# A class function on the mapping class group of an orientable surface and the Meyer cocycle

MASATOSHI SATO

In this paper we define a $\mathbf{QP}^1$–valued class function on the mapping class group $\mathcal{M}_{g,2}$ of a surface $\Sigma_{g,2}$ of genus $g$ with two boundary components. Let $E$ be a $\Sigma_{g,2}$–bundle over a pair of pants $P$. Gluing to $E$ the product of an annulus and $P$ along the boundaries of each fiber, we obtain a closed surface bundle over $P$. We have another closed surface bundle by gluing to $E$ the product of $P$ and two disks. The sign of our class function cobounds the 2–cocycle on $\mathcal{M}_{g,2}$ defined by the difference of the signature of these two surface bundles over $P$.

57N13, 55R40; 57M07

## 1 Introduction

Let $\Sigma_{g,r}$ be a compact oriented surface of genus $g$ with $r$ boundary components. The mapping class group $\mathcal{M}_{g,r}$ is $\pi_0\mathrm{Diff}_+(\Sigma_{g,r},\partial\Sigma_{g,r})$ where $\mathrm{Diff}_+(\Sigma_{g,r},\partial\Sigma_{g,r})$ is the group of orientation preserving diffeomorphisms of $\Sigma_{g,r}$ which restrict to the identity on the boundary $\partial\Sigma_{g,r}$. We simply denote $\Sigma_g := \Sigma_{g,0}$ and $\mathcal{M}_g := \mathcal{M}_{g,0}$. Harer [4] proved that

$$H^2(\mathcal{M}_{g,r};\mathbf{Z}) \cong \mathbf{Z} \quad g \geq 3,\ r \geq 0,$$

see also Korkmaz and Stipsicz [8].

Meyer [9] defined a cocycle $\tau_g \in Z^2(\mathcal{M}_g;\mathbf{Z})$ $(g \geq 0)$ called the Meyer cocycle which represents four times generator of the second cohomology class when $g \geq 3$. Let $D_1$, $D_2$, and $D_3$ be mutually disjoint disks in $S^2$, and $\mathrm{Int}\,D_i$ the interior of $D_i$ for $i = 1, 2, 3$. We denote by $P := S^2 - \amalg_{i=1}^3 \mathrm{Int}\,D_i$ the pair of pants, and $\alpha, \beta, \gamma \in \pi_1(P)$ be the homotopy classes as shown in Figure 1. We consider a $\Sigma_{g,r}$–bundle $E_{g,r}^{\varphi,\psi}$ on the pair of pants $P$ which has monodromies $\varphi$, $\psi$, $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha$, $\beta$, $\gamma \in \pi_1(P)$. The diffeomorphism type of $E_{g,r}^{\varphi,\psi}$ does not depend on the choice of representatives in the mapping classes $\varphi$ and $\psi$. Since $E_{g,r}^{\varphi,\psi}$ is the oriented fiber

*Published: 8 October 2008* DOI: 10.2140/agt.2008.8.1647

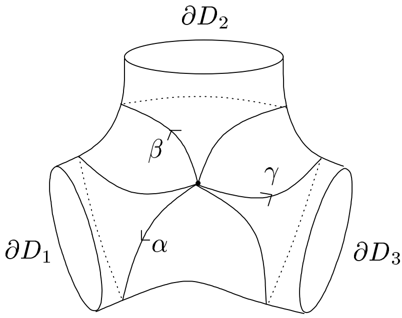


Figure 1

bundle, it has the canonical orientation comes from that of $\Sigma_{g,r}$ and $P$. The Meyer cocycle is defined by

$$\begin{array}{rccccl} \tau_g\colon & \mathcal{M}_g & \times & \mathcal{M}_g & \to & \mathbf{Z}, \\ & (\varphi & , & \psi) & \mapsto & \operatorname{Sign} E_g^{\varphi,\psi} \end{array}$$

where $\operatorname{Sign} E_g^{\varphi,\psi}$ is the signature of the compact oriented 4–manifold $E_g^{\varphi,\psi}$. For $k>0$, it is known as Novikov additivity that when two compact oriented $4k$–manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signature. When a pants decomposition of a closed oriented 2–manifold is given, the signature of a $\Sigma_g$–bundle on the 2–manifold is the sum of the signature of the $\Sigma_g$–bundles restricted to each pair of pants. Therefore, it is important to study the Meyer cocycle to calculate the signature of compact 4–manifolds. For $g=1,2$ the Meyer cocycle $\tau_g$ is a coboundary, and the cobounding function of this cocycle is calculated by several authors, for instance, Meyer [9], Atiyah [1], Kasagawa [6] and Iida [5]. The Meyer cocycle is not a coboundary if genus $g\geq 3$, but the cocycle can be a coboundary when it is restricted to some subgroups. For example, on the subgroup called the hyperelliptic mapping class group, the cobounding function is calculated by Endo [2] and Morifuji [11].

Let $I$ be the unit interval $[0,1]\subset \mathbf{R}$. By sewing a pair of disks onto the surface $\Sigma_{g,2}$ along the boundary, we have $\Sigma_g$. For $h\in \operatorname{Diff}_+(\Sigma_{g,2},\partial\Sigma_{g,2})$, if we extend $h$ by the identity on the pair of disks, we have a self-diffeomorphism of $\Sigma_g$. We denote it by $h\cup id_{\amalg_{i=1}^2 D^2}$. By sewing an annulus $S^1\times I$ onto the surface $\Sigma_{g,2}$ along the boundary, we have $\Sigma_{g+1}$. In the same way, if we extend $h\in \operatorname{Diff}_+(\Sigma_{g,2},\partial\Sigma_{g,2})$ by the identity on the annulus, we have a self-diffeomorphism $h\cup id_{S^1\times I}$.

Define the induced homomorphism on the mapping class group by

$$\begin{array}{rccc} \theta\colon & \mathcal{M}_{g,2} & \to & \mathcal{M}_g \\ & [h] & \mapsto & [h \cup id_{\amalg_{i=1}^2 D^2}] \end{array}$$

and

$$\begin{array}{rccc} \eta\colon & \mathcal{M}_{g,2} & \to & \mathcal{M}_{g+1,0}. \\ & [h] & \mapsto & [h \cup id_{S^1\times I}]. \end{array}$$

Harer [3; 4] shows that $\theta$ and $\eta$ induce an isomorphism on the second homology classes when genus $g \geq 5$, so that $\widetilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$ is a coboundary. Powell [12] proved that the first homology group $H_1(\mathcal{M}_{g,r};\mathbf{Z})$ is trivial for $g \geq 3$ and $r \geq 0$, so by the universal coefficient theorem, it follows that the cobounding function of $\widetilde{\tau}_g$ is unique.

In this paper we define a $\mathbf{QP}^1$–valued class function $m$ on the mapping class group $\mathcal{M}_{g,2}$ in an explicit way by using information of the first homology group of a mapping torus of $[h] \in \mathcal{M}_{g,2}$. For $[p:q] \in \mathbf{QP}^1$, we define the sign of $[p:q]$ by $\operatorname{sign}([p:q]) := \operatorname{sign}(pq)$. We prove that the sign of the function $m$ cobounds the cocycle $\widetilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$. In particular, it turns out that the cocycle $\widetilde{\tau}_g$ is coboundary for any $g \geq 0$.

