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Sign refinement for combinatorial link Floer homology

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Link Floer homology is an invariant for links which has recently been described entirely in a combinatorial way. Originally constructed with mod 2 coefficients, it was generalized to integer coefficients thanks to a sign refinement. In this paper, thanks to the spin extension of the permutation group we give an alternative construction of the combinatorial link Floer chain complex associated to a grid diagram with integer coefficients. In particular we prove that the sign refinement comes from a 2–cohomological class corresponding to the spin extension of the permutation group.

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1 Introduction

Heegaard–Floer homology (Ozsváth–Szabó [9]) is an invariant for closed oriented 3-manifolds which was extended to give an invariant for null-homologous oriented links in such manifolds called link Floer homology (Ozsváth–Szabó [8; 10], Rasmussen [11]). It gives the Seifert genus $g(K)$ of a knot K (Ozsváth–Szabó [7]), detects fibered knots (Ghiggini [2] in the case where $g(K) = 1$ and Ni [6] in general) and its graded Euler characteristic gives the Alexander polynomial [8; 11]. Recently, a combinatorial description of link Floer homology was given (Manolescu–Ozsváth–Sarkar [4]) and its topological invariance was proved in a purely combinatorial way (Manolescu–Ozsváth–Sarkar–Thurston [5]). The purpose of this paper is to give an alternative description of combinatorial link Floer homology with $\mathbb{Z}$ coefficients. This point of view was recently used by Audoux [1] to describe combinatorial Heegaard–Floer homology for singular knots.

Let first recall the context of combinatorial link Floer homology: we follow conventions of [5]. A planar grid diagram G lies in a square on the plane with $n \times n$ squares where n is the complexity of G. Each square is decorated with an X, an O or nothing in such a way that each row and each column contains exactly one X and one O. We number the X and the O from 1 to n and denote $\mathbb{X}$ the set $\{X_i\}_{i=1}^n$ and $\mathbb{O}$ the set $\{O_i\}_{i=1}^n$.

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Given a grid diagram G, we place it in standard position on the plane as follows: the bottom left corner is at the origin and each cell is a square of length one. We construct a planar link projection by drawing horizontal segments from the O to the X in each row and vertical segments from the X to the O in each column. At each intersection point, the vertical segment is over the horizontal one. This gives an oriented link $\vec{L}$ in S^3 and we say that $\vec{L}$ has a grid presentation given by G.

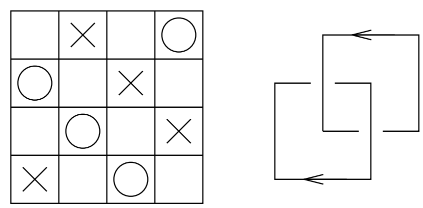

Figure 1: Grid presentation of the Hopf link.

We place the grid diagram on the oriented torus $\mathcal{T}$ by making the usual identification of the boundary of the square. We endow $\mathcal{T}$ with the orientation induced by the planar orientation. Let be the collection of the horizontal circles and the collection of the vertical ones. We associate with G a chain complex (C^-, ∂^-): it is the group ring of $\mathfrak{S}_n$ over $\mathbb{Z}/2\mathbb{Z}[U_{O_1}, \dots, U_{O_n}]$ where $\mathfrak{S}_n$ is the permutation group of n elements. A generator $\mathbf{x} \in \mathfrak{S}_n$ is given on G by its graph: we place dots in points $(i, x(i))$ for $i = 0, \dots, n-1$ (thus the fundamental domain of G is the square minus the right vertical segment and the top horizontal segment).

For A, B two finite sets of points in the plane we define $\mathcal{I}(A, B)$ to be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ such that $a_1 < b_1$ and $a_2 < b_2$. Let $\mathcal{J}(A, B) = (\mathcal{I}(A, B) + \mathcal{I}(B, A))/2$. We provide the set of generators with a Maslov degree M given by

$$M(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$$

where we extend $\mathcal{J}$ by bilinearly over formal sums (or differences) of subsets. Each variable U_{O_i} has a Maslov degree equal to -2 and constants have Maslov degree equal to zero. Let $M_S(\mathbf{x})$ be the same as $M(\mathbf{x})$ with the set S playing the role of $\mathbb{O}$.

We provide the set of generators with an Alexander filtration A given by $A(\mathbf{x}) = (A_1(\mathbf{x}), \dots, A_l(\mathbf{x}))$ with

$$A_i(\mathbf{x}) = \mathcal{J}(\mathbf{x} - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_i - \mathbb{O}_i) - \frac{n_i - 1}{2}$$

where when we number the components of $\overrightarrow{L}$ from 1 to ℓ, $\mathbb{O}_i \subset \mathbb{O}$ (resp. $\mathbb{X}_i \subset \mathbb{X}$) is the subset of $\mathbb{O}$ (resp. $\mathbb{X}$) wich belongs to the i th component of $\overrightarrow{L}$ and n_i is the number of horizontal segments which belongs to the i th component. We let $A(U_{O_j}) = (0, \dots, -1, 0, \dots, 0)$ where -1 corresponds to the i th coordonate if O_j belongs to the i th component of $\overrightarrow{L}$.

