# The curvature of contact structures on 3–manifolds

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We study the sectional curvature of plane distributions on 3–manifolds. We show that if a distribution is a contact structure it is easy to manipulate its curvature. As a corollary we obtain that for every transversally oriented contact structure on a closed 3–dimensional manifold, there is a metric such that the sectional curvature of the contact distribution is equal to $-1$. We also introduce the notion of Gaussian curvature of the plane distribution. For this notion of curvature we get similar results.

53D35; 53B21

## 1 Introduction

The problem of prescribing the curvatures of a manifold is one of the central problems in Riemannian geometry. That is, given a smooth function can it be realized as a scalar (Ricci or sectional) curvature of some Riemannian metric on a manifold. The solution of the Yamabe problem is the best known result in prescribing the scalar curvature on a manifold (cf Lee and Parker [4]). There are several results on prescribing the Ricci curvature of a manifold (cf for example Lohkamp [5]). It is natural to ask to what extent it is possible to prescribe the sectional curvature of the plane distribution on a 3-manifold. It turns out that this problem is closely connected with the contactness of the distribution. In fact we have the following:

**Theorem A**  *Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the sectional curvature of $\xi$.*

If we impose more topological restrictions on the distribution we can obtain an even stronger result:

**Theorem B**  *Let $\xi$ be a transversally orientable contact structure on $M$ with Euler class zero. Then for any smooth function $f$, there is a metric on $M$ such that $f$ is a sectional curvature of $\xi$.*

In [2], Chern and Hamilton studied a similar problem of prescribing the so-called Webster curvature $W$ on a contact three-manifold. The main difference in their approach is that they restrict the class of metrics to the metrics which are adapted to a contact structure, while we deal with the class of all metrics. They prove that in their class one can either find a metric with the constant negative Webster curvature or a metric with strictly positive Webster curvature.

It is a well-known problem whether a foliation on a 3–dimensional manifold admits a simultaneous uniformization of all its leaves. The Reeb stability theorem asserts that on a compact orientable 3–manifold the only foliation with the leaves having positive Gaussian curvature is the foliation of $M = S^2 \times S^1$ by spheres. It is known (see Candel [1]) that if $M$ is atoroidal and aspherical and the foliation is taut, then there is a metric on $M$ such that all leaves have constant negative Gaussian curvature $-1$. In the case of contact structures we ask a similar question. For this we have to introduce the notion of Gaussian curvature of the plane distribution.

We define the Gaussian curvature of the plane distribution as the sum $K_G(\xi) = K(\xi) + K_e(\xi)$ of the sectional and the extrinsic curvatures of the distribution. In the case of integrable $\xi$ this equation is nothing but the Gauss equation.

**Definition 1.1**  Let $\xi$ be a plane distribution on $M$. We say that $\xi$ admits a uniformization if there is a metric on $M$ such that the Gaussian curvature of $\xi$ is constant.

It turns out that unlike the case of foliations, every transversally orientable contact structure on a closed 3–manifold admits a uniformization. We have the following:

**Theorem C**  *Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the Gaussian curvature of $\xi$.*

This paper is organized as follows. In Section 2 we recall basic facts about the geometry of plane distributions. In Section 3 we prove the main technical lemma. Section 4 is devoted to the proof of Theorem A and Theorem B. We prove Theorem C in Section 5.

**Acknowledgment**  I would like to thank Patrick Massot for pointing out Corollary 3.6. This led to a much stronger and natural formulation of Theorem B.

## 2  Basic definitions and notation

Throughout this paper $M$ will be a closed orientable 3–manifold. A distribution on $M$ is a two dimensional subbundle of the tangent bundle of $M$. That is, at each point $p$

in $M$ there is a plane $\xi_p$ in the tangent space $T_p M$. A distribution is called integrable, if there is a foliation on $M$ which is tangent to it. The following Frobenius theorem gives necessary and sufficient conditions for $\xi$ to be integrable.

**Theorem 2.1** Let $\xi$ be a distribution on $M$. Then $\xi$ is integrable if and only if for any two sections $S$ and $T$ of $\xi$ its Lie bracket belongs to $\xi$.

**Definition 2.2** A distribution $\xi$ is called a contact structure if for any linearly independent sections $S$ and $T$ of $\xi$ and for any $p \in M$ the Lie bracket $[S, T]$ at $p$ does not belong to $\xi_p$.

