hyperbolic polyhedron as edge lengths are equal to complements of interior dihedral angle measures. The third condition is again a consequence of convexity, but convexity of the faces of a hyperbolic polyhedron as the face angles of a cone point have measures which are complementary to the face angles of the polyhedron. Note that $\mathcal{C}$ satisfies these three conditions.

Note that if c is a cone point of Q_t and if c is neither w_1 nor w_2, then the only other cone points at distance less than π to c are exactly distance $\pi/2$ away. By conditions (1) and (2), these are the only possible endpoints of edges connecting c.

Suppose c is not v_i or w_i for $i = 1, 2$. If b is a cone point at distance $\pi/2$ from c and if b and c are not connected by an edge in $\mathcal{D}$, then there must exist a 2–cell in $\mathcal{D}$ whose face angle at c has measure π or larger. This contradicts condition (3). Therefore, in $\mathcal{D}$, c must be connected to every other cone point distance $\pi/2$ away, just as in $\mathcal{C}$.

Consider next v_1. The above argument shows that any cone point of distance $\pi/2$ from v_1 which is not w_1 or w_2 must be connected to v_1 by an edge in $\mathcal{D}$. If v_1 and w_1 are not connected by an edge in $\mathcal{D}$, then there must be a 2–cell in $\mathcal{D}$ whose face angle at w_1 has measure π, a contradiction to condition (3). This argument works just as well to show that both v_1 and v_2 are connected to both w_1 and w_2 by edges in $\mathcal{D}$.

So far, it has been shown that every edge in $\mathcal{C}$ with an endpoint other than w_1 or w_2 must also be in $\mathcal{D}$. The only edge in $\mathcal{C}$ not accounted for is the one connecting w_1 and w_2. This must be in $\mathcal{D}$ as well since if it were not, there would be a 2–cell in $\mathcal{D}$ containing a face angle of measure π at both w_1 and w_2 — a contradiction. Therefore, this edge is in $\mathcal{D}$ and so the edges of $\mathcal{C}$ are a subset of the edges of $\mathcal{D}$.

Given conditions (1) and (2), the only other possible edges in $\mathcal{D}$ are those edges not in $\mathcal{C}$ which connect w_1 (or w_2) to a cone point connected to w_2 (or w_1). However, it is rather clear that such an edge would intersect an existing edge in $\mathcal{D}$ in an interior point which is a contradiction to condition (1).

Therefore $\mathcal{C} = \mathcal{D}$ and this proves Theorem 8.2. □

The set of polyhedra P_t, $t \in [0, 1)$ form a 1–parameter family of hyperbolic polyhedra of fixed combinatorial type. Therefore, the following classical result of Schläfli is applicable (see Alekseevskij, Vinberg and Solodovnikov [2] for a proof). It is often referred to as *Schläfli's Differential Formula*.

Theorem 8.5 *If P_t is a 1–parameter family of polyhedra in $\mathbb{H}^n$, $n \geq 2$, then the derivative of the volume* vol *of P_t is given by:*

$$\frac{d\,\mathrm{vol}(P_t)}{dt} = \frac{-1}{n-1} \sum_F \mathrm{vol}(F) \frac{d\theta_F}{dt}$$

where the sum is taken over all codimension 2 faces of P_t and θ_F is the measure of the dihedral angle of P_t at the face F.

This formula implies that as the dihedral angle measure of a hyperbolic polyhedron in any dimension increases, the volume decreases. For the family P_t constructed above, the dihedral angle along the very good edge e is increasing from $\pi/2$ to π. Therefore:

Theorem 8.6 *If P_1 is a right-angled hyperbolic polyhedron gotten by edge surgery along some very good edge of P_0, another right-angled hyperbolic polyhedron, then*

$$\mathrm{vol}(P_0) > \mathrm{vol}(P_1). \qquad \square$$

9 Conclusion

Let P be a right-angled hyperbolic polyhedron. Then Theorem 7.3 implies that either P is a Löbell polyhedron, it is decomposable, or it has a very good edge along which edge surgery can be performed. A similar trichotomy holds for the polyhedron or polyhedra which result after applying decomposition or edge surgery. Note that each of these operations reduce the average number of faces of the polyhedra and so a process of repeated application of them must terminate in a set of Löbell polyhedra after a finite number of steps. Furthermore, by Theorem 6.4 and Theorem 8.6 the total volume of polyhedra does not increase at each step. Therefore, the following result, the main result of this article, follows:

Theorem 9.1 *Let P_0 be a compact right-angled hyperbolic polyhedron. Then there exists a sequence of disjoint unions of right-angled hyperbolic polyhedra $P_1, P_2, \dots, P_k$ such that for $i = 1, \dots, k$, P_i is gotten from P_{i-1} by either a decomposition or edge surgery, and P_k is a set of Löbell polyhedra. Furthermore,*

$$\mathrm{vol}(P_0) \geq \mathrm{vol}(P_1) \geq \mathrm{vol}(P_2) \geq \cdots \geq \mathrm{vol}(P_k). \qquad \square$$

With a blackbox or oracle which is able to compute volumes of right-angled polyhedra, this result would enable one to completely order the volumes of such objects. In particular, the following result is a simple corollary:

Corollary 9.2 *The compact right-angled hyperbolic polyhedron of smallest volume is $L(5)$ (a dodecahedron) and the second smallest is $L(6)$.*

Proof By Theorem 4.2, the smallest Löbell polyhedron is $L(5)$ while the second smallest is $L(6)$. Note that there is no polyhedron which results in $L(5)$ when an edge surgery is performed as $L(5)$ has only pentagonal faces. Furthermore, it is obvious that any composition of Löbell polyhedra will have volume larger than that of $L(5)$, and so the first result follows.

The only possible polyhedron besides $L(5)$ whose volume is possibly not larger than that of $L(6)$ is a composition of two copies of $L(5)$. However, the volume of such an object is $2\,\mathrm{vol}(L(5)) = 8.612... > \mathrm{vol}(L(6)) = 6.023....$ This proves the second result. $\square$

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