Given two generators $\mathbf{x}$ and $\mathbf{y}$ and an immersed rectangle r in the torus whose edges are arcs in the horizontal and vertical circles, we say that r connects $\mathbf{x}$ to $\mathbf{y}$ if $\mathbf{y}.\mathbf{x}^{-1}$ is a transposition, if all four corners of r are intersection points in $\mathbf{x} \cup \mathbf{y}$, and if we traverse each horizontal boundary component of r in the direction dictated by the orientation of r induced by $\mathcal{T}$, then the arc is oriented from a point in $\mathbf{x}$ to the point in $\mathbf{y}$. Let $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ be the set of rectangles connecting $\mathbf{x}$ to $\mathbf{y}$: either it is the empty set or it consists of exactly two rectangles. Here a rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ is said to be empty if there is no point of $\mathbf{x}$ in its interior. Let $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ be the set of empty rectangles connecting $\mathbf{x}$ to $\mathbf{y}$.

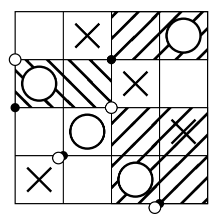

Figure 2: Rectangles. We mark with black dots the generator $\mathbf{x}$ and with white dots the generator $\mathbf{y}$. There are two rectangles in $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ but only the left one is in $\mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$.

The differential $\partial^-\colon C^-(G) \to C^-(G)$ is given on the set of generators by

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathfrak{S}_n} \sum_{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})} U_{O_1}^{O_1(r)} \dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

where $O_i(r)$ is the number of times O_i appears in the interior of r.

Theorem 1.1 (Manolescu–Ozsváth–Sarkar [4]) *$(C^-(G), \partial^-)$ is a chain complex for $CF^-(S^3)$ with homological degree induced by M and filtration level induced by A which coincides with the link filtration of $CF^-(S^3)$.*

In [5], the authors define a sign assigment for empty rectangles $\mathbf{S}\colon \mathrm{Rect}^\circ \to \{\pm 1\}$. Then, by considering $C^-(G)$ the group ring of $\mathfrak{S}_n$ over $\mathbb{Z}[U_{O_1}, \dots, U_{O_n}]$ and the differential $\partial^-\colon C^-(G)\to C^-(G)$ given by

$$\partial^-\mathbf{x}=\sum_{\mathbf{y}\in\mathfrak{S}_n}\sum_{r\in\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})}\mathbf{S}(r).U_{O_1}^{O_1(r)}\dots U_{O_n}^{O_n(r)}.\mathbf{y}$$

they obtain the following result.

Theorem 1.2 (Manolescu–Ozsváth–Szabó–Thurston [5]) *Let $\overrightarrow{L}$ be an oriented link with ℓ components. We number the $\mathbb{O}$ so that $O_1,\dots,O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then the filtered quasi-isomorphism type of $(C^-(G),\partial^-)$ over $\mathbb{Z}[U_{O_1},\dots,U_{O_\ell}]$ is an invariant of the link.*

In this paper, we give a way to refine the complex over $\mathbb{Z}$ thanks to $\widetilde{\mathfrak{S}}_n$ the spin extension of $\mathfrak{S}_n$ which is a non-trivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$. In Section 2 we define the spin extension $\widetilde{\mathfrak{S}}_n$ and make some algebraic calculus. Let z be the unique non-trivial central element of $\widetilde{\mathfrak{S}}_n$ and $\Lambda=\mathbb{Z}[U_{O_1},\dots,U_{O_n}]$. In Section 3 we define a filtered chain complex $(\widetilde{C}^-(G),\widetilde{\partial}^-)$ where $\widetilde{C}^-(G)$ is the quotient module of the free Λ–module with generating set $\widetilde{\mathfrak{S}}_n$ by the submodule generated by $\{z+1\}$. Finally, in Section 4, we prove that our chain complex defines a sign assignment in the sense of [5] and that $(\widetilde{C}^-(G),\widetilde{\partial}^-)$ is filtered quasi-isomorphic to $(C^-(G),\partial^-)$ with coefficients in $\mathbb{Z}$.

