Model structures on the category of small double categories

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In this paper we obtain several model structures on **DblCat**, the category of small double categories. Our model structures have three sources. We first transfer across a categorification-nerve adjunction. Secondly, we view double categories as internal categories in **Cat** and take as our weak equivalences various internal equivalences defined via Grothendieck topologies. Thirdly, **DblCat** inherits a model structure as a category of algebras over a 2–monad. Some of these model structures coincide and the different points of view give us further results about cofibrant replacements and cofibrant objects. As part of this program we give explicit descriptions for and discuss properties of free double categories, quotient double categories, colimits of double categories, horizontal nerve and horizontal categorification.

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1 Introduction

The theory of categories enriched in **Cat**, called 2–categories, has been highly developed over the past 40 years and has found numerous applications. Beginning with Bénabou's bicategories (weak 2–categories) in [4], through Kelly's monograph [54] on enriched categories, and including the more recent companion by Lack [57], as well as many others, we have seen the $n = 2$ case for higher category theory become very well understood. Limits in 2–categories from Kelly [53], 2–monads on 2–categories from Blackwell–Kelly–Power [8] and Kan extensions for 2–functors from Dubuc [26] are now widely known. Model structures on **2–Cat** have also been studied recently by Lack [58; 59] and Worytkiewicz–Hess–Parent–Tonks [83]. Model structures, more generally, have been used in the study of $(\infty, 1)$–categories as a means of comparison by Bergner [5; 6; 7], Joyal–Tierney [52], Rezk [75] and Toën [81].

Recent examples, however, show that 2–categories are not enough and that one must invoke Ehresmann's earlier notion of *double* category [30; 31]. In many mathematical situations one is interested in two types of morphisms, which may or may not interact. Between rings, for example, there are ring homomorphisms as well as bimodules.

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Between manifolds there are differentiable maps and cobordisms, which are both used in field theory. Between categories there are functors as well as adjunctions. The notion of 2–category does not capture both types of morphisms, but the notion of (pseudo) double category certainly does.[1]

Concisely, a *small double category* is an internal category in **Cat**. A small double category consists of a set of objects, a set of horizontal morphisms, a set of vertical morphisms and a set of squares, equipped with various associative and unital compositions satisfying the interchange law. In addition to the early work of Bastiani–Ehresmann [2], Ehresmann–Ehresmann [27; 28; 29], C Ehresmann [30; 31] and Brown–Spencer [18], recent work on double categories has been completed by Brown and collaborators [15; 16; 17], Dawson–Paré [20; 21; 22], Dawson–Paré–Pronk [23], Fiore [34], Garner [38], Grandis–Paré [42; 43; 44; 45], Kock [56], Shulman [76; 77] and others.

Double categories are the $n = 2$ case for n–fold categories, which have been studied and applied for some time now. In the same way that *higher* categories may be defined by iterated enrichment, one may define *wider* categories or *n–fold categories* via iterated internalization. The edge symmetric[2] case has been studied by Brown and Higgins in the concept of *cubical ω–category* and by Grandis in the concept of *symmetric weak cubical categories* in [40] and [41]. Further, n–fold categories internal to the category of groups have been used to model connected homotopy $(n+1)$–types by Loday [63] as summarized in the survey paper by Paoli [70]. Recent work includes Lack–Paoli [61] and Paoli [71]. Applications of versions of the $n = 2$ case of internalized categories include Dawson–Paré–Pronk [25], Fiore [33; 34], Kerler–Lyubashenko [55], May–Sigurdsson [66], Morton [69] and Shulman [76; 77]. Thus, there has been a general trend towards n–fold categories, especially the $n = \omega$ and $n = 2$ cases.

In this article we introduce model categories into the theory of double categories, anticipating a utility in the theory of *wider categories* analogous to that of model structures in the theory of *higher categories*. Already in the $n = 2$ case we see that n–fold categories and n–categories diverge: even though the homotopy theory of 2–categories resembles that of categories, the homotopy theory of double categories is much richer. This results from the numerous ways to view a double category: as an internal category in **Cat**, as a categorical structure with two directions, as certain simplicial objects in **Cat**, as certain bisimplicial sets or as algebras over a 2–monad.

[1] These examples and others can be found in the article of Grandis–Paré [42]. There it is remarked in the introduction: "We conclude with some remarks about the motivation of this work. Its leitmotif can be summarised as follows: arrows which are too relaxed (like profunctors, spans, relations) or too strict (like adjunctions) to have limits, can be studied in a (pseudo) double category, correlating them with more ordinary (horizontal) arrows."

[2] *Edge symmetric* means that the n–morphisms in all $n+1$ directions are the same.

Each point of view suggests different notions of weak equivalence and fibration. The new types of pasting diagrams available in a double category also create new phenomena. We take these various points of view into consideration when constructing the model structures.

Thus, our model structures have three sources. First, we transfer the Thomason diagram structure and categorical diagram structure on the category of simplicial objects in **Cat** to **DblCat** via a horizontal categorification-horizontal nerve adjunction. In the Thomason structure on **Cat** in [80], a functor is a weak equivalence if and only if its nerve is a weak homotopy equivalence of simplicial sets. In the categorical structure on **Cat** of Joyal–Tierney [51] (reproved by Rezk in [74]), a functor is a weak equivalence if and only if it is an equivalence of categories. Both the Thomason structure and the categorical structure on **Cat** are cofibrantly generated, and thus induce cofibrantly generated model structures on simplicial objects in **Cat** where weak equivalences and fibrations are defined levelwise. We apply Kan's Lemma on Transfer of cofibrantly generated model structures (Theorem 7.11) to transfer both of these diagram structures to **DblCat** (Theorem 7.13 and Theorem 7.17). However, the application is not straightforward, and we must make several double categorical preparations, including horizontal categorification and a pushout formula in **DblCat**. We also prove one negative result in Theorem 7.22: it is impossible to transfer the Reedy categorical structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ to **DblCat**. The transfer from bisimplicial sets is treated by Fiore–Paoli in [35]. Even better, [35] contains a Thomason model structure on the category of small n–fold categories, and this model structure is Quillen equivalent to **SSet**.

We arrive at a second source for model structures on **DblCat** when we view double categories as internal categories in **Cat**. In this way we obtain double categorical versions of the categorical structure on **Cat**, where a functor is a weak equivalence if and only if it is fully faithful and essentially surjective. Although the notion of fully faithfulness makes sense internally, essential surjectivity does not, and therefore equivalences of internal categories need further explanation. Model structures on categories internal to a good category **C** have already been developed by Everaert–Kieboom–Van der Linden in [32], and we apply their results to the case $\mathbf{C} = \mathbf{Cat}$. They define essential surjectivity (and hence also weak equivalences) with respect to a Grothendieck topology $\mathcal{T}$ on $\mathbf{C}$. We take simplicially surjective functors and categorically surjective functors as bases for Grothendieck topologies on **Cat**, and obtain two distinct model structures in Sections 8.2 and 8.3. Additionally, we show in Theorem 8.52 that the model structure induced by the trivial topology coincides with the trivial model structure from the 2–category of internal categories.

Third, **DblCat** inherits a model structure as a category of algebras over a 2–monad. This is an application of Lack's algebra structure [60]. The underlying 1–category of a 2–category with finite limits and finite colimits always admits the so-called trivial model structure, whose weak equivalences are equivalences and fibrations are isofibrations. If $\mathcal{K}$ is a locally finitely presentable 2–category equipped with a 2–monad T with rank, then the category of (strict) T–algebras is a model category: a morphism of T–algebras is a weak equivalence or fibration if and only if its underlying morphism is a weak equivalence or fibration in the trivial model structure on $\mathcal{K}$. In our application of [60] in Section 9, $\mathcal{K}$ is the 2–category $\mathsf{Cat}(\mathbf{Graph})$ of internal categories in small nonreflexive graphs, and T is the 2–monad $\overline{M}$ induced by the Cartesian monad M on **Graph** whose algebras are categories.

Depending on the reader's experience, certain model structures will be of more interest than others. Simplicially-minded readers will no doubt find the transfers from $\mathbf{Cat}^{\Delta^{op}}$ most interesting, while categorically-minded readers may find the model structures on $\mathsf{Cat}(\mathbf{Cat})$ arising from [32] more interesting. Universal algebraists may find the third point of view most appealing, namely double categories as algebras for a 2–monad. Nevertheless, certain model structures can be defined from two points of view, and will thus be of interest to readers working in different fields.

In other words, we prove that some of these model structures coincide. The model structure obtained by transferring the categorical diagram structure across the vertical categorification-vertical nerve adjunction is the same as the model structure associated to the simplicially surjective topology on **Cat** (Corollary 8.29). The algebra structure is the same as the model structure associated to the categorically surjective topology on **Cat** (Theorem 9.1).

These two different constructions of the same model structures yield more refined information about cofibrant replacements and cofibrant objects. For example, the cofibrant objects in the algebra structure are known to be precisely the flexible algebras, but from the categorically-surjective-topology structure we see that the flexible double categories are precisely those with object category free on a graph (Corollary 9.4 and Remark 9.7). Such a description allows us to conclude that the flexible 2–categories of Lack [58] are indeed flexible algebras for a 2–monad. Lack's Theorem 4.8 (iv) in [58], which characterizes flexible 2–categories as those 2–categories with underlying 1–category a free category on a graph, now extends to double categories.

We also compare our model structures on **DblCat** with the analogous ones for **Cat** in Propositions 7.16, 7.20 and 8.56. The vertical embedding of **Cat** into **DblCat** preserves and reflects weak equivalences, fibrations and cofibrations from the Thomason structure into the transferred diagram Thomason structure, as well as from the categorical structure into the transferred diagram categorical structure. The horizontal embedding of **Cat** into **DblCat** preserves and reflects weak equivalences, fibrations and cofibrations from the categorical structure into the trivial structure. However, the vertical inclusion of **2–Cat** into **DblCat** preserves neither the weak equivalences nor the cofibrations of the categorical structure into the algebra structure, as shown at the end of Section 9. Nevertheless, a 2–category is cofibrant in **2–Cat** if and only if its vertical embedding into **DblCat** is cofibrant.

In order to build our model structures we prove various general results about double categories, so far not available in the literature. These results are also of independent interest for the theory of double categories in its own right. We develop free double categories, their quotients and colimits of double categories using a double categorical version of Street's 2–categorical notion of *derivation scheme* [78]. In particular we obtain an explicit formula for two pushouts of double categories in Theorem 10.6, which is essential for our application of Kan's Lemma on Transfer in Theorem 7.13 and Theorem 7.17. We also prove that the 2–categories $\mathbf{DblCat_v}$ and $\mathbf{DblCat_h}$ are 2–cocomplete in Theorem 4.2.

Free double categories on reflexive double graphs have been studied by Dawson–Paré in [22]. By *reflexive double graph* we mean a collection of objects, vertical edges, horizontal edges and squares equipped with source and target maps, identity edges and identity squares. In this paper, we will instead use double graphs with 1–identities. A *double graph with* 1*–identities* is like a reflexive double graph, except identity squares are not required. Between double graphs with 1–identities and double categories, there is the intermediate notion of double derivation scheme. A *double derivation scheme* is a double graph with 1–identities in which the horizontal and vertical reflexive 1–graphs are categories. In the free double category on a double derivation scheme, the vertical and horizontal 1–categories are preserved, but nontrivial squares consist of allowable compatible arrangements. Since we are considering compatible arrangements of squares in a double derivation scheme rather than in a double reflexive graph, our *allowable* compatible arrangements are different than the *composable* compatible arrangements of Dawson–Paré [21].

Free double categories on double derivation schemes and their quotients allow us to construct colimits of double categories. First one takes the colimits of the vertical and horizontal 1–categories. These, together with the colimit of the *sets* of squares, form a double derivation scheme. Finally, we mod out the free double category on this double derivation scheme by the smallest congruence which guarantees that the natural maps are double functors, and the result is the colimit in **DblCat**. This colimit formula is the basis of Theorem 10.6 which gives an explicit description of the pushouts of a double functor along two inclusions of external products. This theorem is crucial for our application of Kan's Lemma on Transfer. These two pushouts are special cases of a more general theorem on pushouts along inclusions of external products, which will appear in a separate paper with a comparison to Dawson–Paré–Pronk [24].

Free double categories on double derivation schemes and their quotients find further application in the construction of fundamental double categories of simplicial objects in **Cat**, that is, in our construction of a left adjoint to the horizontal nerve. We obtain an important example of our explicit constructions of fundamental double categories in a second way as well, namely via weighted colimits (see Example 6.6 and Proposition 6.11).

We begin in Section 2 with a review of double categories, including horizontal 2–categories, vertical 2–categories, double functors, horizontal and vertical natural transformations, the external product of 2–categories and Cartesian closedness of the category **DblCat**, as well as the 2–categories $\mathbf{DblCat_v}$ and $\mathbf{DblCat_h}$. Free double categories on double derivation schemes are introduced in Section 3 and are used in Section 4 to describe colimits in **DblCat**. The horizontal nerve is discussed in Section 5 along with its representable definition in terms of external products of finite ordinals. In Section 6, free double categories on double derivation schemes and their quotients are applied to construct the left adjoint to the horizontal nerve. Section 7 focuses on transferring model structures across the horizontal categorification-horizontal nerve adjunction and recalls model structures on **Cat**, smallness issues and Kan's Lemma on Transfer. Section 8 begins with an exposition of the methods of Everaert–Kieboom–Van der Linden [32], and then applies them to obtain model structures on $\mathsf{Cat}(\mathbf{Cat}) = \mathbf{DblCat}$ induced by three Grothendieck topologies on **Cat**: the simplicially surjective topology, the categorically surjective topology, and the trivial topology. The model structure induced by the simplicially surjective topology coincides with the transfer of the diagram categorical structure across the adjunction $c_v \dashv N_v$. In Section 9 we prove that the algebra structure on **DblCat** coincides with the model structure induced by the categorically surjective topology. In Section 10, the Appendix, we obtain an explicit description of certain pushouts in **DblCat**, namely Theorem 10.6. We use this to characterize the behavior of the horizontal nerve on such pushouts in Theorem 10.7. The essential application is to pushouts along the generating acyclic cofibrations in the transfer in Section 7.

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2 Double categories

We first recall the elementary notions of double category theory. In many mathematical contexts there are two interesting types of morphisms; double categories organize them into one structure. For example, between rings there are morphisms of rings as well as bimodules, between objects of any 2–category there are morphisms as well as adjunctions, and so on. Sometimes one would like to distinguish a family of squares, such as the pullback squares among the commutative squares, and double categories are also of use here. The notion of double category is not new, and goes back to Ehresmann [30; 31].

Definition 2.1 A *small double category* $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ is a category object in **Cat**, the category of small categories. This means that $\mathbb{D}_0$ and $\mathbb{D}_1$ are small categories equipped with functors

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "m"] & \mathbb{D}_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathbb{D}_0 \arrow[l, "u"'] \end{tikzcd}$$

that satisfy the usual axioms of a category. We call the objects and morphisms of $\mathbb{D}_0$ respectively the *objects* and *vertical morphisms* of $\mathbb{D}$, and we call the objects and morphisms of $\mathbb{D}_1$ respectively the *horizontal morphisms* and *squares* of $\mathbb{D}$.

When one expands this definition, one sees that a small double category consists of a set of *objects*, a set of *horizontal morphisms*, a set of *vertical morphisms*, and a set of *squares* equipped with various sources, targets, and associative and unital compositions. Further, the horizontal and vertical compositions of squares are required to satisfy the interchange law. Since we only deal with small categories and small double categories, we will usually leave off the adjective *small*. Sources and targets are indicated as follows.

$$\text{(1)} \qquad A \xrightarrow{f} B \qquad \begin{tikzcd} A \arrow[d, "j"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

We denote the set of squares with the boundary

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

by
$$\mathbb{D}\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix}.$$

Then one has the categories

$$(\mathrm{Obj}\,\mathbb{D}, \mathrm{Hor}\,\mathbb{D}) \text{ and } (\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D})$$

under horizontal composition and the categories

$$(\mathrm{Obj}\,\mathbb{D}, \mathrm{Ver}\,\mathbb{D}) \text{ and } (\mathrm{Hor}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D})$$

under vertical composition. We will write $[f\ g]$ for the horizontal composition of horizontal morphisms f and g, and similarly $[\alpha\ \beta]$ for the horizontal composition of squares α and β. We will write $\begin{bmatrix} v \\ w \end{bmatrix}$ for the vertical composition of vertical morphisms v and w, and similarly $\begin{bmatrix} \gamma \\ \delta \end{bmatrix}$ for the vertical composition of squares γ and δ.

There are many examples of double categories. The commutative squares in a given 1–category form the squares of a double category. More generally, for a 2–category $\mathbf{C}$, Bastiani–Ehresmann defined the double category $\mathbb{Q}\mathbf{C}$ of *quintets of* $\mathbf{C}$ (pages 272 and 273 of [2]). The objects of $\mathbb{Q}\mathbf{C}$ are the objects of $\mathbf{C}$, horizontal and vertical morphisms are the morphisms of $\mathbf{C}$, and the squares α as in (1) are the 2–cells $\alpha\colon k \circ f \Longrightarrow g \circ j$. In many situations, one has examples of a slightly more general notion called *pseudo double category*, defined briefly in [2] on page 288 as the notion of *dicategory*, and defined in detail by Grandis–Paré in Section 7 of [42] . A pseudo double category is like a double category, except one direction is a bicategory (weak 2–category) rather than a 2–category. For example, the double category of rings, bimodules, ring homomorphism, and twisted maps of bimodules is weak in one direction. Another example is given by finite sets, Riemann surfaces with labelled analytically parametrized boundary components, bijections of finite sets, and holomorphic maps preserving the given structure. In these two examples we choose the horizontal direction to be weak, so that bimodules respectively Riemann surfaces are the horizontal morphisms. In this paper we work only with strict double categories, though pseudo double categories can also fit into our framework.

The notion of double category contains many familiar structures. If we view a category as an internal category in **Cat** with object and morphism categories discrete, it is equivalent to viewing an ordinary category as a double category with trivial vertical morphisms and trivial squares. Every 2–category $\mathbf{C}$ can be considered a double category in at least four ways: as a double category $\mathbb{H}\mathbf{C}$ with trivial vertical morphisms, as a double category $\mathbb{V}\mathbf{C}$ with trivial horizontal morphisms, as Ehresmann's quintets $\mathbb{Q}\mathbf{C}$, or as the transpose of Ehresmann's quintets $(\mathbb{Q}\mathbf{C})^t$. Any double category $\mathbb{D}$ has an *underlying horizontal* 2*–category* $\mathbf{H}\mathbb{D}$ and an *underlying vertical* 2*–category* $\mathbf{V}\mathbb{D}$: we obtain these substructures as the full subdouble categories with only trivial vertical morphisms or trivial horizontal morphisms respectively. We denote the underlying 1–categories of $\mathbf{H}\mathbb{D}$ and $\mathbf{V}\mathbb{D}$ by $(\mathbf{H}\mathbb{D})_0$ and $(\mathbf{V}\mathbb{D})_0$ respectively. The subscript 0 here means underlying 1–category of a 2–category, and is unrelated to the subscript 0 in Definition 2.1. Though the formula $(\mathbf{V}\mathbb{D})_0 = \mathbb{D}_0$ holds, $(\mathbf{H}\mathbb{D})_0$ is *not* the same as $\mathbb{D}_0$.

Definition 2.2 A *double functor* $F\colon \mathbb{D} \longrightarrow \mathbb{E}$ is an internal functor in **Cat**. Such a functor consists of functions

$$\begin{aligned}
\mathrm{Obj}\ \mathbb{D} &\longrightarrow \mathrm{Obj}\ \mathbb{E}\\
\mathrm{Hor}\ \mathbb{D} &\longrightarrow \mathrm{Hor}\ \mathbb{E}\\
\mathrm{Ver}\ \mathbb{D} &\longrightarrow \mathrm{Ver}\ \mathbb{E}\\
\mathrm{Sq}\ \mathbb{D} &\longrightarrow \mathrm{Sq}\ \mathbb{E}
\end{aligned}$$

which preserve all sources, targets, compositions, and identities.

Internal natural transformations in **Cat** are also called *horizontal natural transformations*. We recall the notion from pages 249 and 250 of Bastiani–Ehresmann's article [2], or 1.4 of Grandis–Paré's article [42].

Definition 2.3 If $F, G\colon \mathbb{D} \longrightarrow \mathbb{E}$ are double functors, then a *horizontal natural transformation* $\theta\colon F \Longrightarrow G$ as in [42] assigns to each object A a horizontal morphism $\theta A\colon FA \longrightarrow GA$ and assigns to each vertical morphism j a square

$$\begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\ FC \arrow[r, "\theta C"'] & GC \end{tikzcd}$$

such that:

(1) For all $A \in \mathbb{D}$, we have $\theta 1^v_A = i^v_{\theta A}$.

(2) For composable vertical morphisms j and k,

$$\begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "F\left[\substack{j\\k}\right]"'] \arrow[dr, phantom, "\theta\left[\substack{j\\k}\right]"] & GA \arrow[d, "G\left[\substack{j\\k}\right]"] \\ FE \arrow[r, "\theta E"'] & GE \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[d, "Gj"] \\ FC \arrow[r, "\theta C"] \arrow[d, "Fk"'] \arrow[dr, phantom, "\theta k"] & GC \arrow[d, "Gk"] \\ FE \arrow[r, "\theta E"'] & GE. \end{tikzcd}$$

(3) For all α as in Diagram (1),

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "Fj"'] \arrow[dr, phantom, "F\alpha"] & FB \arrow[r, "\theta B"] \arrow[d, "Fk"] \arrow[dr, phantom, "\theta k"] & GB \arrow[d, "Gk"] \\ FC \arrow[r, "Fg"'] & FC \arrow[r, "\theta C"'] & GD \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\theta A"] \arrow[d, "Fj"'] \arrow[dr, phantom, "\theta j"] & GA \arrow[r, "Gf"] \arrow[d, "Gj"] \arrow[dr, phantom, "G\alpha"] & GB \arrow[d, "Gk"] \\ FC \arrow[r, "\theta C"'] & GC \arrow[r, "Gg"'] & GD. \end{tikzcd}$$

We also need the analogous notion of vertical natural transformation.

Definition 2.4 If $F, G\colon \mathbb{D} \longrightarrow \mathbb{E}$ are double functors, then a *vertical natural transformation* $\sigma\colon F \Longrightarrow G$ as in [42] assigns to each object A a vertical morphism $\sigma A\colon FA \longrightarrow GA$ and assigns to each horizontal morphism f a square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\sigma A"'] \arrow[dr, phantom, "\sigma f"] & FB \arrow[d, "\sigma B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

such that:

(1) For all objects $A \in \mathbb{D}$, we have $\sigma 1^h_A = i^h_{\sigma A}$.

(2) For all composable horizontal morphisms f and g,

$$\sigma[f \;\; g] = [\sigma f \;\; \sigma g].$$

(3) For all α as in Diagram (1),

$$\begin{bmatrix} F\alpha \\ \sigma g \end{bmatrix} = \begin{bmatrix} \sigma f \\ G\alpha \end{bmatrix}.$$

Thus, double categories form a 2–category in two different ways, depending on the choice of 2–cell. Further, there are useful adjunctions with **2–Cat** and **Cat**. Our notation, introduced before Definition 2.2, follows that found in Section 1.3 of [42].

Proposition 2.5 (Compare page 80 of [29].) *Let* $\mathbf{DblCat_h}$ *respectively* $\mathbf{DblCat_v}$ *denote the* 2*–categories of small double categories, double functors, and horizontal natural transformations respectively vertical natural transformations. Let* **2–Cat** *denote the* 2*–category of small* 2*–categories,* 2*–functors, and* 2*–natural transformations.*[3] *Then the inclusion* 2*–functors*

$$\mathbb{H}\colon \mathbf{2\text{–}Cat} \longrightarrow \mathbf{DblCat_h}$$

$$\mathbb{V}\colon \mathbf{2\text{–}Cat} \longrightarrow \mathbf{DblCat_v}$$

have as right 2*–adjoints the* 2*–functors*

$$\mathbf{H}\colon \mathbf{DblCat_h} \longrightarrow \mathbf{2\text{–}Cat}$$

$$\mathbf{V}\colon \mathbf{DblCat_v} \longrightarrow \mathbf{2\text{–}Cat}$$

[3] In this article we follow the convention that 2–functors and 2–natural transformations are *strict* 2–functors and *strict* 2–natural transformations.

respectively. Moreover, the inclusion 2*–functors*

$$\mathbb{H}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat_h}$$

$$\mathbb{V}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat_v}$$

have as right 2*–adjoints the* 2*–functors*

$$(\mathbf{H}\text{-})_0\colon \mathbf{DblCat_h} \longrightarrow \mathbf{Cat}$$

$$(\mathbf{V}\text{-})_0\colon \mathbf{DblCat_v} \longrightarrow \mathbf{Cat}$$

respectively.

The *external product* of 1–categories was defined by Bastiani–Ehresmann on page 251 of [2], and called there the *square product*. We extend this notion to 2–categories in Definition 2.6, and even double categories in Remark 2.8.

Definition 2.6 If $\mathbf{C}$ and $\mathbf{D}$ are 2–categories, then their *external product* $\mathbf{C} \boxtimes \mathbf{D}$ is the double category with objects $\operatorname{Obj}\mathbf{C} \times \operatorname{Obj}\mathbf{D}$,

vertical morphisms $\quad (j, D)\colon (C, D) \longrightarrow (C', D)$,

horizontal morphisms $\quad (C, f)\colon (C, D) \longrightarrow (C, D')$,

and squares

$$\begin{tikzcd}
(C, D) \arrow[r, "{(C,f)}"] \arrow[d, "{(j,D)}"'] \arrow[dr, phantom, "\alpha"] & (C, D') \arrow[d, "{(k,D')}"] \\
(C', D) \arrow[r, "{(C',g)}"'] & (C', D')
\end{tikzcd}$$

given by pairs $\alpha = (\gamma, \delta)$ of 2–cells $\gamma\colon j \Longrightarrow k$ and $\delta\colon f \Longrightarrow g$ in $\mathbf{C}$ and $\mathbf{D}$ respectively.

We may simplify the foregoing definitions using the operation of transposition, which interchanges the roles of horizontal and vertical.

Definition 2.7 The *transpose* of a double category $\mathbb{D}$ is the double category $\mathbb{D}^t$ with

$$\operatorname{Obj}\mathbb{D}^t = \operatorname{Obj}\mathbb{D}$$

$$\operatorname{Hor}\mathbb{D}^t = \operatorname{Ver}\mathbb{D}$$

$$\operatorname{Ver}\mathbb{D}^t = \operatorname{Hor}\mathbb{D}$$

$$\mathbb{D}^t\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix} = \mathbb{D}\begin{pmatrix} & j & \\ f & & g \\ & k & \end{pmatrix}$$

and the expected compositions and units. Transposition defines 2–functors

$$(\text{-})^t\colon \mathbf{DblCat_h} \longrightarrow \mathbf{DblCat_v}$$

$$(\text{-})^t\colon \mathbf{DblCat_v} \longrightarrow \mathbf{DblCat_h}$$

that are mutually inverse.

Remark 2.8 A vertical natural transformation $\sigma\colon F \Longrightarrow G$ is a horizontal natural transformation $\sigma^t\colon F^t \Longrightarrow G^t$. The transpose of the 2–adjunction $\mathbb{H} \dashv \mathbf{H}$ is the 2–adjunction $\mathbb{V} \dashv \mathbf{V}$. The external product of 2–categories $\mathbf{C}$ and $\mathbf{D}$ is $\mathbf{C} \boxtimes \mathbf{D} = \mathbb{V}\mathbf{C} \times \mathbb{H}\mathbf{D} = (\mathbb{H}\mathbf{C})^t \times \mathbb{H}\mathbf{D}$. More generally, the *external product* of double categories $\mathbb{C}$ and $\mathbb{D}$ is $\mathbb{C} \boxtimes \mathbb{D} := \mathbb{C}^t \times \mathbb{D}$.

Lemma 2.9 *The external product of* 2*–categories is a functor*

$$\boxtimes\colon \mathbf{2\text{–}Cat} \times \mathbf{2\text{–}Cat} \longrightarrow \mathbf{DblCat}.$$

Proof Transpose is functorial. □

Example 2.10 Let $[m]$ denote the partially ordered set $\{0, 1, 2, \ldots, m\}$. Then the double category $[m] \boxtimes [n]$ has the shape

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

with m rows and n columns of squares.

We round off this section with a discussion of Cartesian closedness for **DblCat**, $\mathbf{DblCat_v}$, and $\mathbf{DblCat_h}$.

Proposition 2.11 ($n = 2$ case of Ehresmann–Ehresmann [28]) *The category* **DblCat** *is Cartesian closed. In other words for each* $\mathbb{D}$ *there is an endofunctor* $(\text{-})^{\mathbb{D}}$ *of* **DblCat** *and a bijection of sets*

$$\mathbf{DblCat}(\mathbb{C} \times \mathbb{D}, \mathbb{E}) \cong \mathbf{DblCat}\big(\mathbb{C}, \mathbb{E}^{\mathbb{D}}\big) \tag{2}$$

natural in $\mathbb{C}$ and $\mathbb{E}$.