A distribution $\xi$ is called transversally oriented if there is a globally defined 1–form $\alpha$ such that $\xi = \mathrm{Ker}(\alpha)$. This is equivalent to say that there exists a globally defined vector field $n$ which is transverse to $\xi$. It is an easy consequence of Frobenius Theorem that $\xi$ is a contact structure if and only if

$$\alpha \wedge d\alpha \neq 0.$$

Fix some orientation on $M$. A contact structure is said to be positive (resp. negative) if the orientation induced by $\alpha \wedge d\alpha$ coincides (resp. is opposite to) the orientation on $M$.

A contact structure $\xi$ is called overtwisted, if there is an embedded disk such that $TD|_{\partial D} = \xi|_{\partial D}$. If $\xi$ is not overtwisted, it is called tight.

The Euler class $e(\xi) \in H^2(M, \mathbb{Z})$ of a plane distribution is the Euler class of the bundle $\xi \to M$. It is known that if $\xi$ is a 2–dimensional plane distribution on $M$ with vanishing Euler class then $\xi$ is trivial. Recall, that a framing of $M$ is the presentation of the tangent bundle of $M$ as a product $TM \simeq M \times \mathbb{R}^3$. A framing on $M$ consists of three linearly independent vector fields. It is known that every closed orientable 3–manifold admits a framing.

A bi-contact structure on $M$ is a pair $(\xi, \eta)$ of transverse contact structures which define opposite orientation on $M$.

Assume that $M$ is a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection $\nabla$. Let $n$ be a local unit vector field orthogonal to $\xi$. We are now going to define the second fundamental form of $\xi$. The definition is due to Reinhart [7].

**Definition 2.3** The second fundamental form of $\xi$ is a symmetric bilinear form, which is defined in the following way:

$$B(S, T) = \frac{1}{2}\langle \nabla_S T + \nabla_T S, n \rangle$$

for all sections $S$ and $T$ of $\xi$.

**Remark 2.4**  If $\xi$ is integrable, then $B$ restricted to the leaf of $\xi$ agrees with the second fundamental form of the leaf.

Let $S$ and $T$ be two linearly independent sections of $\xi$.

**Definition 2.5**  We call the function

$$K_e(\xi) = \frac{B(S,S)B(T,T) - B(S,T)^2}{\langle S,S\rangle\langle T,T\rangle - \langle S,T\rangle^2}$$

an extrinsic curvature of $\xi$.

It is easy to verify that $K_e(\xi)$ depends only on $\xi$, not on the actual choice of $S$, $T$ and $n$.

**Definition 2.6**  Consider the function $K(\xi)$ which assigns to a point $p \in M$ the sectional curvature of the plane $\xi_p$. We call this function the sectional curvature of $\xi$.

**Definition 2.7**  We call the sum $K_G(\xi) = K(\xi) + K_e(\xi)$ the Gaussian curvature of $\xi$.

Let $S$, $T$ and $U$ be the local sections of $TM$. Recall the Koszul formula for the Levi-Civita connection of $\langle \cdot, \cdot \rangle$:

$$2\langle \nabla_S T, U\rangle = S\langle T,U\rangle + T\langle U,S\rangle - U\langle S,T\rangle$$
$$+ \langle [S,T], U\rangle - \langle [S,U], T\rangle - \langle [T,U], S\rangle$$

## 3  The deformation of metric

In this section we will give the proof of the main technical results we will need throughout the paper.

Let $\xi$ be a transversally orientable plane distribution on a 3–dimensional Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Fix a unit normal vector field $n$. Suppose $a$ is a strictly positive smooth function on $M$. A stretching of $\langle \cdot, \cdot \rangle$ along $n$ by the function $a$ is the following Riemannian metric on $M$:

$$\langle \cdot, \cdot \rangle_a = a\langle \cdot, \cdot \rangle|_n \oplus \langle \cdot, \cdot \rangle|_\xi$$

Our aim is to calculate the sectional curvature of $\xi$ in the stretched metric in terms of the initial metric.

Consider an open subset $U \subset M$ such that $\xi|_U$ is a trivial fibration. Let $X$ and $Y$ be a pair of orthonormal sections of $\xi|_U$. The triple $(X, Y, n)$ is an orthonormal framing on $U$ with respect to $\langle \cdot, \cdot \rangle$.