2 Algebraic preliminaries

Let $\mathfrak{S}_n$ be the group of bijections of a set with n elements numbered from 0 to $n-1$. It is given in terms of generators and relations where the set of generators is $\{\tau_i\}_{i=0}^{n-2}$ with τ_i the transposition which exchanges i and $i+1$ and relations are

$$\begin{aligned}\tau_i^2&=\mathbf{1} && 0\le i\le n-2\\ \tau_i.\tau_j&=\tau_j.\tau_i && |i-j|>1,\quad 0\le i,j\le n-2\\ \tau_i.\tau_{i+1}.\tau_i&=\tau_{i+1}.\tau_i.\tau_{i+1} && 0\le i\le n-3.\end{aligned}$$

Theorem 2.1 *The group given by generators and relations*

$$\begin{aligned}\widetilde{\mathfrak{S}}_n=<\widetilde{\tau}_0,\dots,\widetilde{\tau}_{n-2},z|\quad & z^2=\widetilde{\mathbf{1}},z\widetilde{\tau}_i=\widetilde{\tau}_i z,\widetilde{\tau}_i^2=z, && 0\le i\le n-2;\\ & \widetilde{\tau}_i.\widetilde{\tau}_j=z\widetilde{\tau}_j.\widetilde{\tau}_i\quad |i-j|>1, && 0\le i,j\le n-2;\\ & \widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i=\widetilde{\tau}_{i+1}.\widetilde{\tau}_i.\widetilde{\tau}_{i+1} && 0\le i\le n-3\ >\end{aligned}$$

is a non-trivial central extension ($n\ge4$) of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$ called the spin extension of $\mathfrak{S}_n$.

Remark 2.2 A proof of this theorem can be found in Karpilovsky [3, Theorem 2.12.3]. To see that it is a non-trivial extension, one can notice the following: let $\mathbb{Q}_8$ be the subgroup of $\widetilde{\mathfrak{S}}_n$ generated by $\widetilde{\tau}_0, \widetilde{\tau}_2, z$. Then $\mathbb{Q}_8$ is isomorphic to the unit sphere in the space of quaternions intersected with the lattice $\mathbb{Z}^4$ by a morphism Φ such that $\Phi(\widetilde{\tau}_0) = i$, $\Phi(\widetilde{\tau}_2) = j$, $\Phi(\widetilde{\tau}_0.\widetilde{\tau}_2) = k$ and $\Phi(z) = -1$. Therefore $\widetilde{\mathfrak{S}}_n$ is non-trivial.

Remark 2.3 Cases $n = 2$ and $n = 3$ are not interesting in our situation: the only knot which can be represented by a grid diagram of complexity 2 or 3 is the trivial knot. Nevertheless, the group given by generators and relations above still exists: in the case $n = 2$, it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, in the case $n = 3$, it is isomorphic to a subgroup of $GL(2, \mathbb{C})$ (see [3, Lemma 2.12.2]).

For $i < j$, define

$$\widetilde{\tau}_{i,j} = \widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\ \dots\ .\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}.\ \dots\ .\widetilde{\tau}_{i+1}.\widetilde{\tau}_i$$

and $\widetilde{\tau}_{j,i} = z\widetilde{\tau}_{i,j}$.

Let $\varepsilon\colon \mathfrak{S}_n \to \{0, 1\}$ be the signature morphism.

Lemma 2.4 *Let $\widetilde{\mathbf{x}} = \widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$ be an element in $\widetilde{\mathfrak{S}}_n$ and $\mathbf{x} = p(\widetilde{\mathbf{x}}) \in \mathfrak{S}_n$. Then for any $0 \leq i \neq j \leq n-1$*

$$\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}.$$

Proof Since $\widetilde{\mathbf{x}} = \widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$, $\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{i_k}.\dots.\widetilde{\tau}_{i_1}$. We prove by induction on $k \geq 1$ that for any $i, j \in \{0, \dots, n-1\}$ we have $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1} = z^{\varepsilon(\mathbf{x})}\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}$.

- **Initialization** Let $\widetilde{\mathbf{x}} = \widetilde{\tau}_l$ and $0 \leq i < j \leq n-1$. So $\widetilde{\tau}_l^{-1} = z\widetilde{\tau}_l$ and $\varepsilon(\mathbf{x}) = 1$. There are several cases.
  - **Case 1:** $l < i-1$ or $l > j$ $\quad \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.z\widetilde{\mathbf{x}} = z\tau_{i,j}$.
  - **Case 2:** $l = i-1$ $\quad \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.z\widetilde{\mathbf{x}} = z\widetilde{\tau}_{i-1}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{i-1} = z\widetilde{\tau}_{i-1,j}$ by definition.
  - **Case 3:** $l = i$ $\quad \widetilde{\tau}_i.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_i = z\widetilde{\tau}_{i+1,j}$.
  - **Case 4:** $i < l < j-1$ $\quad$ We prove by induction on $l - i \geq 1$ for i, j fixed that $\widetilde{\tau}_l.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_l = z\widetilde{\tau}_{\tau(i),\tau(j)}$. For $l = i+1$ then we have

$$\begin{aligned}\widetilde{\tau}_{i+1}.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_{i+1} &= z\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i.\widetilde{\tau}_{i+2,j}.\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i \\ &= z\widetilde{\tau}_i.\widetilde{\tau}_{i+1}.\widetilde{\tau}_{i+2,j}.\widetilde{\tau}_{i+1}.\widetilde{\tau}_i \\ &= z\widetilde{\tau}_{i,j}.\end{aligned}$$