Corollary 2.12 (pages 423–425 of [28], 1.6 of [42]) *The objects of $\mathbb{E}^{\mathbb{D}}$ are double functors $\mathbb{D} \longrightarrow \mathbb{E}$, horizontal morphisms are horizontal natural transformations, vertical morphisms are vertical natural transformations, and squares are modifications.*

Proof In Equation (2), we take $\mathbb{C}$ to be the terminal double category, $\mathbb{H}[1]$, $\mathbb{V}[1]$, or $[1] \boxtimes [1]$. See 1.6 of [42] for the definition of modification. $\square$

Proposition 2.13 *The 2–category $\mathbf{DblCat_v}$ is Cartesian closed. More precisely, the functor $(\text{-})^{\mathbb{D}}$ of Proposition 2.11 and Corollary 2.12 extends to an endo–2–functor of $\mathbf{DblCat_v}$, and there is an isomorphism of categories*

$$\mathbf{DblCat_v}(\mathbb{C} \times \mathbb{D}, \mathbb{E}) \cong \mathbf{DblCat_v}\big(\mathbb{C}, \mathbb{E}^{\mathbb{D}}\big) \tag{3}$$

2–natural in $\mathbb{C}$ and $\mathbb{E}$. Similarly, the 2–category $\mathbf{DblCat_h}$ is Cartesian closed.

Proof Equation (2) is the object part of the isomorphism in Equation (3). For the bijection of morphism sets we have

$$\begin{aligned} \operatorname{Mor} \mathbf{DblCat_v}(\mathbb{C} \times \mathbb{D}, \mathbb{E}) &\cong \mathbf{DblCat}(\mathbb{V}[1] \times \mathbb{C} \times \mathbb{D}, \mathbb{E}) \\ &\cong \mathbf{DblCat}\big(\mathbb{V}[1] \times \mathbb{C}, \mathbb{E}^{\mathbb{D}}\big) \\ &\cong \operatorname{Mor} \mathbf{DblCat_v}\big(\mathbb{C}, \mathbb{E}^{\mathbb{D}}\big). \end{aligned}$$

The proof that this isomorphism of graphs is a 2–natural functor is similar to the analogous proof of the Cartesian closedness of $\mathbf{Cat}$. $\square$

Corollary 2.14 *For a small category $\mathbf{C}$ and small double categories $\mathbb{D}$ and $\mathbb{E}$, we have an isomorphism of categories*

$$\mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{D}, \mathbb{E}) \cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{D}, \mathbb{E}))$$

2–natural in $\mathbf{C}$ and $\mathbb{E}$.

Proof From Equation (3) and the 2–adjunction $\mathbb{V} \dashv (\mathbf{V}\text{-})_0$ of Proposition 2.5 we have a 2–natural isomorphism of categories

$$\begin{aligned} \mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{D}, \mathbb{E}) &\cong \mathbf{DblCat_v}\big(\mathbb{V}\mathbf{C}, \mathbb{E}^{\mathbb{D}}\big) \\ &\cong \mathbf{Cat}\Big(\mathbf{C}, \big(\mathbf{V}(\mathbb{E}^{\mathbb{D}})\big)_0\Big) \\ &\cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{D}, \mathbb{E})). \end{aligned} \qquad \square$$

3 Free double categories and quotients

As expected, there is a notion of free double category and quotient double category. However, the situation is richer than for ordinary categories, as there is an intermediate step between double categories and double graphs, which we call *double derivation schemes*. Double derivation schemes and quotients are crucial in the explicit description of colimits in Section 4, the construction of a left adjoint to horizontal nerve in Section 6, and the computation of pushouts in Theorem 10.6 and Theorem 10.7.

In this section we introduce double analogues to some of Street's concepts in [78]. The special kind of double graphs we will work with have 1–identities but are not equipped with identity squares. This is important because nontrivial squares in a double category may very well have one or more trivial edges. Recall that a *reflexive* graph is a graph equipped with a distinguished *identity* edge $1_A\colon A \longrightarrow A$ for each vertex A. All graphs in this paper are directed and small, so we often leave off the adjectives *directed* and *small*.

Definition 3.1 A *double graph* $\mathbb{A}$ is an internal graph in the category of small graphs. This consists of a set of vertices (objects) $\operatorname{Obj}\mathbb{A}$, a set of horizontal edges $\operatorname{Hor}\mathbb{A}$, a set of vertical edges $\operatorname{Ver}\mathbb{A}$, and a set of squares $\operatorname{Sq}\mathbb{A}$ equipped with source and target maps as in Diagram (1). A *morphism of double graphs* is a morphism of internal graphs in the category of small graphs, or equivalently, a map which preserves the sources and targets of Diagram (1). We denote the horizontal and vertical 2-graphs of a double graph $\mathbb{A}$ by $\mathbf{H}\mathbb{A}$ and $\mathbf{V}\mathbb{A}$.

Definition 3.2 A *double graph with* 1*–identities* is a double graph in which the horizontal and vertical 1–graphs are reflexive graphs. This means for each object A, there is a distinguished horizontal edge $1^h_A\colon A \longrightarrow A$ as well as a distinguished vertical edge $1^v_A\colon A \longrightarrow A$. There are no distinguished squares. A *morphism of double graphs with* 1*–identities* is a morphism of double graphs which preserves the distinguished edges. Double graphs with 1–identities form a category which we denote by **DblGr1-Id**.

A double graph with 1–identities is a double category without any of the compositions and without identity squares. The intermediate structure between double graphs with 1–identities and double categories is analogous to Street's notion of *derivation scheme* in [78].

Definition 3.3 A *double derivation scheme* is a double graph with 1–identities whose vertical reflexive 1–graph and horizontal reflexive 1–graph are categories. A *morphism* *of double derivation schemes* is a morphism of double graphs with 1–identities which is a functor on both the horizontal and vertical 1–categories. Double derivation schemes form a category which we denote by **DblDerSch**. We denote the horizontal and vertical derivation schemes of a double derivation scheme $\mathbb{S}$ by $\mathbf{H}\mathbb{S}$ and $\mathbf{V}\mathbb{S}$, and their underlying categories by $(\mathbf{H}\mathbb{S})_0$ and $(\mathbf{V}\mathbb{S})_0$.

To take a free category on a reflexive graph, one merely takes paths of composable edges and identifies paths which differ only by insertion or deletion of identity edges. However, the 2–dimensional situation is more subtle, as evidenced by Dawson–Paré [22], Johnson [49] and Power [72; 73]. Thus, in the construction of a free double category we need a careful definition of *allowable compatible arrangement*. We use the notion of compatible arrangement from Dawson–Paré [21], and develop it further for our purposes.

Definition 3.4 In a double derivation scheme $\mathbb{S}$, a *compatible arrangement* consists of a subdivision of a rectangle into smaller rectangles and a function which assigns to each vertex an object, to each horizontal line segment a horizontal morphism, to each vertical line segment a vertical morphism, and to each constituent rectangle a square in $\mathbb{S}$, which are compatible in the following sense:

(1) For each horizontal edge in the subdivision, the domain and codomain respectively of the morphism assigned to it are the objects assigned to the left and right vertices respectively.

(2) For each vertical edge in the subdivision, the domain and codomain respectively of the morphism assigned to it are the objects assigned to the top and bottom vertices respectively.

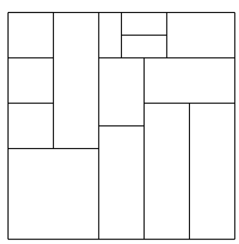
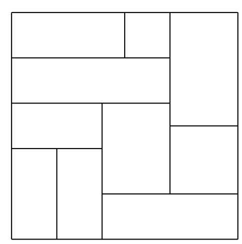
(3) For each constituent rectangle the composition of the morphisms assigned to the edges on
   (a) the left side is the horizontal domain of the square assigned to it;
   (b) the right side is the horizontal codomain of the square assigned to it;
   (c) the top is the vertical domain of the square assigned to it;
   (d) the bottom is the vertical codomain of the square assigned to it.

In the free double category on a double derivation scheme, a square is a compatible arrangement for which the image under any morphism of double derivation schemes into any double category becomes composable to a single square by a sequence of horizontal and vertical compositions. We will call such compatible arrangements *allowable*. However, an image of a compatible arrangement is just a compatible arrangement in the target double category with the same underlying subdivision of the rectangle. So whether a compatible arrangement is allowable in the free double category depends only on its shape, ie, the underlying subdivision of the rectangle.

A *horizontal* (respectively *vertical*) *cut* in a compatible arrangement is a horizontal (respectively vertical) line segment which consists of edges of the underlying subdivision of the rectangle. A horizontal (respectively vertical) cut is *full length* if it stretches from the left (respectively top) edge of the arrangement to the right (respectively bottom) edge of the arrangement. We can use this notion to characterize when a compatible arrangement is allowable.

Definition 3.5 A subdivision of a rectangle is *allowable* if it is either the trivial subdivision, consisting of just the rectangle itself, or contains a full length horizontal or vertical cut which divides it into two allowable subdivisions. A compatible arrangement is *allowable* if its underlying subdivision of the rectangle is allowable.

As an illustration, consider the following two examples of subdivisions of a rectangle.

Allowable:  Not allowable: 

Note that our notion of *allowable* compatible arrangement differs from Dawson–Paré's notion of *composable* compatible arrangement [21] in that a compatible arrangement in a double category $\mathbb{D}$ is *composable* if it is composable to a single square through the use of compositions in $\mathbb{D}$ *and factorizations*. So their notion depends on the ambient double category, not only on the shape of the arrangement. Any allowable compatible arrangement in our sense is composable in the sense of Dawson–Paré.

Proposition 3.6 *A compatible arrangement in a double category is allowable if and only if it can be composed to a single square by a sequence of horizontal and vertical compositions.*

Proof We argue by induction on the number of squares in the arrangement. The statement is trivially true for arrangements consisting of a single square. Now let CA be a compatible arrangement consisting of two or more squares, with an assignment into a double category $\mathbb{D}$ which is composable by a sequence of horizontal and vertical compositions of squares. Consider the last composition used. Without loss of generality, assume that this is a horizontal composition of two squares γ_1 and γ_2 along a vertical morphism v, as below.

$$\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma_1"] & {} \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\gamma_2"] & {} \arrow[d] \\
{} \arrow[r] & {} \arrow[r] & {}
\end{tikzcd}$$

Both γ_1 and γ_2 have been obtained by sequences of horizontal and vertical compositions of squares in CA, so v is a vertical composition of vertical morphisms $v_1, \ldots, v_n$ in CA. The underlying edges of these vertical morphisms form a full length cut in the underlying subdivision of the rectangle for CA. The squares on the left side of this cut form a compatible arrangement, since they form a rectangular subset of a compatible arrangement. Call this arrangement CA_1. It can be composed to γ_1 by a subsequence of the horizontal vertical compositions used for CA. In the same way, the squares on the right side of this cut form a compatible arrangement CA_2 which can be composed to γ_2 by a sequence of horizontal and vertical compositions. Since both CA_1 and CA_2 contain strictly less squares than CA, the induction hypothesis gives that they are both allowable compatible arrangements.

Conversely, suppose that a compatible arrangement CA of two or more squares in a double category $\mathbb{D}$ is allowable. Then it contains a horizontal (respectively vertical) cut into two allowable compatible arrangements CA_1 and CA_2. By induction these arrangements can be composed to single squares in $\mathbb{D}$ by sequences of horizontal and vertical compositions. Now consider the sequence of horizontal and vertical compositions used for CA_1 followed by the one for CA_2 and then one final vertical (respectively horizontal) composition along the cut. This shows that CA is composable to a single square in $\mathbb{D}$ by a sequence of horizontal and vertical compositions of squares. □

For inductive arguments on the number of squares in an allowable compatible arrangement, we need to know that cutting an allowable arrangement along *any* full length cut produces two smaller allowable compatible arrangements.

Proposition 3.7 *If CA is a compatible arrangement which is allowable, then any full length cut divides the arrangement into two allowable compatible arrangements.*

Proof We prove this by induction on the number of squares in the arrangement. It is obviously true for compatible arrangements consisting of a single square. For an arrangement consisting of $n \geq 2$ squares, let $\mathcal{C}_1$ be an arbitrary full length cut as in this proposition and let $\mathcal{C}_2$ be the full length cut used to establish that CA is allowable. Assume without loss of generality that $\mathcal{C}_2$ is horizontal. Let CA_1 and CA_2 be the

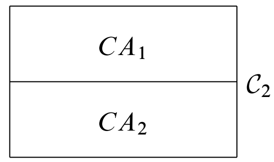


Figure 1: CA cut along $\mathcal{C}_2$

compatible arrangements obtained by cutting CA along $\mathcal{C}_2$, as in Figure 1. Note that both of the arrangements CA_1 and CA_2 are allowable and contain strictly less than n squares.

If $\mathcal{C}_1$ is vertical, the cut $\mathcal{C}_1$ itself gets divided by $\mathcal{C}_2$ into two vertical cuts $\mathcal{C}_{1,1}$ and $\mathcal{C}_{1,2}$, which are full length vertical cuts for CA_1 and CA_2 respectively, as in Figure 2. The cut $\mathcal{C}_{1,1}$ divides CA_1 into compatible arrangements $CA_{1,1}$ and $CA_{1,2}$, and the

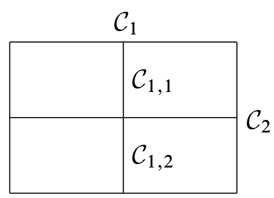


Figure 2: Vertical $\mathcal{C}_1$ cut by $\mathcal{C}_2$ into $\mathcal{C}_{1,1}$ and $\mathcal{C}_{1,2}$

cut $\mathcal{C}_{1,2}$ divides CA_2 into compatible arrangements $CA_{2,1}$ and $CA_{2,2}$, as in Figure 3. By the induction hypothesis, $CA_{1,1}$, $CA_{1,2}$, $CA_{2,1}$, and $CA_{2,2}$ are all allowable. It

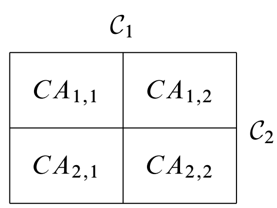


Figure 3: CA_1 and CA_2 cut into allowable compatible arrangements

is clear that the compatible arrangement to the left of $\mathcal{C}_1$ gets divided into $CA_{1,1}$ and $CA_{2,1}$ by the left side of the cut $\mathcal{C}_2$, so the compatible arrangement to the left of $\mathcal{C}_1$ is allowable. In the same way the compatible arrangement to the right of $\mathcal{C}_1$ gets cut into

$CA_{1,2}$ and $CA_{2,2}$ by the right side of the cut $\mathcal{C}_2$, so this compatible arrangement is also allowable, as we wanted to prove.

If $\mathcal{C}_1$ is horizontal, assume without loss of generality that CA_1 contains $\mathcal{C}_1$. By the induction hypothesis, $\mathcal{C}_1$ divides the allowable compatible arrangement CA_1 into two allowable compatible arrangements, say $CA_{1,a}$ and $CA_{1,b}$, as in Figure 4. We see that

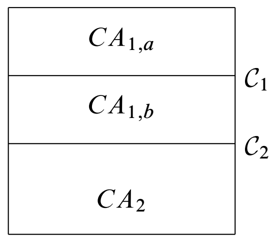


Figure 4: Horizontal $\mathcal{C}_1$ cuts CA_1 into allowable compatible arrangements

$\mathcal{C}_1$ divides the total arrangement CA into two compatible arrangements, $CA_{1,a}$ and $CA_{1,c}$, the latter of which is divided by $\mathcal{C}_2$ into $CA_{1,b}$ and CA_2. Since both $CA_{1,b}$ and CA_2 are allowable, we conclude that both $CA_{1,c}$ and $CA_{1,a}$ are allowable. This completes the proof. □

Proposition 3.8 *The forgetful functors T and U admit left adjoints S and R respectively.*

$$\begin{tikzcd}
\mathbf{DblGr1\text{-}Id} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathbf{DblDerSch} \arrow[l, bend left, "T"] \arrow[r, bend left, "R"] \arrow[r, phantom, "\perp"] & \mathbf{DblCat} \arrow[l, bend left, "U"]
\end{tikzcd}$$

The left adjoint S gives the free double derivation scheme on a double graph with 1–identities, and the left adjoint R gives the free double category on a double derivation scheme. The functor R preserves the horizontal and vertical 1–categories.

Proof For a double graph with 1–identities $\mathbb{A}$, let $S\mathbb{A}$ have vertical and horizontal 1–categories the free 1–categories on the respective reflexive graphs. The set of squares remains the same. It is straightforward to verify that this defines a left adjoint to T.

For a double derivation scheme $\mathbb{S}$, let $R\mathbb{S}$ have vertical and horizontal 1–categories the vertical and horizontal 1–categories of $\mathbb{S}$ respectively. The squares of $R\mathbb{S}$ are allowable compatible arrangements whose constituent squares are squares of $\mathbb{S}$ and identity squares. Such compatible arrangements are composed vertically and horizontally by concatenation. Clearly, composites of allowable compatible arrangements are allowable.

If $J\colon \mathbb{S} \longrightarrow U\mathbb{D}$ is a morphism of double derivation schemes, then it induces a double functor $J'\colon R\mathbb{S} \longrightarrow \mathbb{D}$ which is J on the horizontal and vertical 1–categories. For an allowable compatible arrangement D, the square $J'D$ is the composite in $\mathbb{D}$ of J applied to the constituents of D. Morphisms $R\mathbb{S} \longrightarrow \mathbb{D}$ restrict to morphisms $\mathbb{S} \longrightarrow U\mathbb{D}$, and it is not hard to check that these two operations are inverse. We conclude that $R \dashv U$. □

Now that we have free notions, we also define quotients. Note that the notion of congruence for ordinary categories is an equivalence relation on the morphisms of highest dimension, satisfying certain compatibility properties. We imitate this in our notion of congruence for a double category.

Definition 3.9 A *congruence* on a category $\mathbf{C}$ is an equivalence relation on $\mathbf{C}(a,b)$ for each $a,b \in \mathbf{C}$, such that if $f \sim f'$ and $g \sim g'$, then $gf \sim g'f'$ whenever the composites exist.

Definition 3.10 A *congruence* on a double derivation scheme $\mathbb{S}$ consists of a congruence on the horizontal 1–category and a congruence on the vertical 1–category.

Definition 3.11 A *congruence* on a double category $\mathbb{D}$ consists of an equivalence relation on

$$\mathbb{D}\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix}$$

for each boundary

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

such that if $\alpha \sim \alpha', \beta \sim \beta'$, and $\gamma \sim \gamma'$ then

$$\begin{bmatrix}\alpha & \beta\end{bmatrix} \sim \begin{bmatrix}\alpha' & \beta'\end{bmatrix}$$
$$\begin{bmatrix}\alpha \\ \gamma\end{bmatrix} \sim \begin{bmatrix}\alpha' \\ \gamma'\end{bmatrix}$$

whenever the composites exist. Note that the congruence does not concern the horizontal and vertical morphisms.

Example 3.12 Suppose $K\colon \mathbb{D} \longrightarrow \mathbb{E}$ is a double functor. Then we may define a congruence $\sim_K$ on $\mathbb{D}$ by

$$\alpha \sim_K \alpha' :\Longleftrightarrow K(\alpha) = K(\alpha').$$

Proposition 3.13 *Let $\mathbb{D}$ be a double category equipped with a congruence. If two allowable compatible arrangements CA_1 and CA_2 with the same underlying subdivision of the rectangle have congruent constituent squares, then the composites of CA_1 and CA_2 in $\mathbb{D}$ are congruent.*

Proof By Theorem 1.2 of Dawson–Paré [21], any two composites of a composable compatible arrangement are equal. The compatible arrangements CA_1 and CA_2 are composable since they are allowable. If we compose each of CA_1 and CA_2 using the same sequence of pairwise compositions, then the pairwise composites in each step are congruent. An inductive argument shows that the total composites are then also congruent. □

Definition 3.14 Let $\mathbf{C}$ be a category and $\sim$ a congruence on $\mathbf{C}$. The *quotient category* $\mathbf{C}/\sim$ has the same objects as $\mathbf{C}$ and has homsets $(\mathbf{C}/\sim)(a,b) = \mathbf{C}(a,b)/\sim$. The composition in $\mathbf{C}/\sim$ is induced by the composition in $\mathbf{C}$.

Definition 3.15 Let $\mathbb{S}$ be a double derivation scheme and $\sim$ a congruence on $\mathbb{S}$. The *quotient double derivation scheme* $\mathbb{S}/\sim$ has the same objects and squares as $\mathbb{S}$. The horizontal and vertical 1–categories of $\mathbb{S}/\sim$ are the quotient categories of $(\mathbf{H}\mathbb{S})_0$ and $(\mathbf{V}\mathbb{S})_0$.

Definition 3.16 Let $\mathbb{D}$ be a double category and $\sim$ a congruence on $\mathbb{D}$. The *quotient double category* $\mathbb{D}/\sim$ has the same objects and the same horizontal and vertical 1–categories as $\mathbb{D}$. The set of squares of $\mathbb{D}/\sim$ with the indicated boundary are

$$(\mathbb{D}/\sim)\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix} = \mathbb{D}\begin{pmatrix} & f & \\ j & & k \\ & g & \end{pmatrix}/\sim .$$

The horizontal and vertical compositions of squares in $\mathbb{D}/\sim$ are induced by the horizontal and vertical compositions of squares in $\mathbb{D}$.

These are of course not the most general notions of quotient, but more general quotients can be built from these as follows. All quotients can be characterized by the usual universal properties.

Definition 3.17 Let $\mathbf{C}$ be a category and $\mathbf{R} \subseteq \mathbf{C} \times \mathbf{C}$ a subcategory satisfying the usual axioms of an equivalence relation both on the set of objects and on the set of morphisms. Then the *quotient category* $\mathbf{C}/\mathbf{R}$ is defined as follows. First we obtain a graph with object set $\operatorname{Obj}\mathbf{C}/\operatorname{Obj}\mathbf{R}$ and morphism set $\operatorname{Mor}\mathbf{C}/\operatorname{Mor}\mathbf{R}$. We make this into a reflexive graph by identifying 1_A and 1_B whenever A and B are identified. Let $\mathbf{F}$ be the free category on this reflexive graph. The *quotient category* $\mathbf{C}/\mathbf{R}$ is defined as $\mathbf{F}/\sim$ where $\sim$ is the smallest congruence on the free category $\mathbf{F}$ such that the induced map of reflexive graphs $\mathbf{C} \longrightarrow \mathbf{F}/\sim$ is a functor.

Such quotients of categories have been considered by Boerger [9]. However a counterexample by Boerger [9] and Mersch [67; 68] shows that the quotient functor may identify morphisms which are not equivalent. Isbell completed early work on quotients in [48]. More recently, quotients of categories by generalized congruences have been considered by Bednarczyk–Borzyszkowski–Pawlowski [3].

For general quotients of double categories, we need quotients of double derivation schemes as an intermediate notion.

Definition 3.18 Let $\mathbb{S}$ be a double derivation scheme and $\mathbb{R} \subseteq \mathbb{S} \times \mathbb{S}$ a subdouble derivation scheme satisfying the usual axioms of an equivalence relation on the sets of objects, vertical morphisms, horizontal morphisms, and squares. Then the *quotient double derivation scheme* $\mathbb{S}/\mathbb{R}$ is defined as follows. The horizontal and vertical 1–categories are the quotients of the horizontal and vertical 1–categories of $\mathbb{S}$ as in Definition 3.17. The squares are $\operatorname{Sq}(\mathbb{S}/\mathbb{R}) = (\operatorname{Sq}\mathbb{S})/(\operatorname{Sq}\mathbb{R})$.

Definition 3.19 Let $\mathbb{D}$ be a double category and $\mathbb{R} \subseteq \mathbb{D} \times \mathbb{D}$ a subdouble category satisfying the usual axioms of an equivalence relation on the sets of objects, vertical morphisms, horizontal morphisms, and squares. Then the *quotient double category* $\mathbb{D}/\mathbb{R}$ is defined as follows. First we take the quotient of the underlying double derivation scheme of $\mathbb{D}$ by the underlying double derivation scheme of $\mathbb{R}$ as in Definition 3.18. Let $\mathbb{F}$ be the free double category $\mathbb{F}$ on this double derivation scheme. The *quotient double category* $\mathbb{D}/\mathbb{R}$ is defined as $\mathbb{F}/\sim$ where $\sim$ is the smallest congruence on the free double category $\mathbb{F}$ such that the induced morphism of double derivation schemes $\mathbb{D} \longrightarrow \mathbb{F}/\sim$ is a double functor.

In the above definition, see Definitions 3.11 and 3.16 for congruences and quotients of double categories by congruences. Note that only squares get identified in the last step, since the horizontal and vertical 1–categories of the free double category on a double derivation scheme are the same as the horizontal and vertical 1–categories of the double derivation scheme.

We will make use of free double categories and their quotients in our discussion of categorification in Section 6 as well as in an explicit description of certain pushouts of double categories in Theorem 10.6 and Theorem 10.7. These are essential ingredients in the construction of model structures on **DblCat**. For now it is sufficient to give a colimit formula in **DblCat**.

4 Limits and colimits of double categories

Model structures in general require the existence of limits and colimits. Moreover, in order to transfer model structures along certain adjunctions we will need an explicit formula for certain pushouts of double categories, as in Theorem 10.6 and Theorem 10.7. So in this section we discuss limits and colimits of double categories. We also prove that the 2–categories $\mathbf{DblCat_v}$ and $\mathbf{DblCat_h}$ are 2–cocomplete.

Colimits for categories were described in detail by Gabriel and Zisman. Their construction was extended by Worytkiewicz–Hess–Parent–Tonks [83] to a construction of colimits in **2–Cat**. We extend this further to a construction in **DblCat** which goes roughly as follows. To take the colimit of a functor F from an indexing category I into **DblCat**, first we take the colimit $\mathbb{S}$ of the underlying double derivation schemes, then we take the free double category $\mathbb{F}$ on $\mathbb{S}$, and finally we form the quotient $\mathbb{F}/\sim$ of the free double category $\mathbb{F}$ by the smallest congruence $\sim$ on $\mathbb{F}$ such that the induced maps of double derivation schemes $Fi \longrightarrow \mathbb{F}/\sim$ are double functors. The quotient double category $\mathbb{F}/\sim$ is the colimit of the functor F. The intermediate notion of double derivation scheme allows us to deal with the quotients of morphisms and quotients of squares separately. We present the details in the following theorems.

Theorem 4.1 *The category* **DblCat** *is complete and cocomplete.*

Proof The limits of the sets of objects, horizontal morphisms, vertical morphisms, and squares assemble to form a double category and this double category is the limit. After all, **DblCat** is a category of algebras.

The category **DblCat** is the category of models in **Cat** of a sketch with finite diagrams, and **Cat** is locally finitely presentable, so an application of Proposition 1.53 in Adámek and Rosický's book [1] shows that **DblCat** is locally finitely presentable. Locally finitely presentable categories are cocomplete, so **DblCat** is cocomplete. □

Note that the underlying horizontal and vertical 2–categories of the limit are the limits of the underlying horizontal and vertical 2–categories, since **H** and **V** admit left adjoints by Proposition 2.5. The forgetful functor $\mathbf{2\text{-}Cat} \longrightarrow \mathbf{Cat}$ also admits a left adjoint, so similar comments hold for the horizontal and vertical 1–categories.

Theorem 4.2 *The* 2*–categories* $\mathbf{DblCat_v}$ *and* $\mathbf{DblCat_h}$ *are* 2*–cocomplete.*

Proof We prove that $\mathbf{DblCat_v}$ is 2–cocomplete; the statement for $\mathbf{DblCat_h}$ will follow, since transposition is an isomorphism of 2–categories.

The *cotensor product* $\{\mathbf{C}, \mathbb{E}\}$ of a category $\mathbf{C}$ with a double category $\mathbb{E}$ is $\mathbb{E}^{\mathbb{V}\mathbf{C}}$, since

$$\begin{aligned}\mathbf{DblCat_v}\big(\mathbb{D}, \mathbb{E}^{\mathbb{V}\mathbf{C}}\big) &\cong \mathbf{DblCat_v}(\mathbb{D} \times \mathbb{V}\mathbf{C}, \mathbb{E}) \\ &\cong \mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{D}, \mathbb{E}) \\ &\cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{D}, \mathbb{E}))\end{aligned}$$

is an isomorphism of categories 2–natural in $\mathbb{D}$ by Proposition 2.13 and Corollary 2.14. Thus $\mathbf{DblCat_v}$ is *cotensored*, and by the dual of a statement on page 50 of Kelly's monograph [54], the existence of conical colimits in the 2–category $\mathbf{DblCat_v}$ is equivalent to the existence of ordinary conical colimits in its underlying 1–category $\mathbf{DblCat}$. But ordinary conical colimits exist in $\mathbf{DblCat}$ by Theorem 4.1, so that the 2–category $\mathbf{DblCat_v}$ admits conical colimits.

The *tensor product* $\mathbf{C} * \mathbb{D}$ of a category $\mathbf{C}$ with a double category $\mathbb{D}$ is $\mathbb{V}\mathbf{C} \times \mathbb{D}$, since

$$\mathbf{DblCat_v}(\mathbb{V}\mathbf{C} \times \mathbb{D}, \mathbb{E}) \cong \mathbf{Cat}(\mathbf{C}, \mathbf{DblCat_v}(\mathbb{D}, \mathbb{E}))$$

is an isomorphism of categories 2–natural in $\mathbb{E}$ by Corollary 2.14. Thus $\mathbf{DblCat_v}$ is *tensored*.

