Some open 3–manifolds and 3–orbifolds without locally finite canonical decompositions

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We give examples of open 3–manifolds and 3–orbifolds that exhibit pathological behavior with respect to splitting along surfaces (2–suborbifolds) with nonnegative Euler characteristic.

57M50, 57M99; 57M10

Introduction

Much of the theory of compact 3–manifolds relies on decompositions into canonical pieces, in particular the Kneser–Milnor prime decomposition [12; 16] and the Jaco–Shalen–Johannson characteristic splitting [10; 11]. These have led to important developments in group theory (see Rips and Sela [22], Dunwoody and Sageev [7], Fujiwara and Papasoglu [9] and Scott and Swarup [24]) and form the background of W Thurston's geometrization conjecture, which has recently been proved by G Perelman [20; 21; 19].

For open 3–manifolds, by contrast, there is not even a conjectural description of a general 3–manifold in terms of geometric ones. Such a description would be all the more useful that noncompact hyperbolic 3–manifolds are now increasingly well-understood, thanks in particular to the recent proofs of the ending lamination conjecture (see Minsky [17] and Brock, Canary and Minsky [4]) and the tameness conjecture (see Calegari and Gabai [5] and Agol [1]).

The goal of this paper is to present a series of examples which show that naive generalizations to open 3–manifolds of the canonical decomposition theorems of compact 3–manifold theory are false. This contrasts with our positive results in [15; 13] which give decompositions under various hypotheses.

We now describe our examples and their properties in more detail. All manifolds and orbifolds in the following discussion are connected, orientable, and without boundary.

An embedded 2–sphere S in a 3–manifold M is called *compressible* if S bounds a 3–ball in M. If all 2–spheres in M are compressible, we say that M is irreducible. A *spherical decomposition* $\mathcal{S}$ of a 3–manifold M is a locally finite collection of (possibly nonseparating) pairwise disjoint embedded 2–spheres in M such that the operation of cutting M along $\mathcal{S}$ and gluing a ball to each boundary component of the resulting

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manifold yields a collection of irreducible manifolds. Note that if $\mathcal{S}$ is a spherical decomposition, then the collection of spheres obtained by removing compressible spheres of $\mathcal{S}$ is still a spherical decomposition.

Kneser's theorem is equivalent to the statement that every compact 3–manifold has a spherical decomposition. The first two examples in this paper show that this result does not generalize to open manifolds. The first relevant example was given by P Scott [23]. Our example M_1 in Section 1 is simpler and has additional properties: for instance, it is a graph manifold and has only one end. Our second example M_2 in Section 2 is closer in spirit to Scott's; its main purpose is to lead to example M_3 in Section 3.

The remaining examples are concerned with generalizing the toric splitting of Jaco–Shalen and Johannson. The correct definition of a JSJ-splitting for open 3–manifolds is still open to debate; however we make a few observations. Call an embedded torus T in a 3–manifold M *incompressible* if it is π_1–injective, and *canonical* if it is incompressible and any incompressible torus in M is homotopically disjoint from T. The version of the toric splitting theorem proved by Neumann and Swarup [18] asserts that if one takes a collection $\mathcal{T}$ of pairwise disjoint representatives of all homotopy classes of canonical tori in a compact manifold M (which is always possible, for instance by taking least area surfaces in some generic Riemannian metric), then $\mathcal{T}$ splits M into submanifolds that are either Seifert-fibered or atoroidal. This approach can be generalized to 3–orbifolds (see Boileau, Maillot and Porti [2, Chapter 3]). Here tori are replaced by *toric* 2–orbifolds, ie finite quotients of tori. Those include *pillows*, ie spheres with four cone points of order 2.

Let N_1 be an orientable 3–manifold whose boundary is an annulus, which does not contain any essential tori and open annuli, and is not homeomorphic to $S^1\times\mathbf{R}\times[0,+\infty)$. Let N_2 be the product of S^1 with an orientable surface of infinite genus whose boundary is a line. Since N_2 is a Seifert fiber space of infinite topological complexity, it contains many incompressible tori, none of which is canonical. By gluing N_1 and N_2 along their boundaries, one obtains an open 3–manifold M containing again many incompressible tori, neither of which are canonical. It is easy to construct infinite families of disjoint incompressible tori which cannot be made locally finite by any isotopy. However, there is in our view nothing pathological about this example: the "JSJ-splitting" in this case should consist of the single open annulus A, which splits M into a Seifert part and an atoroidal part.

This discussion makes plausible the idea that every irreducible open 3–manifold (or 3–orbifold) M could have a JSJ-splitting consisting of a representative of each class of canonical tori (toric suborbifolds), plus some properly embedded, incompressible open annuli (annular suborbifolds), splitting M into pieces which are either atoroidal, or maximal Seifert submanifolds (orbifolds). Furthermore, each incompressible torus (toric suborbifold) in M should be homotopic into some Seifert piece.

However, if one wishes to stick to locally finite splittings, this simple idea does not work, as examples 3–5 show. Our example M_3 is an irreducible open 3–manifold which contains an infinite collection $\{T'(v)\}$ of pairwise nonisotopic canonical tori and a compact set X' that meets every torus isotopic to some $T'(v)$. In particular, it is impossible to select a representative in each isotopy class of canonical tori to form a locally finite collection.

The manifold M_3 is constructed as a finite cover of a 3–orbifold $\mathcal{O}_3$ which contains infinitely many isotopy classes of canonical pillows, but such that no infinite collection of pairwise nonisotopic canonical pillows is locally finite. Moreover, all incompressible toric 2–suborbifolds in $\mathcal{O}_3$ are pillows, and they are all canonical.