Suppose it is proved until rank $(l-1)-i$. Then for $\widetilde{\mathbf{x}}=\widetilde{\tau}_l$ with $l<j-1$ we have

$$\begin{aligned}\widetilde{\mathbf{x}}.\widetilde{\tau}_i.z\widetilde{\mathbf{x}} &= z\widetilde{\tau}_l.\widetilde{\tau}_{i,j}.\widetilde{\tau}_l\\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-2}).(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).(\widetilde{\tau}_{l-2}.\dots.\widetilde{\tau}_i)\\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-2}).(\widetilde{\tau}_{l-1}.\widetilde{\tau}_l.\widetilde{\tau}_{l-1}).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_{l-1}.\widetilde{\tau}_l.\widetilde{\tau}_{l-1}).(\widetilde{\tau}_{l-2}.\dots.\widetilde{\tau}_i)\\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-1}.\widetilde{\tau}_l).\widetilde{\tau}_{l-1,j}.(\widetilde{\tau}_l.\widetilde{\tau}_{l-1}.\dots.\widetilde{\tau}_i) \text{ by induction}\\ &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{l-1}).\widetilde{\tau}_{l,j}.(\widetilde{\tau}_{l-1}.\dots.\widetilde{\tau}_i) \text{ by induction}\\ &= z\widetilde{\tau}_{i,j} \text{ by case 2.}\end{aligned}$$

– **Case 5:** $l=j-1$

$$\begin{aligned}\widetilde{\tau}_{j-1}.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_{j-1} &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{j-3}).\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.\widetilde{\tau}_{j-2}.\widetilde{\tau}_{j-1}.(\widetilde{\tau}_{j-3}.\dots.\widetilde{\tau}_i)\\ &= z\widetilde{\tau}_{i,j-1}.\end{aligned}$$

– **Case 6:** $l=j$

$$\begin{aligned}\widetilde{\tau}_j.\widetilde{\tau}_{i,j}.z\widetilde{\tau}_j &= z(\widetilde{\tau}_i.\dots.\widetilde{\tau}_{j-2}).\widetilde{\tau}_j.\widetilde{\tau}_{j-1}.\widetilde{\tau}_j.(\widetilde{\tau}_{j-2}.\dots.\widetilde{\tau}_i)\\ &= z\widetilde{\tau}_{i,j+1}.\end{aligned}$$

- **Heredity** Suppose the property is true until rank k. Let $\widetilde{\mathbf{x}}=\widetilde{\tau}_{i_1}.\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$ and $\widetilde{\tau}_{i,j}$ be two elements in $\widetilde{\mathfrak{S}}_n$. Denote $\widetilde{\mathbf{y}}=\widetilde{\tau}_{i_2}.\dots.\widetilde{\tau}_{i_k}$. Then $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1}=\widetilde{\tau}_{i_1}.\widetilde{\mathbf{y}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{y}}^{-1}.z\widetilde{\tau}_{i_1}$. By induction hypothesis,

$$\widetilde{\mathbf{y}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{y}}^{-1}=z^{\varepsilon(\mathbf{y})}.\widetilde{\tau}_{\mathbf{y}(i),\mathbf{y}(j)}.$$

So, $\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1}=\widetilde{\tau}_{i_1}.z^{\varepsilon(\mathbf{y})}.\widetilde{\tau}_{\mathbf{y}(i),\mathbf{y}(j)}.z\widetilde{\tau}_{i_1}$. By induction hypothesis one more time,

$$\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\mathbf{x}}^{-1}=z^{\varepsilon(\mathbf{y})+1}\widetilde{\tau}_{\tau_{i_1}.\mathbf{y}(i),\tau_{i_1}.\mathbf{y}(j)}=z^{\varepsilon(\mathbf{x})}.\widetilde{\tau}_{\mathbf{x}(i),\mathbf{x}(j)}.$$ □

The group $\widetilde{\mathfrak{S}}_n$ has another presentation in terms of generators and relations. Take $\{z'\}\cup\{\widetilde{\tau}'_{i,j}\}_{i\neq j}$ where $0\le i,j\le n-1$ as the set of generators with the following relations:

(2–1) $z'.z'=\widetilde{\mathbf{1}}'$ $z'\widetilde{\tau}'_{i,j}=\widetilde{\tau}'_{i,j}z'$ $\widetilde{\tau}'_{i,j}=z'\widetilde{\tau}'_{j,i}$ $\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{i,j}=z'$ for any i,j

(2–2) $\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{k,l}=z'\widetilde{\tau}'_{k,l}.\widetilde{\tau}'_{i,j}$ for any i,j,k,l if $\{i,j\}\cap\{k,l\}=\varnothing$

(2–3) $\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{j,k}.\widetilde{\tau}'_{i,j}=\widetilde{\tau}'_{j,k}.\widetilde{\tau}'_{i,j}.\widetilde{\tau}'_{j,k}=\widetilde{\tau}'_{i,k}$ for any i,j,k.