In the stretched metric this frame is orthogonal, vector fields $X$ and $Y$ are unit and the length of $n$ is equal to $\sqrt{a}$. Denote by $\nabla$ the Levi-Civita connection of $\langle \cdot, \cdot \rangle_a$.

**Lemma 3.1** *The sectional curvature of* $\xi$ *with respect to* $\langle \cdot, \cdot \rangle_a$ *can be calculated by the following formula:*

$$K(\xi) = -\frac{3}{4}a\langle [X, Y], n \rangle^2 + P + \frac{1}{a}Q$$

where
$$P = X\langle [X, Y], Y \rangle - Y\langle [X, Y], X \rangle - \langle [X, Y], X \rangle^2 - \langle [X, Y], Y \rangle^2$$
$$+ \frac{1}{2}\langle [X, Y], n \rangle (-\langle [n, Y], X \rangle + \langle [n, X], Y \rangle))$$

and
$$Q = \frac{1}{4}(\langle [X, n], Y \rangle + \langle [Y, n], X \rangle)^2 - \langle [Y, n], Y \rangle \langle [X, n], X \rangle$$

**Proof** Since $X$ and $Y$ are unit, the sectional curvature of $\xi$ is calculated by the formula:

$$K(\xi) = \langle R(X, Y)Y, X \rangle_a = \langle \nabla_X \nabla_Y Y, X \rangle_a - \langle \nabla_Y \nabla_X Y, X \rangle_a - \langle \nabla_{[X,Y]} Y, X \rangle_a$$

The first summand can be rewritten:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = X\langle \nabla_Y Y, X \rangle_a - \langle \nabla_Y Y, \nabla_X X \rangle_a$$

Apply the Koszul formula to $X\langle \nabla_Y Y, X \rangle_a$. We get:

$$X\langle \nabla_Y Y, X \rangle_a = \frac{1}{2}X(2Y\langle Y, X \rangle_a - X\langle Y, Y \rangle_a + \langle [Y, Y], X \rangle_a - 2\langle [Y, X], Y \rangle_a)$$
$$= -X\langle [Y, X], Y \rangle_a = -X\langle [Y, X], Y \rangle$$

Decompose the vector field $\nabla_Y Y$ with respect to the frame $(X, Y, n/\sqrt{a})$ orthonormal in the stretched metric $\langle \cdot, \cdot \rangle_a$:

$$\nabla_Y Y = \langle \nabla_Y Y, \frac{n}{\sqrt{a}} \rangle_a \frac{n}{\sqrt{a}} + \langle \nabla_Y Y, Y \rangle_a Y + \langle \nabla_Y Y, X \rangle_a X$$

Substituting these expressions into $\langle \nabla_X \nabla_Y Y, X \rangle_a$, we obtain:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = -X\langle [Y, X], Y \rangle - \langle \langle \nabla_Y Y, n \rangle_a \frac{n}{a} + \langle \nabla_Y Y, Y \rangle_a Y$$
$$+ \langle \nabla_Y Y, X \rangle_a X, \nabla_X X \rangle_a$$

Since $X$ and $Y$ are of unit length this reduces to:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle \nabla_Y Y, n \rangle_a \langle \nabla_X X, n \rangle_a$$

Apply the Koszul formula to the term $\langle \nabla_Y Y, n \rangle_a \langle \nabla_X X, n \rangle_a$. Finally, we have:

$$\langle \nabla_X \nabla_Y Y, X \rangle_a = -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle [Y, n], Y \rangle_a \langle [X, n], X \rangle_a$$

$$= -X \langle [Y, X], Y \rangle - \frac{1}{a} \langle [Y, n], Y \rangle \langle [X, n], X \rangle$$

The second summand is equal to:

$$-\langle \nabla_Y \nabla_X Y, X \rangle_a = -Y \langle \nabla_X Y, X \rangle_a + \langle \nabla_X Y, \nabla_Y X \rangle_a$$

$$= Y \langle Y, \nabla_X X \rangle_a + \left\langle \langle \nabla_X Y, n \rangle_a \frac{n}{a} + \langle \nabla_X Y, Y \rangle_a Y \right.$$

$$\left. + \langle \nabla_X Y, X \rangle_a X, \nabla_Y X \right\rangle_a$$

$$= -Y \langle [X, Y], X \rangle_a + \frac{1}{a} \langle \nabla_X Y, n \rangle_a \langle \nabla_Y X, n \rangle_a$$