Since $\mathbf{DblCat_v}$ admits conical colimits and tensor products, we conclude from the dual of Theorem 3.73 in [54] that $\mathbf{DblCat_v}$ is 2–cocomplete. □

We work towards an explicit description of colimits in $\mathbf{DblCat}$ which mimics Gabriel and Zisman's calculation of colimits in $\mathbf{Cat}$ below.

Theorem 4.3 (Colimit formula in $\mathbf{Cat}$ of [37]) *The colimit of a functor* $F\colon I \longrightarrow \mathbf{Cat}$ *is calculated as follows. Let* $\mathbf{F}$ *be the free category on the colimit of the underlying reflexive graphs. The colimit of* F *is the quotient* $\mathbf{F}/\sim$ *of the free category* $\mathbf{F}$ *by the smallest congruence* $\sim$ *on* $\mathbf{F}$ *such that the induced morphisms of reflexive graphs* $Fi \longrightarrow \mathbf{F}/\sim$ *are functors.*

Lemma 4.4 *The horizontal and vertical* 1*–categories of a colimit of double derivation schemes are the colimits of the underlying horizontal and vertical* 1*–categories. Similarly, the horizontal and vertical* 1*–categories of a colimit of double categories are the colimits of the underlying horizontal and vertical* 1*–categories.*

Proof The right adjoint to the forgetful functor

$$\begin{aligned} \mathbf{DblDerSch} &\longrightarrow \mathbf{Cat} \\ \mathbb{S} &\longmapsto (\mathbf{H}\mathbb{S})_0 \end{aligned}$$

assigns to a category $\mathbf{E}$ the double derivation scheme $\mathbb{E}$ with horizontal 1–category $\mathbf{E}$, a unique vertical morphism between any two objects, and a unique square for each boundary. Similarly, the forgetful functor $\mathbb{S} \longmapsto (\mathbf{V}\mathbb{S})_0$ admits a right adjoint. Since left adjoints preserve colimits, the statement for double derivation schemes follows.

The same argument works for **DblCat** in place of **DblDerSch**. □

Theorem 4.5 (Colimit formula in **DblDerSch**) *The colimit $\mathbb{S}$ of a functor*

$$F\colon I \longrightarrow \mathbf{DblDerSch}$$

is calculated in the following way. Let $\mathbb{F}$ be the free double derivation scheme on the colimit of the underlying double graphs with 1–identities. The colimit $\mathbb{S}$ of F is the quotient $\mathbb{F}/\!\sim$ of the free double derivation scheme $\mathbb{F}$ by the smallest congruence $\sim$ on $\mathbb{F}$ such that the induced morphisms of double graphs with 1–identities $Fi \longrightarrow \mathbb{F}/\!\sim$ are morphisms of double derivation schemes.

Proof Suppose $\mathbb{S}'$ is a double derivation scheme and $\beta_i\colon Fi \longrightarrow \mathbb{S}'$ are natural morphisms of double derivations schemes. We define a unique factorization

$$\begin{tikzcd} Fi \arrow[r, "\beta_i"] \arrow[d, "\alpha_i"'] & \mathbb{S}' \\ \mathbb{S} \arrow[ur, dotted] & \end{tikzcd}$$

on horizontal and vertical 1–categories by the universal property of Lemma 4.4, and on squares by the universal property of the colimit of the sets $\mathrm{Sq}\, Fi$. The set of squares in the free double derivation scheme on a double graph with identities is the same as the set of squares in the double graph with identities by Proposition 3.8. □

Theorem 4.6 (Colimit formula in **DblCat**) *The colimit $\mathbb{C}$ of a functor*

$$F\colon I \longrightarrow \mathbf{DblCat}$$

is calculated as follows. Let $\mathbb{S}$ be the colimit in **DblDerSch** *of the underlying double derivation schemes, and $\mathbb{F}$ the free double category on $\mathbb{S}$. The colimit $\mathbb{C}$ of F is* *the quotient $\mathbb{F}/\!\sim$ of $\mathbb{F}$ by the smallest congruence $\sim$ such that the induced natural morphisms of double derivation schemes*

$$Fi \xrightarrow{\alpha_i} \mathbb{S} \xrightarrow{p} \mathbb{F} \longrightarrow \mathbb{F}/\!\sim \tag{4}$$

are double functors. Note that the horizontal and vertical 1–categories of $\mathbb{S}, \mathbb{F}$, and $\mathbb{C}$ are the same, in particular the horizontal and vertical 1–categories of $\mathbb{C}$ are the colimits of the horizontal and vertical 1–categories of the Fi.

Proof Let $q\colon \mathbb{S} \longrightarrow \mathbb{C}$ denote the morphism of double derivation schemes defined as the composite of the inclusion p with the quotient double functor from $\mathbb{F}$ to $\mathbb{C}$. Then $q \circ \alpha_i$ is a double functor for all $i \in I$. Suppose $\mathbb{C}'$ is a double category and $\beta_i\colon Fi \longrightarrow \mathbb{C}'$ are natural double functors. Then by Theorem 4.5 there exists a unique morphism J of double derivation schemes that makes the upper left triangle commute:

$$\begin{tikzcd}
Fi \arrow[r, "\beta_i"] \arrow[d, "\alpha_i"'] & \mathbb{C}' \\
\mathbb{S} \arrow[r, "q"'] \arrow[ur, dotted, "\exists!\, J"] & \mathbb{C}. \arrow[u, dotted, "\exists!\, L"']
\end{tikzcd}$$

The morphism J induces a double functor $K\colon \mathbb{F} \longrightarrow \mathbb{C}'$ since $\mathbb{F}$ is free on $\mathbb{S}$. Since $K \circ p \circ \alpha_i = \beta_i$ is a double functor for all i, the induced morphisms of double derivations schemes

$$Fi \longrightarrow \mathbb{F}/\!\sim_K$$

analogous to (4) are double functors. Here $\sim_K$ is defined as in Example 3.12. Since K preserves $\sim_K$ and $\sim_K$ contains $\sim$, the double functor K also preserves $\sim$ and induces a unique functor L which makes the lower right triangle commute. Therefore the square commutes, and further L is the unique double functor such that the square commutes by the uniqueness of the two fillers. □

Recall that filtered colimits in **Cat** are particularly simple to calculate: the filtered colimit of the underlying reflexive graphs is already a category and this category is the filtered colimit in **Cat**. Similarly, one does not need to use free constructions and quotients to calculate filtered colimits in **DblCat**.

Theorem 4.7 *A filtered colimit of double categories is calculated by simply taking the filtered colimits of the underlying reflexive double graphs.*

Proof The filtered colimit of the underlying reflexive double graphs admits all the associative and unital compositions necessary for a double category by the corresponding result in **Cat**. The interchange law holds because it is possible to find representatives of all four squares in a single stage, where the interchange law is known to hold. □

5 Nerves of double categories

Grothendieck's full and faithful nerve $N\colon \mathbf{Cat} \longrightarrow \mathbf{SSet}$ has been of tremendous use in higher category theory. One can expect that its n–fold version will similarly be of use. In fact, Brown–Higgins [11; 12; 13; 14] have studied edge symmetric n–fold categories from the point of view of cubical sets, and also Grandis [40; 41]. A double category is a 2–truncated cubical set. We introduce in this section *simplicial* nerves of double categories. The simplicial nerve will be of use in Section 7 where we transfer model structures on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ to **DblCat** via a horizontal categorification-horizontal nerve adjunction. The *bisimplicial nerve*, its left adjoint, and the associated transfer of model structure, are treated in [35]. The horizontal nerve, which we consider now, is really an internal notion.

Definition 5.1 Let $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ be a double category. Then the *horizontal nerve of* $\mathbb{D}$ is the simplicial object $N_h\mathbb{D}$ in **Cat** defined as

$$\begin{aligned}
(N_h\mathbb{D})_0 &= \mathbb{D}_0 \\
(N_h\mathbb{D})_1 &= \mathbb{D}_1 \\
(N_h\mathbb{D})_n &= \underbrace{\mathbb{D}_1 \; {}_t\!\times_s \mathbb{D}_1 \; {}_t\!\times_s \cdots {}_t\!\times_s \mathbb{D}_1}_{n \text{ copies of } \mathbb{D}_1}.
\end{aligned}$$

$$\mathrm{Obj}(N_h\mathbb{D})_n : \quad \begin{tikzcd} {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

$$\mathrm{Mor}(N_h\mathbb{D})_n : \quad \begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

In other words,

$$\begin{aligned}
\mathrm{Obj}\,(N_h\mathbb{D})_n &= \mathbf{Cat}([n], (\mathrm{Obj}\,\mathbb{D}, \mathrm{Hor}\,\mathbb{D})) \\
\mathrm{Mor}\,(N_h\mathbb{D})_n &= \mathbf{Cat}([n], (\mathrm{Ver}\,\mathbb{D}, \mathrm{Sq}\,\mathbb{D})).
\end{aligned}$$

Composition in $(N_h\mathbb{D})_n$ is vertical.

There is of course the analogous notion of *vertical nerve of* $\mathbb{D}$ denoted $N_v\mathbb{D}$.

Example 5.2 If $\mathbf{C}$ is a category, then the simplicial set $N_h(\mathbb{H}\mathbf{C})$ is the usual nerve of $\mathbf{C}$. The horizontal nerve N_h is preferred for this reason. The horizontal embedding $\mathbb{H}$, in turn, is preferred because it is the functor

$$\mathbf{Cat} = \mathsf{Cat}(\mathbf{Set}) \longrightarrow \mathsf{Cat}(\mathbf{Cat}) = \mathbf{DblCat}$$

induced by the embedding $\mathbf{Set} \longrightarrow \mathbf{Cat}$. In other words, considering a category $\mathbf{C}$ as a double category with discrete object category and discrete morphism category is the same as identifying it with $\mathbb{H}\mathbf{C}$.

Like the nerve of a category, the horizontal nerve of a double category has a representable definition. Recall that $\mathbf{DblCat_v}$ denotes the 2–category of small double categories, double functors, and vertical natural transformations.

Proposition 5.3 *For every double category $\mathbb{D}$, the simplicial category*

$$[n] \mapsto \mathbf{DblCat_v}(\mathbb{H}[n], \mathbb{D})$$

is isomorphic to the horizontal nerve $N_h\mathbb{D}$. Equivalently, the object simplicial set of the horizontal nerve is

$$[n] \mapsto \mathbf{DblCat}([0] \boxtimes [n], \mathbb{D})$$

and the morphism simplicial set of the horizontal nerve is

$$[n] \mapsto \mathbf{DblCat}([1] \boxtimes [n], \mathbb{D}).$$

Proof The double categories $\mathbb{H}[n]$ and $[0] \boxtimes [n]$ are isomorphic, and vertical natural transformations between double functors $\mathbb{H}[n] \longrightarrow \mathbb{D}$ are the same as double functors

$$[1] \boxtimes [n] = (\mathbb{H}[1])^t \times \mathbb{H}[n] \longrightarrow \mathbb{D}$$

as pointed out by Grandis–Paré [42]. □

Proposition 5.3 makes the functoriality of N_h immediate. Even more, if we make $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ into a 2–category with 2–cells the modifications, then N_h becomes a 2–functor.

Corollary 5.4 *The horizontal nerve is a 2–functor*

$$N_h\colon \mathbf{DblCat_v} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}.$$

In Section 6 we construct the left adjoint to the horizontal nerve explicitly, but for now we observe that a left adjoint exists. Recall the Enriched Lemma from Kan, which follows from Theorem 4.51 of Kelly's monograph [54].

Theorem 5.5 (Enriched Lemma from Kan) *Let $\mathcal{V}$ be a symmetric monoidal closed category with small homsets. Suppose $\mathbf{A}$ is a small $\mathcal{V}$–category, $\mathbf{B}$ is a cocomplete $\mathcal{V}$–category, and $J\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a $\mathcal{V}$–functor. Then the enriched left Kan extension of J along the Yoneda embedding exists and is the enriched left adjoint of the singular functor*

$$\begin{aligned} J_*\colon \mathbf{B} &\longrightarrow \mathcal{V}^{\mathbf{A}^{\mathrm{op}}} \\ B &\longmapsto \mathbf{B}(J(-), B). \end{aligned}$$

Theorem 5.6 *The horizontal nerve $N_h\colon \mathbf{DblCat} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ admits a left 2–adjoint c_h called horizontal categorification.*

Proof Let $\mathcal{V}$ be $\mathbf{Cat}$, and let $J\colon \mathbf{A} \longrightarrow \mathbf{B}$ be the $\mathbf{Cat}$–functor

$$\mathbb{H}\colon \Delta \longrightarrow \mathbf{DblCat_v}\ .$$

By Proposition 5.3 the horizontal nerve is J_*.

Since the 2–category $\mathbf{DblCat_v}$ is 2–cocomplete by Theorem 4.2, and Δ is small, we may now apply Theorem 5.5 to obtain the left 2–adjoint c_h. □

Theorem 5.7 *The horizontal nerve N_h preserves filtered colimits.*

Proof It follows from Theorem 4.7 that the category of horizontal morphisms and squares of a filtered colimit of double categories is the filtered colimit of the categories of horizontal morphisms and squares. Since filtered colimits commute with finite limits, in particular iterated pullbacks, N_h preserves filtered colimits. □

The horizontal nerve is also well behaved with respect to external products.

Proposition 5.8 *Let $\sigma\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ denote the constant functor. Let*

$$\nu\colon \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$$

be the inclusion induced by the functor $\mathbf{Set} \longrightarrow \mathbf{Cat}$ which takes a set to the corresponding discrete category. If $\mathbf{A}$ and $\mathbf{B}$ are categories, then $N_h(\mathbf{A} \boxtimes \mathbf{B}) = \sigma\mathbf{A} \times \nu N\mathbf{B}$. In other words, $N_h(\mathbf{A} \boxtimes \mathbf{B})_k = \mathbf{A} \times N\mathbf{B}_k$ where we view the set $N\mathbf{B}_k$ as a discrete category.

Like the traditional nerve, the horizontal nerve is fully faithful.

Proposition 5.9 *The horizontal nerve* $N_h\colon \mathbf{DblCat_v} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ *is fully faithful in the 2–categorical sense, that is, the functors*

$$\mathbf{DblCat_v}(\mathbb{D},\mathbb{E}) \xrightarrow{(N_h)_{\mathbb{D},\mathbb{E}}} \mathbf{Cat}^{\Delta^{\mathrm{op}}}(N_h\mathbb{D}, N_h\mathbb{E})$$

are isomorphisms of categories.

Proof The data of a double functor F and a vertical natural transformation σ are encoded entirely in N_hF and $N_h\sigma$, so $(N_h)_{\mathbb{D},\mathbb{E}}$ is injective on objects and injective on morphisms.

If $F'\colon N_h\mathbb{D} \longrightarrow N_h\mathbb{E}$ is a morphism in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$, then the functors F'_0 and F'_1 give the data for a double functor F, and compatibility with face and degeneracy maps guarantees compatibility of F with horizontal composition and units. From $(N_hF)_0 = F'_0$ and $(N_hF)_1 = F'_1$ it follows that $N_hF = F'$ using the compatibility with the injective maps $e_{i,i+1}\colon \{0,1\} \longrightarrow \{0,\dots,n\}$ defined by

$$e_{i,i+1}(0) = i$$
$$e_{i,i+1}(1) = i+1$$

for $0 \le i \le n-1$. Similarly, if σ' is a 2–cell in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$, we can construct a vertical natural transformation σ from σ'_0 and σ'_1 such that $N_h\sigma = \sigma'$. Then $(N_h)_{\mathbb{D},\mathbb{E}}$ is surjective on objects and surjective on morphisms. □

Also like the traditional nerve, the horizontal nerve is 2–coskeletal, which means that the component $N_h\mathbb{D} \longrightarrow \mathrm{csk}_2\,\mathrm{tr}_2\,N_h\mathbb{D}$ of the unit for the adjunction

$$\begin{tikzcd} \mathbf{Cat}^{\Delta^{\mathrm{op}}} \arrow[r, bend left, "\mathrm{tr}_2"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}^{\Delta_2^{\mathrm{op}}} \arrow[l, bend left, "\mathrm{csk}_2"] \end{tikzcd}$$

is an isomorphism of simplicial objects in $\mathbf{Cat}$. To prove this, we need a proposition from enriched category theory, the first part of which is Theorem 5.13 of [54].

Proposition 5.10 (Proposition 1.1 of Lack–Paoli [62]) *Let* $\mathcal{V}$ *be a symmetric monoidal closed category which is complete and cocomplete. If*

$$\mathcal{A} \xrightarrow{I} \mathcal{B} \xrightarrow{J} \mathcal{C}$$

are $\mathcal{V}$*–functors, and* J *is fully faithful, then* J *is dense provided that* JI *is so, and then the identity* $JI = JI$ *exhibits* J *as the left Kan extension of* JI *along* I*. Furthermore,* *the singular functor* $\mathcal{C}(J,1)\colon \mathcal{C} \longrightarrow [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ *can then be obtained by first applying the singular functor*

$$\mathcal{C}(JI,1)\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$$

and then right Kan extending along $I\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$.

Proposition 5.11 *The horizontal nerve of a small double category is* 2*–coskeletal.*

Proof For I and J in Proposition 5.10, we take

$$\Delta_2 \xrightarrow{I} \Delta \xrightarrow{J} \mathbf{DblCat_v}$$

where I is the inclusion of the full subcategory Δ_2 of Δ on the objects [0], [1] and [2], and $J = \mathbb{H}$ as in the proof of Theorem 5.6. Clearly, J is fully faithful in the 2–categorical sense. We denote by N_h^2 the 2–truncation of the horizontal nerve, which is the singular functor $\mathbf{DblCat_v}(JI,1)$. By the same argument as in Proposition 5.9, N_h^2 is fully faithful in the 2–categorical sense, which is equivalent to the density of JI according to Theorem 5.1 (ii) of [54]. By Proposition 5.10, the horizontal nerve $N_h = \mathbf{DblCat_v}(J,1)$ is 2–naturally isomorphic to the composite

$$\mathbf{DblCat_v} \xrightarrow{N_h^2} \mathbf{Cat}^{\Delta_2^{\mathrm{op}}} \xrightarrow{\mathrm{Ran}_I} \mathbf{Cat}^{\Delta^{\mathrm{op}}} .$$

Evaluating this 2–natural isomorphism at a double category $\mathbb{D}$, we obtain an isomorphism

$$N_h\mathbb{D} \longrightarrow \mathrm{csk}_2\, \mathrm{tr}_2\, N_h\mathbb{D}$$

between $N_h\mathbb{D}$ and a 2–coskeletal simplicial object. From the naturality of the unit, it follows that $N_h\mathbb{D}$ is also 2–coskeletal. □

Definition 5.12 The *classifying space functor* B is the composite

$$\mathbf{DblCat} \xrightarrow{N_h} \mathbf{Cat}^{\Delta^{\mathrm{op}}} \xrightarrow{N_*} \mathbf{SSet}^{\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{diag}} \mathbf{SSet} \xrightarrow{|\cdot|} \mathbf{Top},$$

where N_* is the levelwise nerve induced by the ordinary nerve N, diag is induced by the diagonal functor, and $|\cdot|$ is the geometric realization. The bisimplicial set $N_*(N_h(\mathbb{D}))$ is isomorphic to the *bisimplicial nerve* $N_d\mathbb{D}$ of the double category $\mathbb{D}$.

6 Horizontal categorification

We now construct a left adjoint c_h to the horizontal nerve N_h by analogy with the usual nerve N. Horizontal categorification c_h is appropriately compatible with external products, as we show in Example 6.6 and Proposition 6.11.

We recall the left adjoint $c\colon \mathbf{SSet} \longrightarrow \mathbf{Cat}$ to the nerve functor N constructed by Gabriel–Zisman [37]. For a simplicial set X, the category cX is the *fundamental category of* X, or *categorification of* X. It is the free category on the reflexive graph (X_0, X_1) modulo the smallest congruence such that for every $\tau \in X_2$ with edges

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=4, "\tau"] & & \cdot \end{tikzcd}$$

we have $g \circ f \sim h$. The following proof is our guideline for the left adjoint c_h to N_h.

Proposition 6.1 *Categorification c is left adjoint to the nerve functor N.*

Proof We need to construct a natural bijection

$$\mathbf{Cat}(cX, \mathbf{A}) \cong \mathbf{SSet}(X, N\mathbf{A}).$$

Suppose we have a map $G\colon X \longrightarrow N\mathbf{A}$ of simplicial sets. The 1–truncation is a morphism of reflexive graphs, so there is a unique functor J making the upper left triangle commute.

$$\begin{tikzcd} (X_0, X_1) \arrow[r, "(G_0, G_1)"] \arrow[d] & \mathbf{A} \\ \mathrm{FreeCat}(X_0, X_1) \arrow[r] \arrow[ur, dotted, "\exists!\, J"] & cX \arrow[u, dotted, "\exists!\, G'"'] \end{tikzcd}$$

Since J comes from a morphism of simplicial sets, the functor J takes congruent morphisms to equal ones. Therefore there exists a unique functor G' making the lower right triangle commute.

For the converse, given a functor $G'\colon cX \longrightarrow \mathbf{A}$, we compose it with the morphism of reflexive graphs

$$(X_0, X_1) \longrightarrow \mathrm{FreeCat}(X_0, X_1) \longrightarrow cX$$

to obtain a morphism of reflexive graphs $(G_0, G_1)\colon (X_0, X_1) \longrightarrow \mathbf{A}$. We define $G_2\colon X_2 \longrightarrow (N\mathbf{A})_2$ on $\sigma \in X_2$ as

$$\xrightarrow{G_1(d_2\sigma)} \; \xrightarrow{G_1(d_0\sigma)} \; .$$

By definition G_2 is compatible with d_2 and d_0, and the quotient in the definition of cX makes G_2 compatible with d_1. This, together with the simplicial identities relating

X_1 and X_2

$$d_0 s_0 = \mathrm{id}_{X_1} \qquad d_2 s_0 = s_0 d_1 \qquad d_0 s_1 = s_0 d_0 \qquad d_2 s_1 = \mathrm{id}_{X_1},$$

implies that G_2 is also compatible with the degeneracies s_0 and s_1. Since $N\mathbf{A}$ is 2–coskeletal this morphism (G_0, G_1, G_2) of 2–truncated simplicial sets induces a morphism $G\colon X \longrightarrow N\mathbf{A}$ of simplicial sets:

$$\mathbf{Set}^{\Delta_2^{\mathrm{op}}}(\mathrm{tr}_2\, X, \mathrm{tr}_2\, N\mathbf{A}) \cong \mathbf{SSet}(X, \mathrm{csk}_2\, \mathrm{tr}_2\, N\mathbf{A}) \cong \mathbf{SSet}(X, N\mathbf{A}).$$

The two procedures $G \mapsto G'$ and $G' \mapsto G$ are inverse to one another. □

Remark 6.2 Any morphism of simplicial sets $G\colon X \longrightarrow N\mathbf{A}$ is completely determined by its 1-truncation (G_0, G_1) as follows. We let $e_{i,i+1}\colon \{0,1\} \longrightarrow \{0,\dots,n\}$ be the injective map defined by

$$e_{i,i+1}(0) = i$$
$$e_{i,i+1}(1) = i+1$$

for $0 \leq i \leq n-1$, and we let $e_i\colon \{0\} \longrightarrow \{0,\dots,n\}$ be the injective map defined by

$$e_i(0) = i$$

for $0 \leq i \leq n$. If σ is an n–simplex, then $G(\sigma)$ is the string of n morphisms in $\mathbf{A}$

$$\xrightarrow{G(e_{0,1}^*(\sigma))} \xrightarrow{G(e_{1,2}^*(\sigma))} \cdots \qquad\qquad \cdots \xrightarrow{G(e_{n-1,n}^*(\sigma))}$$

where the source and target of $G(e_{i,i+1}^*(\sigma))$ are $G(e_i^*(\sigma))$ and $G(e_{i+1}^*(\sigma))$.

We turn next to the left adjoint of the horizontal nerve. We will exhibit two proofs that the horizontal categorification of the product of a category with a simplicial set is an external product of the category with the fundamental category of the simplicial set. This is done in Example 6.6 using the definition of horizontal categorification, while it is done in Proposition 6.11 using weighted colimits.

Definition 6.3 Let $X \in \mathbf{Cat}^{\Delta^{\mathrm{op}}}$. We define a double category $c_h X$ called the *horizontal categorification* or *fundamental double category of* X as follows. First we define a double derivation scheme $\mathbb{S}$ with vertical 1–category X_0 and with horizontal 1–category the fundamental category of the simplicial set $\mathrm{Obj}\, X$. The squares of $\mathbb{S}$ are the morphisms of X_1. We equip the free double category $\mathbb{F}$ on the double derivation scheme $\mathbb{S}$ with the smallest congruence $\sim$ such that:

(1) If $\alpha, \beta \in \mathrm{Mor}\, X_1$ are composable in X_1, then the vertical composite $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ in $\mathbb{F}$ is congruent to the composite of β and α in X_1.

(2) For all $\tau \in \operatorname{Mor} X_2$ with boundary

$$\begin{tikzcd} & \cdot \arrow[dr, "\beta"] & \\ \cdot \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] \arrow[rr, phantom, shift left=3, "\tau"] & & \cdot \end{tikzcd}$$

we have $[\alpha\beta] \sim \gamma.$

(3) For any vertical morphism j, that is, for any $j \in \operatorname{Mor} X_0$, the horizontal identity i^h_j is congruent to the degeneracy of j in $\operatorname{Mor} X_1$.

(4) For any $f \in \operatorname{Obj} X_1$, the vertical identity square i^v_f on the image of f in the horizontal 1–category of $\mathbb{S}$ is congruent to the identity on f in the category X_1.

We define $c_h X$ as the quotient of $\mathbb{F}$ by the congruence $\sim$. The horizontal and vertical 1–categories of $c_h X$ are the horizontal and vertical 1–categories of $\mathbb{S}$.

Remark 6.4 In the definition of horizontal categorification it is not necessary to mod out by additional relations to make the identity squares functorial. If $g \circ f \sim h$ in the horizontal 1–category because of $\tau \in \operatorname{Obj} X_2$, then the identity morphism on τ in the category X_2 implies we have $i^v_h \sim [i^v_f \, i^v_g]$ (the face maps are functors and we have (2) and (4)). For vertically composable morphisms j and k, we have

$$i^h_{\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]} \sim \begin{bmatrix} i^h_j \\ i^h_k \end{bmatrix}$$

because degeneracy is a functor and by (1) and (3).

Example 6.5 If X is a simplicial set, then $c_h \nu X = \mathbb{H} c X$. By definition, the horizontal 1–category is cX, and the vertical 1–category is the discrete category X_0. Since X_1 is also discrete, there are no nontrivial squares.

Example 6.6 Recall from Proposition 5.8 that $\sigma\colon \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ denotes the constant functor and $\nu\colon \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ denotes the inclusion. If $\mathbf{A}$ is a category and Y is a simplicial set, then the horizontal categorification of the simplicial category $\sigma\mathbf{A} \times \nu Y$ is $\mathbf{A} \boxtimes cY$. In fact, the horizontal 1–category of the double derivation scheme $\mathbb{S}$ is

$$c(\operatorname{Obj}(\sigma\mathbf{A} \times \nu Y)) = c(\operatorname{Obj}\mathbf{A} \times Y) = (\operatorname{Obj}\mathbf{A}) \times cY,$$

which is the same as the horizontal 1–category of $\mathbf{A} \boxtimes cY$. The vertical 1–category of $\mathbb{S}$ is

$$(\sigma\mathbf{A} \times \nu Y)_0 = \mathbf{A} \times Y_0,$$

which is the same as the vertical 1–category of $\mathbf{A} \boxtimes cY$. The squares of $\mathbb{S}$ are

$$\operatorname{Mor}(\sigma\mathbf{A} \times \nu Y)_1 = (\operatorname{Mor}\mathbf{A}) \times Y_1.$$

The congruence on $\mathbb{F}$ corresponds precisely to the relations in $\mathbf{A} \boxtimes cY$ for pairwise compositions of squares and identity squares. We present an alternative conceptual proof of this example in Proposition 6.11.

As we have seen in Theorem 5.6, the horizontal nerve N_h admits a left 2–adjoint. We verify that c_h as in Definition 6.3 is the underlying 1–functor of the left 2–adjoint.

Proposition 6.7 *Horizontal categorification c_h is left adjoint to the horizontal nerve N_h.*

Proof We use the notation of Definition 6.3 and construct a natural bijection

$$\mathbf{DblCat}(c_h X, \mathbb{D}) \cong \mathbf{Cat}^{\Delta^{\mathrm{op}}}(X, N_h\mathbb{D}).$$

Suppose $G\colon X \longrightarrow N_h\mathbb{D}$ is a morphism of simplicial objects in **Cat**. This induces a morphism of double derivation schemes $\mathbb{S} \longrightarrow \mathbb{D}$ and a unique double functor J making the upper left triangle commute:

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & \mathbb{D} \\
\mathbb{F} \arrow[r] \arrow[ur, dotted, "\exists!\, J"] & c_h X . \arrow[u, dotted, "\exists!\, G'"']
\end{tikzcd}$$

Since G is a morphism of simplicial objects in **Cat**, J takes congruent squares to equal squares, and there exists a unique double functor G' making the lower right triangle commute.