The next example $\mathcal{O}_4$ is another open, irreducible 3–orbifold with the property that there are infinitely many isotopy classes of canonical pillows, but no infinite, locally finite collection of representatives (see Section 4). Again, all of its incompressible toric 2–suborbifolds are pillows. However, unlike $\mathcal{O}_3$, it also contains infinitely many classes of *noncanonical* incompressible pillows. Such pillows come in pairs and can be used to produce Seifert suborbifolds bounded by canonical pillows, which also accumulate in an essential way, and hence do not give a maximal Seifert suborbifold which would contain all incompressible toric 2–suborbifolds up to isotopy. Since the underlying space of $\mathcal{O}_4$ is $\mathbf{R}^3$, it is somewhat easier to visualize than M_3 and $\mathcal{O}_3$.

Lastly, the example $\mathcal{O}_5$ is an open, irreducible 3–orbifold which contains infinitely many isotopy classes of noncanonical incompressible pillows, but not a single canonical toric 2–suborbifold (see Section 5). Its pathological character comes from the fact that it is impossible to find a Seifert suborbifold that contains all incompressible pillows up to isotopy. Instead, one finds an infinite collection of incompatible Seifert suborbifolds which can be made pairwise disjoint, but all intersect essentially some fixed compact subset of $\mathcal{O}_5$. This shows that strange things can occur even without canonical toric suborbifolds.

For terminology and background on 3–orbifolds and their geometric decompositions, we refer to Boileau, Maillot and Porti [2].

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1 Example 1

Our first example is a one-ended open 3–manifold M_1 that does not have any spherical decomposition.

Let F be the orientable surface with one end, infinite genus and one boundary component homeomorphic to the circle. Set $N := S^1 \times F$. Let M_1 be the 3–manifold obtained by gluing a solid torus $V \cong D^2 \times S^1$ to N so that the boundary of the D^2 factor of V is glued to the S^1 factor of N, and the S^1 factor of V is glued to ∂F.

We observe that the universal cover of $\operatorname{Int} N$ is $\mathbf{R}^3$. We know from Alexander's theorem that $\mathbf{R}^3$ is irreducible, so by an elementary argument, N is also irreducible.

Let us prove by contradiction that M_1 cannot have a spherical decomposition $\mathcal{S}$. We may assume that there are no compressible spheres in $\mathcal{S}$. Since N is irreducible, all spheres in $\mathcal{S}$ must intersect V. Since V is compact and $\mathcal{S}$ is locally finite, we deduce that $\mathcal{S}$ must be finite.

Hence there is a compact subsurface $X \subset F$ such that every sphere in $\mathcal{S}$ lies in $V \cup S^1 \times X$. Since F has infinite genus, we can find a properly embedded arc $\alpha \subset F$ and a simple closed curve $\beta \subset F - X$ which intersect transversally in a single point. We then obtain an embedded 2–sphere $S \subset M$ by taking the annulus $S^1 \times \alpha$ and gluing a meridian disk to each boundary component.

We may assume that S is in general position with respect to $\mathcal{S}$. After finitely many isotopies and surgeries along disks in $\bigcup \mathcal{S}$, we get a finite collection $S_1, \dots, S_n$ of embedded 2–spheres in M_1 such that $[S] = \sum_i [S_i] \in H_2(M_1)$ and each S_i is disjoint from $\bigcup \mathcal{S}$. Since $\mathcal{S}$ is a spherical decomposition, each S_i either bounds a ball, or cobounds a punctured 3–sphere with some members of $\mathcal{S}$.

As a result, the homology class $[S] \in H_2(M_1)$ can be written as a finite sum of classes $[S_j]$ with $S_j \in \mathcal{S}$. Now the intersection number of each S_j with β is zero, while that of S with β is one. This is a contradiction.

Remark The first example of an open 3–manifold without a spherical decomposition was given by P Scott. His construction is quite intricate and his example is simply connected. Our example is simpler; it is far from simply connected however, in fact its fundamental group is an infinitely generated free group. Our example has one end; it is easy to modify the construction to give any number of ends.

Remark Our example is a graph-manifold: this is of some interest since those manifolds arise in the theory of collapsing sequences of manifolds in Riemannian geometry.

Remark There is an alternative description of M_1 as the double of the manifold $H = I \times F$. Any properly embedded arc in F gives a properly embedded 2–disk in H, which gives a sphere in the double. One readily sees that all those spheres have to intersect the annulus $I \times \partial F$. I owe this remark to Saul Schleimer.

2 Example 2

Here we give another example of a manifold without any spherical decomposition. The main interest of this construction is that a simple modification of it will give a manifold which behaves pathologically with respect to essential tori.

First we define an open 3*–orbifold* $\mathcal{O}_2$. In the following construction, all local groups will be cyclic of order 2. We let X be a 3–ball with singular locus a trivial 2–tangle, whose components are denoted by σ_l and σ_r. Let Y be a thrice punctured 3–sphere with singular locus consisting of six unknotted arcs, as in Figure 1. The boundary components of Y are denoted by $\partial_u Y$, $\partial_l Y$, and $\partial_r Y$ (where the letters u, l, r stand for "up", "left" and "right" respectively). There are two arcs σ_l^1, σ_l^2 connecting $\partial_u Y$ to $\partial_l Y$, two arcs σ_r^1, σ_r^2 connecting $\partial_u Y$ to $\partial_r Y$, and two arcs σ_m^1, σ_m^2 connecting $\partial_l Y$ to $\partial_r Y$.