Write the equations for the terms $\langle \nabla_X Y, n \rangle_a$ and $\langle \nabla_Y X, n \rangle_a$:

$$2 \langle \nabla_X Y, n \rangle_a = \langle [X, Y], n \rangle_a - \langle [X, n], Y \rangle_a - \langle [Y, n], X \rangle_a$$

$$= a \langle [X, Y], n \rangle - \langle [X, n], Y \rangle - \langle [Y, n], X \rangle$$

$$2 \langle \nabla_Y X, n \rangle_a = \langle [Y, X], n \rangle_a - \langle [Y, n], X \rangle_a - \langle [X, n], Y \rangle_a$$

$$= a \langle [Y, X], n \rangle - \langle [Y, n], X \rangle - \langle [X, n], Y \rangle$$

Inserting the above equations into the second summand we have:

$$-\langle \nabla_Y \nabla_X Y, X \rangle_a = -Y \langle [X, Y], X \rangle_a + \frac{1}{4a} \left( -a \langle [X, Y], n \rangle + \langle [X, n], Y \rangle + \langle [Y, n], X \rangle \right)$$

$$\cdot \left( -a \langle [Y, X], n \rangle + \langle [Y, n], X \rangle + \langle [X, n], Y \rangle \right)$$

The last summand is:

$$-\langle \nabla_{[X,Y]} Y, X \rangle_a = -\langle \nabla_{\langle [X,Y], n \rangle n + \langle [X,Y], X \rangle X + \langle [X,Y], Y \rangle Y} Y, X \rangle_a$$

$$= -\langle [X, Y], n \rangle \langle \nabla_n Y, X \rangle_a - \langle [X, Y], X \rangle \langle \nabla_X Y, X \rangle_a$$

$$- \langle [X, Y], Y \rangle \langle \nabla_Y Y, X \rangle_a$$

The term $\langle \nabla_n Y, X \rangle_a$ is equal to

$$\langle \nabla_n Y, X \rangle_a = -\frac{1}{2}\left(-\langle [n, Y], X \rangle_a + \langle [n, X], Y \rangle_a + \langle [Y, X], n \rangle_a\right)$$

$$= -\frac{1}{2}\left(-\langle [n, Y], X \rangle + \langle [n, X], Y \rangle + a\langle [Y, X], n \rangle\right)$$

which gives us:

$$-\langle \nabla_{[X,Y]}Y, X \rangle_a = -\langle [X, Y], n \rangle\langle \nabla_n Y, X \rangle_a - \langle [X, Y], X \rangle\langle \nabla_X Y, X \rangle_a$$
$$- \langle [X, Y], Y \rangle\langle \nabla_Y Y, X \rangle_a$$

$$= \frac{1}{2}\langle [X, Y], n \rangle\left(-\langle [n, Y], X \rangle + \langle [n, X], Y \rangle + a\langle [Y, X], n \rangle\right)$$
$$- \langle [X, Y], X \rangle^2 - \langle [X, Y], Y \rangle^2$$

Summing this up, the sectional curvature of $\xi$ is equal to:

$$K(\xi) = -X\langle [Y, X], Y \rangle - \frac{1}{a}\langle [Y, n], Y \rangle\langle [X, n], X \rangle$$

$$- \left(Y\langle [X, Y], X \rangle - \frac{1}{4a}\left(-a\langle [X, Y], n \rangle + \langle [X, n], Y \rangle + \langle [Y, n], X \rangle\right)\right.$$
$$\left. \cdot\left(-a\langle [Y, X], n \rangle + \langle [Y, n], X \rangle + \langle [X, n], Y \rangle\right)\right)$$

$$- \left(-\frac{1}{2}\langle [X, Y], n \rangle\left(-\langle [n, Y], X \rangle + \langle [n, X], Y \rangle + a\langle [Y, X], n \rangle\right)\right.$$
$$\left. + \langle [X, Y], X \rangle^2 + \langle [X, Y], Y \rangle^2\right)$$