On the other hand, given a double functor $G'\colon c_h X \longrightarrow \mathbb{D}$ we compose it with

$$(X_0, X_1) \longrightarrow \mathbb{F} \longrightarrow c_h X$$

to obtain a morphism (G_0, G_1) of 1-truncated simplicial objects in **Cat**. By the same argument as in the proof of Proposition 6.1, we obtain maps of simplicial sets

$$G^{\mathrm{Obj}}\colon \operatorname{Obj} X \longrightarrow \operatorname{Obj} N_h\mathbb{D}$$

$$G^{\mathrm{Mor}}\colon \operatorname{Mor} X \longrightarrow \operatorname{Mor} N_h\mathbb{D} .$$

The maps $(G_0^{\mathrm{Obj}}, G_0^{\mathrm{Mor}})$ and $(G_1^{\mathrm{Obj}}, G_1^{\mathrm{Mor}})$ are already known to be functors, and X is a simplicial object in **Cat**.

We claim that $(G_2^{\mathrm{Obj}}, G_2^{\mathrm{Mor}})$ is also a functor. If $\sigma, \sigma' \in \operatorname{Mor} X_2$ are composable we denote their composite as

$$\left[\frac{\sigma}{\sigma'}\right].$$

We similarly denote composites in $(N_h\mathbb{D})_2$. To indicate a composite of a horizontal path of squares in $\mathbb{D}$, we separate the squares by a dot. Then G_2 preserve compositions

$$\begin{aligned} G_2\left[\frac{\sigma}{\sigma'}\right] = d_2G_2\left[\frac{\sigma}{\sigma'}\right]\cdot d_0G_2\left[\frac{\sigma}{\sigma'}\right] &= G_1d_2\left[\frac{\sigma}{\sigma'}\right]\cdot G_1d_0\left[\frac{\sigma}{\sigma'}\right] \\ = \left[\frac{G_1d_2\sigma}{G_1d_2\sigma'}\right]\cdot\left[\frac{G_1d_0\sigma}{G_1d_0\sigma'}\right] &= \left[\frac{G_1d_2\sigma\cdot G_1d_0\sigma}{G_1d_2\sigma'\cdot G_1d_0\sigma'}\right] \\ &= \left[\frac{G_2\sigma}{G_2\sigma'}\right], \end{aligned}$$

and G_2 preserves units similarly.

By Proposition 5.11, $N_h\mathbb{D}$ is 2–coskeletal, so the 2–truncated morphism (G_0, G_1, G_2) induces a morphism $X \longrightarrow N_h\mathbb{D}$, and by Remark 6.2, this morphism must be $(G^{\mathrm{Obj}}, G^{\mathrm{Mor}})$ from above.

The two procedures $G \mapsto G'$ and $G' \mapsto G$ are inverse to one another. $\square$

Proposition 6.8 *Consider* **Cat** *embedded into* $\mathbf{Cat}^{\Delta^{op}}$ *as the constant simplicial objects, and consider* **Cat** *embedded vertically into* **DblCat**. *Then the adjunction* $c_h \dashv N_h$ *restricts to the identity adjunction on these full subcategories.*

We now move towards a conceptual proof of Example 6.6 in Proposition 6.11.

Remark 6.9 Recall that if S is a set and A is an object of a category, then the *copower* $S\cdot A$ is the coproduct of A with itself S times. In some categories, the copower has a simple description. For example, if $\mathbf{C}$ is a category, then the copower in **Cat** is

$$S\cdot\mathbf{C} = \coprod_S \mathbf{C} = S\times\mathbf{C}.$$

If X is a simplicial set and $Y \in \mathbf{Cat}^{\Delta^{op}}$, then $X\cdot Y$ is the simplicial object in **Cat**

$$[n] \longmapsto X_n\cdot Y_n = \textstyle\coprod_{X_n} Y_n = X_n\times Y_n\ ,$$

which is the same as $\nu X\times Y$.

Lemma 6.10 *If X and Y are simplicial objects in* $\mathbf{Cat}$, *then $X \times Y$ is the weighted colimit $X * G$ of the* $\mathbf{Cat}$*–functor*

$$\begin{aligned} G\colon \Delta &\longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}} \\ [n] &\longmapsto Y \times \nu\Delta[n] \end{aligned}$$

with weighting $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.

Proof Since $(\mathbf{Cat}, \times)$ is symmetric monoidal closed, it follows from a general fact that $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ has a tensor product

$$(Y \otimes Z)_n := Y_n \times Z_n = (Y \times Z)_n$$

and an internal hom

$$\begin{aligned} [Y, Z]_n : &= \mathbf{Cat}^{\Delta^{\mathrm{op}}}(\Delta[n] \cdot Y, Z) \\ &\cong \mathbf{Cat}^{\Delta^{\mathrm{op}}}(Y \times \nu\Delta[n], Z) \end{aligned}$$

for all $Y, Z \in \mathbf{Cat}^{\Delta^{\mathrm{op}}}$.

For any $Z \in \mathbf{Cat}^{\Delta^{\mathrm{op}}}$,

$$\mathbf{Cat}^{\Delta^{\mathrm{op}}}(G([n]), Z)$$

is the n–th category of the internal hom $[Y, Z]$. Thus we have a natural isomorphism

$$\mathbf{Cat}^{\Delta^{\mathrm{op}}}(X, \mathbf{Cat}^{\Delta^{\mathrm{op}}}(G(-), Z)) \cong \mathbf{Cat}^{\Delta^{\mathrm{op}}}(X \times Y, Z)$$

and $X \times Y$ satisfies the universal property of the weighted colimit $X * G$. □

We finish the conceptual proof of Example 6.6.

Proposition 6.11 *If* $\mathbf{A}$ *is a category and Y is a simplicial set, then the horizontal categorification of the simplicial category $\sigma\mathbf{A} \times \nu Y$ is $\mathbf{A} \boxtimes cY$ where cY is the traditional categorification of Y.*

Proof By Lemma 6.10, $\sigma\mathbf{A} \times \nu Y$ is the weighted colimit $\sigma\mathbf{A} * G$ of

$$\begin{aligned} G\colon \Delta &\longrightarrow \mathbf{Cat}^{\Delta^{\mathrm{op}}} \\ [n] &\longmapsto \nu Y \times \nu\Delta[n] \end{aligned}$$

with weighting $\sigma\mathbf{A}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$.

Let $J\colon \Delta \longrightarrow \mathbf{DblCat}$ be the horizontal embedding. Then by Theorem 4.51 of Kelly's monograph [54], for each $Z \in \mathbf{Cat}^{\Delta^{\mathrm{op}}}$, $c_h(Z) \cong Z * J$. Hence

$$\begin{aligned} c_h(\sigma\mathbf{A} \times \nu Y) &= c_h(\sigma\mathbf{A} * G) \\ &\cong (\sigma\mathbf{A} * G) * J \\ &\cong \sigma\mathbf{A} * (G * J) \end{aligned} \tag{5}$$

by the general Fubini Theorem, which is equation (3.23) in [54]. The functor

$$G * J\colon \Delta \longrightarrow \mathbf{DblCat}$$

in the last line takes $[n]$ to

$$G([n]) * J \cong c_h(G([n])) = c_h(\nu Y \times \nu\Delta[n]).$$

From Example 6.5 and the fact that c preserves finite products, we have

$$c_h(\nu Y \times \nu\Delta[n]) = \mathbb{H}c(Y \times \Delta[n]) \cong \mathbb{H}cY \times \mathbb{H}[n].$$

We conclude that (5) has the form

$$c_h(\sigma\mathbf{A} \times \nu Y) \cong \sigma\mathbf{A} * (\mathbb{H}cY \times \mathbb{H}[-]). \tag{6}$$

We claim that the right hand side of (6) is isomorphic to $\mathbb{V}\mathbf{A} \times \mathbb{H}cY$. In fact, Corollary 2.14 and the adjunction $sk_0 \dashv \mathrm{tr}_0$ give, for all $\mathbb{E} \in \mathbf{DblCat_v}$

$$\begin{aligned} \mathbf{DblCat_v}(\mathbb{V}\mathbf{A} \times \mathbb{H}cY, \mathbb{E})) &\cong \mathbf{Cat}(\mathbf{A}, \mathbf{DblCat_v}(\mathbb{H}cY, \mathbb{E})) \\ &\cong \mathbf{Cat}(\mathbf{A}, \mathrm{tr}_0\ \mathbf{DblCat_v}(\mathbb{H}cY \times \mathbb{H}[-], \mathbb{E})) \\ &\cong \mathbf{Cat}^{\Delta^{\mathrm{op}}}(\sigma\mathbf{A}, \mathbf{DblCat_v}(\mathbb{H}cY \times \mathbb{H}[-], \mathbb{E})). \end{aligned}$$

The claim follows now from the definition of weighted colimit. Hence, (6) implies that

$$c_h(\sigma\mathbf{A} \times \nu Y) \cong \mathbb{V}\mathbf{A} \times \mathbb{H}cY = \mathbf{A} \boxtimes cY. \qquad \square$$

Remark The vertical categorification of a simplicial object X in **Cat** is the transpose of $c_h X$.

7 Model structures arising from $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$

Now that we have the adjunction $c_h \dashv N_h$ in place we can use it to transfer model structures from $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ to **DblCat** using Kan's Lemma on Transfer (Theorem 7.11). This theorem says that one can lift a model structure across an adjunction under a certain pushout condition and certain smallness conditions, which guarantee functorial factorizations. This is our first method for constructing model structures on **DblCat**.

In Section 8 we will adopt the point of view of double categories as internal categories and apply the results of Everaert–Kieboom–Van der Linden [32]. In Section 9 we will consider **DblCat** as a category of algebras for a 2–monad and use Lack's algebra structure [60].

The category $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ has four model structures of interest to us. These arise as diagram structures and Reedy structures associated to two cofibrantly generated model structures on **Cat**: the Thomason structure and the categorical structure. In Sections 7.1–7.4 we review some material for the reader's convenience: model structures on **Cat**, their associated diagram structures, smallness arguments, and Kan's Lemma on Transfer. After these preliminaries, we turn to our new results. In Sections 7.5 and 7.6 we transfer the diagram structures to **DblCat** across the horizontal categorification-horizontal nerve adjunction, and show that the transferred structures on **DblCat** extend the Thomason structure and categorical structure on the vertically embedded subcategory $\mathbf{Cat}$. In the proofs of our transfer results we crucially need to know the behavior of certain pushouts, and these are treated in Theorem 10.6 and Theorem 10.7 of the Appendix. We show in Section 7.7 that the Reedy categorical structure cannot transfer.

Recall the notion of cofibrantly generated model category.

Definition 7.1 A model category $\mathbf{C}$ is *cofibrantly generated* if there exist sets of morphism I and J in $\mathbf{C}$ such that:

(1) The domains of I are small with respect to I–cell as defined in Definition 7.5 and Definition 7.8.

(2) The domains of J are small with respect to J–cell.

(3) The class of fibrations is precisely the class of morphisms with the right lifting property with respect J.

(4) The class of acyclic fibrations is precisely the class of morphisms with the right lifting property with respect to I.

In this case, I is the set of *generating cofibrations* and J is the set of *generating acyclic cofibrations*.

7.1 Model structures on Cat

In the *Thomason structure* on **Cat** in [80] a functor F is a weak equivalence (respectively fibration) if and only if $\mathrm{Ex}^2 NF$ is a weak equivalence (respectively fibration) of simplicial sets. The functor Ex^2 is superfluous for weak equivalences, as Thomason proved that F is a weak equivalence if and only if NF is. The functor $\mathrm{Ex}\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$

is the right adjoint to barycentric subdivision $\mathrm{Sd}\colon \mathbf{SSet}\longrightarrow\mathbf{SSet}$, which we recall below. The Thomason structure is cofibrantly generated. The generating cofibrations are the inclusions of categorical boundaries

$$c\,\mathrm{Sd}^2\,\partial\Delta[m]\longrightarrow c\,\mathrm{Sd}^2\,\Delta[m],$$

while the generating acyclic cofibrations are the inclusions of categorical horns

$$c\,\mathrm{Sd}^2\,\Lambda^k[m]\longrightarrow c\,\mathrm{Sd}^2\,\Delta[m].$$

We now recall the definition of *barycentric subdivision* Sd. The simplicial sets $\mathrm{Sd}\,\Delta[m]$ and $\mathrm{Sd}\,\Lambda^k[m]$ are respectively the nerves of the posets of nondegenerate simplices of $\Delta[m]$ and $\Lambda^k[m]$. The ordering is the face relation. Thus a q–simplex of $\mathrm{Sd}\,\Delta[m]$ is a tuple $(v_0,\dots,v_q)$ of nondegenerate simplices (faces) of $\Delta[m]$ such that v_i is a face of v_{i+1} for all $0\leq i\leq q-1$. Such a tuple is a q–simplex of $\mathrm{Sd}\,\Lambda^k[m]$ if and only if all $v_0,\dots,v_q$ are in $\Lambda^k[m]$. A p–simplex u is a face of a q–simplex v in $\mathrm{Sd}\,\Delta[m]$ if and only if

$$\{u_0,\dots,u_p\}\subseteq\{v_0,\dots,v_q\}.$$

A p–simplex u of $\mathrm{Sd}\,\Delta[m]$ is nondegenerate if and only if all u_i are distinct.

The *barycentric subdivision* of a simplicial set Y is by definition

$$\operatorname*{colim}_{\Delta[n]\to Y}\mathrm{Sd}\,\Delta[n]$$

where the colimit is indexed over the category of simplices of Y. It follows from page 311 of [80] that $c\,\mathrm{Sd}^2\,\Delta[m]$ and $c\,\mathrm{Sd}^2\,\Lambda^k[m]$ are respectively the posets of nondegenerate simplices of $\mathrm{Sd}\,\Delta[m]$ and $\mathrm{Sd}\,\Lambda^k[m]$ and the generating acyclic cofibration $c\,\mathrm{Sd}^2\,\Lambda^k[m]\longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$ is the inclusion of these posets.

The other model structure on **Cat** is the *categorical structure*, or *natural structure*, of Joyal–Tierney [51]. In the categorical structure a functor is a weak equivalence if and only if it is an equivalence of categories. A functor $F\colon \mathbf{A}\longrightarrow\mathbf{B}$ is a fibration if and only if for each isomorphism $\beta\colon b\cong Fa$ in $\mathbf{B}$ there is an isomorphism $\alpha\colon a'\cong a$ in $\mathbf{A}$ such that $Fa'=b$ and $F\alpha=\beta$. These fibrations of categories are also called *isofibrations*. A cofibration is a functor that is injective on objects. The categorical structure on **Cat** is also cofibrantly generated. There are three generating cofibrations:

$$\varnothing\hookrightarrow\{1\}$$

$$\{0,1\}\hookrightarrow\{0\to 1\}$$

$$\{0\rightrightarrows 1\}\longrightarrow\{0\to 1\}$$

and one generating acyclic cofibration:

$$\{1\} \hookrightarrow \{0 \cong 1\} = \mathbf{I} \, .$$

7.2 Diagram model structures on $\mathbf{Cat}^{\Delta^{op}}$

Given a model category $\mathbf{M}$ and a small category $\mathbf{C}$, one might hope that the category $\mathbf{M}^{\mathbf{C}}$ of functors $\mathbf{C} \longrightarrow \mathbf{M}$ is also a model category with levelwise weak equivalences and levelwise fibrations. By this we mean that a natural transformation is a weak equivalence (respectively fibration) if and only if each of its components is. Unfortunately, this definition does not always give rise to a model structure on $\mathbf{M}^{\mathbf{C}}$. However, if $\mathbf{M}$ is a *cofibrantly generated* model category, Theorem 7.2 guarantees that this definition does indeed give rise to a model structure on $\mathbf{M}^{\mathbf{C}}$, which is even cofibrantly generated.

Theorem 7.2 (Theorem 11.6.1 in [46]) *Let $\mathbf{C}$ be a small category and $\mathbf{M}$ a cofibrantly generated model category with I the set of generating cofibrations and J the set of generating acyclic cofibrations. Then $\mathbf{M}^{\mathbf{C}}$ is a cofibrantly generated model category with levelwise weak equivalences and levelwise fibrations. The generating cofibrations are natural transformations of the form*

$$\coprod_{\mathbf{C}(C,-)} A \xrightarrow{\coprod_{\mathbf{C}(C,-)} f} \coprod_{\mathbf{C}(C,-)} B$$

for $f\colon A \longrightarrow B$ in I. The generating acyclic cofibrations are defined similarly with f in J. A morphism in $\mathbf{M}^{\mathbf{C}}$ is a cofibration if and only if it is a retract of a transfinite composition of pushouts of generating cofibrations. The components of a cofibration are also cofibrations.

Thus, the category $\mathbf{Cat}^{\Delta^{op}}$ inherits two model structures from Section 7.1. In the *diagram Thomason structure* on $\mathbf{Cat}^{\Delta^{op}}$, a natural transformation α is a weak equivalence (respectively fibration) if and only if $\mathrm{Ex}^2 N\alpha_i$ is a weak equivalence (respectively fibration) of simplicial sets for each $i \geq 0$. In the *diagram categorical structure* on $\mathbf{Cat}^{\Delta^{op}}$, a natural transformation α is a weak equivalence (respectively fibration) if and only if α_i is an equivalence of categories (respectively isofibration) for all $i \geq 0$.

If $\mathbf{C}$ is a *Reedy category*, then a model structure on $\mathbf{M}$ also induces a *Reedy model structure* on $\mathbf{M}^{\mathbf{C}}$ (see for example Hirschhorn [46] or Hovey [47]). The category Δ^{op} is a Reedy category, so the Thomason and categorical structures on $\mathbf{Cat}$ also give rise to two more model structures on $\mathbf{Cat}^{\Delta^{op}}$. However, we do not study these in more detail because of the following Theorem and also because of Theorem 7.22.

Theorem 7.3 (Theorem 15.6.4 in [46]) *If* $\mathbf{C}$ *is a Reedy category and* $\mathbf{M}$ *is a cofibrantly generated model category, then the identity functor of* $\mathbf{M}^{\mathbf{C}}$ *is a left Quillen equivalence from the cofibrantly generated diagram model structure to the Reedy model structure, and a right Quillen equivalence in the opposite direction.*

7.3 Smallness

We will need some knowledge about smallness to use Kan's Lemma on Transfer. We recall some of the relevant notions described in Hovey's monograph [47]. Appropriate smallness conditions also allow us to conclude that a transfinite composition of weak equivalences is a weak equivalence.

Definition 7.4 Let κ be a cardinal. An ordinal λ is *κ–filtered* if it is a limit ordinal and, if $A \subseteq \lambda$ and $|A| \leq \kappa$, then $\sup A < \lambda$.

Definition 7.5 Let $\mathbf{C}$ be a category with all small colimits and κ a cardinal. An object A of $\mathbf{C}$ is called *κ–small* if for all κ–filtered ordinals λ and all colimit-preserving functors $X\colon \lambda \longrightarrow \mathbf{C}$ the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathbf{C}(A, X_\beta) \longrightarrow \mathbf{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta) \tag{7}$$

is a bijection. An object A is said to be *small* if it is κ–small for some cardinal κ. An object A is said to be *finite* if it is κ–small for a finite cardinal κ, that is, for *any* limit ordinal λ and colimit-preserving functor X, the map (7) is a bijection. We say the concepts hold *relative to a class of morphisms* $\mathbf{D}$ *in* $\mathbf{C}$ if they hold true for all X with $X_\beta \longrightarrow X_{\beta+1}$ in $\mathbf{D}$ for all $\beta + 1 < \lambda$.

For example, categories are small as follows, and we conclude similarly that double categories are small.

Proposition 7.6 *Any category* $\mathbf{A}$ *is* κ*–small where*

$$\kappa = |\operatorname{Obj} \mathbf{A}| + |\operatorname{Mor} \mathbf{A}| + |\operatorname{Mor} \mathbf{A}\, {}_s\!\times_t \operatorname{Mor} \mathbf{A}|.$$

In particular, if $\operatorname{Mor} \mathbf{A}$ *is a finite set, then* $\mathbf{A}$ *is finite as an object of* $\mathbf{Cat}$.

Proof Let $\mathbf{X}\colon \lambda \longrightarrow \mathbf{Cat}$ be a colimit-preserving functor from a κ–filtered ordinal λ. Recall that ordinals are filtered categories and filtered colimits of categories are formed by simply taking the filtered colimits of the object set and the morphism set.

Suppose $F\colon \mathbf{A} \longrightarrow \operatorname{colim} \mathbf{X}$ is a functor. For each $A \in \operatorname{Obj} \mathbf{A}$ and $f \in \operatorname{Mor} \mathbf{A}$ there are ordinals $\alpha_1(A)$ and $\alpha_2(f)$ such that $F(A)$ and $F(f)$ are in the image of $\mathbf{X}_{\alpha_1(A)}$

and $\mathbf{X}_{\alpha_2(A)}$. Let β be the supremum of all the $\alpha_1(A)$ and $\alpha_2(f)$. Then $\beta < \lambda$ and we obtain maps of sets

$$G^{\mathrm{Obj}}\colon \operatorname{Obj}\mathbf{A} \longrightarrow \operatorname{Obj}\mathbf{X}_\beta$$
$$G^{\mathrm{Mor}}\colon \operatorname{Mor}\mathbf{A} \longrightarrow \operatorname{Mor}\mathbf{X}_\beta$$

which factor the functor F. There exists for each $f \in \operatorname{Mor}\mathbf{A}$ an index $\gamma(f)$ such that $s(G(f)) = G(s(f))$ and $t(G(f)) = G(t(f))$ in $\mathbf{X}_{\gamma(f)}$. For each $A \in \operatorname{Obj}\mathbf{A}$ there is an index $\delta(A)$ such that $G(1_A) = 1_{G(A)}$ in $\mathbf{X}_{\delta(A)}$. For each $(\ell,k) \in \operatorname{Mor}\mathbf{A}\ {}_s{\times}_t \operatorname{Mor}\mathbf{A}$ there exists an index $\epsilon(\ell,k)$ such that $G(\ell\circ k) = G(\ell)\circ G(k)$ in $\mathbf{X}_{\epsilon(\ell,k)}$. Let ζ be the supremum of all these indices γ,δ,ϵ. Then $\zeta < \lambda$ and G induces a functor $\mathbf{A}\longrightarrow\mathbf{X}_\zeta$ which factors F. Hence (7) is onto.

Suppose $M\colon \mathbf{A}\longrightarrow\mathbf{X}_\alpha$ and $N\colon \mathbf{A}\longrightarrow\mathbf{X}_\beta$ are functors that become equal in the colimit. Then for each $A \in \operatorname{Obj}\mathbf{A}$ and each $f \in \operatorname{Mor}\mathbf{A}$ there are indices $\gamma(A)$ and $\delta(f)$ such that $M(A) = N(A)$ and $M(f) = N(f)$ in $\mathbf{X}_{\gamma(A)}$ and $\mathbf{X}_{\delta(f)}$ respectively. Let $\zeta < \lambda$ be the supremum of all these indices $\gamma(A)$ and $\delta(f)$. Then M and N become equal at the stage ζ and the map (7) is injective. □

Proposition 7.7 *Let $\mathbb{D}$ be a double category and s^h, s^v, t^h, t^v the horizontal and vertical source and target maps. Then $\mathbb{D}$ is κ–small where*

$$\begin{aligned}\kappa =& |\operatorname{Obj}\mathbb{D}| + |\operatorname{Hor}\mathbb{D}| + |\operatorname{Hor}\mathbb{D}\ {}_{s^h}{\times}_{t^h} \operatorname{Hor}\mathbb{D}| \\ &+ |\operatorname{Ver}\mathbb{D}| + |\operatorname{Ver}\mathbb{D}\ {}_{s^v}{\times}_{t^v} \operatorname{Ver}\mathbb{D}| \\ &+ |\operatorname{Sq}\mathbb{D}| + |\operatorname{Sq}\mathbb{D}\ {}_{s^v}{\times}_{t^v} \operatorname{Sq}\mathbb{D}| \\ &+ |\operatorname{Sq}\mathbb{D}\ {}_{s^h}{\times}_{t^h} \operatorname{Sq}\mathbb{D}|.\end{aligned}$$

In particular, if $\operatorname{Sq}\mathbb{D}$ *is a finite set, then* $\mathbb{D}$ *is finite as an object of* **DblCat**.

Proof We first obtain a map of the underlying quadruple of sets, and then we go out far enough to make it into a double functor by considering the various compositions and identities as in Proposition 7.6. □

Note that this proposition easily generalizes to n–fold categories.

One useful application of finiteness is to transfinite compositions of weak equivalences.

Definition 7.8 If $\mathbf{C}$ is a category with all small colimits, λ is an ordinal, $\mathbf{D}$ is a class of morphisms in $\mathbf{C}$, and $X\colon \lambda\longrightarrow\mathbf{C}$ is a colimit preserving functor such that $X_\beta\longrightarrow X_{\beta+1}$ is in $\mathbf{D}$ for all $\beta+1<\lambda$, then the morphism

$$X_0 \longrightarrow \operatorname{colim} X$$

is called a *transfinite composition of morphisms in* $\mathbf{D}$. If I is a class of morphisms in $\mathbf{C}$, then a transfinite composition of pushouts of elements of I is called a *relative I–cell complex*. The class of relative I–cell complexes is denoted *I–cell*.

Proposition 7.9 (Corollary 7.4.2 in [47]) *Suppose $\mathbf{C}$ is a cofibrantly generated model category in which the domains and codomains of the generating cofibrations and generating acyclic cofibrations are finite. Then every transfinite composition of weak equivalences is a weak equivalence.*

Example 7.10 In both the Thomason structure and the categorical structure on **Cat**, every transfinite composition of weak equivalences is a weak equivalence, as the domains and codomains of the generating cofibrations and generating acyclic cofibrations only have finitely many morphisms. Since weak equivalences and colimits in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ are levelwise, every transfinite composition of weak equivalences in the diagram structures is also a weak equivalence.

7.4 Kan's Lemma on Transfer

Our first main tool for constructing model structures on **DblCat** is Kan's Lemma on Transfer. The form we will use is Corollary 7.12.

Theorem 7.11 (Kan's Lemma on Transfer, Theorem 11.3.2 in [46]) *Let $\mathbf{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Suppose $\mathbf{D}$ is complete and cocomplete, and that*

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd} \tag{8}$$

is an adjunction. Assume the following.

(1) *For every $i \in I$, $\operatorname{dom} Fi$ is small with respect to FI–cell. For every $j \in J$, $\operatorname{dom} Fj$ is small with respect to FJ–cell.*

(2) *The functor G maps every relative FJ–complex to a weak equivalence in $\mathbf{C}$.*

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ with generating cofibrations FI and generating acyclic cofibrations FJ. Further, f is a weak equivalence in $\mathbf{D}$ if and only $G(f)$ is a weak equivalence in $\mathbf{C}$, and f is a fibration in $\mathbf{D}$ if and only $G(f)$ is a fibration in $\mathbf{C}$.

Along the lines of Proposition 3.4.1 of Worytkiewicz–Hess–Parent–Tonks [83], we have the following corollary.

Corollary 7.12 *Let* $\mathbf{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Suppose* $\mathbf{D}$ *is complete and cocomplete, and that* $F \dashv G$ *is an adjunction as in* (8)*. Assume the following.*

(1) *For every* $i \in I$ *and* $j \in J$*, the objects* $\operatorname{dom} Fi$ *and* $\operatorname{dom} Fj$ *are small with respect to the entire category* $\mathbf{D}$.

(2) *For any ordinal* λ *and any colimit preserving functor* $X \colon \lambda \longrightarrow \mathbf{C}$ *such that* $X_\beta \longrightarrow X_{\beta+1}$ *is a weak equivalence, the transfinite composition*

$$X_0 \longrightarrow \operatorname{colim} X$$

is a weak equivalence.

(3) G *preserves filtered colimits.*

(4) *If* j' *is a pushout of* $F(j)$ *in* $\mathbf{D}$ *for* $j \in J$*, then* $G(j')$ *is a weak equivalence in* $\mathbf{C}$.

Then there exists a cofibrantly generated model structure on $\mathbf{D}$ *with generating cofibrations* FI *and generating acyclic cofibrations* FJ*. Further,* f *is a weak equivalence in* $\mathbf{D}$ *if and only* $G(f)$ *is a weak equivalence in* $\mathbf{C}$*, and* f *is a fibration in* $\mathbf{D}$ *if and only* $G(f)$ *is a fibration in* $\mathbf{C}$.

Proof Clearly, (1) of Theorem 7.11 follows from the hypotheses. To see (2), we recall that a relative FJ–complex is a transfinite composition of pushouts of morphisms Fj where $j \in J$. If a relative FJ–complex f is a transfinite composition of $Y \colon \lambda \longrightarrow \mathbf{D}$, then Gf is the transfinite composition of $X = G \circ Y$. Since Gf is a transfinite composition of weak equivalences, Gf is also a weak equivalence. Hence G takes relative FJ–complexes to weak equivalences in $\mathbf{C}$. □

7.5 Transfer of the diagram Thomason structure on $\mathbf{Cat}^{\Delta^{op}}$

With these preliminaries and our free constructions on double categories, we can transfer the diagram Thomason structure to **DblCat**. Recall the diagram Thomason structure on $\mathbf{Cat}^{\Delta^{op}}$ from Section 7.2.

Theorem 7.13 *There is a cofibrantly generated model structure on* **DblCat** *such that a double functor* K *is a weak equivalence (respectively fibration) if and only if* $N_h K$ *is levelwise a weak equivalence (respectively fibration) in the Thomason structure on* **Cat**.