Then we take a countable collection of copies of Y indexed by the vertices of the regular rooted binary tree $\mathcal{T}$. We glue them together according to the following rule: each copy Y_u of Y has two sons, a left son Y_l and a right son Y_r. Then we glue $\partial_l Y_u$ to $\partial_u Y_l$ so that $\sigma_l^i(Y_u)$ is glued to $\sigma_l^i(Y_l)$ and $\sigma_m^i(Y_u)$ is glued to $\sigma_r^i(Y_l)$ for $i = 1, 2$. Likewise we glue $\partial_r Y_u$ to $\partial_u Y_r$ so that $\sigma_r^i(Y_u)$ is glued to $\sigma_r^i(Y_r)$ and $\sigma_m^i(Y_u)$ is glued to $\sigma_l^i(Y_r)$ for $i = 1, 2$.

In this way we get a noncompact 3–orbifold N with a single boundary component which is the upper boundary of the ancestor Y_0. We glue in a copy of X so that $\sigma_l(X)$ is glued to $\sigma_l^i(Y_0)$ and $\sigma_r(X)$ is glued to $\sigma_r^i(Y_0)$. We call $\mathcal{O}_2$ the resulting open 3–orbifold.

For future reference, we note:

Lemma 2.1 *The orbifold N is irreducible.*

Proof Since the gluing of the various copies of Y occurs along incompressible suborbifolds, we only need to show that Y is irreducible. Arguing by contradiction, let S be an incompressible spherical 2–suborbifold of Y. Since any incompressible sphere in $|Y|$ meets the singular locus Σ_Y in at least four points, $|S|$ is compressible in $|Y|$, ie bounds a 3–ball B. Since $Y \setminus \Sigma_Y$ is irreducible, the only possibility is that B intersects Σ_Y in a knotted arc α. One could then extend α to produce a knotted component of Σ_Y, contradicting the definition of Y. □

We claim that $\mathcal{O}_2$ is homeomorphic to a connected sum of two copies of itself. Indeed, let $D_1 \subset X$ be a properly embedded nonsingular 2–disk separating σ_l from σ_r. Then ∂D_1 bounds a 2–disk $D_2 \subset Y_0$ intersecting the singular locus transversally in two points, one on σ_m^1 and one on σ_m^2. The union of D_1 and D_2 is a football S. Splitting

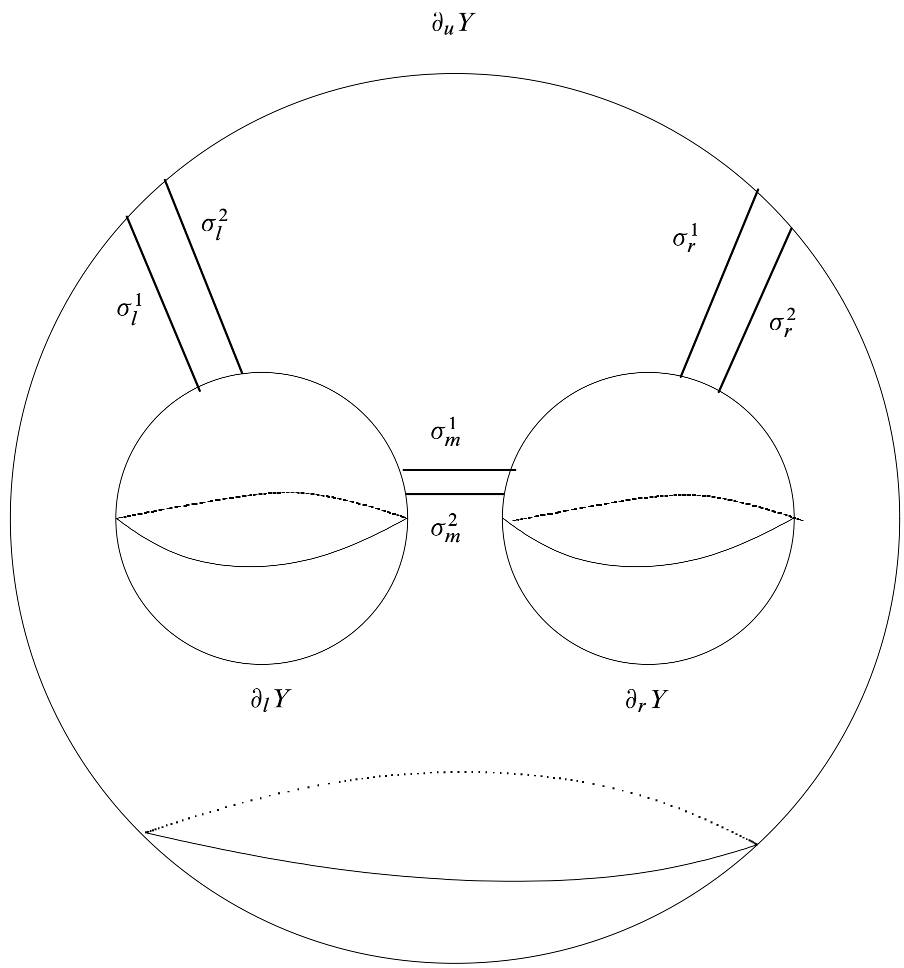

Figure 1: Y

$\mathcal{O}_2$ along S and capping off the corresponding discal 3–orbifolds, we get two copies of $\mathcal{O}_2$.

We now define our manifold M_2: by repeatedly applying the Seifert–van Kampen Theorem [2, Corollary 2.3], we see that the fundamental group of $\mathcal{O}_2$ has an infinite presentation with generators $m_1, m_2, \ldots$ and relations $r_1, r_2, \ldots$ as follows: each generator m_i corresponds to a meridian; for each i one has a relation $m_i^2 = 1$; all other relations are of the form $m_i m_j m_k m_l = 1$ corresponding to some boundary component of some copy of Y. Hence there is a well-defined group epimorphism $\phi\colon \pi_1\mathcal{O}_2 \to \mathbf{Z}/2\mathbf{Z}$ sending each m_i to the generator of $\mathbf{Z}/2\mathbf{Z}$. The kernel of ϕ is a torsion free index two subgroup of $\pi_1\mathcal{O}_2$. Let M_2 be the corresponding covering space of $\mathcal{O}_2$ and p be the covering map.