It is straightforward to verify that this gives us the desired expression. $\qquad\square$

**Lemma 3.2** *The extrinsic curvature $K_e(\xi)$ with respect to $\langle \cdot, \cdot \rangle_a$ can be calculated by the following formula:*

$$K_e(\xi) = \frac{1}{a}\left(\langle [X, n], X \rangle\langle [Y, n], Y \rangle - \frac{1}{4}\left(\langle [X, n], Y \rangle + \langle [Y, n], X \rangle\right)^2\right)$$

**Proof** Since $X$ and $Y$ are unit vectors, the extrinsic curvature is given by:

$$K_e(\xi) = B(X, X)B(Y, Y) - B(X, Y)^2$$

By the definition of $B$, the extrinsic curvature is equal to:

$$K_e(\xi) = \langle \nabla_X X, \frac{n}{\sqrt{a}} \rangle_a\langle \nabla_Y Y, \frac{n}{\sqrt{a}} \rangle_a - \frac{1}{4}\langle \nabla_X Y + \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a^2$$

Apply the Koszul formula to

$$\langle \nabla_X X, \frac{n}{\sqrt{a}} \rangle_a, \quad \langle \nabla_Y Y, \frac{n}{\sqrt{a}} \rangle_a \quad \text{and} \quad \langle \nabla_X Y + \nabla_Y X, \frac{n}{\sqrt{a}} \rangle_a$$

to obtain:

$$
\begin{aligned}
K_e(\xi) &= \frac{1}{a}\Big( \langle [X,n], X \rangle_a \langle [Y,n], Y \rangle_a - \frac{1}{4}\big( \frac{1}{2}\langle [X,Y], n \rangle_a - \frac{1}{2}\langle [X,n], Y \rangle_a \\
&\qquad - \frac{1}{2}\langle [Y,n], X \rangle_a - \frac{1}{2}\langle [X,Y], n \rangle_a - \frac{1}{2}\langle [Y,n], X \rangle_a - \frac{1}{2}\langle [X,n], Y \rangle_a \big)^2 \Big) \\
&= \frac{1}{a}\Big( \langle [X,n], X \rangle \langle [Y,n], Y \rangle - \frac{1}{4}\big( \langle [X,n], Y \rangle + \langle [Y,n], X \rangle \big)^2 \Big)
\end{aligned}
$$

Summing the extrinsic curvature of $\xi$ with the sectional curvature gives us the Gaussian curvature of the plane distribution $\xi$. □

**Lemma 3.3** *The Gaussian curvature $K_G(\xi)$ can be calculated by the formula:*

$$
\begin{aligned}
K_G(\xi) &= K(\xi) + K_e(\xi) \\
&= -\frac{3}{4}a\langle [X,Y], n \rangle^2 + \big( X\langle [X,Y], Y \rangle - Y\langle [X,Y], X \rangle \\
&\qquad\qquad\qquad\qquad - \langle [X,Y], X \rangle^2 - \langle [X,Y], Y \rangle^2 \big) \\
&\qquad + \frac{1}{2}\langle [X,Y], n \rangle\big( -\langle [n,Y], X \rangle + \langle [n,X], Y \rangle \big)
\end{aligned}
$$

**Remark 3.4** If $\xi$ is integrable then $\langle [X,Y], n \rangle = 0$ and

$$K_G(\xi) = X\langle [X,Y], Y \rangle - Y\langle [X,Y], X \rangle - \langle [X,Y], X \rangle^2 - \langle [X,Y], Y \rangle^2$$

is nothing else as the expression of the Gaussian curvature of the leaves of $\xi$ written in the local frame tangent to the leaves.