Proof We apply Corollary 7.12 to the adjunction $F = c_h \dashv N_h = G$. First we point out that

$$\begin{aligned} c_h\Big(\coprod_{\Delta^{\mathrm{op}}([n],-)} c\,\mathrm{Sd}^2\,\Lambda^k[m] \Big) &= c_h(c\,\mathrm{Sd}^2\,\Lambda^k[m] \times \Delta[n]) \\ &= (c\,\mathrm{Sd}^2\,\Lambda^k[m]) \boxtimes c\,\Delta[n] \\ &= (c\,\mathrm{Sd}^2\,\Lambda^k[m]) \boxtimes [n] \end{aligned}$$

by Example 6.6 or Proposition 6.11 (for simplicity we suppress σ and ν). Similarly,

$$c_h\Big(\coprod_{\Delta^{\mathrm{op}}([n],-)} c\,\mathrm{Sd}^2\,\Delta[m] \Big) = (c\,\mathrm{Sd}^2\,\Delta[m]) \boxtimes [n]$$

and the horizontal categorifications of the generating acyclic cofibrations $j = f \times 1_{\Delta[n]}$ in Theorem 7.2 are the inclusions $f \boxtimes 1_{[n]}$ for the inclusions

$$f\colon c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$$

and $m \geq 1$, $n \geq 0$.

(1) The double categories $(c\,\mathrm{Sd}^2\,\partial\Delta[m]) \boxtimes [n]$ and $(c\,\mathrm{Sd}^2\,\Lambda^k[m]) \boxtimes [n]$ have a finite number of squares, hence they are finite by Proposition 7.7.

(2) A transfinite composition of weak equivalences in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ is a weak equivalence by Example 7.10.

(3) The horizontal nerve N_h preserves filtered colimits by Theorem 5.7.

(4) Consider the pushout in **DblCat**:

$$\begin{tikzcd} (c\,\mathrm{Sd}^2\,\Lambda^k[m]) \boxtimes [n] \arrow[r] \arrow[d, "c_h(j) = f \boxtimes 1_{[n]}"'] & \mathbb{D} \arrow[d, "j'"] \\ (c\,\mathrm{Sd}^2\,\Delta[m]) \boxtimes [n] \arrow[r] & \mathbb{P}. \end{tikzcd}$$

Then by Proposition 5.8 $N_h c_h(j)$ is the acyclic cofibration j, and by Theorem 10.7 the diagram

$$\begin{tikzcd} (c\,\mathrm{Sd}^2\,\Lambda^k[m]) \times \Delta[n] \arrow[r] \arrow[d, "N_h c_h(j) = j"'] & N_h\mathbb{D} \arrow[d, "N_h(j')"] \\ (c\,\mathrm{Sd}^2\,\Delta[m]) \times \Delta[n] \arrow[r] & N_h\mathbb{P} \end{tikzcd}$$

is a pushout in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$. Hence $N_h(j')$ is an acyclic cofibration, and in particular a weak equivalence in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$. $\square$

We may compare the transferred diagram Thomason model structure on **DblCat** to the Thomason model structure on **Cat** as follows.

Proposition 7.14 *The functor* $(\mathbf{V}\text{-})_0\colon \mathbf{DblCat} \longrightarrow \mathbf{Cat}$ *maps the weak equivalences and fibrations of the transferred diagram Thomason model structure on* **DblCat** *to weak equivalences and fibrations in the Thomason model structure on* **Cat**. *In particular,* $(\mathbf{V}\text{-})_0$ *is a right Quillen functor.*

Proof As functors, $(\mathbf{V}\text{-})_0$ and $(N_h\text{-})_0$ are the same. □

Corollary 7.15 *The adjunction*

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "\mathbb{V}"] \arrow[r, phantom, "\perp"] & \mathbf{DblCat} \arrow[l, bend left, "(\mathbf{V}-)_0"] \end{tikzcd}$$

is a Quillen adjunction.

Proof An adjunction is a Quillen adjunction if and only if the right adjoint is a right Quillen functor, so the Corollary follows immediately from Proposition 7.14. □

Proposition 7.16 *The functor* $\mathbb{V}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat}$ *preserves and reflects weak equivalences, fibrations, and cofibrations. In other words, a functor* F *is a weak equivalence (respectively fibration, respectively cofibration) in the Thomason model structure on* **Cat** *if and only if* $\mathbb{V}F$ *is a weak equivalence (respectively fibration, respectively cofibration) in the transferred diagram Thomason model structure on* **DblCat**. *As a consequence, the transferred diagram Thomason model structure on* **DblCat** *extends the Thomason model structure on* **Cat** *as a vertically embedded subcategory.*

Proof For a functor F, the morphism $N_h\mathbb{V}F$ of simplicial objects in **Cat** is F in every degree. Thus $\mathbb{V}$ preserves and reflects weak equivalences and fibrations.

By Corollary 7.15, $\mathbb{V}$ preserves cofibrations. It also reflects cofibrations as follows. If F is a functor such that $\mathbb{V}F$ is a cofibration, and G is an acyclic fibration in **Cat**, then any diagram in **DblCat**

$$\begin{tikzcd} \mathbb{V}\mathbf{C} \arrow[r, "K"] \arrow[d, "\mathbb{V}F"'] & \mathbb{V}\mathbf{C}' \arrow[d, "\mathbb{V}G"] \\ \mathbb{V}\mathbf{D} \arrow[r, "L"'] & \mathbb{V}\mathbf{D}' \end{tikzcd}$$

admits a lift, as $\mathbb{V}G$ is an acyclic fibration by the above. Since $\mathbb{V}$ is fully faithful, this lift gives us a lift in **Cat**. Hence F is a cofibration. □

7.6 Transfer of the diagram categorical structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$

Our preparations allow us to also quickly transfer the diagram categorical structure. Recall the diagram categorical structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ from Section 7.2. In Section 8.2 we will show that the vertical analogue of this transferred model structure on **DblCat** coincides with the model structure induced by the simplicially surjective topology τ on **Cat** using the methods of Everaert–Kieboom–Van der Linden [32]. An important reason for interest in the equality of these two structures lies in the fact that the second construction yields an explicit form for the cofibrant replacement, which is not at all transparent using only the transferred structure.

Theorem 7.17 *There is a cofibrantly generated model structure on* **DblCat** *such that a double functor K is a weak equivalence (respectively fibration) if and only if $N_h K$ is levelwise a weak equivalence (respectively fibration) in the categorical structure on* **Cat**.

Proof We apply Corollary 7.12 to the adjunction $F = c_h \dashv N_h = G$. All generating acyclic cofibrations j in Theorem 7.2 for the categorical diagram structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ are natural transformations of the form

$$\coprod_{\Delta^{\mathrm{op}}([n],-)} \{1\} \xrightarrow{\coprod_{\Delta^{\mathrm{op}}([n],-)} f} \coprod_{\Delta^{\mathrm{op}}([n],-)} \mathbf{I}$$

where f is the inclusion $\{1\} \longrightarrow \mathbf{I}$ and $[n]$ is an object of Δ^{op}. These generating acyclic cofibrations have horizontal categorification

$$\{1\} \boxtimes [n] \longrightarrow \mathbf{I} \boxtimes [n]$$

by Example 6.6 or Proposition 6.11 (for simplicity we suppress σ and ν).

(1) The double categories $\varnothing \boxtimes [n]$, $\{0,1\} \boxtimes [n]$, $\{0 \rightrightarrows 1\} \boxtimes [n]$, and $\{1\} \boxtimes [n]$ have a finite number of squares, hence they are finite by Proposition 7.7.

(2) A transfinite composition of weak equivalences in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ is a weak equivalence by Example 7.10.

(3) The horizontal nerve N_h preserves filtered colimits by Theorem 5.7.

(4) Consider the pushout in **DblCat**:

$$\begin{tikzcd} \{1\} \boxtimes [n] \arrow[r] \arrow[d, "c_h(j)"'] & \mathbb{D} \arrow[d, "j'"] \\ \mathbf{I} \boxtimes [n] \arrow[r] & \mathbb{P}\,. \end{tikzcd}$$

Then by Proposition 5.8 $N_h c_h(j)$ is the acyclic cofibration j, and by Theorem 10.7 the diagram

$$\begin{tikzcd}
\{1\} \times \Delta[n] \arrow[r] \arrow[d, "N_h c_h(j)=j"'] & N_h \mathbb{D} \arrow[d, "N_h(j')"] \\
\mathbf{I} \times \Delta[n] \arrow[r] & N_h \mathbb{P}
\end{tikzcd}$$

is a pushout in $\mathbf{Cat}^{\Delta^{op}}$. Hence $N_h(j')$ is an acyclic cofibration, and in particular a weak equivalence in $\mathbf{Cat}^{\Delta^{op}}$. $\square$

We may compare the transferred diagram categorical model structure on **DblCat** to the categorical model structure on **Cat** as follows.

Proposition 7.18 *The functor* $(\mathbf{V}\text{-})_0\colon \mathbf{DblCat} \longrightarrow \mathbf{Cat}$ *maps the weak equivalences and fibrations of the transferred diagram categorical model structure on* **DblCat** *to weak equivalences and fibrations in the categorical model structure on* **Cat**. *In particular,* $(\mathbf{V}\text{-})_0$ *is a right Quillen functor.*

Proof As functors, $(\mathbf{V}\text{-})_0$ and $(N_h\text{-})_0$ are the same. $\square$

Corollary 7.19 *The adjunction*

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\mathbb{V}"] \arrow[r, phantom, "\perp"] & \mathbf{DblCat} \arrow[l, bend left, "(\mathbf{V}-)_0"]
\end{tikzcd}$$

is a Quillen adjunction.

Proof This follows immediately from Proposition 7.18. $\square$

Proposition 7.20 *The functor* $\mathbb{V}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat}$ *preserves and reflects weak equivalences, fibrations, and cofibrations. In other words, a functor* F *is a weak equivalence (respectively fibration, respectively cofibration) in the categorical model structure on* **Cat** *if and only if* $\mathbb{V}F$ *is a weak equivalence (respectively fibration, respectively cofibration) in the transferred diagram categorical model structure on* **DblCat**. *As a consequence, the transferred diagram categorical model structure on* **DblCat** *extends the categorical model structure on* **Cat** *as a vertically embedded subcategory.*

Proof The proof is completely analogous to the proof of Proposition 7.16. $\square$

7.7 No transfer of the Reedy categorical structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$

In this section we consider the category $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ of simplicial objects in **Cat** equipped with the Reedy model structure associated with the categorical model structure on **Cat**. The weak equivalences in this Reedy model structure are the levelwise equivalences of categories and the fibrations are the Reedy fibrations. (For further details, see Hirschhorn [46].) In this section we show that it is impossible to transfer this model structure to **DblCat** via the adjunction $c_h \dashv N_h$, where N_h is the horizontal nerve and c_h is the horizontal categorification. We will need the following theorem.

Theorem 7.21 (Theorem 1 of Joyal–Street [50]) *For a given functor $G\colon \mathbf{B} \longrightarrow \mathbf{C}$, the canonical comparison functor from the pullback of F along G to the pseudo pullback of F along G is an equivalence of categories for all functors $F\colon \mathbf{A} \longrightarrow \mathbf{C}$ if and only if G is an isofibration.*

Now we turn to the objective of this subsection.

Theorem 7.22 *There does not exist a model structure on* **DblCat** *such that a double functor K is a weak equivalence (respectively fibration) if and only if $N_h K$ is a weak equivalence (respectively fibration) in the Reedy model structure on $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$ associated to the categorical structure on* **Cat**.

Proof Suppose that such a transferred model structure on **DblCat** does exist. Then (c_h, N_h) is a Quillen pair. Let $\mathbb{D}$ be a double category and consider a Reedy cofibrant fibrant replacement $r\colon N_h\mathbb{D} \longrightarrow V_\bullet$ in $\mathbf{Cat}^{\Delta^{\mathrm{op}}}$, that is, $V_\bullet$ is a Reedy fibrant object and r is an acyclic cofibration in the Reedy structure. Our strategy is to prove that the existence of such a transferred model structure implies a false statement, namely that (14) is an equivalence of categories for every double category $\mathbb{D}$. We then exhibit a double category for which (14) is not an equivalence.

Since $V_\bullet$ is Reedy fibrant, the map (d_0, d_1) is an isofibration in **Cat**, which implies that the composite functors d_i

$$\begin{tikzcd} V_1 \arrow[r, "(d_0,d_1)"'] \arrow[rr, bend left, "d_i"] & V_0 \times V_0 \arrow[r, "\pi_i"'] & V_0 \end{tikzcd}$$

are themselves isofibrations (the two projections $\pi_i\colon V_0 \times V_0 \longrightarrow V_0$ are clearly isofibrations). By Theorem 7.21 this implies that the canonical functor

$$V_1 \times_{V_0} V_1 \longrightarrow V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1 \tag{9}$$

from the pullback to the pseudo pullback is an equivalence of categories.

Next we similarly show that (12) is an equivalence of categories. By Definition 6.3 $(c_h V)_0 = V_0$, the set of objects of V_1 maps to the set of horizontal morphisms of $c_h V$, and the set of morphisms of V_1 maps to the set of squares of $c_h V$. We claim that the functor

$$(c_h V)_1 \xrightarrow{(d_0,d_1)} (c_h V)_0 \times (c_h V)_0 \tag{10}$$

is also an isofibration. First note that $V_1 \longrightarrow V_0 \times V_0$ is an isofibration precisely when any diagram in $c_h V$ of the special form

$$\begin{tikzcd} \cdot \arrow[d, "v"', "\wr\|"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

with v, w isomorphisms in V_0 and g an object of V_1, can be filled with an isomorphism α in V_1 (a "generating vertically invertible square" in $c_h V$):

$$\begin{tikzcd} \cdot \arrow[r, "g'"] \arrow[d, "v"', "\wr\|"] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}.$$

Now consider a general such diagram in $c_h(V)$. This has the form

$$\begin{tikzcd} \cdot \arrow[d, "v"', "\wr\|"] & & & & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdots & \cdot \arrow[r, "g_n"'] & \cdot \end{tikzcd}, \tag{11}$$

where the bottom edge is an equivalence class of a path of composable horizontal morphisms by Definition 6.3. We next insert vertical identity morphisms and fill in the individual squares to obtain the following allowable compatible arrangement

$$\begin{tikzcd} \cdot \arrow[r, "g_1'"] \arrow[d, "v"', "\wr\|"] \arrow[dr, phantom, "\alpha_1"] & \cdot \arrow[r, "g_2'"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_2"] & \cdot \arrow[d, equal] \cdots & \cdot \arrow[r, "g_n'"] \arrow[d, equal] \arrow[dr, phantom, "\alpha_n"] & \cdot \arrow[d, "w", "\|\wr"'] \\ \cdot \arrow[r, "g_1"'] & \cdot \arrow[r, "g_2"'] & \cdot \cdots & \cdot \arrow[r, "g_n"'] & \cdot \end{tikzcd}$$

(this is possible because $V_1 \longrightarrow V_0 \times V_0$ is an isofibration). The equivalence class in $c_h(V)$ of this allowable compatible arrangement gives the required filling for (11).

So (10) is indeed an isofibration. Reasoning as for (9), this implies that the following functor is an equivalence of categories.

$$(c_h V)_1 \times_{(c_h V)_0} (c_h V)_1 \longrightarrow (c_h V)_1 \overset{\mathrm{ps}}{\times}_{(c_h V)_0} (c_h V)_1 \tag{12}$$

We claim that the unit component η_V is a weak equivalence. First we observe that every component of the counit $\varepsilon\colon c_h N_h \Longrightarrow \mathrm{Id}_{\mathbf{DblCat}}$ is an isomorphism of double categories, since the nerve functor N_h is fully faithful. Also, one of the triangle identities states that $N_h \varepsilon_{\mathbb{D}} \cdot \eta_{N_h \mathbb{D}} = \mathrm{id}$, so $\eta_{N_h\mathbb{D}}$ is an isomorphism. The naturality of the unit η therefore gives us a commutative diagram

$$\begin{tikzcd}
N_h\mathbb{D} = N_h c_h N_h \mathbb{D} \arrow[d, "N_h c_h r"'] & N_h\mathbb{D} \arrow[l, "\eta_{N_h\mathbb{D}}"'] \arrow[d, "r"] \\
N_h c_h V & V \arrow[l, "\eta_V"]
\end{tikzcd}$$

in which the morphism $\eta_{N_h\mathbb{D}}$ is a levelwise equivalence (since it is an isomorphism), and the morphism r is a levelwise equivalence (by hypothesis). The morphism $N_h c_h r$ is one as well, since $c_h r$ is a weak equivalence ($c_h r$ is an acyclic cofibration since c_h was assumed to be a left Quillen functor and r is an acyclic cofibration). By the 2-out-of-3 property, it follows that η_V is a levelwise equivalence of categories, as claimed.

Consider the following commutative diagram in **Cat**.

$$\begin{tikzcd}
(N_h\mathbb{D})_2 \arrow[r, "\mathrm{Segal}", "\cong"'] \arrow[d, "r_2"', "\wr"] \arrow[dr, phantom, "(D)"] & (N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \arrow[r] \arrow[d, "(r_1, r_1)"] \arrow[dr, phantom, "(C)"] & (N_h\mathbb{D})_1 \overset{\mathrm{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \arrow[d, "(r_1, r_1)"] \\
V_2 \arrow[r, "\mathrm{Segal}"] \arrow[d, "(\eta_V)_2"', "\wr"] \arrow[dr, phantom, "(B)"] & V_1 \times_{V_0} V_1 \arrow[r, "\sim"] \arrow[d, "((\eta_V)_1, (\eta_V)_1)"] \arrow[dr, phantom, "(A)"] & V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1 \arrow[d, "((\eta_V)_1, (\eta_V)_1)"] \\
(N_h c_h V)_2 \arrow[r, "\cong", "\mathrm{Segal}"'] & (N_h c_h V)_1 \times_{(N_h c_h V)_0} (N_h c_h V)_1 \arrow[r, "\sim"] & (N_h c_h V)_1 \overset{\mathrm{ps}}{\times}_{(N_h c_h V)_0} (N_h c_h V)_1
\end{tikzcd}$$

Note that (B) and (D) commute by the definition of Segal maps, while the commutativity of (A) and (C) follows from the universal property of the pseudo pullbacks. The vertical functors r_2 and $(\eta_V)_2$ are equivalences of categories, since r and η_V are weak equivalences from above. The bottom edge of (C) is an equivalence, since it is (9). The bottom edge of (A) is an equivalence as it is (12) (recall that $(N_h\mathbb{E})_0 = \mathbb{E}_0$ and $(N_h\mathbb{E})_1 = \mathbb{E}_1$ for any double category $\mathbb{E}$).

We claim that the top edge of (C) is an equivalence of categories. Since r_0 and r_1 are equivalences, the right vertical functor

$$(r_1, r_1)\colon (N_h\mathbb{D})_1 \overset{\mathrm{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow V_1 \overset{\mathrm{ps}}{\times}_{V_0} V_1$$

is an equivalence. Moreover, since η_V is a levelwise equivalence, the 2-out-of-3 property and the commutativity of (A) imply that the functor

$$V_1 \times_{V_0} V_1 \longrightarrow (N_h c_h V)_1 \times_{(N_h c_h V)_0} (N_h c_h V)_1$$

is an equivalence. Also, the commutativity of (B) and the 2-out-of-3 property imply that $V_2 \longrightarrow V_1 \times_{V_0} V_1$ is an equivalence. The commutativity of (D) then implies that

$$(N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow V_1 \times_{V_0} V_1$$

is an equivalence. Finally, the commutativity of (C) implies that the canonical map

$$(N_h\mathbb{D})_1 \times_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \longrightarrow (N_h\mathbb{D})_1 \overset{\mathrm{ps}}{\times}_{(N_h\mathbb{D})_0} (N_h\mathbb{D})_1 \tag{13}$$

is an equivalence of categories, as claimed.

The map (13) is nothing but

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1 \overset{\mathrm{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1. \tag{14}$$

The objects of $\mathbb{D}_1 \overset{\mathrm{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1$ are diagrams of the form

$$\begin{tikzcd} & {} \arrow[d, "\cong"] \arrow[r, "g"] & {} \\ {} \arrow[r, "f"'] & {} & \end{tikzcd} \tag{15}$$

and morphisms of $\mathbb{D}_1 \overset{\mathrm{ps}}{\times}_{\mathbb{D}_0} \mathbb{D}_1$ are diagrams of the form

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\alpha"] & {} \arrow[d] & {} \arrow[l, "\cong"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & {} \arrow[d] \\ {} \arrow[r, "f'"'] & {} & {} \arrow[l, "\cong"] \arrow[r, "g'"'] & {} \end{tikzcd} \tag{16}$$

where the middle square is a *commutative square of vertical morphisms*. The objects and morphisms of $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ are those of (15) and (16) where the isomorphisms are identities. The canonical functor (14) is given by this inclusion.

We now exhibit a double category $\mathbb{D}$ where the canonical functor (14) is not an equivalence, which then implies that our original assumption on the existence of a transferred Reedy structure on **DblCat** is false. Let $\mathbb{D}$ be the double category with four distinct objects A, B, C, D and only the following nontrivial morphisms.

$$\begin{tikzcd} & C \arrow[r, "g"] \arrow[d, "\cong"] & D \\ A \arrow[r, "f"'] & B & \end{tikzcd}$$

There are no nontrivial squares. Suppose that the canonical functor (14) is essentially surjective. Then there exist objects X, Y, Z and morphisms as in the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\alpha"] & B \arrow[d] & C \arrow[l, "\cong"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\beta"] & D \arrow[d] \\ X \arrow[r, "f'"'] & Y \arrow[r, equal] & Y \arrow[r, "g'"'] & Z \end{tikzcd}$$

with α and β vertically invertible squares. However, since all squares in $\mathbb{D}$ are trivial, we conclude that $B = Y = C$, a contradiction. Hence, the canonical functor (14) is not essentially surjective and is not an equivalence.

We conclude that it is impossible to transfer the categorical Reedy model structure on $\mathbf{Cat}^{\Delta^{op}}$ to **DblCat**. □

8 Model structures arising from Grothendieck topologies

Until now we have considered model structures transferred from $\mathbf{Cat}^{\Delta^{op}}$. But one can also view double categories as internal categories, and for these a homotopy theory has already been developed. Model structures on internal categories in a category $\mathbf{C}$ satisfying certain hypotheses have been studied by Everaert–Kieboom–Van der Linden [32]. As they point out, there are various notions of internal equivalence of internal categories. The notions full and faithful representably make sense for internal functors as in Definition 8.8, but notions of essential surjectivity depend on a class of morphisms $\mathcal{E}$ in $\mathbf{C}$. If this class of morphisms is the class $\mathcal{E}_{\mathcal{T}}$ of $\mathcal{T}$–epimorphisms for a Grothendieck topology $\mathcal{T}$ on $\mathbf{C}$, then the internal equivalences are the weak equivalences for a model structure on $\mathsf{Cat}(\mathbf{C})$ in good cases. The classes $\mathsf{we}(\mathcal{T})$, $\mathsf{fib}(\mathcal{T})$, and $\mathsf{cof}(\mathcal{T})$ are defined in [32] so that the following theorem holds. We will recall the classes $\mathsf{we}(\mathcal{T})$, $\mathsf{fib}(\mathcal{T})$, and $\mathsf{cof}(\mathcal{T})$ below.

Theorem 8.1 (5.5 of [32]) *Let $\mathbf{C}$ be a finitely complete category such that $\mathsf{Cat}(\mathbf{C})$ is finitely complete and finitely cocomplete and $\mathcal{T}$ is a Grothendieck topology on $\mathbf{C}$. If the class $\mathsf{we}(\mathcal{T})$ of $\mathcal{T}$–equivalences has the 2-out-of-3 property and $\mathbf{C}$ has enough $\mathcal{T}$–projectives, then*

$$(\mathsf{Cat}(\mathbf{C}), \mathsf{we}(\mathcal{T}), \mathsf{fib}(\mathcal{T}), \mathsf{cof}(\mathcal{T}))$$

is a model category.[4]

We apply this theorem to $\mathbf{C} = \mathbf{Cat}$ for various Grothendieck topologies in this section. In Section 8.2 we show that the model structure associated to the simplicially surjective topology τ is the same as the vertically transferred diagram categorical structure. This second construction using [32] is advantageous, as it gives us more information about the model structures, such as simple descriptions of cofibrations and cofibrant replacements. We will show in Section 9 that the model structure associated to the categorically surjective basis τ' in Section 8.3 turns out to be the same as the model structure on **DblCat** viewed as a category of algebras over a 2–monad. We also show that the trivial topology induces the trivial model structure associated to the 2–category $\mathbf{DblCat_h}$.

8.1 Homotopy theory of internal categories according to Everaert–Kieboom–Van der Linden [32]

First we recall the notions and results of Everaert–Kieboom–Van der Linden [32], specialized to the case of internal categories in $\mathbf{C} = \mathbf{Cat}$.

Definition 8.2 Let $\mathsf{iso}\colon \mathsf{Cat}(\mathbf{Cat}) \longrightarrow \mathsf{Grpd}(\mathbf{Cat})$ be the right adjoint to the inclusion $\mathsf{Grpd}(\mathbf{Cat}) \longrightarrow \mathsf{Cat}(\mathbf{Cat})$. For $\mathbb{B} \in \mathsf{Cat}(\mathbf{Cat})$, this means that $\mathsf{iso}(\mathbb{B})_1$ has objects the invertible horizontal morphisms of $\mathbb{B}$ and morphisms the horizontally invertible squares. The category $\mathsf{iso}(\mathbb{B})_1$ is a subcategory of $\mathbb{B}_1$. Composition is the vertical composition of squares.

[4] As Tim Van der Linden pointed out to us, the factorizations in this model structure are functorial if there exists a functor $P\colon \mathbf{C} \longrightarrow \mathbf{C}$ and a natural transformation $\eta\colon P \Longrightarrow \mathrm{Id}_{\mathbf{C}}$ such that $P(C)$ is $\mathcal{T}$–projective and η_C is a $\mathcal{T}$–epimorphism for all objects C of $\mathbf{C}$. This is the case with the τ–topology, the τ'–topology, and the trivial topology we consider in this paper.

Definition 8.3 If $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a double functor, then the *mapping path object* is the category $(\mathbb{P}_F)_0$ defined as the pullback below:

$$\begin{tikzcd}
(\mathbb{P}_F)_0 \arrow[r, "\overline{F}_0"] \arrow[d, "\overline{t}"'] & \mathsf{iso}(\mathbb{B})_1 \arrow[d, "t"] \\
\mathbb{A}_0 \arrow[r, "F_0"'] & \mathbb{B}_0 .
\end{tikzcd}$$

The functor $t\colon \mathsf{iso}(\mathbb{B})_1 \longrightarrow \mathbb{B}_0$ is the target for horizontal composition, exactly as in Definition 2.1. The objects of $(\mathbb{P}_F)_0$ are

$$(a,\, f\colon b \xrightarrow{\cong} F_0 a)$$

for a an object of $\mathbb{A}$ and f a horizontal isomorphism of $\mathbb{B}$. The morphisms are pairs

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} ,\ \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) \tag{17}$$

where k is a vertical morphism in $\mathbb{A}$ and α is a horizontally invertible square in $\mathbb{B}$. Composition in $(\mathbb{P}_F)_0$ comes from the vertical composition in $\mathbb{A}$ and $\mathbb{B}$.

Definition 8.4 Let $\mathcal{T}$ be a topology on **Cat**. We denote by $Y_{\mathcal{T}}$ the composition of the Yoneda embedding Y with the sheafification functor.

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "Y"'] \arrow[rr, bend left, "Y_{\mathcal{T}}"] & \mathbf{Set}^{\mathbf{Cat}^{\mathrm{op}}} \arrow[r] & \mathrm{Sh}(\mathbf{Cat}, \mathcal{T})
\end{tikzcd}$$

A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is a $\mathcal{T}$*–epimorphism* if $Y_{\mathcal{T}}(p)$ is an epimorphism. In this case, we often simply say that p is $\mathcal{T}$*–epi*. We denote the class of $\mathcal{T}$–epimorphisms by $\mathcal{E}_{\mathcal{T}}$.

To show that a functor is $\mathcal{T}$–epi, we will use the following characterization of $\mathcal{T}$–epimorphisms.

Proposition 8.5 (Corollary III.7.5 and III.7.6 of Mac Lane–Moerdijk [65], Proposition 2.12 in Everaert–Kieboom–Van der Linden [32]) *Let $\mathcal{T}$ be a topology on a small category. A morphism $p\colon E \longrightarrow B$ is $\mathcal{T}$–epi if and only if for every morphism*

$g\colon X \longrightarrow B$ there exists a covering sieve $\{f_i\colon U_i \longrightarrow X\}_i$ and a family of morphisms $\{u_i\colon U_i \longrightarrow E\}_i$ such that for every $i \in I$ the following diagram commutes.

(18)
$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, "u_i"'] & X \arrow[d, "g"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

Remark 8.6 Suppose K is a basis for the topology $\mathcal{T}$ in Proposition 8.5 and such g and p are given. Then there exists a covering *sieve* $\{f_i\colon U_i \longrightarrow X\}_i$ in $\mathcal{T}$ and a family of morphisms $\{u_i\colon U_i \longrightarrow E\}_i$ making (18) commute if and only if there exists a covering *family* $\{g_j\colon V_j \longrightarrow X\}_j$ in K and a family of morphisms $\{v_j\colon V_j \longrightarrow E\}_j$ making (18) commute. Thus, in Proposition 8.5 one could equivalently replace the phrase "covering sieve" by the phrase "covering family in a given basis".