Proof Let $\widetilde{\mathfrak{S}}_n$ the group with z and $\widetilde{\tau}_i$ as generators and $\widetilde{\mathfrak{S}}'_n$ the other one. Define $\phi\colon\widetilde{\mathfrak{S}}_n\to\widetilde{\mathfrak{S}}'_n$ given on generators by $\phi(\widetilde{\tau}_i)=\widetilde{\tau}'_{i,i+1}$, $\phi(z)=z'$. For $i<j$, let $\phi(\widetilde{\tau}_{i,j})=\widetilde{\tau}'_{i,j}$. By definition, (2–1) is verified. Lemma 2.4 gives equations (2–2) and (2–3). So the map ϕ extends to a group isomorphism. □

In what follows, we drop the prime exponent and only refer to $\widetilde{\tau}_{i,j}$ and z ($\widetilde{\tau}_i$ means $\widetilde{\tau}_{i,i+1}$).

3 The chain complex

Let G be a grid presentation with complexity n of the link $\overrightarrow{L}$. Let Λ denote the ring $\mathbb{Z}[U_{O_1},\dots,U_{O_n}]$. We define $\widetilde{C}^-(G)$ to be the free Λ–module with generating set $\widetilde{\mathfrak{S}}_n$ quotiented by the submodule generated by $\{z+1\}$ ie

$$\widetilde{C}^-(G) = \Lambda[\widetilde{\mathfrak{S}}_n]/ < z+1 > .$$

Considered as module, $\widetilde{C}^-(G)$ coincides with the free Λ–module with generating set $\mathfrak{S}_n$. But we can also consider the structure of algebra of $\widetilde{C}^-(G)$ over Λ. In this case, one can think of $\widetilde{C}^-(G)$ as the group algebra of $\mathfrak{S}_n$ over Λ where the product is twisted by a non-trivial 2–cocycle (see Section 4).

We endow the set of generators with a Maslov grading M and an Alexander filtration A given by $M(\widetilde{\mathbf{x}}) = M(\mathbf{x})$ and $A(\widetilde{\mathbf{x}}) = A(\mathbf{x})$.

Let $\widetilde{\mathbf{x}}$ be an element of $\widetilde{\mathfrak{S}}_n$ and let $\mathrm{Rect}(\widetilde{\mathbf{x}})$ be the set of rectangles starting at $\widetilde{\mathbf{x}}$: by definition it is the set $\{\widetilde{\tau}_{i,j}\}_{0\le i\neq j\le n-1}$. If we consider the set $\mathrm{Rect}(\widetilde{\mathbf{x}},\widetilde{\mathbf{y}})$ of rectangles connecting $\mathbf{x}$ to $\mathbf{y}$ (where $\mathbf{y}=\mathbf{x}.\tau_{i,j}$) as in [5], either it is the empty set, or it consists of two rectangles. We interpret the rectangle $\widetilde{\tau}_{i,j}$ in the oriented torus $\mathcal{T}$ as the rectangle whose bottom left corner belongs to the ith vertical circle. So in the case where $\mathrm{Rect}(\mathbf{x},\mathbf{y}) = \{r_1, r_2\}$ the two corresponding rectangles are $\widetilde{\tau}_{i,j}$ and $\widetilde{\tau}_{j,i}$ Let r be the rectangle of $\mathrm{Rect}(\mathbf{x},\mathbf{y})$ corresponding to $\widetilde{r}$. A rectangle $\widetilde{r}\in\mathrm{Rect}(\widetilde{\mathbf{x}})$ is said to be empty if the corresponding rectangle $r\in\mathrm{Rect}(\mathbf{x},\mathbf{y})$ is empty. The set of empty rectangles starting at $\widetilde{\mathbf{x}}$ is denoted $\mathrm{Rect}^\circ(\widetilde{\mathbf{x}})$.

We endow $\widetilde{C}^-(G)$ with a differential $\widetilde{\partial}^-$ given on elements of $\widetilde{\mathfrak{S}}_n$ by:

$$\widetilde{\partial}^-\widetilde{\mathbf{x}} = \sum_{\widetilde{r}\in\mathrm{Rect}^\circ(\widetilde{\mathbf{x}})} U_{O_1}^{O_1(\widetilde{r})}\dots U_{O_n}^{O_n(\widetilde{r})}.\widetilde{\mathbf{x}}.\widetilde{r}$$

where $O_k(\widetilde{r})$ is the number of times O_k appears in the interior of r.