**Lemma 3.5** *Let $(X, Y, n)$ be a framing on $M$. Assume that distribution spanned by $n$ and $Y$ is a contact structure. Then there is a metric on $M$ such that extrinsic curvature of the distribution spanned by $X$ and $Y$ is strictly less than zero.*

**Proof** Fix a metric $\langle \cdot, \cdot \rangle$ such that the framing is orthonormal. Let $\xi$ be a distribution spanned by vector fields $X$ and $Y$. Stretch the metric along $X$ by a constant factor $\lambda^2$ and along $Y$ by a constant factor $1/\lambda^2$. Let's denote this metric by $\langle \cdot, \cdot \rangle_\lambda$. Calculate

the extrinsic curvature of $\xi$ with respect to this metric:

$$
\begin{aligned}
K_e(\eta) &= \langle [n, X], X \rangle_\lambda \langle [n, Y], Y \rangle_\lambda - \frac{1}{4} \big( \langle [n, X], Y \rangle_\lambda + \langle [n, Y], X \rangle_\lambda \big)^2 \\
&= \lambda^2 \langle [n, X], X \rangle \frac{1}{\lambda^2} \langle [n, Y], Y \rangle - \frac{1}{4} \big( \frac{1}{\lambda^2} \langle [n, X], Y \rangle + \lambda^2 \langle [n, Y], X \rangle \big)^2 \\
&= \langle [n, X], X \rangle \langle [n, Y], Y \rangle - \frac{1}{4} \big( \frac{1}{\lambda^2} \langle [n, X], Y \rangle + \lambda^2 \langle [n, Y], X \rangle \big)^2 \\
&= \langle [n, X], X \rangle \langle [n, Y], Y \rangle - \frac{1}{4\lambda^4} \langle [n, X], Y \rangle^2 - \frac{1}{2} \langle [n, X], Y \rangle \langle [n, Y], X \rangle \\
&\quad - \frac{\lambda^4}{4} \langle [n, Y], X \rangle^2
\end{aligned}
$$

Since $M$ is compact there is a positive constant $C$ such that:

$$
\left| \langle [n, X], X \rangle \langle [n, Y], Y \rangle - \frac{1}{2} \langle [n, X], Y \rangle \langle [n, Y], X \rangle \right| < C
$$

We assumed that distribution spanned by vector fields $n$ and $Y$ is a contact structure. The form $\alpha(*) = \langle *, X \rangle$ is a contact form of this distribution, so $\langle [n, Y], X \rangle = \alpha([n, Y]) \neq 0$. Since $M$ is compact there is an $\varepsilon$ such that:

$$
\left| \langle [n, Y], X \right| > \varepsilon
$$

This means that
$$
K_e(\eta) < C - \frac{\lambda^4 \varepsilon^2}{4}.
$$

This expression is strictly negative for some sufficiently large $\lambda$. □

**Corollary 3.6** *Assume that $\xi$ is a transversally orientable contact structure with the Euler class zero on $M$. Then there is a metric on $M$ such that the extrinsic curvature of $\xi$ is a strictly negative function.*

**Proof** Let $n$ be a vector field on $M$ transverse to $\xi$. Since $e(\xi) = 0$, the distribution $\xi$ is trivial and has two nowhere zero sections, say $X$ and $Y$.

Choose some positive number $\varepsilon$ and consider a distribution $\eta$ spanned by the vector fields $X$ and $Y + \varepsilon n$. It is obvious that for all $\varepsilon$ the distribution $\eta$ is transverse to $\xi$ and is a contact structure for some sufficiently small $\varepsilon$. Therefore, we can apply Lemma 3.5 to the framing $(X, Y, Y + \varepsilon n)$ to get a desired metric. □

# 4  Prescribing the sectional curvature of ξ

**Theorem A**  *Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the sectional curvature of $\xi$.*

**Proof**  Since $\xi$ is transversally orientable, there is a globally defined vector field $n$ which is transverse to $\xi$. Fix some Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$ such that $n$ is a unit normal vector field. Consider a finite cover of $M$ by the open sets $U_\alpha$ such that for each $\alpha$ there is an open set $U'_\alpha$ for which $\overline{U}_\alpha \subset U'_\alpha$ and $\xi|_{U'_\alpha}$ is a trivial fibration.

In each $U'_\alpha$ choose an orthonormal framing $(X_\alpha, Y_\alpha, n|_{U'_\alpha})$. Consider the stretching $\langle \cdot, \cdot \rangle_a$ of $\langle \cdot, \cdot \rangle$ along $n$ by a positive function $a$.

According to Lemma 3.1 the sectional curvature $K(\xi)$ on $U'_\alpha$ can be rewritten in the following way:

$$K(\xi) = -\frac{3}{4}a\langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a}Q_\alpha$$

where $P_\alpha$ and $Q_\alpha$ are functions on $U'_\alpha$ independent of $a$.