Proof A sieve S is a covering sieve in the topology $\mathcal{T}$ generated by the basis K if and only if it contains a covering family R from the basis K. Suppose such a covering sieve $\{f_i\}_i$ with morphisms $\{u_i\}_i$ is given. Then this covering sieve contains a covering family in L for which (18) commutes. Conversely, given such a covering family $\{g_j\}_j$ with morphisms $\{v_j\}_j$, we may take the sieve

$$\{g_j \circ w | w \text{ a morphism such that } g_j \circ w \text{ exists}\}_j$$

generated by the family $\{v_j\}_j$. Then the family $\{v_j \circ w\}$ makes (18) commute. □

Example 8.7 Suppose $\mathcal{T}$ is a topology on a small category. If a morphism p admits a right inverse q, then p is a $\mathcal{T}$–epimorphism. To see this using Proposition 8.5, take any covering family $\{f_i\}$ of X and the morphisms $u_i = qgf_i$.

Definition 8.8 Let $s\colon \mathsf{iso}(\mathbb{B})_1 \longrightarrow \mathbb{B}_0$ be the source map for horizontal composition, as in Definition 2.1. A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is *essentially $\mathcal{T}$–surjective* if the functor

$$(\mathbb{P}_F)_0 \xrightarrow{\overline{F}_0} \mathsf{iso}(\mathbb{B})_1 \xrightarrow{s} \mathbb{B}_0$$

given by

$$s \circ \overline{F}_0(a, b \xrightarrow{\cong} F_0 a) = b,$$

and

$$s \circ \overline{F}_0 \left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) = \begin{tikzcd} b \arrow[d, "j"'] \\ b' \end{tikzcd}$$

is a $\mathcal{T}$–epimorphism. If F is additionally *fully faithful* in the sense of Bunge–Paré [19], that is, if

$$\begin{tikzcd} \mathbb{A}_1 \arrow[r, "F_1"] \arrow[d, "(s,t)"'] & \mathbb{B}_1 \arrow[d, "(s,t)"] \\ \mathbb{A}_0 \times \mathbb{A}_0 \arrow[r, "F_0 \times F_0"'] & \mathbb{B}_0 \times \mathbb{B}_0 \end{tikzcd}$$

is a pullback square in **Cat**, then F is called a *$\mathcal{T}$–equivalence*. We denote the class of $\mathcal{T}$–equivalences by $\mathsf{we}(\mathcal{T})$. Note that a double functor F is fully faithful if and only if its restrictions to the two functors

$$(\mathrm{Obj}\,\mathbb{A}, \mathrm{Hor}\,\mathbb{A}) \longrightarrow (\mathrm{Obj}\,\mathbb{B}, \mathrm{Hor}\,\mathbb{B})$$
$$(\mathrm{Ver}\,\mathbb{A}, \mathrm{Sq}\,\mathbb{A}) \longrightarrow (\mathrm{Ver}\,\mathbb{B}, \mathrm{Sq}\,\mathbb{B})$$

are both fully faithful.

Remark 8.9 If **A** and **B** are 1–categories, then a functor $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ is essentially surjective in the usual sense if and only if $s \circ \overline{F}_0$ is surjective. The functor F is fully faithful in the sense of Definition 8.8 if and only if it is fully faithful in the usual sense. The notions of essential surjectivity and fully faithfulness can be found in any standard reference on category theory, such as pages 19 and 115 of Borceux [10] or pages 14, 15 and 93 of Mac Lane [64].

Definition 8.10 Let $\mathcal{E}$ be a class of functors. We say that a category **P** is *projective with respect to the class* $\mathcal{E}$ if for every functor $G\colon \mathbf{Q} \longrightarrow\!\!\!\!\to \mathbf{R}$ in $\mathcal{E}$ and every functor

$H\colon \mathbf{P} \longrightarrow \mathbf{R}$ there exists a functor $F\colon \mathbf{P} \longrightarrow \mathbf{Q}$ such that $GF = H$:

$$\begin{tikzcd} & \mathbf{P} \arrow[dl, dashed, "\exists F"'] \arrow[d, "H"] \\ \mathbf{Q} \arrow[r, two heads, "G"'] & \mathbf{R}. \end{tikzcd}$$

The double arrow head signifies that G is in $\mathcal{E}$. If $\mathcal{E}$ is the class $\mathcal{E}_{\mathcal{T}}$ of $\mathcal{T}$–epi functors, then a projective category $\mathbf{P}$ is called *$\mathcal{T}$–projective*.[5]

Definition 8.11 We say that there are *enough $\mathcal{T}$–projectives* in $\mathbf{Cat}$ if for every category $\mathbf{C}$ there exists a $\mathcal{T}$–projective category $\mathbf{P}$ and a $\mathcal{T}$–epi functor $\mathbf{P} \twoheadrightarrow \mathbf{C}$.

Definition 8.12 A double functor $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ is a *$\mathcal{T}$–fibration* if the induced morphism $(r_F)_0$ in the diagram below is a $\mathcal{T}$–epimorphism:

(19)
$$\begin{tikzcd} \mathsf{iso}(\mathbb{E})_1 \arrow[dr, dashed, "(r_F)_0"] \arrow[drr, bend left, "\mathsf{iso}\,(F)_1"] \arrow[ddr, bend right, "t"'] & & \\ & (\mathbb{P}_F)_0 \arrow[r, "\bar{F}_0"] \arrow[d, "\bar{t}"'] & \mathsf{iso}(\mathbb{B})_1 \arrow[d, "t"] \\ & \mathbb{E}_0 \arrow[r, "F_0"'] & \mathbb{B}_0 . \end{tikzcd}$$

On objects, $(r_F)_0$ is

$$\left(f'\colon b' \xrightarrow{\cong} e \right) \mapsto \left(e,\ Ff'\colon Fb' \xrightarrow{\cong} Fe \right).$$

Remark 8.13 If $\mathbf{E}$ and $\mathbf{B}$ are 1–categories, then $(r_F)_0$ is surjective if and only if F is an isofibration. Recall from Section 7.1 that a functor $F\colon \mathbf{E} \longrightarrow \mathbf{B}$ is said to be an *isofibration* if for any object e of $\mathbf{E}$ and any isomorphism $f\colon b \cong Fe$ in $\mathbf{B}$, there exists a lift to an isomorphism $f'\colon b' \cong e$ in $\mathbf{E}$.

Example 8.14 (Example 5.2 of [32]) If $\mathbf{1}$ denotes the terminal double category, then the unique double functor $\mathbb{D} \longrightarrow \mathbf{1}$ is a $\mathcal{T}$–fibration for every topology $\mathcal{T}$ on $\mathbf{Cat}$. Hence, in the model structure of Theorem 8.1, every object is fibrant.

[5] This was also called *$\mathcal{E}_{\mathcal{T}}$–projective* in [32].

Proof In diagram (19), the functor $\overline{t}$ is the identity, so that $(r_F)_0$ is simply t. A right inverse assigns horizontal identities to objects, and horizontal identity squares to vertical morphisms. By Example 8.7, the functor $(r_F)_0$ is a $\mathcal{T}$–epimorphism. □

Proposition 8.15 (Proposition 5.6 of [32]) *A double functor $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ is an acyclic $\mathcal{T}$–fibration if and only if it is fully faithful and F_0 is a $\mathcal{T}$–epi functor.*

Definition 8.16 A double functor is a $\mathcal{T}$*–cofibration* if it has the left lifting property with respect to all acyclic $\mathcal{T}$–fibrations.

Proposition 8.17 (Proposition 5.9 of [32]) *A double functor $J\colon \mathbb{A} \longrightarrow \mathbb{X}$ is a $\mathcal{T}$–cofibration if and only if J_0 has the left lifting property with respect to all $\mathcal{T}$–epi functors.*

Corollary 8.18 *A double category $\mathbb{X}$ is cofibrant in the $\mathcal{T}$–model structure if and only if $\mathbb{X}_0$ is $\mathcal{T}$–projective.*

Proof By Proposition 8.17, $\mathbb{X}$ is cofibrant if and only if for any $\mathcal{T}$–epi functor G and any functor H, a lift ℓ

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \mathbf{Q} \arrow[d, "G"] \\ \mathbb{X}_0 \arrow[r, "H"'] \arrow[ur, dashed, "\ell"] & \mathbf{R} \end{tikzcd}$$

exists, or equivalently, $\mathbb{X}_0$ is $\mathcal{T}$–projective. □

Remark 8.19 These results allow us to construct a *cofibrant replacement* $\mathbb{E}$ for a double category $\mathbb{B}$ in the $\mathcal{T}$–model structure. Let $\mathbb{E}_0$ be a $\mathcal{T}$–projective category and $K_0\colon \mathbb{E}_0 \twoheadrightarrow \mathbb{B}_0$ a $\mathcal{T}$–epimorphism (we will explicitly give $\mathbb{E}_0$ and K_0 in the τ– and τ'–structures in Remark 8.31 and Remark 8.48). Let $\mathbb{E}_1$ be the following pullback in **Cat**:

$$\begin{tikzcd} \mathbb{E}_1 \arrow[r, "K_1"] \arrow[d, "(s,t)"'] & \mathbb{B}_1 \arrow[d, "(s,t)"] \\ \mathbb{E}_0 \times \mathbb{E}_0 \arrow[r, "K_0 \times K_0"'] & \mathbb{B}_0 \times \mathbb{B}_0 . \end{tikzcd}$$

Then the double graph $\mathbb{E}$ carries a unique double category structure such that $K = (K_0, K_1)$ is a double functor by Lemma 5.14 of [32]. Since K is fully faithful and K_0 is $\mathcal{T}$–epi, $K\colon \mathbb{E} \longrightarrow \mathbb{B}$ is an acyclic $\mathcal{T}$–fibration by Proposition 8.15. By Corollary 8.18, $\mathbb{E}$ is a cofibrant double category, and hence a cofibrant replacement for $\mathbb{B}$ in the $\mathcal{T}$–model structure.

Remark 8.20 We conclude from Proposition 8.5 that if $\mathcal{T}' \subseteq \mathcal{T}$ are Grothendieck topologies, then every $\mathcal{T}'$–epimorphism is a $\mathcal{T}$–epimorphism. Thus we conclude from Definitions 8.8, 8.12 and 8.16 that if $\mathcal{T}' \subseteq \mathcal{T}$ then $\mathsf{we}(\mathcal{T}') \subseteq \mathsf{we}(\mathcal{T})$, $\mathsf{fib}(\mathcal{T}') \subseteq \mathsf{fib}(\mathcal{T})$, and $\mathsf{cof}(\mathcal{T}') \supseteq \mathsf{cof}(\mathcal{T})$.[6]

8.2 Model structure from the simplicially surjective basis

For a category $\mathbf{C}$, we write $\mathbf{C}_k$ for the k–th set of the nerve $N\mathbf{C}$. Similarly for a functor F we write $(NF)_k = F_k$. We say that a functor F is *simplicially surjective* if F_k is surjective for all $k \geq 0$. We prove that the associated topology on **Cat** induces a model structure on **DblCat** which coincides with the vertical analogue of the transferred diagram categorical structure of Section 7.6. This second construction gives additional information about (the vertical analogue of) the transferred diagram categorical structure, including an explicit form for the cofibrant replacement functor.

Lemma 8.21 *For a category* $\mathbf{C}$ *define*

$$K(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} \mid F \text{ a simplicially surjective functor}\}.$$

Then K *is a basis for a Grothendieck topology* τ *on* **Cat**.

Proof

(1) If F is an isomorphism, then NF is an isomorphism and each F_k is bijective.

(2) If $\{F\} \in K(\mathbf{C})$ and $G\colon \mathbf{C}' \longrightarrow \mathbf{C}$ is any functor, consider the pullback $\pi_2\colon \mathbf{D} \times_{\mathbf{C}} \mathbf{C}' \longrightarrow \mathbf{C}'$ in **Cat** of F along G. Since the nerve functor preserves limits, $N\pi_2$ is the pullback of NF along NG. Then $N\pi_2$ is simplicially surjective, since limits of simplicial sets are formed pointwise.

(3) If $G \circ F$ exists and F_k and G_k are surjective for all $k \geq 0$, then clearly $G_k \circ F_k$ is surjective for all $k \geq 0$, and $\{G \circ F\}$ is a covering family. □

Lemma 8.22 *A functor* $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ *is* τ*–epi for the Grothendieck topology* τ *if and only if* p *is simplicially surjective.*

Proof If p is τ–epi, then take $g = 1_{\mathbf{B}}$ in Proposition 8.5 with Remark 8.6 to obtain $pu = f$ for some covering family $\{f\}$ in K. Then f_k is surjective for all $k \geq 0$. Hence p is simplicially surjective.

If p is simplicially surjective, then $\{p\}$ is a covering family in K, and so is the pullback π_2 of p along g. Applying Proposition 8.5 with Remark 8.6 again, we see that p is τ–epi. □

[6] We thank Joachim Kock for posing this question.

Recall that the objects of the k–th category $((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) = (N_v\mathbb{A})_k$ of the vertical nerve are composable strings of k vertical morphisms, and the morphisms are vertically composable strings of k squares. The composition is horizontal composition of vertical strings of squares. Fully faithful double functors and τ–equivalences have a useful characterization in terms of the vertical nerve.

Proposition 8.23 *A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is fully faithful if and only if for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is fully faithful.

Proof Since the nerve functor preserves pullbacks, and pullbacks of simplicial sets are formed pointwise, it follows from Definition 8.8 that F is fully faithful if and only if each $((F_0)_k, (F_1)_k)$ is fully faithful. □

Proposition 8.24 *A double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a τ–equivalence if and only if for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an equivalence of categories.

Proof The double functor F is essentially τ–surjective if and only if $s \circ \overline{F}_0$ is τ–epi. But this occurs if and only if $(s \circ \overline{F}_0)_k$ is surjective for each k, which is equivalent to the essential surjectivity of $((F_0)_k, (F_1)_k)$ by Remark 8.9. Fully faithfulness follows from Proposition 8.23. □

Corollary 8.25 *The class* $\mathsf{we}(\tau)$ *of τ–equivalences has the 2-out-of-3 property.*

Proposition 8.26 **Cat** *has enough τ–projectives.*

Proof We first construct a τ–projective category $\mathbf{P}$ from a category $\mathbf{C}$. Let

$$\mathbf{P} := \coprod_{n \geq 0} \mathbf{C}_n \cdot [n] = \Big(\coprod_{c \in \mathbf{C}_0} [0] \Big) \coprod \Big(\coprod_{n \geq 1} \coprod_{(f_1, \ldots, f_n) \in \mathbf{C}_n} [n] \Big)$$

where $[n] = \{0, 1, \ldots, n\}$ is the $(n+1)$–element ordinal viewed as a category and $\mathbf{C}_n \cdot [n]$ denotes the copower of the category $[n]$ with the set $\mathbf{C}_n$, as recalled in Remark 6.9.

Suppose we have functors

$$\begin{tikzcd} & \mathbf{P} \arrow[d, "H"] \\ \mathbf{Q} \arrow[r, two heads, "G"'] & \mathbf{R}, \end{tikzcd}$$

and G is τ–epi. We denote H on the $(f_1,\dots,f_n)$–summand of $\mathbf{P}$ by

$$H_{(f_1,\dots,f_n)}\colon [n] \longrightarrow \mathbf{R}$$

and by H_c on the c–summand. If $H_{(f_1,\dots,f_n)}(j-1\le j)=r_j$ for $1\le j\le n$, then there exists $(q_1,\dots,q_n)\in\mathbf{Q}_n$ such that $G_n(q_1,\dots,q_n)=(r_1,\dots,r_n)$ since G is τ–epi. We define a functor

$$F_{(f_1,\dots,f_n)}\colon [n] \longrightarrow \mathbf{Q}$$

$$F_{(f_1,\dots,f_n)}(j-1\le j) := q_j$$

for $1\le j\le n$. Similarly, for $c\in\mathbf{C}_0$, there exists $d\in\mathbf{Q}_0$ such that $G_0(d)=H_c(0)$. We define $F_c(0)=d$. Putting these F's together, we obtain a functor $F\colon \mathbf{P}\longrightarrow\mathbf{Q}$ such that $GF=H$, and we conclude that $\mathbf{P}$ is τ–projective.

Next we construct a τ–epi functor $L\colon \mathbf{P}\twoheadrightarrow\mathbf{C}$. On the $(f_1,\dots,f_n)$–summand of $\mathbf{P}$ define L as

$$L_{(f_1,\dots,f_n)}(j-1\le j) := f_j$$

for $1\le j\le n$. Similarly on the c summand, $L_c(0):=c$. We claim that for each $k\ge 0$, $L_k\colon \mathbf{P}_k\longrightarrow\mathbf{C}_k$ is surjective. Note that

$$\mathbf{P}_k=\Big(\coprod_{c\in\mathbf{C}_0}[0]_k\Big)\coprod\Big(\coprod_{n\ge 1}\ \coprod_{(f_1,\dots,f_n)\in\mathbf{C}_n}[n]_k\Big).$$

If $k\ge 1$, and $(f_1,\dots,f_k)\in\mathbf{C}_k$, then L_k maps $(0\le 1, 1\le 2,\dots,k-1\le k)$ in the $(f_1,\dots,f_k)$–component of $\mathbf{P}_k$ to $(f_1,\dots,f_k)$. Similarly if $c\in\mathbf{C}_0$, then L_0 maps 0 in the c–component of $\mathbf{P}_0$ to c. Hence L_k is surjective for all $k\ge 0$, L is τ–epi, and **Cat** has enough τ–projectives. □

Theorem 8.27 *The simplicially surjective topology τ on* **Cat** *determines a model structure*

$$(\mathsf{Cat}(\mathbf{Cat}), \mathsf{we}(\tau), \mathsf{fib}(\tau), \mathsf{cof}(\tau)).$$

Proof The category $\mathsf{Cat}(\mathbf{Cat})$ is complete and cocomplete by Theorem 4.1. The class of τ–equivalences has the 2-out-of-3 property by Corollary 8.25 and **Cat** has enough τ–projectives by Proposition 8.26, so we can apply Theorem 8.1. □

We now give a more explicit description of the fibrations, acyclic fibrations, fibrant objects, and cofibrant replacement.

Proposition 8.28 *Let $F\colon \mathbb{E} \longrightarrow \mathbb{B}$ be a double functor.*

(1) *F is a τ–fibration if and only if for each $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{E}_0)_k, (\mathbb{E}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an isofibration.

(2) *F is an acyclic τ–fibration if and only if the functor $((F_0)_k, (F_1)_k)$ is fully faithful and surjective on objects for each $k \geq 0$.*

Proof

(1) Applying the nerve to Diagram (19), we see that F is a τ–fibration if and only if $(r_F)_{0k}$ is surjective for all $k \geq 0$. By Remark 8.13, this is the case if and only if for each $k \geq 0$ the functor $((F_0)_k, (F_1)_k)$ is an isofibration. Here $(\mathsf{iso}(\mathbb{B})_1)_k = \mathsf{iso}((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$ is the category with objects composable strings of k vertical morphisms and with morphisms vertical strings of vertically composable squares that are each horizontally invertible.

(2) From Proposition 8.23, F is fully faithful if and only if each $((F_0)_k, (F_1)_k)$ is fully faithful. Since F_0 is τ–epi if and only if $(F_0)_k$ is surjective for each $k \geq 0$, the statement follows from Proposition 8.15. $\square$

Corollary 8.29 *The model structure on* **DblCat** *induced by the simplicially surjective topology τ on* **Cat** *coincides with the model structure obtained by transferring the diagram categorical structure across the vertical categorification-vertical nerve adjunction $c_v \dashv N_v$. The transfer across $c_v \dashv N_v$ is completely analogous to the transfer across $c_h \dashv N_h$ in Section 7.6.*

Proof From Propositions 8.24 and 8.28 we see that the weak equivalences and fibrations of the two model structures coincide. $\square$

Remark 8.30 By Example 8.14, every double category is fibrant in the τ–model structure.

Remark 8.31 We can now easily construct a cofibrant replacement $\mathbb{E}$ for a double category $\mathbb{B}$ in the τ–model structure. Let $\mathbb{E}_0$ be the τ–projective category associated to $\mathbb{B}_0$ with projection $K_0 := L$ as in the proof of Proposition 8.26. Then $\mathbb{E}$ and K as defined in Remark 8.19 are a cofibrant replacement for $\mathbb{B}$ in the τ–model structure.

Proposition 8.32 *Let F be a τ–equivalence. Then BF, as in Definition 5.12, is a weak homotopy equivalence of topological spaces.*

Proof By Proposition 8.24, F is a τ–equivalence if and only if $(N_v F)_k$ is an equivalence of categories for each $k \geq 0$. Since $((N_v F)_k)_\ell = (N_d F)_{k\ell}$, we see that $(N_d F)_{k*}$ is a weak equivalence of simplicial sets for each $k \geq 0$. Hence $\mathrm{diag}(N_d F)$ is a weak equivalence of simplicial sets, and $BF = |\mathrm{diag}(N_d F)|$ is a weak homotopy equivalence. □

Remark 8.33 For each $m \in \mathbb{N}$, the assignment

$$\mathbf{C} \mapsto K_m(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} | F_k \text{ surjective for all } 0 \leq k \leq m\}$$

is a basis for a Grothendieck topology τ_m on **Cat**. We obtain a τ_m–model structure as above, though τ_m–equivalences will of course not necessarily be weak homotopy equivalences of classifying spaces.

8.3 Model structure from the categorically surjective basis

A functor is said to be *categorically surjective* if it is surjective on objects and full. It is straightforward to check that a basis for a Grothendieck topology on **Cat** is given by declaring a covering family to be a single categorically surjective functor. We call this topology τ'. In this section we study the model structure on **DblCat** induced by τ'. In Section 9 we show that this model structure is the model structure on **DblCat** viewed as a category of algebras over a 2–monad.

As before we start with a characterization of the τ'–epi functors. We will use this to prove a 2-out-of-3 property for the τ'–equivalences.

Proposition 8.34 *A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is τ'–epi if and only if there is a subcategory $\mathbf{H} \hookrightarrow \mathbf{E}$ such that $p|_{\mathbf{H}}\colon \mathbf{H} \longrightarrow \mathbf{B}$ is surjective on objects and full. Thus, a τ'–epi functor is not necessarily categorically surjective.*

Proof Suppose that p is τ'–epi. Then by Proposition 8.5 and Remark 8.6 there is a commutative square

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[d, "u"'] & \mathbf{B} \arrow[d, "1_{\mathbf{B}}"] \\ \mathbf{E} \arrow[r, "p"'] & \mathbf{B}, \end{tikzcd}$$

where f is surjective on objects and full. For each pair of objects x, y in $\mathbf{U}$, we have a commutative triangle:

$$\begin{tikzcd} \mathbf{U}(x,y) \arrow[rr, "f(x,y)"] \arrow[dr, "u(x,y)"'] & & \mathbf{B}(fx,fy) \\ & \mathbf{E}(ux,uy) \arrow[ur, "p(ux,uy)"'] & \end{tikzcd}$$

Since $f(x,y)$ is surjective, so is $p(ux,uy)$. Let $\mathbf{H} = \mathrm{im}(u)$. Thus, $p|_{\mathbf{H}}$ is surjective on objects and full.

Conversely, let $\mathbf{H} \overset{\ell}{\hookrightarrow} \mathbf{E}$ be a subcategory such that $p|_{\mathbf{H}}$ is surjective on objects and full, and let $g\colon \mathbf{X} \longrightarrow \mathbf{B}$ be any functor. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{H} \times_{\mathbf{B}} \mathbf{X} \arrow[r, "p'"] \arrow[d, "s"'] \arrow[dr, phantom, "\text{pb}"] & \mathbf{X} \arrow[dd, "g"] \\ \mathbf{H} \arrow[d, "\ell"'] \arrow[dr, "p|_{\mathbf{H}}"] & \\ \mathbf{E} \arrow[r, "p"'] & \mathbf{B}. \end{tikzcd}$$

Then p' is surjective on objects and full since $p|_{\mathbf{H}}$ is. Further, $gp' = p\ell s$. By Proposition 8.5 and Remark 8.6, it follows that p is τ'–epi. $\square$

Even though the τ'–epi functors do not coincide with the categorically surjective functors, they do give rise to the same projective objects.

Corollary 8.35 *A category* $\mathbf{P}$ *is* τ'*–projective if and only if it is projective with respect to categorically surjective functors.*

Proof We use the same notation as in Definition 8.10. If $\mathbf{P}$ is τ'–projective, then $\mathbf{P}$ is projective with respect to categorically surjective functors because every categorically surjective functor is τ'–epi by Proposition 8.34. For the converse, suppose $\mathbf{P}$ is projective with respect to categorically surjective functors, and suppose G is τ'–epi. Then by Proposition 8.34 again, there exists an inclusion $\ell\colon \mathbf{Q}' \longrightarrow \mathbf{Q}$ such that $G\ell$ is a categorically surjective functor. Thus there exists an F' such that $G\ell F' = H$. If we let $F = \ell F'$ then we see that $\mathbf{P}$ is τ'–projective. $\square$

Even better, we can characterize the τ'–projective objects.

Proposition 8.36 *A category is τ'–projective if and only if it is a free category on a directed graph.*

Proof Let $\mathbf{C}$ be a free category on a directed graph Γ; we show that $\mathbf{C}$ is τ'–projective. Suppose $G\colon \mathbf{Q} \twoheadrightarrow \mathbf{R}$ is τ'–epi and $H\colon \mathbf{C} \longrightarrow \mathbf{R}$ is a functor. Let $\mathbf{Q}' \subseteq \mathbf{Q}$ be a subcategory such that $G|_{\mathbf{Q}'}\colon \mathbf{Q}' \longrightarrow \mathbf{R}$ is surjective on objects and full. Let $U\colon \mathbf{Cat} \longrightarrow \mathbf{Graph}$ denote the forgetful functor. Then there is a map of directed graphs which makes the following diagram commute:

$$\begin{tikzcd} & \Gamma \arrow[dl, dashed] \arrow[d, "U(H)|_\Gamma"] \\ U\mathbf{Q}' \arrow[r, "U(G|_{\mathbf{Q}'})"'] & U\mathbf{R} \end{tikzcd}$$

and induces a functor F such that

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, dashed, "F"'] \arrow[d, "H"] \\ \mathbf{Q}' \arrow[r, "G|_{\mathbf{Q}'}"'] & \mathbf{R} \end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd} & \mathbf{Q}' \arrow[dl, hook] & \mathbf{C} \arrow[l] \arrow[d, "H"] \\ \mathbf{Q} \arrow[rr, two heads, "G"'] & & \mathbf{R} \end{tikzcd}$$

commutes and $\mathbf{C}$ is τ'–projective.

For the converse, suppose $\mathbf{P}$ is a τ'–projective category. Let $\mathbf{Q}$ be the free category on the underlying directed graph of $\mathbf{P}$ with the identity arrows omitted. Note that every arrow of $\mathbf{Q}$ is either a path $\langle f_1, \dots, f_n\rangle$ of nonidentity arrows f_i in $\mathbf{P}$ or an empty path $\langle\rangle_A$ (forming the identity arrow 1_A on the object A in $\mathbf{Q}$). We define an identity-on-objects functor $G\colon \mathbf{Q} \longrightarrow \mathbf{P}$ by $G(\langle\rangle_A) = 1_A$ and $G(\langle f_1, \dots, f_n\rangle) = f_n \circ \cdots \circ f_1$. The functor G is clearly categorically surjective.

Consider the diagram

$$\begin{tikzcd} & \mathbf{P} \arrow[dl, dashed, "F"'] \arrow[d, "1_{\mathbf{P}}"] \\ \mathbf{Q} \arrow[r, two heads, "G"'] & \mathbf{P}. \end{tikzcd}$$

Since $\mathbf{P}$ is τ'–projective, there exists a functor F which makes the diagram commute as indicated. When $F(f) = \langle f_1, \ldots, f_n \rangle$, we say that $F(f)$ *has length* n. Note that since F is a functor, $F(1_A)$ has length 0. By the commutativity of the diagram, the length of $F(f)$ is greater than or equal to 1 for any nonidentity arrow f.

Now let f be any nonidentity arrow in $\mathbf{P}$, and suppose $F(f) = \langle f_1, \ldots, f_n \rangle$ as well as $F(f_i) = \langle f_{i1}, \ldots, f_{im_i} \rangle$. Since $GF = 1_{\mathbf{P}}$, we know that $f = f_n \circ \cdots \circ f_1$. Since F is a functor, we also know that $F(f) = F(f_n) \circ \cdots \circ F(f_1) = \langle f_{11}, \ldots, f_{nm_n} \rangle$. Thus

$$\langle f_1, \ldots, f_n \rangle = \langle f_{11}, \ldots, f_{nm_n} \rangle.$$

Since all the f_i and f_{ij} are nonidentity arrows, it follows that $m_i = 1$ and $f_{i1} = f_i$ for all $i \in \{1, \ldots, n\}$. Summarizing, if $F(f) = \langle f_1, \ldots, f_n \rangle$, then $F(f_i)$ has length 1 for each i.

Let $\mathbf{C}$ be the free category on the directed graph with the objects of $\mathbf{P}$ as vertices and as edges those arrows f of $\mathbf{P}$ for which the length of $F(f)$ is 1. By the argument above, the functor F factors through a functor $\widetilde{F}\colon \mathbf{P} \longrightarrow \mathbf{C}$ as in the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[d, "\mathrm{incl}"'] & \mathbf{P} \arrow[l, "\widetilde{F}"'] \arrow[d, "1_{\mathbf{P}}"] \arrow[dl, dashed, "F"'] \\
\mathbf{Q} \arrow[r, "G"'] & \mathbf{P}.
\end{tikzcd}$$

Let $\widetilde{G}$ be the restriction of G to $\mathbf{C}$. It is obvious that $\widetilde{G}\widetilde{F} = 1_{\mathbf{P}}$.