Proposition 3.1 *The differential $\widetilde{\partial}^-$ drops the Maslov degree by one and respect the Alexander filtration.*

Proof It is a straightforward consequence of calculus done in [5]. □

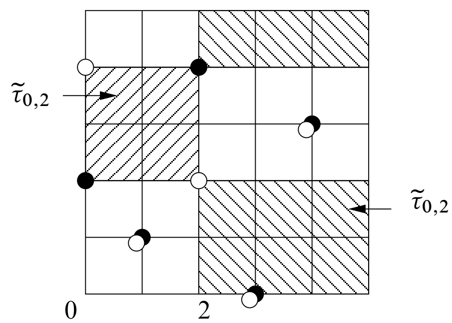

Figure 3: Rectangles. Black dots represent $\mathbf{x}$ and white dots $\mathbf{y}$. The two hatched regions correspond to rectangles $\widetilde{\tau}_{0,2} \in \mathrm{Rect}(\widetilde{\mathbf{x}})$ and $\widetilde{\tau}_{2,0} \in \mathrm{Rect}(\widetilde{\mathbf{x}})$. The rectangle $\widetilde{\tau}_{0,2}$ is an empty rectangle while $\widetilde{\tau}_{2,0}$ is not.

Proposition 3.2 *The endomorphism $\widetilde{\partial}^-$ of $\widetilde{C}^-(G)$ is a differential, ie*

$$\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0.$$

Proof Let $\widetilde{\mathbf{x}} = s(\mathbf{x}) \in \widetilde{\mathfrak{S}}_n$, viewed as a generator of $\widetilde{C}^-(G)$. Then

$$\widetilde{\partial}^- \circ \widetilde{\partial}^-(\widetilde{\mathbf{x}}) = \sum_{\widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{r}_1)} \ \sum_{\widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}})} U_{O_1}^{O_1(\widetilde{r}_1)+O_1(\widetilde{r}_2)} \dots U_{O_n}^{O_n(\widetilde{r}_1)+O_n(\widetilde{r}_2)} .\widetilde{\mathbf{x}}.\widetilde{r}_1.\widetilde{r}_2.$$

There are different cases which are illustrated by Figure 4.

Cases 1,2 The rectangles corresponding to $\widetilde{\tau}_{i,j}$ and $\widetilde{\tau}_{k,l}$ give the elements $\widetilde{\mathbf{z}}_1 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l}.\widetilde{\tau}_{i,j}$ and $\widetilde{\mathbf{z}}_2 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{k,l}$. By equation (2–2) contribution to $\widetilde{\partial}^- \circ \widetilde{\partial}^-(\widetilde{\mathbf{x}})$ is null.

Case 3 Supports of the rectangles have a common edge. The two corresponding elements are $\widetilde{\mathbf{z}}_1 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j}.\widetilde{\tau}_{j,k}$ and $\widetilde{\mathbf{z}}_2 = \widetilde{\mathbf{x}}.\widetilde{\tau}_{i,k}.\widetilde{\tau}_{i,j}$ with $i < j < k$. By equation (2–3), $\widetilde{\mathbf{z}}_1 = z\widetilde{\mathbf{z}}_2$ and so the contribution is null. Other cases work in a similar way.

Case 4 The vertical annulus is of width 1 and corresponds to $\widetilde{\mathbf{z}}_1 = U_{O_m}.\widetilde{\mathbf{x}}.\widetilde{\tau}_i.\widetilde{\tau}_i$ (it is a consequence of the condition on rectangles to be empty).

To this vertical annulus corresponds the horizontal annulus of height 1 which contains O_m. This horizontal annulus contributes for $U_{O_m}.\widetilde{\mathbf{x}}.\widetilde{\tau}_{l,k}.\widetilde{\tau}_{k,l} = U_{O_m}.\widetilde{\mathbf{x}}$ for a pair $k < l \in \{0, \dots, n-1\}$. So, the contribution of each vertical annulus is canceled by the corresponding horizontal annulus. The global contribution to $\widetilde{\partial}^- \circ \widetilde{\partial}^-(\widetilde{\mathbf{x}})$ is null. □