This function makes a little bit easy to evaluate the Meyer cocycle on the subgroups consists of mapping classes that fix a curve on the surface. For example, consider the case $g = 1, 2$. We denote by $\phi_1$ and $\phi_2$ the cobounding functions of $\tau_1$ and $\tau_2$. Since $H_1(\mathcal{M}_{g,2};\mathbf{Q}) = 0$, the equation $\eta^*\tau_{g+1} = \theta^*\tau_g + \delta m$ means $\eta^*\phi_{g+1} = \theta^*\phi_g + m$ for $g = 1, 2$. In particular, the function $\phi_1$ is described explicitly in Meyer [9]. Therefore, our function $m$ helps to describe the cobounding function of the Meyer cocycle for genus 2 and 3 on the subgroup.

In Section 2, we construct a class function $m$, prove some properties of this function, and calculate the image of the function. In Section 3, we prove that the sign of this function cobounds the difference $\widetilde{\tau}_g = \eta^*\tau_{g+1} - \theta^*\tau_g$. By the definition of the Meyer cocycle $\tau_g$, $\widetilde{\tau}_g(\varphi,\psi)$ is just the difference $\operatorname{Sign} E_{g+1}^{\eta(\varphi),\eta(\psi)} - \operatorname{Sign} E_g^{\theta(\varphi),\theta(\psi)}$, so that we calculate the difference by using the sign of the function $m$. Moreover we compute the other differences of signature $\operatorname{Sign}(E_{g,2}^{\varphi,\psi}) - \operatorname{Sign}(E_g^{\theta(\varphi),\theta(\psi)})$ and $\operatorname{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)}) - \operatorname{Sign}(E_{g,2}^{\varphi,\psi})$ by the function $m$.

## 2 Class function $m\colon \mathcal{M}_{g,2} \to \mathbf{QP}^1$

In this section we define the class function on the mapping class group $\mathcal{M}_{g,2}$ stated in the introduction and describe some properties of the function including the nontriviality.

For $[p:q]$, $[r:s] \in \mathbf{QP}^1$, we define an addition in $\mathbf{QP}^1$ by

$$[p:q]+[r:s] = \begin{cases} [pr:ps+qr], & \text{if } [p:q] \neq [0:1] \text{ or } [r:s] \neq [0:1] \\ [0:1], & \text{if } [p:q]=[r:s]=[0:1]. \end{cases}$$

The projective line $\mathbf{QP}^1$ forms an additive monoid under this operation with $[1:0]$ the zero element.

In this section, all (co)homology groups are with $\mathbf{Q}$ coefficients.

## 2.1 Construction of the class function

Before constructing the function, we prepare a fact about homology groups of compact 3–manifolds. Let $Y$ be a compact oriented connected 3–manifold with boundary $\partial Y$ and $i\colon \partial Y \hookrightarrow Y$ the inclusion map. Consider the commutative diagram

$$\begin{tikzcd} H^1(Y) \arrow[r, "i^*"] \arrow[d, "\cap[Y]"] & H^1(\partial Y) \arrow[r, "\delta^*"] \arrow[d, "\cap[\partial Y]"] & H^2(Y,\partial Y) \arrow[d, "\cap[Y]"] \\ H_2(Y,\partial Y) \arrow[r, "\partial_*"] & H_1(\partial Y) \arrow[r, "i_*"] & H_1(Y), \end{tikzcd}$$

where the upper and lower rows are the exact sequences of a pair $(Y,\partial Y)$, and the vertical maps are the cap products with the (relative) fundamental classes of $Y$ and $\partial Y$. By the diagram and Poincaré Duality, it follows that the image of $i^*$ is just its own annihilator with respect to the cup product of $H^1(\partial Y)$

$$\operatorname{Im} i^* = \operatorname{Ann}(\operatorname{Im} i^*).$$

In particular, we have

$$\dim \operatorname{Ker} i_* = \dim \operatorname{Im} i^* = \frac{1}{2} \dim H_1(\partial Y).$$

We define the mapping torus of $\varphi = [h] \in \mathcal{M}_{g,r}$ by

$$X^\varphi := \Sigma_{g,r} \times I/\sim, \quad (x,1) \sim (h(x),0),$$

and $\pi\colon X^\varphi \to I/\partial I = S^1$ by the projection $\pi([x,t]) = [t]$, where $[x,t] \in X^\varphi$ is the equivalent class of $(x,t) \in \Sigma_{g,r} \times I$, and $[t] \in I/\partial I = S^1$ the equivalent class of $t \in I$.

The diffeomorphism type of the mapping torus $X^\varphi$ does not depend on the choice of the representative $h$. We fix the orientation on $X^\varphi$ given by the product orientation on $\Sigma_{g,r} \times I$. Let $i_\varphi\colon \partial X^\varphi \hookrightarrow X^\varphi$ be the inclusion map. In this subsection we denote $\Sigma := \Sigma_{g,2}$, and if we fix $\varphi \in \mathcal{M}_{g,2}$, then we write simply $X := X^\varphi$ and $i := i_\varphi$. Let $S_1$ and $S_2$ be the two boundary components of $\Sigma$, and $[S_k]$ $(k=1,2)$ the image under the inclusion homomorphism $H_1(S_k) \to H_1(\Sigma)$ of the fundamental homology class.

We consider $\Sigma$ as a subspace of $X$ by the embedding $\iota\colon \Sigma \hookrightarrow X$ by $x \mapsto [x,0]$. We choose points $p_1 \in S_1$, $p_2 \in S_2$, and $p \in S^1$, and orientation-preserving homeomorphisms $\iota_1\colon S^1 \to S_1$ and $\iota_2\colon S^1 \to S_2$. We define singular chains $f_k\colon I \to (S_1 \amalg S_2) \times S^1 = \partial X$ $(k = 1,2,3,4)$ by

$f_1(t) = (\iota_1(t), p)$, $f_2(t) = (\iota_2(t), p)$, $f_3(t) = (p_1, t)$ and $f_4(t) = (p_2, t)$ respectively.

Let $e_k \in H_1(\partial X)$ be the homology class of $f_k$ $(k = 1,2,3,4)$. Then the set $\{e_1, e_2, e_3, e_4\}$ forms a basis for $H_1(\partial X)$, and the intersection number

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } j = i+2, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1,2$ and $j = 3,4$. Now we describe the kernel of the homomorphism $i_*\colon H_1(\partial X) \to H_1(X)$. Since $e_1$ and $e_2$ lie in the kernel of $(\pi|_{\partial X})_*$ and $\pi_*(e_3) = \pi_*(e_4) = [S^1] \in H_1(S^1)$, we have

$$\text{Ker } i_* \subset \text{Ker } (\pi_* i_*) = \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4).$$

By the definition of the map $f_k$, $(i \circ f_k)_*[S^1] = \iota_*[S_k]$, and so $i_*(e_1 + e_2) = \iota_*([S_1] + [S_2]) \in H_1(X)$. Since $S_1 \cup S_2$ is the boundary of $\Sigma$, we have $[S_1] + [S_2] = 0 \in H_1(\Sigma)$. Hence

$$\mathbf{Q}(e_1 + e_2) \subset \text{Ker } i_*.$$

As we saw at the beginning of this subsection, $\dim \text{Ker } i_* = \frac{1}{2} \dim H_1(\partial X) = 2$. It follows that $\text{Ker}\, i_* = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + q e_1)$ for some $p, q \in \mathbf{Q}$. Now we can define a class function.