Now consider the other composition, $\widetilde{F}\widetilde{G}$. This is obviously the identity on objects. For a morphism $\langle f_1, \ldots, f_n \rangle$ in $\mathbf{C}$, we have

$$\begin{aligned}
\widetilde{F}\widetilde{G}(\langle f_1, \ldots, f_n \rangle) &= \widetilde{F}(f_n \circ \cdots \circ f_1) \\
&= \widetilde{F}(f_n) \circ \cdots \circ \widetilde{F}(f_1) \\
&= \langle f_n \rangle \circ \cdots \circ \langle f_1 \rangle \\
&= \langle f_1, \ldots, f_n \rangle,
\end{aligned}$$

where the second to last equality follows from the fact that the f_i are edges in the graph on which $\mathbf{C}$ is free, in other words $\widetilde{F}(f_i)$ has length 1. So $\widetilde{F}\widetilde{G}(\langle f_1, \ldots, f_n \rangle) = \langle f_1, \ldots, f_n \rangle$. We conclude that $\mathbf{P} \cong \mathbf{C}$, so $\mathbf{P}$ is a free category on a directed graph. □

Proposition 8.37 *If a functor $p\colon \mathbf{A} \longrightarrow \mathbf{B}$ is τ'–epi, then $p_k\colon \mathbf{A}_k \longrightarrow \mathbf{B}_k$ is surjective for all $k \geq 0$.*

Proof By Proposition 8.5 and Remark 8.6 there exists a functor f surjective on objects and full such that $1_{\mathbf{B}} \circ f = p \circ u$ for some u. Since $f_k = p_k \circ u_k$ is surjective, so is p_k. □

Proposition 8.38 *If a double functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a τ'–equivalence, then for every $k \geq 0$ the functor*

$$((F_0)_k, (F_1)_k)\colon ((\mathbb{A}_0)_k, (\mathbb{A}_1)_k) \longrightarrow ((\mathbb{B}_0)_k, (\mathbb{B}_1)_k)$$

is an equivalence of categories.

Proof Since F is fully faithful, $((F_0)_k, (F_1)_k)$ is fully faithful for all $k \geq 0$ by Proposition 8.23. Since F is essentially τ'–surjective, $s \circ \overline{F}_0$ is τ'–epi and hence $(s \circ \overline{F}_0)_k = (s)_k \circ (\overline{F}_0)_k$ is surjective for all $k \geq 0$ by Proposition 8.37. Remark 8.9 then implies that $((F_0)_k, (F_1)_k)$ is essentially surjective for all $k \geq 0$. □

Lemma 8.39 *Suppose $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ are double functors and two of GF, G, or F are τ'–equivalences. Then the third double functor is fully faithful.*

Proof By Proposition 8.38 the vertical nerves of the two τ'–equivalences are levelwise equivalences of categories. Hence the vertical nerve of the third double functor is also levelwise an equivalence of categories, and in particular levelwise fully faithful. By Proposition 8.23, this implies that the third functor is fully faithful. □

Lemma 8.40 *Suppose $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ are double functors and GF and F are τ'–equivalences. Then G is essentially τ'–surjective.*

Proof We need to show that $s \circ \overline{G}_0$ is τ'–epi. Let $\mathbf{H}_F \subseteq (\mathbb{P}_F)_0$ and $\mathbf{H}_{GF} \subseteq (\mathbb{P}_{GF})_0$ be subcategories such that $s \circ \overline{F}_0|_{\mathbf{H}_F}$ and $s \circ \overline{(GF)}_0|_{\mathbf{H}_{GF}}$ are surjective on objects and full. Define a full subcategory $\mathbf{H}_G$ of $(\mathbb{P}_G)_0 = \mathbb{B}_0 \times_{\mathbb{C}_0} \mathsf{iso}(\mathbb{C})_1$ by applying F_0 to the first coordinate of $\mathbf{H}_{GF}$ as follows. For any object $(a, c \overset{\cong}{\Rightarrow} G_0 F_0 a)$ in $\mathbf{H}_{GF}$, we have an object $(F_0 a, c \overset{\cong}{\Rightarrow} G_0(F_0 a))$ in $\mathbf{H}_G$. For any morphism

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} \;,\; \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 F_0 a \arrow[d, "G_0 F_0 k"] \\ c' \arrow[r, "\cong"'] & G_0 F_0 a' \end{tikzcd} \right)$$

in $\mathbf{H}_{GF}$ we have a morphism

$$\left(\begin{tikzcd} F_0 a \arrow[d, "F_0 k"'] \\ F_0 a' \end{tikzcd} , \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0(F_0 a) \arrow[d, "G_0(F_0 k)"] \\ c' \arrow[r, "\cong"'] & G_0(F_0 a') \end{tikzcd} \right)$$

in $\mathbf{H}_G$. Then we see as follows that $s \circ \overline{G}_0|_{\mathbf{H}_G}\colon \mathbf{H}_G \to \mathbb{C}_0$ is surjective on objects and full. If $c \in \mathbb{C}_0$, there exists an object $(a, c \overset{\cong}{\Rightarrow} G_0 F_0 a)$ in $\mathbf{H}_{GF}$, with $(F_0 a, c \overset{\cong}{\Rightarrow} G_0(F_0 a))$ in $\mathbf{H}_G$ and $s \circ \overline{G}_0((F_0 a, c \overset{\cong}{\Rightarrow} G_0 F_0 a)) = c$. So $s \circ \overline{G}_0|_{\mathbf{H}_G}$ is surjective on objects. If $c \xrightarrow{j} c'$ is a morphism in $\mathbb{C}_0$ and $(F_0 a, c \overset{\cong}{\Rightarrow} G_0(F_0 a))$ and $(F_0 a', c' \overset{\cong}{\Rightarrow} G_0(F_0 a'))$ are objects of $\mathbf{H}_G$, then there exists a morphism

$$\left(\begin{tikzcd} a \arrow[d, "k"'] \\ a' \end{tikzcd} , \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 F_0 a \arrow[d, "G_0 F_0 k"] \\ c' \arrow[r, "\cong"'] & G_0 F_0 a' \end{tikzcd} \right)$$

in $\mathbf{H}_{GF}$ which gives rise to a morphism

$$\left(\begin{tikzcd} F_0 a \arrow[d, "F_0 k"'] \\ F_0 a' \end{tikzcd} , \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0(F_0 a) \arrow[d, "G_0(F_0 k)"] \\ c' \arrow[r, "\cong"'] & G_0(F_0 a') \end{tikzcd} \right)$$

in $\mathbf{H}_G$ that maps to j under $s \circ \overline{G}_0|_{\mathbf{H}_G}$. We conclude that $s \circ \overline{G}_0|_{\mathbf{H}_G}$ is surjective on objects and full and therefore $s \circ \overline{G}_0$ is τ'–epi. This implies that G is essentially τ'–surjective. □

Lemma 8.41 *Suppose $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ are double functors and GF and G are τ'–equivalences. Then F is essentially τ'–surjective.*

Proof We need to show that $s \circ \overline{F}_0$ is τ'–epi. Let $\mathbf{H}_{GF} \subseteq (\mathbb{P}_{GF})_0$ be a subcategory such that $s \circ \overline{GF}_0|_{\mathbf{H}_{GF}}$ is surjective on objects and full. Define a full subcategory $\mathbf{H}_F$ of $(\mathbb{P}_F)_0 = \mathbb{A}_0 \times_{\mathbb{B}_0} \mathsf{iso}(\mathbb{B})_1$ with object set

$$\operatorname{Obj} \mathbf{H}_F := \{(a, b \overset{\cong}{\Rightarrow} F_0 a) | (a, G_0(b \overset{\cong}{\Rightarrow} F_0 a)) \in \operatorname{Obj} \mathbf{H}_{GF}\}.$$

Then we can see as follows that $s \circ \overline{F}_0|_{\mathbf{H}_F}\colon \mathbf{H}_F \to \mathbb{B}_0$ is surjective on objects and full. If $b \in \mathbb{B}_0$, then $G_0 b \in \mathbb{C}_0$, and there is an object $(a, G_0 b \overset{\cong}{\Rightarrow} G_0 F_0 a) \in \mathbf{H}_{GF}$, because $s \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is surjective on objects. However, $((G_0)_0, (G_1)_0)$ is fully faithful, that is, G restricted to the objects of $\mathbb{B}$ and the horizontal morphisms of $\mathbb{B}$ is a fully faithful functor of categories. So there is a unique isomorphism $b \overset{\cong}{\Rightarrow} F_0 a$ whose image under G is $G_0 b \overset{\cong}{\Rightarrow} G_0 F_0 a$. Hence, $(a, b \overset{\cong}{\Rightarrow} F_0 a) \in \mathbf{H}_F$ and this object maps to b under $s \circ \overline{F}_0|_{\mathbf{H}_F}$. We conclude that $s \circ \overline{F}_0|_{\mathbf{H}_F}$ is surjective on objects.

Moreover, if $b \xrightarrow{j} b'$ is a morphism in $\mathbb{B}_0$ and $(a, b \overset{\cong}{\Rightarrow} F_0 a)$ and $(a', b' \overset{\cong}{\Rightarrow} F_0 a')$ are objects of $\mathbf{H}_F$, then $G_0 j$ is a morphism of $\mathbb{C}_0$, and since $s \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is full, there is a morphism of the form

$$\left(\begin{array}{c} a \\ k \big\downarrow \\ a' \end{array} \;,\; \begin{tikzcd} G_0 b \arrow[r, "G(\cong)"] \arrow[d, "G_0 j"'] \arrow[dr, phantom, "\alpha"] & G_0 F_0 a \arrow[d, "G_0 F_0 k"] \\ G_0 b' \arrow[r, "G(\cong)"'] & G_0 F_0 a' \end{tikzcd} \right)$$

in $\mathbf{H}_{GF}$. However, the functor $((G_0)_1, (G_1)_1)$ is fully faithful, so there is a unique square β, such that

$$G \left(\begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right) = \alpha.$$

Moreover, β is also horizontally invertible. Hence

$$\left(\begin{array}{c} a \\ k \big\downarrow \\ a' \end{array} \;,\; \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd} \right)$$

is a morphism in $\mathbf{H}_F$ which maps to j under $s \circ \overline{F}_0$. We conclude that $s \circ \overline{F}_0|_{\mathbf{H}_F}$ is surjective on objects and full, so $s \circ \overline{F}_0$ is τ'–epi and F is essentially τ'–surjective. □

Lemma 8.42 *Suppose that* $\mathbb{A} \xrightarrow{F} \mathbb{B} \xrightarrow{G} \mathbb{C}$ *are double functors, and* F *and* G *are* τ'*–equivalences. Then* $G \circ F$ *is essentially* τ'*–surjective.*

Proof We need to show that $s \circ (\overline{GF})_0$ is τ'–epi. Let $\mathbf{H}_F \subseteq (\mathbb{P}_F)_0$ and $\mathbf{H}_G \subseteq (\mathbb{P}_G)_0$ be subcategories such that $s \circ \overline{F}_0|_{\mathbf{H}_F}$ and $s \circ \overline{G}_0|_{\mathbf{H}_G}$ are surjective on objects and full.

Let $\mathbf{H}_{GF}$ be the full subcategory of $(\mathbb{P}_{GF})_0 = \mathbb{A}_0 \times_{\mathbb{C}_0} \mathsf{iso}(\mathbb{C})_1$, with objects

$$\begin{aligned}\operatorname{Obj}\mathbf{H}_{GF} := \{&(a, c \xrightarrow{\cong} G_0 b \xrightarrow{\cong} G_0 F_0 a) \mid \\ &(b, c \xrightarrow{\cong} G_0 b) \in \mathbf{H}_G, (a, b \xrightarrow{\cong} F_0 a) \in \mathbf{H}_F\}.\end{aligned}$$

Suppose that $c \in \mathbb{C}_0$, then there are objects $(b, c \xrightarrow{\cong} G_0 b) \in \mathbf{H}_G$ and $(a, b \xrightarrow{\cong} F_0 a) \in \mathbf{H}_F$, since $s \circ \overline{G}_0|_{\mathbf{H}_G}$ and $s \circ \overline{F}_0|_{\mathbf{H}_F}$ are surjective on objects. So $(a, c \xrightarrow{\cong} G_0 b \xrightarrow{\cong} G_0 F_0 a) \in \mathbf{H}_{GF}$, and this object maps to c under $s \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$.

Next, suppose that $c \xrightarrow{j} c'$ is a morphism of $\mathbb{C}_0$ and that $(a, c \xrightarrow{\cong} G_0 b \xrightarrow{\cong} G_0 F_0 a)$ and $(a', c' \xrightarrow{\cong} G_0 b' \xrightarrow{\cong} G_0 F_0 a')$ are objects of $\mathbf{H}_{GF}$. Then there exist morphisms

$$\left(\begin{tikzcd} b \arrow[d, "k_b"'] \\ b' \end{tikzcd}, \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 b \arrow[d, "G_0 k_b"] \\ c' \arrow[r, "\cong"'] & G_0 b' \end{tikzcd}\right)$$

in $\mathbf{H}_G$, and

$$\left(\begin{tikzcd} a \arrow[d, "k_a"'] \\ a' \end{tikzcd}, \begin{tikzcd} b \arrow[r, "\cong"] \arrow[d, "k_b"'] \arrow[dr, phantom, "\beta"] & F_0 a \arrow[d, "F_0 k_a"] \\ b' \arrow[r, "\cong"'] & F_0 a' \end{tikzcd}\right)$$

in $\mathbf{H}_F$, and therefore

$$\left(\begin{tikzcd} a \arrow[d, "k_a"'] \\ a' \end{tikzcd}, \begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "j"'] \arrow[dr, phantom, "\alpha"] & G_0 b \arrow[r, "\cong"] \arrow[d, "G_0 k_b"] \arrow[dr, phantom, "G\beta"] & G_0 F_0 a \arrow[d, "G_0 F_0 k_a"] \\ c' \arrow[r, "\cong"'] & G_0 b' \arrow[r, "\cong"'] & G_0 F_0 a' \end{tikzcd}\right)$$

is a morphism of $\mathbf{H}_{GF}$ that maps to $c \xrightarrow{j} c'$ under $s \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$. So we have proved that $s \circ (\overline{GF})_0|_{\mathbf{H}_{GF}}$ is surjective on objects and full. We conclude $s \circ (\overline{GF})_0$ is τ'–epi and GF is essentially τ'–surjective. □

The previous four lemmas are summarized in the following theorem.

Theorem 8.43 *The class* $\mathsf{we}(\tau')$ *of* τ'*–equivalences has the 2-out-of-3 property.*

Proposition 8.44 **Cat** *has enough* τ'*–projectives.*

Proof Suppose $\mathbf{C}$ is a category. Let $\mathbf{P}$ be the free category on the underlying directed graph of $\mathbf{C}$. This is τ'–projective by Proposition 8.36. The functor $\mathbf{P} \longrightarrow \mathbf{C}$, which is the identity on objects and defined by composition on paths of morphisms, is surjective on objects and full, so it is τ'–epi. Thus **Cat** has enough τ'–projectives. $\square$

Theorem 8.45 *The categorically surjective topology τ' determines a model structure*

$$(\mathsf{Cat}(\mathbf{Cat}), \mathsf{we}(\tau'), \mathsf{fib}(\tau'), \mathsf{cof}(\tau')).$$

Proof The category $\mathsf{Cat}(\mathbf{Cat})$ is complete and cocomplete by Theorem 4.1. The class of τ'–equivalences has the 2-out-of-3 property by Theorem 8.43 and **Cat** has enough τ'–projectives by Proposition 8.44, so we can apply Theorem 8.1. $\square$

Remark 8.46 By Example 8.14, every double category is fibrant in the τ'–model structure.

Proposition 8.47 *A double category $\mathbb{X}$ is cofibrant in the τ'–model structure if and only if $\mathbb{X}_0$ is a free category on a directed graph.*

Proof By Corollary 8.18 a double category is cofibrant if and only if $\mathbb{X}_0$ is projective with respect to τ'–epi functors. But by Proposition 8.36, $\mathbb{X}_0$ is projective with respect to τ'–epis if and only if it is a free category on a directed graph. $\square$

Remark 8.48 We can now easily construct a cofibrant replacement $\mathbb{E}$ for a double category $\mathbb{B}$ in the τ'–model structure. Let $\mathbb{E}_0$ be the free category on the underlying directed graph of $\mathbb{B}_0$, and $K_0\colon \mathbb{E}_0 \longrightarrow \mathbb{B}_0$ the functor which is the identity on objects and composition on paths of morphisms. Then $\mathbb{E}_0$ is τ'–projective, and K_0 is a τ'–epimorphism as in the proof of Proposition 8.44. Then $\mathbb{E}$ and K as defined in Remark 8.19 are a cofibrant replacement for $\mathbb{B}$ in the τ'–model structure.

As an immediate consequence of Propositions 8.24 and 8.38, we see that every τ'–equivalence is a τ–equivalence. This also follows from Remark 8.20, since the categorically surjective τ'–topology is contained in the simplicially surjective τ–topology. An interesting question is whether or not a condition slightly stronger than simplicial surjectivity but also slightly weaker than categorical surjectivity would give rise to a model structure with weak equivalences between those of the τ'–structure and the τ–structure. For example, such a condition on a functor is to be *U–split*. However, this condition recovers the τ'–topology instead of something new. In fact, this condition only gives a different basis for the τ'–topology which will be of use in Section 9.

Definition 8.49 Let $U\colon \mathbf{Cat} \longrightarrow \mathbf{Graph}$ be the forgetful functor from categories to directed graphs. We say that a functor p is *U–split* if there exists a morphism q of directed graphs such that $(Up) \circ q = \mathrm{id}$.

Lemma 8.50 *A functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ is U–split if and only if there is a subcategory $\mathbf{H} \hookrightarrow \mathbf{E}$ such that $p|_{\mathbf{H}}\colon \mathbf{H} \longrightarrow \mathbf{B}$ is surjective on objects and full.*

Proof Suppose p is U–split. Then there exists a morphism of directed graphs q such that $Up \circ q = \mathrm{id}$. Let $\mathbf{H}$ be the full subcategory $\mathbf{E}$ whose objects are in the image of q. Then $p|_{\mathbf{H}}$ is surjective on objects and full, as one sees using the directed graph section q.

Conversely, suppose there exists a subcategory $\mathbf{H}$ of $\mathbf{E}$ such that $p|_{\mathbf{H}}$ is categorically surjective. Then $p|_{\mathbf{H}}$ is U–split, and $\mathrm{id} = U(p|\mathbf{H}) \circ q = Up \circ q$ so that p is also U–split. $\square$

Proposition 8.51 *The assignment*

$$\mathbf{C} \mapsto L(\mathbf{C}) := \{\{F\colon \mathbf{D} \longrightarrow \mathbf{C}\} \mid F \text{ is } U\text{–split}\}$$

is a basis for the τ'–topology on **Cat**.

Proof We omit the proof that this is a basis.

Recall that a sieve is a covering sieve in the topology induced by a basis if and only if it contains a covering family from the basis. If S is a τ'–covering sieve, it contains a categorically surjective functor, and hence a U–split functor by Lemma 8.50, so that S is also a covering sieve in the topology induced by L.

Conversely, suppose S is a sieve on $\mathbf{B} \in \mathbf{Cat}$ containing a U–split functor $p\colon \mathbf{E} \longrightarrow \mathbf{B}$ and $p|_{\mathbf{H}}$ is categorically surjective. Then $p \circ i \in S$ for the inclusion $i\colon \mathbf{H} \longrightarrow \mathbf{E}$, and S is a covering sieve in the τ'–topology. $\square$

8.4 Model structure from the trivial topology

On the underlying category of any 2–category $\mathcal{K}$ with finite limits and finite colimits there is the *trivial model structure* as proved by Lack [60] using pseudo limits. A weak equivalence (respectively fibration) in this model structure is a morphism $f\colon A \longrightarrow B$ such that $\mathcal{K}(E, f)\colon \mathcal{K}(E, A) \longrightarrow \mathcal{K}(E, B)$ is a weak equivalence (respectively fibration) for all E in the categorical model structure on **Cat**. Thus f is a weak equivalence if and only if there is a morphism $g\colon B \longrightarrow A$ such that gf and fg are isomorphic via 2–cells to the respective identities. A morphism f is a fibration, or *isofibration*, if and only if for all morphisms $a\colon E \longrightarrow A$ and $b\colon E \longrightarrow B$ and any invertible 2–cell $\beta\colon b \cong fa$, there exists a morphism $a'\colon E \longrightarrow A$ and an invertible 2–cell $\alpha\colon a' \cong a$ with $fa' = b$ and $f\alpha = \beta$ (the notion of isofibration at the end of Section 7.1 is a special case of this). If the 2–category $\mathcal{K}$ is merely a 1–category, then the trivial model structure agrees with the usual trivial model structure: the weak equivalences are exactly the isomorphisms and all morphisms are both fibrations and cofibrations. The trivial model structure on the underlying category of a 2–category $\mathcal{K}$ is compatible with the **Cat**–enrichment as proved in [60].

Thus **DblCat** admits three trivial model structures, depending on whether we take as 2–cells the horizontal natural transformations, the vertical natural transformations, or only trivial 2–cells. When we say *trivial model structure on* **DblCat** we mean the one arising from the 2–category $\mathbf{DblCat_h} = \mathsf{Cat}(\mathbf{Cat})$ which has horizontal natural transformations as its 2–cells.

The following theorem summarizes Section 7 of Everaert–Kieboom–Van der Linden's paper [32] and Theorem 3.3 of Lack's paper [60] applied to $\mathcal{K} = \mathsf{Cat}(\mathbf{C})$ to conclude that the τ_{triv}–model structure coincides with the trivial model structure on the underlying category of the 2–category $\mathsf{Cat}(\mathbf{C})$.

Theorem 8.52 *Let $\mathbf{C}$ be a finitely complete category such that $\mathsf{Cat}(\mathbf{C})$ is finitely complete and finitely cocomplete. Let τ_{triv} denote the trivial topology[7] on $\mathbf{C}$. Then the following hold.*

(1) *A morphism p in $\mathbf{C}$ is τ_{triv}–epi if and only if there exists a morphism q such that $pq = \mathrm{id}$.*

(2) *An internal functor F is an acyclic τ_{triv}–fibration if and only if there exists an internal functor G such that $FG = \mathrm{id}$ and $GF \cong \mathrm{id}$.*

(3) *Every object of $\mathbf{C}$ is τ_{triv}–projective, and hence $\mathbf{C}$ has enough projectives.*

(4) *Assume for the rest of this theorem that the τ_{triv}–equivalences have the 2-out-of-3 property. Then we have the τ_{triv}–model structure on $\mathsf{Cat}(\mathbf{C})$ of Theorem 8.1.*

(5) *Every object of $\mathsf{Cat}(\mathbf{C})$ is fibrant and cofibrant.*

(6) *The weak equivalences in the τ_{triv}–model structure are precisely the equivalences in the 2–category $\mathsf{Cat}(\mathbf{C})$.*

(7) *The τ_{triv}–model structure coincides with the trivial model structure of [60] on the underlying category of the 2–category $\mathcal{K} = \mathsf{Cat}(\mathbf{C})$.*

---

[7] In the trivial topology the only covering sieve on an object is the maximal sieve.

Proof

(1) This follows from Proposition 8.5.

(2) This is Proposition 7.1 of [32], and it follows from (1) along with Proposition 8.15.

(3) This follows immediately from (1).

(4) The model structure follows from (3) and the hypotheses we have made.

(5) Every object is fibrant by Example 8.14, and every object is cofibrant by (2).

(6) Since every object is fibrant and cofibrant, the weak equivalences are precisely the homotopy equivalences. In Section 3 of [32], a cocylinder on $\mathsf{Cat}(\mathbf{C})$ is given such that two internal functors are homotopic if and only if they are naturally isomorphic. Hence, the weak equivalences are precisely the equivalences.

(7) By Theorem 3.3 of [60], the weak equivalences in the trivial model structure on $\mathsf{Cat}(\mathbf{C})$ are the equivalences. By 3.4 of [60], the acyclic fibrations in the trivial model structure are the morphisms in (2). Since the classes of weak equivalences and acyclic fibrations in the two model structures are the same, we conclude that the two model structures coincide. □

Remark 8.53 The assumption that the τ_{triv}–equivalences have the 2-out-of-3 property can be removed by proving directly that the τ_{triv}–equivalences are the equivalences in the 2–category $\mathsf{Cat}(\mathbf{C})$ using (1) and fully faithfulness.

The trivial model structure on a category of internal categories is much like the Strøm structure of [79]. This analogy is made precise in Section 7 of [32].

We finish this section by comparing the categorical model structure on **Cat** with the trivial model structure on **DblCat** that arises from $\mathbf{DblCat_h}$. Our comparison uses the horizontal embedding $\mathbb{H}$ of **Cat** into **DblCat**, which is the same as the embedding

$$\mathsf{Cat}(\mathbf{Set}) \longrightarrow \mathsf{Cat}(\mathbf{Cat})$$

induced by the embedding $\mathbf{Set} \longrightarrow \mathbf{Cat}$.

Proposition 8.54 *The functor* $(\mathbf{H}\text{-})_0\colon \mathbf{DblCat} \longrightarrow \mathbf{Cat}$ *maps the weak equivalences and fibrations of the trivial model structure on* **DblCat** *to weak equivalences and fibrations in the categorical model structure on* **Cat**. *In particular,* $(\mathbf{H}\text{-})_0$ *is a right Quillen functor.*

Proof The functor $(\mathbf{H}\text{-})_0$ preserves weak equivalences because it is the underlying functor of the 2–functor $\mathbf{H}\colon \mathbf{DblCat_h} \longrightarrow \mathbf{Cat}$, which maps equivalences to equivalences.

To prove that $(\mathbf{H}\text{-})_0$ preserves fibrations, we use the characterization of isofibrations (in a 2–category in general, and in **Cat** in particular) at the beginning of Section 8.4. If F is a double functor that is an isofibration, and $\beta\colon b \Longrightarrow (\mathbf{H}F)_0 \circ a$ is a natural isomorphism in **Cat**, then we obtain the required α by applying $\mathbb{H}$ to the lifting problem, and then applying $(\mathbf{H}\text{-})_0$ to the solution α' of the new lifting problem in **DblCat**. $\square$

Corollary 8.55 *The adjunction*

$$\begin{tikzcd} \mathbf{Cat} \arrow[rr, bend left, "\mathbb{H}"] \arrow[rr, phantom, "\perp"] & & \mathbf{DblCat} \arrow[ll, bend left, "(\mathbf{H}-)_0"] \end{tikzcd}$$

is a Quillen adjunction.

Proof This follows immediately from Proposition 8.54. $\square$

Proposition 8.56 *The functor $\mathbb{H}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat}$ preserves and reflects weak equivalences, fibrations, and cofibrations. In other words, a functor F is a weak equivalence (respectively fibration, respectively cofibration) in the categorical model structure on **Cat** if and only if $\mathbb{H}F$ is a weak equivalence (respectively fibration, respectively cofibration) in the trivial model structure on **DblCat** that arises from the 2–category $\mathbf{DblCat_h}$. As a consequence, the trivial model structure on **DblCat** from $\mathbf{DblCat_h}$ extends the categorical model structure on **Cat** as a horizontally embedded subcategory.*

Proof The 2–functor $\mathbb{H}\colon \mathbf{Cat} \longrightarrow \mathbf{DblCat}$ is fully faithful in the 2–categorical sense, so it preserves and reflects equivalences (=weak equivalences).

The functor $\mathbb{H}$ preserves and reflects isofibrations because of the characterization of isofibrations at the beginning of Section 8.4 and the fact that horizontal natural transformations between functors $\mathbb{E} \longrightarrow \mathbb{H}\mathbf{C}$ are in bijective correspondence with natural transformations between the underlying 1–functors.

Cofibrations are preserved by $\mathbb{H}$ by Corollary 8.55. The functor $\mathbb{H}$ reflects cofibrations as follows. If F is a functor such that the double functor $\mathbb{H}F$ is a cofibration, and G

is an acyclic fibration in **Cat**, then any diagram in **DblCat**

$$\begin{tikzcd} \mathbb{H}\mathbf{C} \arrow[r, "K"] \arrow[d, "\mathbb{H}F"'] & \mathbb{H}\mathbf{C}' \arrow[d, "\mathbb{H}G"] \\ \mathbb{H}\mathbf{D} \arrow[r, "L"'] & \mathbb{H}\mathbf{D}' \end{tikzcd}$$

admits a lift, as $\mathbb{H}G$ is an acyclic fibration by the above. Since $\mathbb{H}$ is fully faithful, this lift gives us a lift in **Cat**. Hence the functor F is a cofibration as well. □

After this discussion of model structures on **DblCat** as a category of internal categories in Section 8, we now turn to a model structure on **DblCat** as a category of algebras and show that this model structure is the same as the categorically surjective model structure. We will make use of the trivial model structure on the underlying category of the 2–category $\mathsf{Cat}(\mathbf{Graph})$.