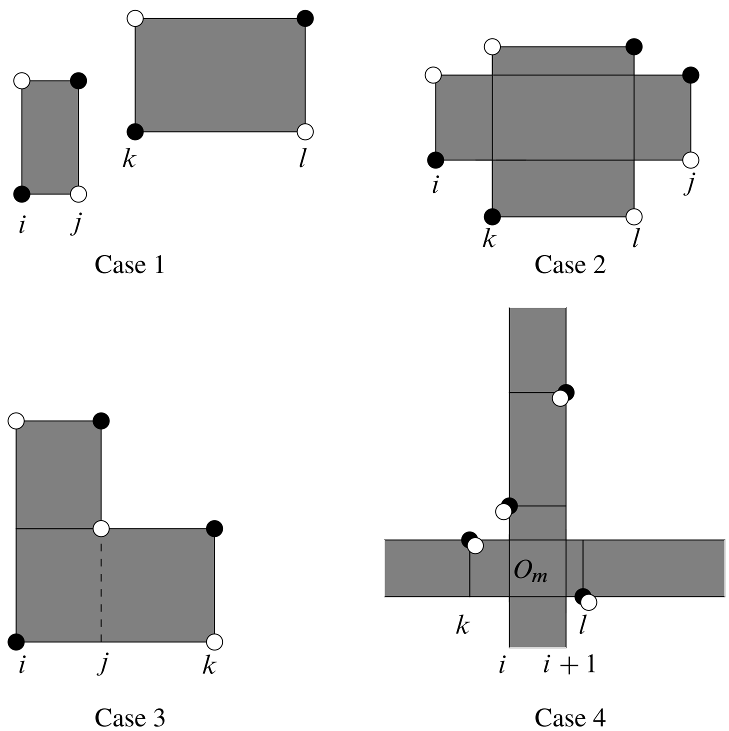

Figure 4: $\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0$.

4 Sign assignment induced by the complex

In this section we prove that the chain complex $\widetilde{C}^-(G)$ coincides with the chain complex $C^-(G)$ over $\mathbb{Z}$ after a choice of a sign assignment.

Definition 4.1 A sign assigment is a function $\mathbf{S}\colon \mathrm{Rect}^\circ \to \{\pm 1\}$ such that

(Sq) for any distincts $r_1, r_2, r_1', r_2' \in \mathrm{Rect}^\circ$ such that $r_1 * r_2 = r_1' * r_2'$ we have

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -\mathbf{S}(r_1').\mathbf{S}(r_2'),$$

(V) if $r_1, r_2 \in \mathrm{Rect}^\circ$ are such that $r_1 * r_2$ is a vertical annulus then

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -1,$$

(H) if $r_1, r_2 \in \mathrm{Rect}^\circ$ are such that $r_1 * r_2$ is a horizontal annulus then

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = +1.$$

Let $s\colon \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ be a section of the map p that is $p \circ s = \mathrm{id}_{\mathfrak{S}_n}$.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \overset{i}{\hookrightarrow} \widetilde{\mathfrak{S}}_n \underset{s}{\overset{p}{\rightleftarrows}} \mathfrak{S}_n \longrightarrow 1$$

To define the sign assignment we need the 2–cocycle $c \in C^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ associated to the map s given by

$$s(\mathbf{x}).s(\mathbf{y}) = (i \circ c(\mathbf{x},\mathbf{y}))s(\mathbf{x}.\mathbf{y}). \tag{4–1}$$

The cohomological class of c measures how s fails to be a group morphism. In particular, it is non-trivial ($n \geq 4$) since $\widetilde{\mathfrak{S}}_n$ is a non-trivial central extension of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$.

We say that a rectangle r is horizontally torn if given the coordinates (i_{bl}, j_{bl}) of its bottom left corner and (i_{tr}, j_{tr}) of its top right corner then $i_{bl} > i_{tr}$. Otherwise, r is said to be not horizontally torn.

Lemma 4.2 *The complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ induces a sign assignment in the sense of Definition 4.1: for all $(\mathbf{x},\mathbf{y}) \in \mathfrak{S}_n^2$ and all $r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$*

$$\mathbf{S}(r) = \varepsilon(r).c(\mathbf{x}^{-1}.\mathbf{y},\mathbf{x}) \tag{4–2}$$

where $\varepsilon(r) = +1$ if r is a rectangle not horizontally torn and $\varepsilon(r) = -1$ otherwise.

Remark The sign assignment in the sense of Definition 4.1 is unique up to a 1–coboundary: if $\mathbf{S}_1$ and $\mathbf{S}_2$ are two sign assignments then there exists an application $f\colon \mathfrak{S}_n \to \{\pm 1\}$ such that for all rectangles $r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$, $\mathbf{S}_1(r) = f(\mathbf{x}).f(\mathbf{y}).\mathbf{S}_2(r)$. It is a consequence of the fact that the central extension corresponds to a 2–cohomological class in $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ (compare with [5, Theorem 4.2]). Here, we construct explicitely a map $s\colon \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ such that $p \circ s = \mathrm{id}$ which means making a choice of a representative of this class, another choice must differ by a 1–coboundary.