**Definition 2.1** For $\varphi \in \mathcal{M}_{g,2}$, we take $p, q \in \mathbf{Q}$ such that $\text{Ker } i_{\varphi *} = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}(p(e_3 - e_4) + q e_1)$.

We define $m\colon \mathcal{M}_{g,2} \to \mathbf{QP^1}$ by $m(\varphi) = [p : q]$.

**Lemma 2.2** *For $\varphi, \psi \in \mathcal{M}_{g,2}$,*

$$m(\psi\varphi\psi^{-1}) = m(\varphi).$$

**Proof** Define $\Psi\colon X^\varphi \to X^{\psi\varphi\psi^{-1}}$ by $\Psi(x,t) = (\psi(x), t)$. Then $\Psi$ maps $e_i$ as defined in $H_1(X^\varphi)$ to the corresponding $e_i$ as defined in $H_1(X^{\psi\varphi\psi^{-1}})$, and the following diagram commutes

$$\begin{array}{ccc} H_1(\partial X^{\varphi}) & \xrightarrow{i_{\varphi *}} & H_1(X^{\varphi}) \\ \downarrow{\scriptstyle \Psi_*} & & \downarrow{\scriptstyle \Psi_*} \\ H_1(\partial X^{\psi\varphi\psi^{-1}}) & \xrightarrow{i_{\psi\varphi\psi^{-1}*}} & H_1(X^{\psi\varphi\psi^{-1}}). \end{array}$$

As we see from the diagram, $\Psi_*$ gives the natural isomorphism between the kernels $\operatorname{Ker}(H_1(\partial X^{\varphi}) \to H_1(X^{\varphi}))$ and $\operatorname{Ker}(H_1(\partial X^{\psi\varphi\psi^{-1}}) \to H_1(X^{\psi\varphi\psi^{-1}}))$. Hence we have $m(\psi\varphi\psi^{-1}) = m(\varphi)$. □

## 2.2 Some properties and the nontriviality of the class function

By the Serre spectral sequence of the $\Sigma$–bundle $\pi\colon X \to S^1$, we have the exact sequence

$$0 \longrightarrow \operatorname{Coker}(\varphi_* - 1) \xrightarrow{\iota_*} H_1(X) \xrightarrow{\pi_*} H_1(S^1) \longrightarrow 0,$$

where $\operatorname{Coker}(\varphi_* - 1)$ is the cokernel of the homomorphism $\varphi_* - 1\colon H_1(\Sigma) \to H_1(\Sigma)$.

Then we have a unique homomorphism $j_\varphi\colon \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) \to \operatorname{Coker}(\varphi_* - 1)$ such that the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Q}e_1 \oplus \mathbf{Q}e_2 \oplus \mathbf{Q}(e_3 - e_4) & \longrightarrow & H_1(\partial X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j_\varphi} & & \downarrow{\scriptstyle i_*} & & \| & & \\ 0 & \longrightarrow & \operatorname{Coker}(\varphi_* - 1) & \xrightarrow{\iota_*} & H_1(X) & \xrightarrow{\pi_*} & H_1(S^1) & \longrightarrow & 0 \end{array}$$

commutes. By the diagram, we have

$$\operatorname{Ker} i_* = \operatorname{Ker} j_\varphi \text{ and}$$
$$j_\varphi(e_1) = -j_\varphi(e_2) = [S_1] \in \operatorname{Coker}(\varphi_* - 1).$$

Now we introduce a cochain $\omega_l \in C^1(\mathcal{M}_{g,2}; H_1(\Sigma))$ defined by Kawazumi [7]. On the fiber $\Sigma = \pi^{-1}(0) \subset X$, pick a path $l$ such that $l(0) \in S_2$ and $l(1) \in S_1$. Define $\omega_l$ by

$$\omega_l(\varphi) := [\varphi(l) - l] \in H_1(\Sigma).$$

Then we have the following lemma.

**Lemma 2.3**

$$j_\varphi(e_3 - e_4) = [\omega_l(\varphi)] \in \operatorname{Coker}(\varphi_* - 1).$$

**Proof** Define a 2–chain $L\colon I \times I \to X$ by $L(s,t) = [l(s), t]$. Its boundary is given by $-i_*(e_3) + \varphi(l) + i_*(e_4) - l \in B_1(X)$. Hence,

$$i_*(e_3 - e_4) = \iota_*([\varphi(l) - l]) \in H_1(X)$$

Since $\iota_*$ is injective, the lemma follows. □

From the lemma, we see the homology class $[\omega_l(\varphi)] \in \mathrm{Coker}\,(\varphi_* - 1)$ is independent of the choice of the path $l$. If $\omega_l(\varphi) = 0$, then $j_\varphi(e_3 - e_4) = 0$.

**Remark 2.4** If there exists a path $l$ from a point in $S_2$ to a point in $S_1$ which has no common point with the support of a representative of $\varphi \in \mathcal{M}_{g,2}$, then $m(\varphi) = [1:0]$. In particular, $m(id) = [1:0]$, the zero element of the monoid $\mathbf{QP}^1$.

Define the subgroups $\mathcal{I}' := \mathrm{Ker}\,(\mathcal{M}_{g,2} \to \mathrm{Aut}\,(H_1(\Sigma_{g,2}; \mathbf{Z}))$ and $\mathcal{I} := \mathrm{Ker}\,(\mathcal{M}_{g,2} \to \mathrm{Aut}\,(H_1(\Sigma_{g,2}, \partial\Sigma_{g,2}; \mathbf{Z}))$. For $\varphi \in \mathcal{I}'$, $m(\varphi) = [p:q]$ means $p(\varphi(l) - l) + qe_1 = 0 \in H_1(\Sigma_{g,2}; \mathbf{Z})$. This shows that $m$ is homomorphic on $\mathcal{I}'$. For $\varphi \in \mathcal{I}$, $\omega(\varphi) = 0 \in H_1(\Sigma_{g,2}; \mathbf{Z})$. This shows that $m(\varphi) = [1:0]$ for all $\varphi \in \mathcal{I}$.

**Remark 2.5** The restriction of $m$ on $\mathcal{I}$ is trivial, and the restriction of $m$ on $\mathcal{I}'$ is a nontrivial monoid homomorphism.

At the beginning of this section, we defined the commutative monoid structure on $\mathbf{QP}^1$. So integral multiples of $m(\varphi)$ are well-defined.

**Proposition 2.6** *If $\varphi \in \mathcal{M}_{g,2}$ and $k \in \mathbf{Z}$, then*

$$m(\varphi^k) = km(\varphi).$$

**Proof** The proposition is trivial for $k = 0$ and $k = 1$. Assume $k \geq 2$.

Let $m(\varphi) = [p:q]$. By the definition of $j_\varphi$, $pj_\varphi(e_3 - e_4) = -q[S_1] \in \mathrm{Coker}\,(\varphi_* - 1)$. Hence, there exists $v \in H_1(\Sigma)$ such that

$$p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v \in H_1(\Sigma).$$

Apply $\varphi^i$ $(i = 0, 1, \ldots, k-1)$ to the both sides of the equation and sum over $i$. Then

$$\sum_{i=0}^{k-1} p[\varphi^{i+1}(l) - \varphi^i(l)] = \sum_{i=0}^{k-1} \{-q[S_1] + (\varphi_*^{i+1}(v) - \varphi_*^i(v))\},$$

that is

$$p[\varphi^k(l) - l] = -kq[S_1] + (\varphi_*^k - 1)v.$$

Hence, $m(\varphi^k) = [p : kq] = km(\varphi)$ for $k \geq 0$.