Then M_2 is a manifold which is a connected sum of two copies of itself. Now N is irreducible (Lemma 2.1,) hence by [14, Theorem 10.1], $p^{-1}(N)$ is irreducible. Since $M_2 \setminus p^{-1}(N)$ has compact closure, we conclude that any spherical decomposition of M_2 with only essential spheres would have to be finite.

Next we prove that there is no such spherical decomposition: let Z be a compact submanifold of M_2 containing all spheres in a putative spherical decomposition. Let v be a vertex of $\mathcal{T}$ which is lower than any vertex v' such that $p(Z) \cap Y_{v'} \neq \varnothing$. There is a football $F \subset \mathcal{O}_2$ intersecting Y_v in a disk with two cone points, one on σ_m^1 and the other on σ_2^m, each $Y_{v'}$ in a nonsingular annulus (for v' above v) and X in a nonsingular disk. Then $S := p^{-1}(F)$ is a 2–sphere embedded in M_2. One can find a properly embedded line $L \subset M_2$ missing Z and hitting S transversally in a single point. As before we get a contradiction.

3 Example 3

Here is an example of an irreducible open 3–manifold which contains infinitely many isotopy classes of incompressible tori, all of which are canonical, but such that there is no infinite, locally finite collection of canonical tori.

Let $\mathcal{O}_3$ be the open 3–orbifold obtained by the following modification of the previous construction, where every singular arc is "doubled".

Let X be a 3–ball with singular locus a trivial 4–tangle, whose components are denoted by $\sigma_l, \sigma_l', \sigma_r, \sigma_r'$. Let Y be a thrice punctured 3–sphere with singular locus consisting of twelve unknotted arcs: four arcs $\sigma_l^1, \sigma_l^2, \sigma_l^{1\prime}, \sigma_l^{2\prime}$ connecting $\partial_u Y$ to $\partial_l Y$, four arcs $\sigma_r^1, \sigma_r^2, \sigma_r^{1\prime}, \sigma_r^{2\prime}$ connecting $\partial_u Y$ to $\partial_r Y$, and four arcs $\sigma_m^1, \sigma_m^2, \sigma_m^{1\prime}, \sigma_m^{2\prime}$ connecting $\partial_l Y$ to $\partial_r Y$.

We remark (cf [6]) that X and Y are irreducible and atoroidal, Y has incompressible boundary, and the only essential annular 2–suborbifolds in Y are nonsingular annuli connecting two distinct boundary components.

As before, let $\mathcal{T}$ be the regular rooted binary tree. To each vertex v of $\mathcal{T}$, we assign a copy $Y(v)$ of Y. With the same notation as in Section 2, we glue $\partial_l Y_u$ to $\partial_u Y_l$ and $\partial_r Y_u$ to $\partial_u Y_r$ so that $\sigma_l^i(Y_u)$ (resp. $\sigma_l^{i\prime}(Y_u)$) is glued to $\sigma_l^i(Y_l)$ (resp. $\sigma_l^{i\prime}(Y_l)$), etc. We again call N the resulting 3–orbifold with boundary. We glue in a copy of X so that $\sigma_l(X)$ (resp. $\sigma_l'(X)$) is glued to $\sigma_l^i(Y_0)$ (resp. $\sigma_l^{i\prime}(Y_0)$), and $\sigma_r(X)$ (resp. $\sigma_r'(X)$) is glued to $\sigma_r^i(Y_0)$ (resp. $\sigma_r^{i\prime}(Y_0)$.)

We call $\mathcal{O}_3$ the resulting open 3–orbifold, and P the union of all boundaries of all $Y(v)$'s.

Proposition 3.1 (1) *$\mathcal{O}_3$ is irreducible.*

(2) *There are infinitely many incompressible pillows up to isotopy.*

(3) *All incompressible toric 2–suborbifolds in $\mathcal{O}_3$ are pillows. Furthermore, they are all canonical, and they all meet the compact 3–suborbifold X.*

Proof (1) Suppose there is an essential 2–suborbifold $S \subset \mathcal{O}_3$ with positive Euler characteristic. Take among all of them one that intersects P minimally. Since X and Y are irreducible, $S \cap P$ cannot be empty. Let v be a vertex of $\mathcal{T}$ such that S meets $Y(v)$, but avoids all $Y(v')$'s with v' below v. Let F be a component of $S \cap Y(v)$. By minimality of $\#S \cap P$, F must be essential. However, Y does not contain any essential 2–suborbifold whose boundary is nonempty and contained in $\partial_u Y(v)$ (this can be seen by embedding Y into a product orbifold and applying [6, Proposition 5].) This is a contradiction.

(2) Pick any vertex v of $\mathcal{T}$. Let $v = v_0, v_1, v_2, \dots, v_n$ be a path in $\mathcal{T}$ connecting v to the ancestor. We define a pillow $T(v) \subset \mathcal{O}_3$ as follows: Let $F \subset Y(v)$ be a disk with four cone points, with boundary in $\partial_u Y(v)$, and intersecting each of $\sigma_m^1, \sigma_m^2, \sigma_m^{1\prime}, \sigma_m^{2\prime}$ exactly once. Define inductively a family $\{A_i\}_{1 \le i \le n}$ such that:

(i) A_i is an essential nonsingular annulus in $Y(v_i)$;

(ii) One boundary component of A_1 is ∂F;

(iii) For every i, one boundary component of A_i is equal to some boundary component of A_{i+1};

(iv) One boundary component of A_n lies in $\partial_u Y(v_n)$.