Proof Since c is 2–cocycle we have $\delta c = 1$ ie for all $(\mathbf{x},\mathbf{y},\mathbf{z}) \in \mathfrak{S}_n^3$

$$\delta c(\mathbf{x},\mathbf{y},\mathbf{z}) = c(\mathbf{y},\mathbf{z}).c(\mathbf{x}.\mathbf{y},\mathbf{z}).c(\mathbf{x},\mathbf{y}.\mathbf{z}).c(\mathbf{x},\mathbf{y}) = 1.$$

By definition we have $c(\mathbf{x},\mathbf{1}) = c(\mathbf{1},\mathbf{x}) = 1$ and $c(\tau_{i,j},\tau_{i,j}) = -1$. Let's prove that $\mathbf{S}$ satisfy properties (Sq), (V) et (H).

(Sq) Let any four distincts rectangles S $r_1, r_2, r_1', r_2' \in \mathrm{Rect}^\circ$ such that $r_1 * r_2 = r_1' * r_2'$. Suppose $\widetilde{\tau}_{i,j} = \widetilde{r}_1 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}})$ corresponds to r_1 and $\widetilde{\tau}_{k,l} = \widetilde{r}_2 \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j})$ corresponds to r_2. Then $\widetilde{r}_1' = \widetilde{\tau}_{k,l} \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}})$ corresponds to r_1' and $\widetilde{r}_2' = \widetilde{\tau}_{i,j} \in$

$\mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{k,l})$ corresponds to r'_2. There are several cases to verify, as for the proof of $\widetilde{\partial}^- \circ \widetilde{\partial}^- = 0$ but all cases can be verified in a similar way. We verify the case $i < j < k < l$. We calculate $\delta c(\tau_{k,l}, \tau_{i,j}, \mathbf{x})$ and $\delta c(\tau_{i,j}, \tau_{k,l}, \mathbf{x})$. With equalities $c(\tau_{i,j}.\tau_{k,l}, \mathbf{x}) = c(\tau_{k,l}.\tau_{i,j}, \mathbf{x})$ and $c(\tau_{i,j}, \tau_{k,l}) = -c(\tau_{k,l}, \tau_{i,j})$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -\mathbf{S}(r'_1).\mathbf{S}(r'_2).$$

(V) Let $r_1, r_2 \in \mathrm{Rect}^\circ$ such that $r_1 * r_2$ is a vertical annulus. Suppose that $\widetilde{r}_1 = \widetilde{\tau}_i \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}})$ corresponds to r_1 and $\widetilde{r}_2 = \widetilde{\tau}_i \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_i)$ corresponds to r_2. We calculate $\delta c(\tau_i, \tau_i, \mathbf{x})$ and with equalities $c(\mathbf{x}, \mathbf{1}) = 1$, $c(\tau_i, \tau_i) = -1$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = -1.$$

(H) Let $r_1, r_2 \in \mathrm{Rect}^\circ$ such that $r_1 * r_2$ is a horizontal annulus (of height one). Suppose $\widetilde{r}_1 = \widetilde{\tau}_{i,j} \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}})$ corresponds to r_1 and $\widetilde{r}_2 = \widetilde{\tau}_{j,i} \in \mathrm{Rect}^\circ(\widetilde{\mathbf{x}}.\widetilde{\tau}_{i,j})$ corresponds to r_2. We calculate $\delta c(\tau_{i,j}, \tau_{i,j}, \mathbf{x})$ and with equalities $c(\mathbf{x}, \mathbf{1}) = 1$, $c(\tau_{i,j}, \tau_{i,j}) = -1$ we get

$$\mathbf{S}(r_1).\mathbf{S}(r_2) = +1. \qquad \square$$

Proposition 4.3 *The filtered chain complex $(\widetilde{C}^-(G), \widetilde{\partial}^-)$ is filtered isomorphic to the filtered chain complex $(C^-(G), \partial^-)$.*

Proof The map $s\colon \mathfrak{S}_n \to \widetilde{\mathfrak{S}}_n$ extends linearly with respect to $\mathbb{Z}[U_1, \dots, U_n]$ uniquely to a map $s\colon C^-(G) \to \widetilde{C}^-(G)$ which is an isomorphism of modules. It commutes with the differentials ie $s \circ \partial^- = \widetilde{\partial}^- \circ s$ where the sign assignment $\mathbf{S}$ is given by equation (4–2). By definition, s respects the Alexander filtration and the Maslov grading. So s defines a filtered isomorphism between the complexes $(C^-(G), \partial^-)$ and $(\widetilde{C}^-(G), \widetilde{\partial}^-)$. $\square$

A consequence of the above proposition and [5, Theorem 1.2] is the following.

Corollary 4.4 *Let $\overrightarrow{L}$ be an oriented link with ℓ components. We number the $\mathbb{O}$ so that $O_1, \dots, O_\ell$ correspond to the different components of $\overrightarrow{L}$. Then the filtered quasi-isomorphism type of $(\widetilde{C}^-(G), \partial^-)$ over $\mathbb{Z}[U_{O_1}, \dots, U_{O_\ell}]$ is an invariant of the link.*

Remark The proof of this theorem can also be done by adaptating the original proof in [5], sometimes with slightly simplified arguments.

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