By applying $\varphi^{-1}$ to the equation $p[\varphi(l) - l] = -q[S_1] + (\varphi_* - 1)v$, we have

$$p[\varphi^{-1}(l) - l] = q[S_1] + (\varphi_*^{-1} - 1)v \in H_1(\Sigma).$$

Hence, $m(\varphi^{-1}) = [p : -q] = -m(\varphi)$. Since $m(\varphi^{-k}) = -m(\varphi^k) = -km(\varphi)$ for $k > 0$, the proposition follows for the case $k < 0$. □

Now we compute the image of the function $m$. In particular, we see that $m$ is nontrivial.

**Proposition 2.7** *For $g \geq 1$, $m$ is surjective. For $g = 0$, $\mathrm{Im}\,(m) = [1 : \mathbf{Z}]$.*

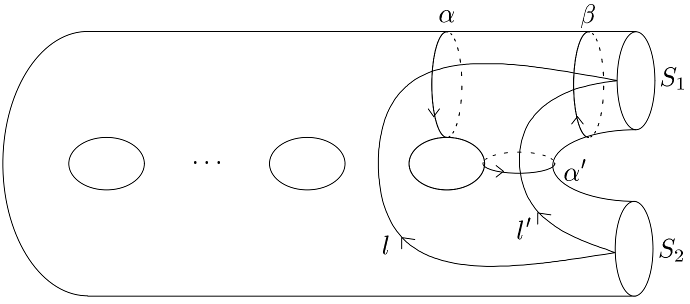


Figure 2

**Proof** Suppose $g \geq 1$. We choose oriented simple closed curves $\alpha$, $\alpha'$, and $\beta$ and paths $l$ and $l'$ as shown in Figure 2. We denote the Dehn twists along a simple closed curve $C \subset \Sigma$ by $t_C$, and the homology class of $C$ by $[C]$. Then $[\alpha] + [\alpha'] + [\beta] = 0 \in H_1(\Sigma)$ since they bound a 2–chain. For $p \in \mathbf{Z}$, if we denote $\varphi := t_\alpha^p t_{\alpha'} t_\beta^{-1}$, then

$$\begin{aligned} j_\varphi((p+1)(e_3 - e_4)) &= \omega_l(\varphi) + p\omega_{l'}(\varphi) \\ &= [(t_\alpha^p t_{\alpha'} t_\beta^{-1})(l) - l] + p[(t_\alpha^p t_{\alpha'} t_\beta^{-1})(l') - l'] \\ &= p([\alpha] + [\alpha'] + [\beta]) + [\beta] = [\beta] = [S_1]. \end{aligned}$$

Hence, $j_\varphi((p+1)(e_3 - e_4) - e_1) = 0$, so that

$$m(\varphi) = [p + 1 : -1].$$

By Proposition 2.6, we have

$$m(\varphi^{-q}) = -q[p+1 : -1] = \begin{cases} [p+1 : q], & \text{if } p \neq -1 \\ [0 : 1], & \text{if } p = -1. \end{cases} \qquad (q \in \mathbf{Z})$$

Since $p$ and $q$ can run over all integers, we see $m$ is surjective for $g \geq 1$.

For $g = 0$, $\mathcal{M}_{0,2}$ is the infinite cyclic group generated by $t_\beta$. Since $m(t_\beta^{-q}) = [1:q]$, we have $\mathrm{Im}\,(m) = [1:\mathbf{Z}]$. □

## 3 The difference of two Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$

In this section (co)homology groups are with $\mathbf{Z}$ coefficient unless specified.

Let $g \geq 0$ be a positive integer. In the introduction, we defined the homomorphisms $\eta\colon \mathcal{M}_{g,2} \to \mathcal{M}_{g+1,0}$ and $\theta\colon \mathcal{M}_{g,2} \to \mathcal{M}_g$ to be the induced maps by sewing a pair of disks and by sewing an annulus onto the surface $\Sigma_{g,2}$ along their boundaries respectively. We denote the Meyer cocycle on the mapping class group of genus $g$ closed orientable surface $\mathcal{M}_g$ by $\tau_g \in Z^2(\mathcal{M}_g)$ and define $\widetilde{\tau}_g \in Z^2(\mathcal{M}_{g,2})$ to be the difference between the Meyer cocycles

$$\widetilde{\tau}_g := \eta^*\tau_{g+1} - \theta^*\tau_g.$$

Let $P := S^2 - \amalg_{i=1}^3 D^2$. In this section, we prove the main theorem and calculate the changes of signature associated with sewing a pair of trivial disk bundles $P \times \amalg_{i=1}^2 D^2$ and sewing a trivial annulus bundles $P \times (S^1 \times I)$ onto $\Sigma_{g,2}$–bundle on the pair of pants $P$ along their boundaries. To state the main theorem, we define the sign of $[p:q] \in \mathbf{QP^1}$ by

$$\mathrm{sign}\,([p:q]) := \mathrm{sign}\,(pq) = \begin{cases} 1 & \text{if } pq > 0, \\ 0 & \text{if } pq = 0, \\ -1 & \text{if } pq < 0. \end{cases}$$

**Theorem 3.1** *For $\varphi, \psi \in \mathcal{M}_{g,2}$, we define*

$$\widetilde{\phi}_g(\varphi) := \mathrm{sign}\,(m(\varphi)).$$

*Then $\widetilde{\phi}_g$ cobounds the difference $\widetilde{\tau}_g$ between the Meyer cocycles $\eta^*\tau_{g+1}$ and $\theta^*\tau_g$*

$$\begin{aligned}\widetilde{\tau}_g(\varphi,\psi) &= \delta\widetilde{\phi}_g(\varphi,\psi) \\ &= \mathrm{sign}\,(m(\varphi)) + \mathrm{sign}\,(m(\psi)) + \mathrm{sign}\,(m((\varphi\psi)^{-1})).\end{aligned}$$

**Remark 3.2** Let $k$ be an integer. By Lemma 2.2 and Proposition 2.6, $\widetilde{\phi}_g$ has the properties

$$\begin{aligned}\widetilde{\phi}_g(\psi\varphi\psi^{-1}) &= \widetilde{\phi}_g(\varphi) \text{ and} \\ \widetilde{\phi}_g(\varphi^k) &= \mathrm{sign}\,(k)\widetilde{\phi}_g(\varphi)\end{aligned}$$

for any $g \geq 0$.

## 3.1 Proof of Main Theorem

In this subsection we prove Theorem 3.1.

In the introduction, we defined compact oriented 4–manifold $E^{\varphi,\psi}_{g,r}$ as a $\Sigma_{g,r}$–bundle on the pair of pants $P$ which has monodromies $\varphi$, $\psi$, and $(\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha$, $\beta$, and $\gamma \in \pi_1(P)$ respectively, and in Section 2.1, we defined compact oriented 3–manifold $X^{\varphi}_{g,r}$ by the mapping torus of $\Sigma_{g,r} \times I/\sim$ where $(x,1) \sim (h(x),0)$ for $\varphi = [h] \in \mathcal{M}_{g,r}$.