Since $\xi$ is a contact structure and $U_\alpha$ has a compact closure, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is bounded below by some positive constant $\varepsilon$ and the functions $P_\alpha$ and $Q_\alpha$ are bounded from above. Therefore there is a sufficiently large $D_\alpha$ such that the equation

$$-\frac{3}{4}a\langle [X_\alpha, Y_\alpha], n \rangle^2 + P_\alpha + \frac{1}{a}Q_\alpha = fD_\alpha$$

has a strictly positive solution $a_\alpha(D_\alpha)$. Notice, that for any $D > D_\alpha$ this equation still has a positive solution $a_\alpha(D)$. Let $D_0 = \max_\alpha\{D_\alpha\}$. Solve the equation above for $D_0$ in each chart $U_\alpha$. Let $a_\alpha = a_\alpha(D_0)$.

We claim that $a_\alpha$ constructed this way does not depend on the choice of the orthonormal framing $(X_\alpha, Y_\alpha, n|_{U_\alpha})$. Let $(X'_\alpha, Y'_\alpha, n|_{U_\alpha})$ be any other orthonormal framing on $\xi|_{U_\alpha}$. This defines a map

$$\phi_\alpha \colon U_\alpha \to O(2)$$

which maps a point $p \in U_\alpha$ to the transition matrix $\phi_\alpha(p)$ between two framings $(X'_\alpha, Y'_\alpha)$ and $(X_\alpha, Y_\alpha)$ on $\xi$. We have

$$\langle [X'_\alpha, Y'_\alpha], n \rangle^2 = (d\eta(X'_\alpha, Y'_\alpha))^2 = (d\eta(\phi_\alpha X_\alpha, \phi_\alpha Y_\alpha))^2 = \det\phi_\alpha^2 (d\eta(X_\alpha, Y_\alpha))^2$$
$$= \det\phi_\alpha^2 \langle [X_\alpha, Y_\alpha], n \rangle^2 = \langle [X_\alpha, Y_\alpha], n \rangle^2,$$

where $\eta$ is a 1–form defined by $\eta(*) = \langle *, n \rangle$. Therefore, $\langle [X_\alpha, Y_\alpha], n \rangle^2$ is independent of the choice of orthonormal framing. The expression $(1/a)Q_\alpha = -K_e(\xi)$ also does

not depend on the choice of the trivialization. Finally the sectional curvature $K(\xi)$ is independent of the framing. It is obvious that the right hand side of

$$P_\alpha = K(\xi) - \frac{1}{a}Q_\alpha + \frac{3}{4}a\langle[X_\alpha, Y_\alpha], n\rangle^2$$

does not depend on the choice of framing, so does $P_\alpha$.

Therefore, the functions $a_\alpha$ agree on the overlaps and define a global function $a$ on $M$. The sectional curvature of $\xi$ in the metric $\langle\cdot,\cdot\rangle_a$ is $fD_0$. Consider the metric $\langle\cdot,\cdot\rangle_0 = (1/D_0)\langle\cdot,\cdot\rangle_a$. It is easy to calculate, that the sectional curvature of $\xi$ in this metric is equal to $f$. □

**Corollary 4.1** *For any transversally orientable contact structure on a closed orientable 3–manifold, there is a metric on $M$, such that the sectional curvature of $\xi$ in this metric is equal to $-1$.*

**Theorem B** *Let $\xi$ be a transversally orientable contact structure on $M$ with Euler class zero. Then for any smooth function $f$, there is a metric on $M$ such that $f$ is a sectional curvature of $\xi$.*

**Proof** Since the Euler class of $\xi$ is zero, there is a contact structure $\eta$, which is transverse to $\xi$. According to the Corollary 3.6, there is a metric $\langle\cdot,\cdot\rangle$ in which the extrinsic curvature of $\xi$ is a strictly negative function. Let $n$ be a unit normal vector field with respect to this metric.

Consider the stretching of $\langle\cdot,\cdot\rangle$ along $n$ by a positive function $a$. According to Lemma 3.1, we have to find $a$ to satisfy the equation

$$-\frac{3}{4}a\langle[X, Y], n\rangle^2 + P - \frac{1}{4a}K_e(\xi) = f$$

where $P$ is a function on $M$ which is independent of $a$.