Finally let D be a disk in X with ∂D equal to the other boundary component of A_n. Then $F \cup A_1 \cup \cdots \cup A_n \cup D$ is an embedded pillow in $\mathcal{O}_3$, which we denote by $T(v)$.

The four singular points of $T(v)$ belong to four distinct components L_1, L_2, L_3, L_4 of $\Sigma_{\mathcal{O}_3}$, which are properly embedded lines. There exists a properly embedded nonsingular annulus A which separates $\mathcal{O}_3$ in two components Z_1, Z_2 such that Z_1 contain the L_i's and is homeomorphic to the product of a disk with four cone points of order two with the real line. By the Seifert–van Kampen theorem, $\pi_1\mathcal{O}_3$ is an amalgamated product of $\pi_1 Z_1$ with $\pi_1 Z_2$ over $\pi_1 A$. The fundamental group of Z_1 can be expressed as the free product of four order two cyclic subgroups generated by meridians m_1, m_2, m_3, m_4 of L_1, L_2, L_3, L_4 respectively. Now $\partial_l Y(v)$ intersects A in a circle, which is essential on A, but bounds a nonsingular disk in $\overline{Z}_2$. Hence $\pi_1 A$, has trivial image in $\pi_1\mathcal{O}_3$. As a result, the image of $\pi_1 Z_1$ in $\pi_1\mathcal{O}_3$ is the quotient of $\pi_1 Z_1$ by the single relation $m_1 m_2 m_3 m_4 = 1$. This implies that $T(v)$ is π_1–injective, hence incompressible.

If $v \neq v'$, then up to exchanging v and v', we can find a line in $\Sigma_{\mathcal{O}_3}$ which intersects $T(v)$ transversally in a single point, and does not meet $T(v')$. As a consequence, $T(v)$ and $T(v')$ are not isotopic.

(3) Let T be an incompressible toric 2–suborbifold of $\mathcal{O}_3$. Then T could be a nonsingular torus, a pillow, or a Euclidean turnover. Any sphere in $|\mathcal{O}_3|$ which is transverse to $\Sigma_{\mathcal{O}_3}$ meets it in an even number of points, so $\mathcal{O}_3$ does not contain any turnover. Let T be a nonsingular torus in $\mathcal{O}_3$. Since T is compact, there is a compact suborbifold $Z \subset \mathcal{O}_3$ consisting of X and finitely many $Y(v)$'s such that $T \subset Z$. The orbifold Z is homeomorphic to S^3 minus a finite union of disjoint balls, with planar singular locus, so $\pi_1(|Z| \setminus \Sigma_Z)$ is a free group. Thus T is compressible in $|Z| \setminus \Sigma_Z$, hence in $\mathcal{O}_3$. As a consequence, all incompressible toric 2–suborbifolds of $\mathcal{O}_3$ are pillows.

Our next goal is to prove that the collection $\{T(v)\}_{v\in\mathcal{T}}$ actually contains *all* incompressible pillows up to isotopy. Indeed, let T be an incompressible pillow. Assume after isotopy that T intersects P minimally. Since X is atoroidal, T meets some $Y(v)$. Choose v_0 so that T meets $Y(v_0)$ and does not meet any $Y(v)$ with v below v_0. Then an argument similar to that used to prove assertion (1) shows that $T \cap Y(v_0)$ must consist of a disk D intersecting each of $\sigma_m^1, \sigma_m^2, {\sigma_m^1}', {\sigma_m^2}'$ in exactly one point. Let v_1 be the father of v_0. The intersection of T with $Y(v_1)$ must be an essential nonsingular annulus A, one of whose boundary components is ∂D.

The other component of ∂A cannot be on $\partial_l Y(v_1)$ or $\partial_r Y(v_1)$, for otherwise by carrying on the same argument, we would get a string of nonsingular annuli going down the tree, and we would never be able to close up. Hence the other component of ∂A lies on $\partial_u Y(v_1)$. We can repeat the argument until we arrive at Y_0. The upshot is that T is isotopic to $T(v_0)$. Finally, since the $T(v)$'s can be realized so as to be pairwise disjoint, it follows that they are all canonical. □

As before, we consider the homomorphism $\phi\colon \pi_1\mathcal{O}_3 \to \mathbf{Z}/2\mathbf{Z}$ which sends meridians to the generator. The corresponding regular cover M_3 is a good orbifold with torsion-free fundamental group, hence a manifold. We let $p\colon M_3 \to \mathcal{O}_3$ denote the covering map, and set $X' := p^{-1}(X)$, $P' := p^{-1}(P)$, and $Y'(v) := p^{-1}(Y(v))$ for all $v \in \mathcal{T}$.

Proposition 3.2 (1) M_3 *is irreducible;*

(2) *There is an infinite collection $\{T'(v)\}_{v\in\mathcal{T}}$ of pairwise nonisotopic canonical tori, all of which essentially intersect X'.*

Proof (1) follows from irreducibility of $\mathcal{O}_3$ and [14, Theorem 10.1].

To establish (2), we define $T'(v) = p^{-1}(T(v))$. This gives us an infinite collection of tori in M_3. If two of them were isotopic, then they would be homologous, and so would be the underlying spaces of their projections to $\mathcal{O}_3$, which is not the case. By the equivariant Dehn Lemma, each T'_v is incompressible.

Fix $v \in \mathcal{T}$. To show that $T'(v)$ is canonical, it suffices to prove that every incompressible annulus properly embedded in M_3 cut along $T'(v)$ is parallel to an annulus in $T'(v)$.