Gluing to $E^{\eta(\varphi),\eta(\psi)}_{g,2}$ the trivial annulus bundle on $P$ along the boundaries of each fiber, we obtain

$$E^{\eta(\varphi),\eta(\psi)}_{g+1} = E^{\varphi,\psi}_{g,2} \cup (-S^1 \times I \times P).$$

Similarly, glue to $X^{\eta(\varphi)}_{g,2}$ the trivial annulus bundle on $S^1$. Then we have

$$X^{\eta(\varphi)}_{g+1} = X^{\varphi}_{g,2} \cup (-S^1 \times I \times S^1).$$

Define

$$\begin{array}{rccc} G\colon & \partial D^2 \times I & \to & \{1\} \times S^1 \times I. \\ & (x,t) & \mapsto & (1, x, \frac{1+t}{3}). \end{array}$$

By the map $G$, we can glue $D^2 \times I$ to $I \times S^1 \times I$ as shown in Figure 3.

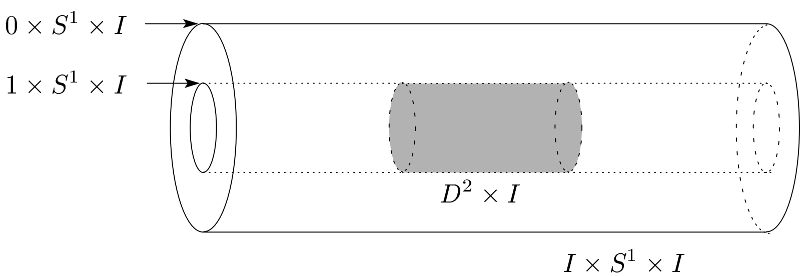


Figure 3: Gluing map $G$

Glue $D^2 \times I \times P$ to $I \times E^{\eta(\varphi),\eta(\psi)}_{g+1} = (I \times E^{\varphi,\psi}_{g,2}) \cup (-I \times S^1 \times I \times P)$ with the gluing map $G \times id_P\colon \partial D^2 \times I \times P \to \{1\} \times S^1 \times I \times P$. In the same way, glue $D^2 \times I \times S^1$

to $I \times X_{g+1}^{\eta(\varphi)} = (I \times X_{g,2}^{\varphi}) \cup (-I \times S^1 \times I \times S^1)$ with $G \times id_{S^1} \partial D^2 \times I \times S^1 \to \{1\} \times S^1 \times I \times S^1$. Namely, we construct two manifolds

$$\widetilde{E}^{\varphi,\psi} := (I \times E_{g+1}^{\eta(\varphi),\eta(\psi)}) \cup_{G \times id_P} (D^2 \times I \times P)$$

and

$$\widetilde{X}^{\varphi} := (I \times X_{g+1}^{\eta(\varphi)}) \cup_{G \times id_{S^1}} (D^2 \times I \times S^1).$$

Fix the orientations of these manifolds induced from the product orientations of $I \times E_{g+1}^{\eta(\varphi),\eta(\psi)}$ and $I \times X_{g+1}^{\eta(\varphi)}$. To prove main theorem, it suffices to prove Lemma 3.3 and Lemma 3.4 below.

**Lemma 3.3**

$(\eta^* \tau_{g+1} - \theta^* \tau_g)(\varphi, \psi) = \operatorname{Sign} \widetilde{X}^{\varphi} + \operatorname{Sign} \widetilde{X}^{\psi} + \operatorname{Sign} \widetilde{X}^{(\varphi\psi)^{-1}}$ *for* $\varphi, \psi \in \mathcal{M}_{g,2},\ g \geq 0$.

**Lemma 3.4**

$$\operatorname{Sign} \widetilde{X}^{\varphi} = \operatorname{sign}(m(\varphi)) \ \ \textit{for}\ \varphi \in \mathcal{M}_{g,2}, \quad g \geq 0.$$

**Proof of Lemma 3.3** Note that

$$\widetilde{X}^{\varphi} = \widetilde{E}^{\varphi,\psi}|_{\partial D_1}.$$

Then we can see

$$\begin{aligned}\partial \widetilde{E}^{\varphi,\psi} &= (\widetilde{E}^{\varphi,\psi}|_{\partial D_1} \cup \widetilde{E}^{\varphi,\psi}|_{\partial D_2} \cup \widetilde{E}^{\varphi,\psi}|_{\partial D_3}) \cup E_g^{\theta(\varphi),\theta(\psi)} \cup -E_{g+1}^{\eta(\varphi),\eta(\psi)} \\ &= (\widetilde{X}^{\varphi} \cup \widetilde{X}^{\psi} \cup \widetilde{X}^{(\psi\varphi)^{-1}}) \cup E_g^{\theta(\varphi),\theta(\psi)} \cup -E_{g+1}^{\eta(\varphi),\eta(\psi)}.\end{aligned}$$

Since the Signature is a bordism invariant (for example, see Milnor and Stasheff [10, Lemma 17.3]), we have $\operatorname{Sign} \partial \widetilde{E}^{\varphi,\psi} = 0$. By Novikov Additivity, we see that

$$\operatorname{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)}) - \operatorname{Sign}(E_g^{\theta(\varphi),\theta(\psi)}) = \operatorname{Sign} \widetilde{X}^{\varphi} + \operatorname{Sign} \widetilde{X}^{\psi} + \operatorname{Sign} \widetilde{X}^{(\psi\varphi)^{-1}}.$$

Notice that $\widetilde{X}^{(\psi\varphi)^{-1}}$ is diffeomorphic to $\widetilde{X}^{(\varphi\psi)^{-1}}$, so that $\operatorname{Sign} \widetilde{X}^{(\psi\varphi)^{-1}} = \operatorname{Sign} \widetilde{X}^{(\varphi\psi)^{-1}}$. By the definition of the Meyer cocycle, we have

$$\operatorname{Sign}(E_{g+1}^{\eta(\varphi),\eta(\psi)}) = \eta^* \tau_{g+1}(\varphi, \psi), \text{ and } \operatorname{Sign}(E_g^{\theta(\varphi),\theta(\psi)}) = \theta^* \tau_g(\varphi, \psi).$$

Define $\widetilde{\phi}(\varphi) = \operatorname{Sign}(\widetilde{X}^{\varphi})$; then we have $\delta \widetilde{\phi} = \eta^* \tau_{g+1} - \theta^* \tau_g$. We get the cobounding function $\widetilde{\phi}$. □

**Proof of Lemma 3.4** Write simply $X := X^{\eta(\varphi)}_{g+1}$, $X' := X^{\varphi}_{g,2}$, and $Y := \widetilde{X}^{\varphi} = (I \times X) \cup_{G \times id_{S^1}} (D^2 \times I \times S^1)$.