But since $-K_e(\xi) > 0$ this equation always has a strictly positive solution $a$. This completes the proof of the theorem. □

**Remark 4.2** In the proof of Theorem B it is crucial that $\xi$ is a contact structure. At points where $\langle[X, Y], n\rangle = 0$ the equation may not have any positive solutions.

**Example 4.3** (Propeller construction [6]) Consider the following pair of contact structures on $\mathbb{T}^3$:

$$\xi = \mathrm{Ker}(\alpha = \cos z\,dx - \sin z\,dy + dz)$$
$$\eta = \mathrm{Ker}(\beta = \cos z\,dx + \sin z\,dy)$$

It is easy to verify, that $\xi$ is transverse to $\eta$ and we get a bi-contact structure. From Theorem B, there is a metric on $\mathbb{T}^3$ such that $\xi$ has a positive sectional curvature. This is an example of a tight contact structure of positive sectional curvature.

**Example 4.4** (Overtwisted contact structures of positive sectional curvature)  Let $\xi$ be any contact structure with the Euler class zero on $M$. It is known (see Geiges [3]) that if we apply a full Lutz twist to this contact structure, the resulting contact structure is overtwisted and has Euler class zero. From Theorem B, it has a positive sectional curvature for some choice of metric on $M$.

## 5  Uniformization of contact structures on 3–manifolds

The same technique as in Theorem A can be applied to the Gaussian curvature of contact structures on three-manifolds.

**Theorem C**  *Let $\xi$ be a transversally orientable contact structure on a closed orientable 3–manifold $M$. For any smooth strictly negative function $f$, there is a metric on $M$ such that $f$ is the Gaussian curvature of $\xi$.*

**Proof**  Same as Theorem A. The only difference is that in the present case the equation which needs to be solved in each trivializing chart is:

$$K_G(\xi) = -\frac{3}{4}a\langle [X_\alpha, Y_\alpha], n\rangle^2 + P_\alpha = fD_0 \qquad\qquad \square$$

**Corollary 5.1**  (Uniformization of contact structures)  *For every transversally orientable contact structure $\xi$ on $M$, there is a metric such that $K_G(\xi) = -1$.*

**Example 5.2**  (Contact structure with $K_G(\xi) = 1$)  Consider the unit sphere $S^3 \subset \mathbb{C}^2$ with a bi-invariant metric. The standard contact structure on $S^3$ is defined as the kernel of the 1–form

$$\alpha = \sum_{i=1}^{2}(x_i dy_i - y_i dx_i),$$

restricted from $\mathbb{C}^2$ to $S^3$. This contact structure is orthogonal to a left-invariant vector field and therefore is left-invariant. Let $(X, Y)$ be a pair of orthonormal left-invariant sections of $\xi$. Since the metric is bi-invariant,

$$\nabla_S T = \frac{1}{2}[S, T]$$

for any left-invariant vector fields on $S^3$. Therefore the second fundamental form of $\xi$ vanishes and $K_G(\xi) = K(\xi) = 1$.

# References

[1]  **A Candel**, *Uniformization of surface laminations*, Ann. Sci. École Norm. Sup. (4) 26 (1993) 489–516  MR1235439

[2]  **S S Chern**, **R S Hamilton**, *On Riemannian metrics adapted to three-dimensional contact manifolds*, from: "Workshop (Bonn, 1984)", Lecture Notes in Math. 1111, Springer, Berlin (1985) 279–308  MR797427 With an appendix by A Weinstein

[3]  **H Geiges**, *Contact geometry*, from: "Handbook of differential geometry. Vol. II", Elsevier/North-Holland, Amsterdam (2006) 315–382  MR2194671

[4]  **J M Lee**, **T H Parker**, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) 17 (1987) 37–91  MR888880

[5]  **J Lohkamp**, *Metrics of negative Ricci curvature*, Ann. of Math. (2) 140 (1994) 655–683  MR1307899

[6]  **Y Mitsumatsu**, *Anosov flows and non-Stein symplectic manifolds*, Ann. Inst. Fourier (Grenoble) 45 (1995) 1407–1421  MR1370752

[7]  **B L Reinhart**, *The second fundamental form of a plane field*, J. Differential Geom. 12 (1977) 619–627 (1978)  MR512930

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