We hence need to deduce this from the corresponding property for $T(v) \subset \mathcal{O}_3$. This can probably done by elementary topology, but we prefer to use a direct minimal surface argument suggested by Peter Scott. Notice first that for all v, v', the pairs $(\mathcal{O}_3, T(v))$ and $(\mathcal{O}_3, T(v'))$ are homeomorphic. Passing to the double cover, this implies that $(M_3, T'(v))$ and $(M_3, T'(v'))$ are homeomorphic for all v, v'. Hence we only give the proof when v is the ancestor v_0.

Let Z be a component of M_3 cut along $T'(v_0)$, and let A be an incompressible annulus properly embedded in Z. Assume that A has been isotoped so as to intersect P' minimally.

If A intersects P' at all, then for some v one component of $A \cap Y'(v)$ is an essential annulus whose boundary is contained in a single component of $\partial Y'(v)$. However, we have the following lemma:

Lemma 3.3 *Every essential annulus in $Y'(v)$ meets two different components of $\partial Y'(v)$.*

Proof To shorten notation set $Y := Y(v)$ and $Y' := Y'(v)$. Put a sufficiently convex Riemannian metric on Y and give Y' the lifted metric. Assume the lemma is false, so that there is a component U of $\partial Y'$ and an essential, properly embedded annulus $A \subset Y'$ with both boundary components in U. By [8], one finds an annulus A_0 of least area with this property. Let A_0' be the translate of A_0 by the deck transformation group of the double cover $p\colon Y' \to Y$. If $A_0 = A_0'$ or $A_0 \cap A_0' = \varnothing$, then $p(A_0)$ is a properly embedded, essential, annular 2–suborbifold of Y with both boundary components in $p(U)$. This is impossible.

So generically A_0 and A_0' intersect in a finite family of curves and arcs. By standard exchange/roundoff arguments (cf [8; 14]) one obtains a contradiction. This proves Lemma 3.3. □

We return to the proof of Proposition 3.2. By Lemma 3.3, our annulus A, does not intersect P'. Hence it is contained in $X' \cap Z$. But this manifold does not contain any essential annulus with both boundary components in $X' \cap T'_{v_0}$, by an argument entirely similar to that used in the proof of Lemma 3.3. This contradiction completes the proof of Proposition 3.2. □

4 Just another brick in the wall

Let $\mathcal{O}_4$ be the orbifold whose underlying space is $\mathbf{R}^3$ and whose singular locus is the trivalent graph shown in Figure 2, where all edges should be labeled with the number 2.

A key property of the graph $\Sigma_{\mathcal{O}_4}$ is that it is planar. In particular, every properly embedded line in $\Sigma_{\mathcal{O}_4}$ is unknotted as a subset of $\mathbf{R}^3$. We choose an unknotted arc α connecting the two components of $\Sigma_{\mathcal{O}_4}$ as in Figure 3.

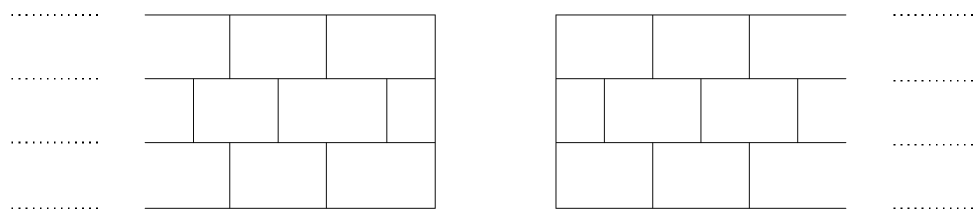

Figure 2: The singular locus of $\mathcal{O}_4$

Proposition 4.1 (1) *$\mathcal{O}_4$ is irreducible.*

(2) *All incompressible toric 2–suborbifolds are pillows, and intersect α.*

(3) *There are infinitely many isotopy classes of canonical pillows.*

Proof (1) Let S be a general position 2–sphere in $\mathbf{R}^3$. Let $B \subset \mathbf{R}^3$ be the 3–ball bounded by S. If S avoids $\Sigma_{\mathcal{O}_4}$, then so does B. Otherwise S intersects $\Sigma_{\mathcal{O}_4}$ in at least two points. If $\# S \cap \Sigma_{\mathcal{O}_4} = 2$, then the two intersection points lie on the same edge of $\Sigma_{\mathcal{O}_4}$ and B intersects $\Sigma_{\mathcal{O}_4}$ in an arc. This arc cannot be knotted inside B since it extends to a properly embedded singular line in $\Sigma_{\mathcal{O}_4}$, which cannot be knotted in $\mathbf{R}^3$ as remarked above. If $\# S \cap \Sigma_{\mathcal{O}_4} = 3$, then B intersects $\Sigma_{\mathcal{O}_4}$ in a Y-shaped graph, which must be unknotted for a similar reason. In each case, the 2–suborbifold whose underlying sphere is S is compressible.

(2) Recall (cf [2, Chapter 2]) that all Euclidean turnovers have at least one cone point of order different from 2. Hence $\mathcal{O}_4$ contains no Euclidean turnovers. Any nonsingular torus $T \subset \mathcal{O}_4$ lies in a 3–ball intersecting $\Sigma_{\mathcal{O}_4}$ in a planar graph. Hence one can find a handlebody in $|\mathcal{O}_4 \setminus \Sigma_{\mathcal{O}_4}|$ containing T. This shows that T is compressible in $|\mathcal{O}_4 \setminus \Sigma_{\mathcal{O}_4}|$, hence *a fortiori* in $\mathcal{O}_4$.