For $i = 0, 1$, define

$$\begin{array}{rcccl} j_i\colon & X & \to & I \times X & \hookrightarrow Y, \\ & x & \mapsto & (i, x) & \end{array}$$

where $I \times X \hookrightarrow Y$ is a natural embedding. We will prove there is a exact sequence

$$H_2(X') \xrightarrow{j_{0*}=j_{1*}} H_2(Y) \longrightarrow \operatorname{Ker}(H_1(\partial X') \to H_1(X')) \longrightarrow 0.$$

Define the submanifolds $Y_1 := I \times X'$ and $Y_2 := Y - \operatorname{Int} Y_1 = (-I \times S^1 \times I \times S^1) \cup_{G \times S^1} (D^2 \times I \times S^1)$. Then we see that

$$Y_1 \simeq X', Y_2 \simeq S^1, Y_1 \cap Y_2 \simeq \partial X' = (S_1 \amalg S_2) \times S^1.$$

By the Meyer–Vietoris exact sequence, we have the exact sequence

$$\begin{array}{ccccccc} H_2(Y_1) \oplus H_2(Y_2) & \longrightarrow & H_2(Y) & \xrightarrow{\partial_*} & H_1(Y_1 \cap Y_2) & \longrightarrow & H_1(Y_1) \oplus H_1(Y_2). \\ \| & & & & \| & & \| \\ H_2(X') \oplus 0 & & & & H_1(\partial X') & & H_1(X') \oplus H_1(S^1). \end{array}$$

Denote the map $H_1(\partial X') \to H_1(X') \oplus H_1(S^1)$ in the above diagram by $h$. the projection $H_1(\partial X') \to H_1(S^1)$ to the second entry of $h$ is the composite of inclusion homomorphism $H_1(\partial X') \to H_1(X')$ and $\pi_*\colon H_1(X') \to H_1(S^1)$. Therefore,

$$\operatorname{Ker}(H_1(\partial X') \to H_1(X') \oplus H_1(S^1)) = \operatorname{Ker}(H_1(\partial X') \to H_1(X')).$$

So the sequence is exact.

Next, we will construct the splitting

$$H_2(Y;\mathbf{Q}) = j_{i*} H_2(X';\mathbf{Q}) \oplus \operatorname{Ker}(H_1(\partial X';\mathbf{Q}) \to H_1(X';\mathbf{Q})).$$

Note that there exist $p, q \in \mathbf{Q}$ such that

$$\operatorname{Ker}(H_1(\partial X';\mathbf{Q}) \to H_1(X';\mathbf{Q})) = \mathbf{Q}(e_1 + e_2) \oplus \mathbf{Q}\{p(e_3 - e_4) + q e_1\}$$

as in Section 2. To construct the splitting, we choose elements of inverse images of $e_1 + e_2$, $p(e_3 - e_4) + q e_1$ under $H_2(Y) \to H_1(\partial X')$. Define $\iota_Y\colon \Sigma_{g+1} \to Y$ by

$$\begin{array}{ccccccc} \Sigma_{g+1} & \to & X & \to & I \times X & \hookrightarrow & Y. \\ x & \mapsto & (x, 0) & \mapsto & (0, x, 0). & & \end{array}$$

By the Meyer–Vietoris exact sequence as above, we have

$$\begin{array}{ccccc} H_2(Y) & \to & H_1(Y_1 \cap Y_2) & \to & H_1(\partial X'), \\ \iota_{Y*}[\Sigma_{g+1}] & \mapsto & \partial_* \iota_{Y*}[\Sigma_{g+1}] & \mapsto & e_1 + e_2 \end{array}$$

so we choose $\iota_{Y*}[\Sigma_{g+1}]$ as an element of the inverse image of $e_1 + e_2$.

Next, we choose an element of the inverse image of $p(e_3 - e_4) + qe_1$. Since $p(e_3 - e_4) + qe_1 \in \mathrm{Ker}\,(H_1(\partial X'; \mathbf{Q}) \to H_1(X'; \mathbf{Q}))$, there exists a singular 2–chain $s \in C_2(X'; \mathbf{Q})$ such that

$$\partial s = p(f_3 - f_4) + qf_1 \in B_1(X'; \mathbf{Q}).$$

For $i = 0, 1$, define $s'_{0i}\colon I \times S^1 \to I \times S^1 \times I \times S^1 \hookrightarrow Y_2$ by $s'_{0i}(t, u) = (i, 0, t, u)$. Then we see that

$$[\partial s'_{0i}] = [j_i f_3 - j_i f_4] \in H_1(Y_1 \cap Y_2; \mathbf{Q}).$$

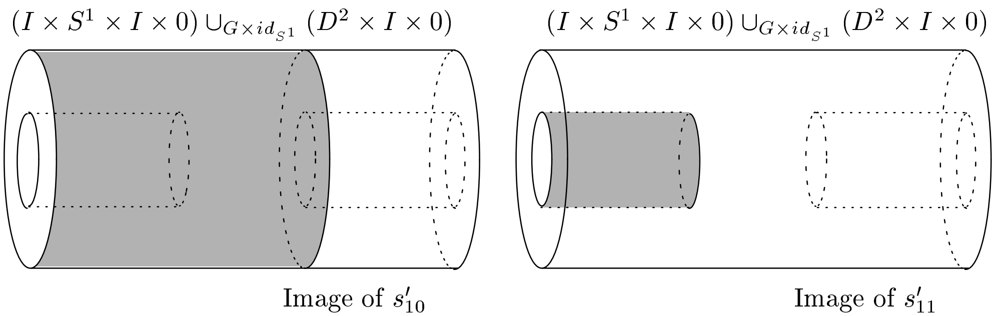


Figure 4: Images of $s'_{10}$ and $s'_{11} \subset Y_2$.

Define $s'_{1i}\colon D^2 \to Y_2 = (-I \times S^1 \times I \times S^1) \cup_{G \times S^1} (D^2 \times I \times S^1) \subset Y$ as shown in Figure 4 by

$$s'_{10}(x) = \begin{cases} (6x, 1, 0) & \in D^2 \times I \times S^1 \quad (\|x\| \le \frac{1}{6}), \\ (2 - 6\|x\|, \frac{x}{\|x\|}, \frac{2}{3}, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{1}{6} \le \|x\| \le \frac{1}{3}), \\ (0, \frac{x}{\|x\|}, 1 - \|x\|, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{1}{3} \le \|x\| \le 1), \end{cases}$$

$$s'_{11}(x) = \begin{cases} (\frac{3}{2}x, 0, 0) & \in D^2 \times I \times S^1 \quad (\|x\| \le \frac{2}{3}), \\ (1, \frac{x}{\|x\|}, 1 - \|x\|, 0) & \in I \times S^1 \times I \times S^1 \quad (\frac{2}{3} \le \|x\| \le 1). \end{cases}$$

Then, we have $[\partial s'_{1i}] = [j_i f_1] \in H_1(Y_1 \cap Y_2; \mathbf{Q})$.

The chain $s'_i := ps'_{0i} + qs'_{1i}$ satisfies

$$[\partial s'_i] = [j_i(p(f_3 - f_4) + qf_1)] \in H_1(Y_1 \cap Y_2; \mathbf{Q}),$$

so that we have $[\partial(j_i s - s'_i)] = 0 \in H_1(Y_1 \cap Y_2; \mathbf{Q})$.

We see

$$\begin{array}{ccccc} H_2(Y;\mathbf{Q}) & \to & H_1(Y_1 \cap Y_2;\mathbf{Q}) & \to & H_1(\partial X';\mathbf{Q}), \\ [j_i s - s'_i] & \mapsto & \partial_*[j_i s - s'_i] & \mapsto & p(e_3 - e_4) + q e_1 \end{array}$$

so that we can choose $[j_i s - s'_i]$ as an element of the inverse image of $p(e_3 - e_4) + q e_1$.