Let P be a pillow in $\mathcal{O}_4$. Assume that P misses α. Then there are two cases: either one can find two edges e_1, e_2 of $\Sigma_{\mathcal{O}_4}$ such that $P \cap \Sigma_{\mathcal{O}_4}$ consists of two points of e_1 and two points of e_2, or $|P|$ is the boundary of a regular neighborhood of an edge of $\Sigma_{\mathcal{O}_4}$. In either case, the compact 3–suborbifold bounded by P is a solid pillow (cf [2, Figure 5, p 33].) This implies that P is compressible.

(3) Let P_1 be the pillow depicted on Figure 3. Let X_1, X_2 be the 3–suborbifolds bounded by P_1. Then any 2–disk properly embedded in $|X_1|$ or $|X_2|$ intersects $\Sigma_{\mathcal{O}_4}$ in at least two points unless it is parallel rel $\Sigma_{\mathcal{O}_4}$ to some disk in P_1. Hence P_1 is incompressible. If there were an incompressible pillow P_2 meeting P_1 essentially, then after isotopy $X_1 \cap P_2$ and $X_2 \cap P_2$ would consist of essential annular 2–orbifolds with underlying space a 2–disk and two singular points. Now the only such annular suborbifolds are shown in Figure 3; their boundaries are not isotopic, hence they cannot

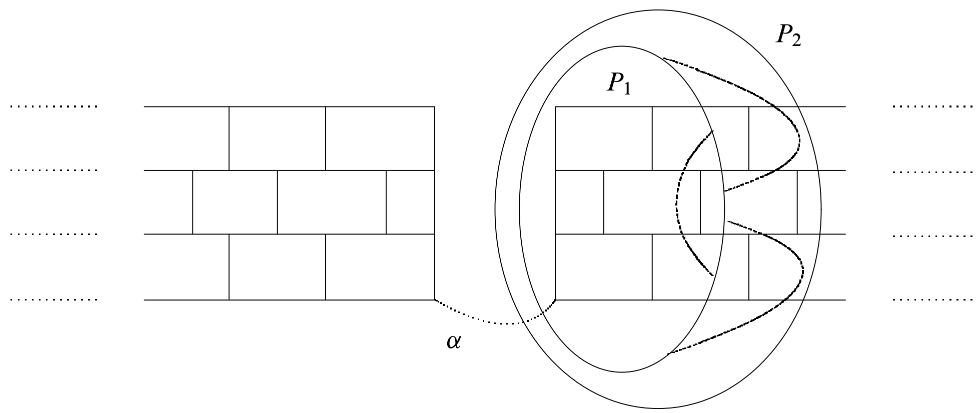

Figure 3: Two canonical pillows in $\mathcal{O}_4$

be glued together to give a pillow that would intersect P_1 essentially. This shows that P_1 is canonical.

The same argument works for the pillow P_2 on the same figure. It is easy to see that one can find in this way an infinite family of canonical pillows $P_1, P_2, P_3, \ldots$, where P_{n+1} is separated from P_n by three vertical bars. □

Remark This orbifold $\mathcal{O}_4$ also contains noncanonical incompressible pillows. For instance, let X be the 3–suborbifold bounded by $P_1 \cup P_2$ and observe that P_2 can be obtained from P_1 by moving to the right and "crossing" three vertical arcs in $\Sigma_{\mathcal{O}_4}$. If one "crosses" only one of these arcs, one obtains another incompressible pillow, consisting of an annular suborbifold of P_1 together with one of the dotted annular suborbifolds in X. There are two such pillows. With a little more work, one can show that X has a Seifert fibration where those two pillows are vertical, and their projections to the base orbifold intersect essentially. Hence by [3] those two pillows intersect essentially. In particular, they are noncanonical.

5 Jacob's nightmare

In order to motivate the example $\mathcal{O}_5$ constructed later in this section, we first consider an example of a 3–orbifold which we do not view as pathological. Its underlying space is $\mathbf{R}^3$ and its singular locus is as in Figure 4, all meridians having order 2. It consists of two connected trivalent graphs looking like bi-infinite ladders (which we shall call "Jacob ladders".)

The same arguments as for $\mathcal{O}_4$ show that this orbifold is irreducible, and that its only incompressible toric 2–suborbifolds are pillows. Furthermore, around each ladder one

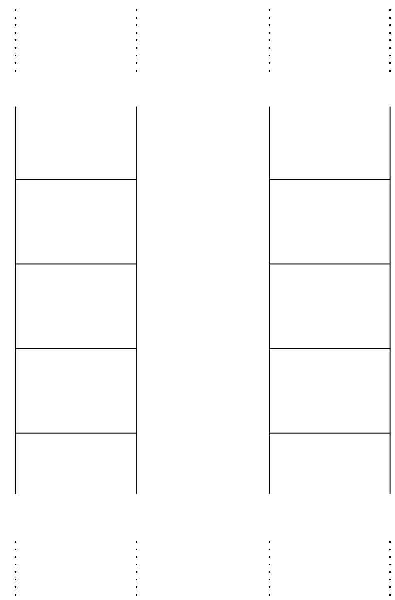

Figure 4: An orbifold with two Jacob ladders as singular locus

can find a properly embedded bi-infinite annulus that bounds a Seifert 3–suborbifold U: one chooses a vertical band Z containing the ladder and foliates $|Z|$ by intervals such that the rungs are leafs. Then those intervals, viewed as 1–suborbifolds, are mirrored intervals, which are fibers of the Seifert fibration on U, the other fibers being circles wrapped around Z (cf [2, p 33].) Thus the orbifold under consideration has a natural "JSJ splitting" consisting of two annuli.

Of course there could be more ladders, or even infinitely many of them. This we still do not consider pathological, since the corresponding infinite collection of annuli would be locally finite.