Now we calculate the intersection form of $H_2(Y;\mathbf{Q})$. Define the subspace $X''_1 = j_1(X) \cup_{G \times id_{S^1}} (D^2 \times 0 \times S^1) \subset Y$. Then we see that $X''_1$ is a deformation retract of $Y$. Hence, every element of $H_2(Y;\mathbf{Q})$ is represented by a cycle in $X''_1$. Therefore, a homology class is included in the annihilator of intersection form in $H_2(Y;\mathbf{Q})$ if it is represented by a cycle which has no common point with $X''_1$. We see

$$j_0(X') \cap X''_1 = \varnothing \text{ and } \iota_Y(\Sigma_{g+1}) \cap X''_1 = \varnothing,$$

so that the preimage of $\mathbf{Q}(e_1 + e_2)$ and $j_{0*} H_2(X';\mathbf{Q})$ are included in the annihilator of intersection form in $H_2(Y;\mathbf{Q})$.

To describe the signature of $Y$, it suffices to calculate the self-intersection number of $[j_i s - s'_i] = p(e_3 - e_4) + q e_1$. The cycle $j_i s - s'_i$ satisfies

$$\begin{aligned} \mathrm{Im}\,(j_0 s) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{01}) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing \\ \mathrm{Im}\,(s'_{00}) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{01}) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing \\ \mathrm{Im}\,(s'_{10}) \cap (\mathrm{Im}\,(j_1 s) \cup \mathrm{Im}\,(s'_{11})) &= \varnothing, \end{aligned}$$

so that

$$\begin{aligned} (j_0 s - s'_0) \cdot (j_1 s - s'_1) &= (j_0 s - (p s'_{00} + q s'_{10})) \cdot (j_1 s - (p s'_{01} + q s'_{11})) \\ &= q s'_{10} \cdot p s'_{01}. \end{aligned}$$

If necessary, perturb the chain $s'_{01}$. Then we see that $s'_{01}$ and $s'_{10}$ intersect only once positively. Hence, we have $\mathrm{Sign}\,(Y) = \mathrm{sign}\,(pq) = \mathrm{sign}\,(m(\varphi))$. □

## 3.2 Wall's non-additivity formula

In the introduction, we stated the Novikov additivity of Signature. Wall derives a formula from this additivity in a more general case, when two compact oriented smooth $4k$–manifolds are glued along common submanifolds of their boundaries. We will give the specific case of his formula for $k = 1$.

Let $Z$ be a closed oriented smooth 2–manifold, $X_-$, $X_0$, $X_+$ compact oriented smooth 3–manifolds with the boundaries $\partial X_- = \partial X_0 = \partial X_+ = Z$, and $Y_-$, $Y_+$ compact oriented smooth 4–manifolds with the boundaries $\partial Y_- = X_- \cup_Z (-X_0)$,

$\partial Y_+ = X_0 \cup_Z (-X_+)$. Here we denote by $M \cup_B (-N)$ the union of two manifolds $M$ and $N$ glued by orientation reversing diffeomorphism of their common boundaries $\partial M = \partial N = B$. Let $Y = Y_- \cup_{X_0} Y_+$ be the union of $Y_-$ and $Y_+$ glued along submanifolds $X_0$ of their boundaries. Suppose $Y$ is oriented by the induced orientation of $Y_-$ and $Y_+$.

Write $V = H_1(Z; \mathbf{R})$; let $A$, $B$, and $C$ be the kernels of the maps on first homology induce by the inclusions of $Z$ in $X_-$, $X_0$ and $X_+$ respectively.

We define

$$W := \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)},$$

and a bilinear form $\Psi$ by

$$\begin{array}{rccccc} \Psi\colon & W & \times & W & \to & \mathbf{R}. \\ & (b & , & b') & \mapsto & b \cdot c'. \end{array}$$

Here $c'$ is an element of $C$ such that there exists an element $a' \in A$ such that $a' + b' + c' = 0$, and $b \cdot c'$ denotes the intersection product of $b$ and $c'$. It is known that $\Psi$ is independent of the choice of $c'$ and well-defined on $W$. Denote the signature of the bilinear form $\Psi$ by $\mathrm{Sign}\,(V; BCA)$ and the signature of the compact oriented 4–manifold $M$ by $\mathrm{Sign}\, M$. We are now ready to state the formula.

**Theorem 3.5** (Wall [13]) $\mathrm{Sign}\, Y = \mathrm{Sign}\, Y_- + \mathrm{Sign}\, Y_+ - \mathrm{Sign}\,(V; BCA)$.

## 3.3 The differences $\mathrm{Sign}\, E_g - \mathrm{Sign}\, E_{g,2}$ and $\mathrm{Sign}\, E_{g+1} - \mathrm{Sign}\, E_{g,2}$

In this subsection, we calculate the difference of signature associated with sewing the trivial Disk bundles onto the $\Sigma_{g,2}$–bundle.

In the introduction, we defined $E_{g,r}^{\varphi,\psi}$ as a oriented $\Sigma_{g,r}$–bundle on $P$ which has monodromies $\varphi, \psi, (\psi\varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha, \beta, \gamma \in \pi_1(P)$. If we fix $\varphi, \psi \in \mathcal{M}_{g,2}$, we denote simply

$$E_{g,2} := E_{g,2}^{\varphi,\psi},\ E_g := E_g^{\theta(\varphi),\theta(\psi)},\ \text{and}\ E_{g+1} := E_{g+1}^{\eta(\varphi),\eta(\psi)}\ (g \geq 0).$$

**Proposition 3.6** $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2}) = -\mathrm{sign}\,(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1}))$ *for* $g \geq 0$.

**Proof** $E_g$ is the union of $E_{g,2}$ and $E_D := (D^2 \amalg D^2) \times P$ glued along their boundaries. Using Non-additivity formula Theorem 3.5, we calculate $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2})$.

Define $Y_-$, $Y_+$, $X_-$, $X_0$, $X_+$, and $Z$ by

$$\begin{aligned} &Y_- := (\amalg_{j=1}^2 D^2)\times P, \quad Y_+ := E_{g,2}, \\ &X_- := (\amalg_{j=1}^2 D^2)\times \partial P, \quad X_+ := E_{g,2}|_{\partial P}, \quad X_0 := (\amalg_{j=1}^2 \partial D^2)\times P, \\ \text{and } &Z := (\amalg_{j=1}^2 \partial D^2)\times \partial P, \quad \text{respectively.} \end{aligned}$$

Here, by the notation stated in Section 2.1,

$$X_+ = E_{g,2}|_{\partial P} \cong X^{\varphi} \amalg X^{\psi} \amalg X^{(\psi\varphi)^{-1}}, \quad Z \cong \partial X^{\varphi} \amalg \partial X^{\psi} \amalg \partial X^{(\psi\varphi)^{-1}}.$$

Define $V$, $A$, $B$, and $C$ as stated in Section 3.1.

Since $X^{\varphi} = X^{\psi} = X^{(\psi\varphi)^{-1}} = S^1 \times S^1$, we can choose the bases of $H_1(\partial X^{\varphi};\mathbf{R})$, $H_1(\partial X^{\psi};\mathbf{R})$, and $H_1(\partial X^{(\psi\varphi)^{-1}};\mathbf{R})$ as stated in Section 2.1. Denote their bases by $\{e_{11}, e_{12}, e_{13}, e_{14}\}$, $\{e_{21}, e_{22}, e_{23}, e_{24}\}$, and $\{e_{31}, e_{32}, e_{33}, e_{34}\}$ respectively.

Since $Z = \partial X^{\varphi} \amalg \partial X^{\psi} \amalg \partial X^{(\psi\varphi)^{-1}}$, we think of $e_{ij}$ as an element of $H_1(Z;\mathbf{R})$.

Denote $m(\varphi) = [a_1 : b_1]$, $m(\psi) = [a_2 : b_2]$, and $m((\psi\varphi)^{-1}) = [a_3 : b_3]$ respectively. Then we have

$$\begin{aligned} &V = H_1(Z,\mathbf{R}) = \bigoplus_{i=1}^{3}\bigoplus_{j=1}^{4} \mathbf{R}e_{ij}, \\ &A = \mathbf{R}e_{11} \oplus \mathbf{R}e_{21} \oplus \mathbf{R}e_{31} \oplus \mathbf{R}e_{12} \oplus \mathbf{R}e_{22} \oplus \mathbf{R}e_{32}, \\ &B = \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{11}-e_{31}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{12}-e_{32}) \\ &\quad \oplus \mathbf{R}(e_{13}+e_{23}+e_{33}) \oplus \mathbf{R}(e_{14}+e_{24}+e_{34}), \\ &C = \bigoplus_{i=1}^{3} \begin{cases} \mathbf{R}(e_{i1}+e_{i2}) \oplus \mathbf{R}(e_{i3}-e_{i4}+m_i e_{i1}) & \text{if } a_i \neq 0 \\ \mathbf{R}e_{i1} \oplus \mathbf{R}e_{i2} & \text{if } a_i = 0. \end{cases} \end{aligned}$$

Here we denote $m_i := \frac{b_i}{a_i}$. Hence,

$$B \cap A = \mathbf{R}(e_{11}-e_{21}) \oplus \mathbf{R}(e_{12}-e_{22}) \oplus \mathbf{R}(e_{11}-e_{31}) \oplus \mathbf{R}(e_{12}-e_{32}),$$

$$B \cap C = \begin{cases} \mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) \oplus \\ \mathbf{R}(e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34} + m_1 e_{11} + m_2 e_{21} + m_3 e_{31}) \\ \text{if } a_i \neq 0 \text{ for } i = 1,2,3 \text{ and } m_1 + m_2 + m_3 = 0, \\ \mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) \\ \text{if } a_i \neq 0 \text{ for } i = 1,2,3 \text{ and } m_1 + m_2 + m_3 \neq 0, \\ \mathbf{R}(e_{11} - e_{21} + e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) \\ \text{if } a_1 = 0, a_2 \neq 0, a_3 \neq 0, \\ \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31} + e_{12} - e_{32}) \\ \text{if } a_1 = a_2 = 0, a_3 \neq 0, \\ \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}) \\ \text{if } a_i = 0 \text{ for } i = 1,2,3, \end{cases}$$

$$B \cap (C + A) = \begin{cases} \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}) \\ \oplus \mathbf{R}(e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34}) \\ \text{if } a_i \neq 0 \text{ for } i = 1,2,3, \\ \mathbf{R}(e_{11} - e_{21}) \oplus \mathbf{R}(e_{12} - e_{22}) \oplus \mathbf{R}(e_{11} - e_{31}) \oplus \mathbf{R}(e_{12} - e_{32}) \\ \text{otherwise.} \end{cases}$$

By computing the signature of $\Psi$, we have

$$\operatorname{Sign}(V; BCA) = \begin{cases} \operatorname{sign}(m_1 + m_2 + m_3) & \text{if } a_i \neq 0 \text{ for } i = 1,2,3, \\ 0 & \text{otherwise.} \end{cases}$$

For example, consider the case when $a_i \neq 0$ for $i = 1,2,3$ and $m_1 + m_2 + m_3 \neq 0$. Then, the space $W$ is generated by the element represented by

$$b := e_{13} + e_{23} + e_{33} - e_{14} - e_{24} - e_{34} \in B \cap (C + A).$$

Choose the elements

$$a := m_1 e_{11} + m_2 e_{21} + m_3 e_{31} \in A \text{ and } c := -\sum_{i=1}^{3} (e_{i3} - e_{i4} + m_i e_{i1}) \in C.$$

Then we see that $a + b + c = 0$ and obtain $\Psi(b, b) = b \cdot c = m_1 + m_2 + m_3$. This shows that $\operatorname{Sign}(V; BCA) = \operatorname{sign}(m_1 + m_2 + m_3)$. The other cases follow in similar ways.

Hence, we obtain

$$\operatorname{Sign}(V; BCA) = \operatorname{sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).$$

By the non-additivity formula, we have

$$\mathrm{Sign}\,(E_g) = \mathrm{Sign}\,(E_D) + \mathrm{Sign}\,(E_{g,2}) - \mathrm{Sign}\,(V; BCA).$$

Since $E_D$ is a trivial bundle $(D^2 \amalg D^2) \times P$, we have $\mathrm{Sign}\,(E_D) = 0$.

This completes the proof of the proposition. □

By Theorem 3.1 and Proposition 3.6, we can calculate the difference of signature $\mathrm{Sign}\,(E_g) - \mathrm{Sign}\,(E_{g,2})$.

**Corollary 3.7** *For $g \geq 0$,*

$$\begin{aligned}\mathrm{Sign}\,(E_{g+1}) - \mathrm{Sign}\,(E_{g,2}) = \mathrm{sign}\,(m(\varphi)) + \mathrm{sign}\,(m(\psi)) + \mathrm{sign}\,(m((\varphi\psi)^{-1}))\\ - \mathrm{sign}\,(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).\end{aligned}$$

## References


[1] **M Atiyah**, *The logarithm of the Dedekind $\eta$–function*, Math. Ann. 278 (1987) 335–380 MR909232

[2] **H Endo**, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. 316 (2000) 237–257 MR1741270

[3] **J L Harer**, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983) 221–239 MR700769

[4] **J L Harer**, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) 121 (1985) 215–249 MR786348

[5] **S Iida**, *Adiabatic limit of $\eta$–invariants and the Meyer function of genus two*, Master's thesis, University of Tokyo (2004)

[6] **R Kasagawa**, *On a function on the mapping class group of a surface of genus 2*, Topology Appl. 102 (2000) 219–237 MR1745445

[7] **N Kawazumi**, *A generalization of the Morita–Mumford classes to extended mapping class groups for surfaces*, Invent. Math. 131 (1998) 137–149 MR1489896

[8] **M Korkmaz**, **A I Stipsicz**, *The second homology groups of mapping class groups of oriented surfaces*, Math. Proc. Cambridge Philos. Soc. 134 (2003) 479–489 MR1981213

[9] **W Meyer**, *Die Signatur von Flächenbündeln*, Math. Ann. 201 (1973) 239–264 MR0331382

[10] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Mathematics Studies 76, Princeton University Press (1974) MR0440554

[11] **T Morifuji**, *On Meyer's function of hyperelliptic mapping class groups*, J. Math. Soc. Japan 55 (2003) 117–129 MR1939188

[12] **J Powell**, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. 68 (1978) 347–350 MR0494115

[13] **C T C Wall**, *Non-additivity of the signature*, Invent. Math. 7 (1969) 269–274 MR0246311

*Graduate School of Mathematical Sciences, The University of Tokyo*
*3-8-1 Komaba Meguro-ku, Tokyo, 153-8914, Japan*

sato@ms.u-tokyo.ac.jp

Received: 20 February 2008 Revised: 30 May 2008