However the 3–orbifold $\mathcal{O}_5$ in Figure 4 fails to have such a decomposition. Its underlying space is again $\mathbf{R}^3$, and its singular locus consists of an infinite sequence of Jacob ladders plus another component J. The precise shape of J is unimportant; the only relevant feature is that it should not create unwanted incompressible 2–suborbifolds. For example, we can take it to be a planar "brickwall" graph similar to each component of $\Sigma_{\mathcal{O}_4}$, also with all labels equal to 2, but with *five* horizontal half-lines instead of four, so that there is an unknotted, properly embedded, nonsingular plane in $\mathcal{O}_5$ separating J from the rest of the singular locus, and having the property that the component containing J is irreducible and does not contain any toric 2–suborbifold.

Most important is the position of J with respect to the ladders: any finite set of ladders can be separated from J by a properly embedded nonsingular plane, but there is an arc α (see Figure 5) which connects J to L_0 and which can be extended to a properly embedded line $\Lambda \subset \mathbf{R}^3$ by adding a half-line in J and a half-line in L_0, such that for all negative n, one can find a circle $Q_n \subset L_n$ (consisting of four arcs: a subarc of each upright plus two rungs connecting them) such that the linking number of Q_n and Λ is 1.

Also, any two ladders are separated by a properly embedded nonsingular plane. For each ladder we assign to each rung an integer so that going up the ladder corresponds to increasing the numbers.

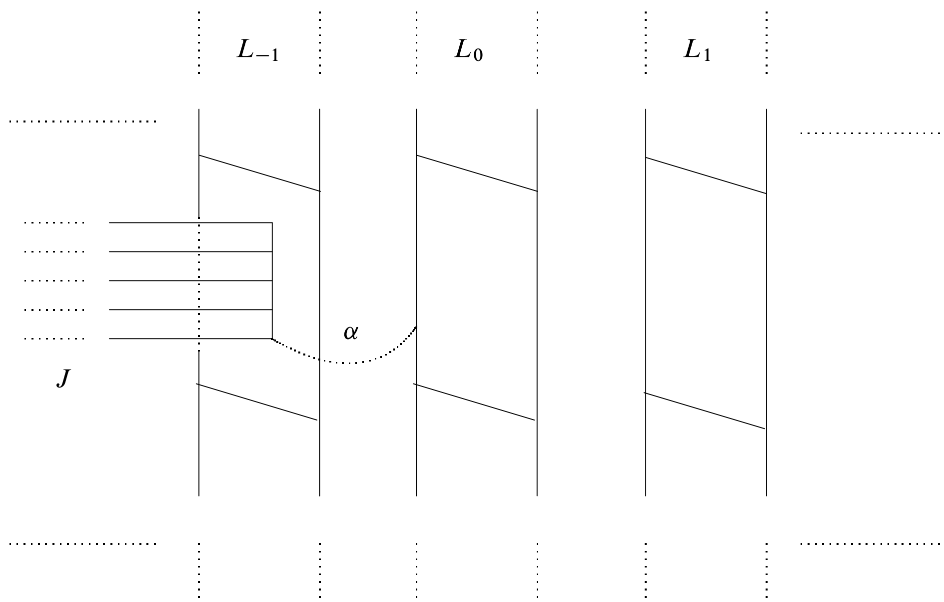

Figure 5: Jacob's nightmare

By arguments similar to those of Section 4, one shows that $\mathcal{O}_5$ is irreducible and that its only incompressible toric 2–suborbifolds are pillows, and meet a single component of the singular locus. By choice of J, this component must be a ladder. More precisely, each incompressible pillow is obtained in the following way: fix a ladder L_n, an integer $p \geq 2$ and a finite sequence of consecutive rungs $r_1 < \cdots < r_p$ of L_n. Then take a sphere S intersecting $\Sigma_{\mathcal{O}_5}$ only on L_n, and such that $S \cap L_n$ consists of four points on the uprights of L_n, two immediately below r_1 and two immediately above r_p. Observe that this pillow is not canonical, since for instance, the pillow associated to the sequence $(r_1 - 1), r_1$ will intersect it essentially. Thus $\mathcal{O}_5$ has no canonical toric 2–suborbifolds at all.

Further observe that for all $n < 0$, there is a pillow T_n associated to L_n such that the intersection number of T_n with α is 1. (In fact, there are infinitely many.) As a consequence, T_n intersects α *essentially*, ie any pillow isotopic to T_n still meets α.

Proposition 5.1 *There exists no Seifert 3–suborbifold $U \subset \mathcal{O}_5$ such that all incompressible pillows can be isotoped into U.*

This is a consequence from the following Claim:

Claim Let T, T' be two incompressible pillows associated to distinct ladders, and U be a Seifert 3–suborbifold containing $T \cup T'$. Then T, T' belong to distinct components of U.

Indeed, applying the Claim to the infinite family of pillows $\{T_n\}_{n<0}$ described above, we see that the compact set α would have to meet infinitely many distinct connected components of U, which is impossible.

Lastly, we prove the claim: if T and T' are contained in a connected Seifert suborbifold V, then they are vertical by [3, Theorem 4]. Therefore, there exist closed curves $c \subset T$ and $c' \subset T'$ such that c and c' are freely isotopic in V, hence freely isotopic in $\mathcal{O}_5$. Now since the ladders are unlinked, there exists a properly embedded nonsingular plane $P \subset \mathcal{O}_5$ separating T from T'. Hence by the Seifert–van Kampen theorem, $\pi_1\mathcal{O}_5$ can be expressed as a free product, with $\pi_1 T$ and $\pi_1 T'$ belonging to distinct factors. Hence the elements of $\pi_1\mathcal{O}_5$ represented by c and c' cannot be conjugate. This is a contradiction.

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