9 A model structure for DblCat as the 2–category of algebras for a 2–monad

Every 2–category of strict algebras over a 2–monad T with rank (that is, which preserves α–filtered colimits for some α) on a locally finitely presentable 2–category $\mathcal{K}$ admits a canonical cofibrantly generated **Cat**-enriched model structure as proved by Lack [60]. It is obtained by transferring the trivial model structure on the 2–category $\mathcal{K}$ described in Section 8.4. A strict morphism of strict T–algebras is a weak equivalence (respectively fibration) if and only if its underlying morphism is an equivalence (respectively isofibration). We prove that the model structure induced by the categorically surjective topology τ' can be recovered in this way.

The interest in having these two different descriptions of the τ'–model structure lies in the fact that they allow a characterization of the flexible double categories (Corollary 9.4 and Remark 9.7). We will see that the cofibrant replacement in the τ'–model structure of Remark 8.48 is left 2–adjoint to the inclusion of strict algebras, strict morphisms, and 2–cells into strict algebras, pseudo morphisms, and 2–cells. In particular, the cofibrant replacement in Remark 8.48 coincides with the cofibrant replacement in the algebra structure of Lack [60]. Another interesting aspect of the two descriptions of the τ'–model structure is that **DblCat** provides a good setting for comparing Lack's categorical model structure on **2–Cat** [58; 59] to a model structure on algebras over a 2–monad.

Recall that the adjunction **Graph** $\dashv$ **Cat** induces a Cartesian monad M on the category **Graph** of small directed graphs. These directed graphs are nonreflexive; a choice of distinguished arrows called identities is *not* part of the data. The algebras of M are precisely the small categories. A *Cartesian monad* is a monad whose underlying functor preserves pullbacks and whose unit and multiplication are Cartesian natural transformations, that is, each of their naturality squares is a pullback square. Since M is Cartesian, it induces a 2–monad $\overline{M}$ on $\mathsf{Cat}(\mathbf{Graph})$, the 2–category of internal categories in $\mathbf{Graph}$. The strict algebras for this 2–monad $\overline{M}$ are pairs $(\mathbb{D}_0, \mathbb{D}_1)$ of M–algebras with source, target, unit, and composition maps compatible with the M–algebra structure. Thus, the strict $\overline{M}$–algebras are precisely double categories. Similarly, strict morphisms and 2–cells for strict $\overline{M}$–algebras are double functors and horizontal natural transformations. The 2–category of strict $\overline{M}$–algebras, strict morphisms, and 2–cells is

$$\overline{M}\text{–}\mathrm{Alg}_{\mathrm{s}} = \mathsf{Cat}(\mathbf{Cat}) = \mathbf{DblCat_h}.$$

The term *algebra* will always mean *strict algebra*, so we occasionally leave off the adjective *strict*.

Let $U\colon \mathbf{Cat} \longrightarrow \mathbf{Graph}$ be the forgetful functor. The functor U induces a 2–functor

$$\bar{U}\colon \mathsf{Cat}(\mathbf{Cat}) \longrightarrow \mathsf{Cat}(\mathbf{Graph})$$

which coincides with the forgetful 2–functor

$$\overline{M}\text{–}\mathrm{Alg}_{\mathrm{s}} \longrightarrow \mathsf{Cat}(\mathbf{Graph}) .$$

An internal category in $\mathbf{Graph}$ is a (nonreflexive) double graph $\mathbb{E}$ with a category structure on $(\mathrm{Obj}\,\mathbb{E}, \mathrm{Hor}\,\mathbb{E})$ and on $(\mathrm{Ver}\,\mathbb{E}, \mathrm{Sq}\,\mathbb{E})$, in other words horizontal compositions are defined in the double graph $\mathbb{E}$ but vertical compositions are not. There are no vertical identity 1–arrows in $\mathbb{E}$, and no vertical identity squares on horizontal 1–morphisms.

Theorem 9.1 *The model structure on* **DblCat** *induced by the* 2*–monad* $\overline{M}$ *is the* τ'*–model structure.*

Proof First we prove that the weak equivalences are the same. Note that a double functor G is fully faithful if and only if $\overline{U}G$ is fully faithful as in Definition 8.8. A double functor G is a weak equivalence as a morphism of algebras if and only if $\overline{U}G$ is a weak equivalence in the trivial model structure on $\mathsf{Cat}(\mathbf{Graph})$, which is the case if and only if $\overline{U}G$ is fully faithful and there exists a morphism q of directed graphs such that $U(s \circ \overline{G}_0) \circ q = \mathrm{id}_{\mathbb{B}_0}$ by Definition 8.8 and Theorem 8.52 (1). That is equivalent to G being fully faithful and $s \circ \overline{G}_0$ being U–split, which is precisely the definition of weak equivalence in the τ'–model structure using Proposition 8.34 and Lemma 8.50. Hence the weak equivalences coincide.

Similarly, a double functor G is a fibration as a morphism of algebras if and only if $\bar{U}G$ is a fibration in the trivial model structure on $\mathsf{Cat}(\mathbf{Graph})$, which is the case if and only if there exists a morphism q of directed graphs such that $(U(r_G)_0) \circ q = \mathrm{id}$ in Diagram (19), which is the case if and only if $(r_G)_0$ is U–split. This is equivalent to G being a fibration in the τ'–model structure. Hence the fibrations coincide. □

Corollary 9.2 *The categorically surjective τ'–model structure is cofibrantly generated and admits an enrichment as a* **Cat***-enriched model category.*

For a 2–monad T on $\mathcal{K}$ as above, let T–Alg_s denote the 2–category of strict T–algebras, strict morphisms, and 2–cells. As usual, we denote by T–Alg the 2–category of strict T–algebras, pseudo morphisms, and 2–cells. As proved by Blackwell–Kelly–Power [8], the inclusion $T\text{–}\mathrm{Alg}_\mathrm{s} \longrightarrow T\text{–}\mathrm{Alg}$ admits a left 2–adjoint denoted $A \mapsto A'$. The counit component $q\colon A' \longrightarrow A$ is a strict morphism, and if q admits a section in T–Alg_s, then A is called *flexible*. The flexible algebras are the closure under flexible colimits of the free algebras. Strict morphisms from A' to B are in bijective correspondence with pseudo morphisms from A to B.

Theorem 9.3 (Theorem 4.12 of Lack [60]) *The cofibrant objects of T–Alg_s are precisely the flexible algebras; in particular, any algebra of the form A' is cofibrant, and is thus a cofibrant replacement for A. Every free algebra is flexible.*

Corollary 9.4 *The cofibrant objects in the τ'–model structure are precisely the flexible double categories. In particular, a double category $\mathbb{X}$ is flexible if and only if $\mathbb{X}_0$ is a free category on a directed graph.*

Proof This follows from Theorem 9.1, Theorem 9.3, and Proposition 8.47. □

We next show that the cofibrant replacement in the τ'–model structure of Remark 8.48 is the same as the left 2–adjoint $\mathbb{A} \mapsto \mathbb{A}'$ to the inclusion $\overline{M}\text{–}\mathrm{Alg}_\mathrm{s} \longrightarrow \overline{M}\text{–}\mathrm{Alg}$. Our method is to verify that the cofibrant replacement has the same 2–universal property as $\mathbb{A}'$. For this we need to identify the pseudo morphisms of strict $\overline{M}$–algebras.

Proposition 9.5 *The pseudo morphism between strict $\overline{M}$–algebras are the pseudo double functors which are strict in the horizontal direction, but weak in the vertical direction.*

Proof Let $\mathbb{A}$ and $\mathbb{B}$ be strict algebras for the 2–monad $\overline{M}$, in other words, $\mathbb{A}$ and $\mathbb{B}$ are internal categories in **Graph** equipped with internal functors $a\colon \overline{M}(\mathbb{A}) \longrightarrow \mathbb{A}$, and $b\colon \overline{M}(\mathbb{B}) \longrightarrow \mathbb{B}$ defining the vertical composition. More specifically, a_0 and b_0 define the vertical compositions of paths of vertical arrows, and a_1 and b_1 define the vertical composition of vertical paths of squares.

A pseudo morphism $(F,\varphi)\colon \mathbb{A} \longrightarrow \mathbb{B}$ consists of an internal functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ in **Graph** and an invertible internal transformation

$$\begin{tikzcd} \overline{M}(\mathbb{A}) \arrow[r, "\overline{M}(F)"] \arrow[d, "a"'] \arrow[dr, phantom, "\varphi"] & \overline{M}(\mathbb{B}) \arrow[d, "b"] \\ \mathbb{A} \arrow[r, "F"'] & \mathbb{B}, \end{tikzcd}$$

given by a morphism of graphs $\varphi\colon (\overline{M}(\mathbb{A}))_0 \longrightarrow \mathbb{B}_1$ which satisfies the coherence conditions from Blackwell–Kelly–Power [8]. It follows from the identity coherence condition that φ sends every object of $\mathbb{A}$ to the corresponding horizontal identity arrow. To every path $\langle u_1, \dots, u_n \rangle$ of compatible vertical arrows in $\mathbb{A}$, φ assigns a horizontally invertible square in $\mathbb{B}$ denoted by

$$\begin{tikzcd} FA_0 \arrow[r, equal] \arrow[d, "Fu_1"'] & FA_0 \arrow[dddd, "F(u_n\circ\cdots\circ u_1)"] \\ FA_1 \arrow[d, dotted] \arrow[r, phantom, "\varphi_{u_1,\dots,u_n}"] & \\ FA_{n-1} \arrow[d, "Fu_n"'] & \\ FA_n \arrow[r, equal] & FA_n . \end{tikzcd}$$

It follows from the second coherence axiom that for any path of paths

$$\langle \langle u_{11}, \dots, u_{1m_1} \rangle, \langle u_{21}, \dots, u_{2m_2} \rangle, \dots, \langle u_{n1}, \dots, u_{nm_n} \rangle \rangle,$$

the pasting in Figure 5 is equal to $\varphi_{u_{11},\dots,u_{1m_1},\dots,u_{n1},\dots,u_{nm_n}}$.

Note that the coherence conditions imply φ is completely determined by its components φ_{u_1,u_2} for composable vertical arrows u_1 and u_2 in $\mathbb{A}$. These φ_{u_1,u_2} are the composition coherence isomorphisms for the underlying vertical pseudo functor of F.

The internal natural transformation φ associates to the empty path on A the unit coherence isomorphism of the underlying vertical pseudo functor of F. The coherence

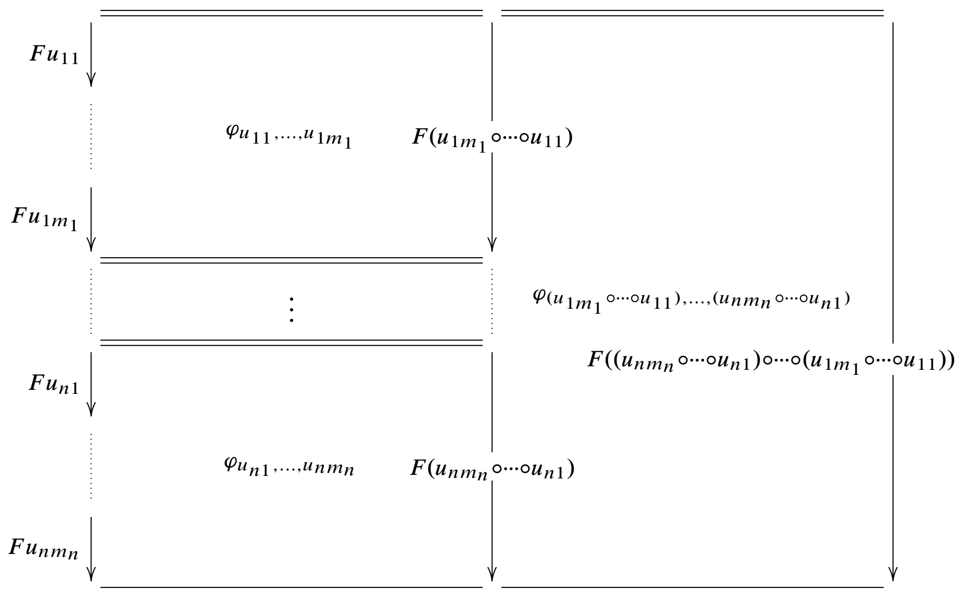


Figure 5: Pasting of associativity coherence squares

conditions on φ contain the coherence conditions for the coherence isomorphisms of a pseudo functor.

Conversely, given a pseudo double functor (weak in the vertical direction) the natural isomorphism φ is defined in terms of the coherence isomorphisms. □

Proposition 9.6 *The cofibrant replacement in the τ'–model structure of Remark 8.48 is isomorphic to the left 2–adjoint $\mathbb{A} \mapsto \mathbb{A}'$ to the inclusion $\overline{M}\text{–Alg}_\mathrm{s} \longrightarrow \overline{M}\text{–Alg}$.*

Proof Let Q denote the cofibrant replacement functor defined on objects in Remark 8.48. Our task is to present a natural isomorphism of categories

$$\overline{M}\text{–Alg}_\mathrm{s}(Q\mathbb{A},\mathbb{B}) \cong \overline{M}\text{–Alg}(\mathbb{A},\mathbb{B}) \tag{20}$$

for strict $\overline{M}$–algebras (double categories) $\mathbb{A}$ and $\mathbb{B}$.

The category $(Q\mathbb{A})_0$ is the free category on the underlying directed graph of $\mathbb{A}_0$, and the category $(Q\mathbb{A})_1$ is the pullback in **Cat**

$$\begin{tikzcd}
(Q\mathbb{A})_1 \arrow[r] \arrow[d, "{(s,t)}"'] & \mathbb{A}_1 \arrow[d, "{(s,t)}"] \\
(Q\mathbb{A})_0 \times (Q\mathbb{A})_0 \arrow[r] & \mathbb{A}_0 \times \mathbb{A}_0 .
\end{tikzcd}$$

In particular, squares of $Q\mathbb{A}$ (the morphisms of $(Q\mathbb{A})_1$) have the form

$$(\langle u_1, \dots, u_m \rangle, \alpha, \langle v_1, \dots, v_n \rangle),$$

where α is a square in $\mathbb{A}$ with the vertical morphism $u = u_m \circ \cdots u_1$ as its horizontal source and the vertical morphism $v = v_n \circ \cdots \circ v_1$ as its horizontal target.

Given a double functor $G\colon Q\mathbb{A} \longrightarrow \mathbb{B}$, we may define a pseudo morphism

$$(F, \phi)\colon \mathbb{A} \longrightarrow \mathbb{B}$$

as follows. On objects, $F(A) = G(A)$; on horizontal morphisms, $F(f) = G(f)$; on vertical morphisms, $F(u) = G(\langle u \rangle)$; and on squares, $F(\alpha) = G(\langle u \rangle, \alpha, \langle v \rangle)$, where u is the horizontal source of α and v is the horizontal target of α. Further, ϕ has components $\phi_{u_1,\dots,u_m} = G(\langle u_1, \dots, u_m \rangle, i^h_{u_m \circ \cdots \circ u_1}, \langle u_m \circ \cdots \circ u_1 \rangle)$ (where $i^h_{u_m \circ \cdots \circ u_1}$ is the horizontal identity square on the vertical morphism $u_m \circ \cdots \circ u_1$).

Given a pseudo morphism $(F, \varphi)\colon \mathbb{A} \longrightarrow \mathbb{B}$, we may define a double functor

$$G\colon Q\mathbb{A} \longrightarrow \mathbb{B}$$

as follows. On objects, $G(A) = F(A)$; on horizontal morphisms, $G(f) = F(f)$; on vertical morphisms,

$$G(\langle u_1, \dots, u_m \rangle) = F(u_m) \circ \cdots \circ F(u_1);$$

and on squares

$$G(\langle u_1, \dots, u_m \rangle, \alpha, \langle v_1, \dots, v_n \rangle) = [\phi_{u_1,\dots,u_m} \quad F\alpha \quad (\varphi_{v_1,\dots,v_m})^{-1}].$$

It is straightforward to see that these two procedures are inverse to each other, and define a bijection on the object sets of Equation (20). We extend this bijection to an isomorphism of categories. If $\theta\colon G_1 \Longrightarrow G_2$ is a horizontal natural transformation of double functors, then we define $\lambda\colon F_1 \Longrightarrow F_2$ by $\lambda_A = \theta_A$ and

$$\lambda_{\langle u_1,\dots,u_m \rangle} = [(\varphi_1)_{u_1,\dots,u_m} \quad \theta_{u_m \circ \cdots \circ u_1} \quad (\varphi_2)^{-1}_{u_1,\dots,u_m}].$$

Conversely, if $\lambda\colon F_1 \Longrightarrow F_2$ is a horizontal natural transformation of pseudo morphisms, then we define $\theta\colon G_1 \Longrightarrow G_2$ by $\theta_A = \lambda_A$ and $\theta_u = \lambda_{\langle u \rangle}$.

Naturality can be easily verified. □

Remark 9.7 Proposition 9.6 allows us to give an alternative proof of Corollary 9.4. In fact, if $\mathbb{A}$ is a cofibrant object in the τ'–model structure, the map $\varnothing \longrightarrow \mathbb{A}$ has the left lifting property with respect to the counit component $q\colon Q\mathbb{A} \longrightarrow \mathbb{A}$, which is an acyclic fibration by Lemma 5.14 of Everaert–Kieboom–Van der Linden [32]. It follows easily that q has a section, and thus $\mathbb{A}$ is flexible. Conversely, if $\mathbb{A}$ is flexible, then q has a section p. In the diagram below, let f be an acyclic fibration in the τ'–model structure and h any map.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \varnothing \arrow[r] \arrow[d] & \mathbb{B} \arrow[d, "f"] \\
Q\mathbb{A} \arrow[r, dashed, "q"] \arrow[urr, dashed, "r"] & \mathbb{A} \arrow[r, "h"'] \arrow[l, bend left, "p"] & \mathbb{C}
\end{tikzcd}$$

Since $Q\mathbb{A}$ is cofibrant, there is a map $r\colon Q\mathbb{A} \longrightarrow \mathbb{B}$ with $fr = hq$. Hence the map $rp\colon \mathbb{A} \longrightarrow \mathbb{B}$ satisfies $frp = hqp = h$. This shows that $\varnothing \longrightarrow \mathbb{A}$ is a cofibration; that is, $\mathbb{A}$ is cofibrant.

The categorical model structure on **2–Cat** of Lack [58; 59] has weak equivalences the strict 2–functors that are biequivalences, fibrations those strict 2–functors with the equivalence lifting property (as defined in [59], not [58]), and cofibrations those strict 2–functors whose underlying functor has the left lifting property with respect to functors that are surjective on objects and full. We can compare this with the τ'–model structure as follows.

Proposition 9.8 *Consider* **2–Cat** *vertically embedded in* **DblCat***. If a* 2*–functor is a cofibration in the* τ'*–model structure on* **DblCat***, then it is a cofibration in the categorical model structure on* **2–Cat***. A* 2*–category is cofibrant in the categorical model structure on* **2–Cat** *if and only if it is cofibrant in the* τ'*–model structure on* **DblCat***. Thus a* 2*–category is flexible in the sense of Lack [58] if and only if it is flexible as an algebra over the* 2*–monad* $\overline{M}$.

Proof Suppose G is a 2–functor such that $\mathbb{V}G$ is a cofibration in the τ'–model structure on **DblCat**. Then $(\mathbb{V}G)_0$ has the left lifting property with respect to τ'–functors by Proposition 8.17. This implies that $(\mathbb{V}G)_0$ has the left lifting property with respect to all functors that are surjective on objects and full by Proposition 8.34. The underlying functor of G is $(\mathbb{V}G)_0$, so G is a cofibration in the categorical structure on **2–Cat**.

A 2–category is cofibrant in the categorical structure on **2–Cat** if and only if its underlying category is projective with respect to all functors that are surjective on objects and full. But this coincides with cofibrant 2–categories in the τ'–model structure on **DblCat** by Corollary 8.18 and Corollary 8.35. □

Remark 9.9 Our characterization of flexible double categories extends Lack's characterization of flexible 2–categories as those 2–categories with underlying category a free category on a graph (Theorem 4.8 (iv) of [58]). See Corollary 9.4 and Proposition 9.8.

The sets of weak equivalences with source and target 2–categories in the two model structures have nontrivial intersection, but neither set of weak equivalences is contained in the other. The $\mathbb{V}$–image of a biequivalence is not necessarily essentially τ'–surjective, though it is fully faithful. For example, consider the inclusion $h\colon \{0\} \longrightarrow \{0 \cong 1\}$. Then h is a biequivalence, and the only object of $(\mathbb{P}_{\mathbb{V}h})_0$ in Definitions 8.3 and 8.8 is $(0, 0 \xrightarrow{\mathrm{id}_0} 0)$. Then $s \circ \overline{\mathbb{V}h}_0$ cannot be surjective on objects, as its target has two objects. By Proposition 8.34, the functor $s \circ \overline{\mathbb{V}h}_0$ is not τ'–epi. Thus $\mathbb{V}h$ is not essentially τ'–surjective.

For a reason why the left adjoint $\mathbb{V}$ in Proposition 9.8 is not a left Quillen functor, consider the 2–functor j_1' given by inclusion of the terminal 2–category $\{1\}$ into the free-living adjoint equivalence E'. The free-living adjoint equivalence E' has objects 0 and 1. Morphisms are

$$f\colon 0 \longrightarrow 1$$
$$g\colon 1 \longrightarrow 0,$$

as well as all concatenations of f and g. There is a unique 2–cell between every parallel pair of morphisms. In particular, every 2–cell of E' is invertible. The 2–functor j_1' is a generating acyclic cofibration for the categorical structure on **2–Cat** as described in [59]. However $\mathbb{V}j_1'$ is not a cofibration in the τ'–model structure on **DblCat**: its underlying functor $(\mathbb{V}j_1')_0$ of object categories does not have the left lifting property with respect to all τ'–epimorphisms. For example, let $\mathbf{C}$ be the smallest category containing the underlying category $(E')_0 = (\mathbb{V}E')_0$ of E' as well as an additional object $1'$ and an arrow $1' \longrightarrow 0$. The projection from $\mathbf{C}$ to E' takes

$1'$ to 1 and is the identity on E', hence it is τ'–epi. Then the commutative diagram

$$\begin{tikzcd}
(\mathbb{V}\{1\})_0 \arrow[r, "1\mapsto 1'"] \arrow[d, "(\mathbb{V} j_1')_0"'] & \mathbf{C} \arrow[d] \\
(\mathbb{V} E')_0 \arrow[r, "\mathrm{id}"'] & (\mathbb{V} E')_0
\end{tikzcd}$$

does not admit a lift. Thus, $\mathbb{V}$ in Proposition 9.8 preserves neither cofibrations nor weak equivalences.

It is interesting to note that the **Cat**-analogue of Theorem 9.1 does not hold. In other words, if we view **Cat** as the category of algebras over the 2–monad M on **Graph**, then the associated model structure on **Cat** is not the model structure associated to the topology of surjective functions on **Set**. A covering family in a basis for this topology is a single surjective function, so that the epimorphisms for this topology are the same as the epimorphisms for the trivial topology by Proposition 8.5 and Remark 8.6, namely the surjective maps themselves. In fact, the trivial topology, simplicially surjective topology, and categorically surjective topology on **Set** all give rise to the categorical model structure on **Cat**, while the algebra structure on **Cat** has weak equivalences the isomorphisms of categories. When we pass to **DblCat** on the other hand, the three model structures associated to these three topologies become distinct, and one of them agrees with the algebra structure.

10 Appendix: Horizontal nerves and pushouts

Though the horizontal nerve preserves filtered colimits, it certainly does not preserve general colimits, not even pushouts. The purpose of this appendix is to explicitly describe the behavior of the horizontal nerve on pushouts in **DblCat** along

$$i \boxtimes 1_{\mathbf{C}}\colon \mathbf{A} \boxtimes \mathbf{C} \longrightarrow \mathbf{B} \boxtimes \mathbf{C}$$

where $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ is either of the following full inclusions from Section 7.1.

$$c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$$
$$\{1\} \longrightarrow \mathbf{I}$$

Theorem 10.7 is the main technical result needed for an application of Kan's Lemma on Transfer (Theorem 7.11) to transfer model structures across the adjunction $c_h \dashv N_h$ in Theorem 7.13 and Theorem 7.17. In the following, we use "$\backslash$" to denote set-theoretic complement. We begin with some pushouts in **Cat** which will aid us in our description of the horizontal and vertical 1–categories of the pushouts in Theorem 10.6. The squares will require an induction argument. Special pushouts in **Cat** have been considered by many people, for example Fritsch–Latch [36], Thomason [80] and Trnková [82].

Lemma 10.1 *If $A \subseteq B$ and D are sets, then the pushout in* **Set**

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & D \arrow[d] \\ B \arrow[r] & P \end{tikzcd}$$

is $P = D \coprod (B \backslash A)$.

Lemma 10.2 *Suppose $\mathbf{A}$ is a full subcategory of $\mathbf{B}$ and*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \arrow[d] \\ \mathbf{B} \arrow[r] & \mathbf{P} \end{tikzcd}$$

is a pushout in **Cat**. *Then the objects of $\mathbf{P}$ are*

$$\mathrm{Obj}\,\mathbf{P} = \mathrm{Obj}\,\mathbf{D} \coprod (\mathrm{Obj}\,\mathbf{B} \backslash \mathrm{Obj}\,\mathbf{A})$$

and morphisms of $\mathbf{P}$ have two forms:

(1) *A morphism $B_0 \xrightarrow{f} B_1$ with $f \in (\mathrm{Mor}\,\mathbf{B} \backslash \mathrm{Mor}\,\mathbf{A})$.*

(2) *A path $X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$ where d is a morphism in $\mathbf{D}$, and $f_1, f_2 \in (\mathrm{Mor}\,\mathbf{B} \backslash \mathrm{Mor}\,\mathbf{A}) \cup \{$identities on $\mathrm{Obj}\,\mathbf{P}\}$. If f_1 is nontrivial, then $D_1 \in \mathbf{A}$. If f_2 is nontrivial, then $D_2 \in \mathbf{A}$.*

Proof To calculate a pushout of categories, one takes the free category on the pushout of the underlying graphs, and then mods out by the relations necessary to make the natural maps from $\mathbf{A}, \mathbf{B}, \mathbf{D}$ to the free category into functors as in Theorem 4.3. Thus the objects of $\mathbf{P}$ are

$$\mathrm{Obj}\,\mathbf{D} \coprod (\mathrm{Obj}\,\mathbf{B} \backslash \mathrm{Obj}\,\mathbf{A})$$

by Lemma 10.1. The edges of the pushout graph are

$$\mathrm{Mor}\,\mathbf{D} \coprod (\mathrm{Mor}\,\mathbf{B} \backslash \mathrm{Mor}\,\mathbf{A}),$$

again by Lemma 10.1. The free category on this consists of finite composable paths of these edges.

Suppose

$$P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} P_2 \cdots\cdots\cdots P_{k-1} \xrightarrow{f_k} P_k$$

is a morphism in the pushout $\mathbf{P}$. Then we can reduce it to the form (1) or (2) using the relations induced by $\mathbf{A},\mathbf{B}$, and $\mathbf{D}$ as follows. Suppose f_{j-1} and f_{j+1} are in $\operatorname{Mor}\mathbf{D}$, while f_j is in $(\operatorname{Mor}\mathbf{B}\backslash \operatorname{Mor}\mathbf{A})$. Then P_{j-1} and P_j must be objects of $\mathbf{A}$. But by the fullness of $\mathbf{A}$, f_j must be in $\operatorname{Mor}\mathbf{A}$, and we have arrived at a contradiction. Thus no morphism of $(\operatorname{Mor}\mathbf{B}\backslash \operatorname{Mor}\mathbf{A})$ can be surrounded by morphisms of $\mathbf{D}$: there exist $0 \leq m \leq n \leq k+1$ such that for all $0 \leq j \leq m$ and all $n \leq j \leq k$ we have $f_j \in (\operatorname{Mor}\mathbf{B}\backslash \operatorname{Mor}\mathbf{A})$, and for all $m < j < n$ we have $f_j \in \operatorname{Mor}\mathbf{D}$. Next we compose the f_j in each range, and we obtain a path of the form (1) or (2). □

Remark 10.3 A morphism f of $\mathbf{B}$ is in $\operatorname{Mor}\mathbf{B}\backslash \operatorname{Mor}\mathbf{A}$ if and only if its source or target is in $\operatorname{Obj}\mathbf{B}\backslash \operatorname{Obj}\mathbf{A}$ by the fullness of $\mathbf{A}$ in $\mathbf{B}$.

Lemma 10.4 *If $A \subseteq B$ are sets and $\mathbf{C}$ and $\mathbf{D}$ are categories, then the pushout in* **Cat**

$$\begin{tikzcd}
A_{\mathrm{disc}} \times \mathbf{C} \arrow[r] \arrow[d, hook, "i\times 1_{\mathbf{C}}"'] & \mathbf{D} \arrow[d] \\
B_{\mathrm{disc}} \times \mathbf{C} \arrow[r] & \mathbf{P}
\end{tikzcd}$$

is $\mathbf{P} = \mathbf{D}\coprod((B\backslash A)_{\mathrm{disc}} \times \mathbf{C})$. (The subscript "disc" means discrete category on a given set.)

Proof Since $B_{\mathrm{disc}} \times \mathbf{C} = A_{\mathrm{disc}} \times \mathbf{C}\coprod((B\backslash A)_{\mathrm{disc}} \times \mathbf{C})$, the pushout of the underlying graphs is

$$\mathbf{D}\coprod((B\backslash A)_{\mathrm{disc}} \times \mathbf{C}) \tag{21}$$

by Lemma 10.1. The free category on this graph, modulo the appropriate relations as in Theorem 4.3, is once again (21).

Alternatively, this Lemma also follows easily from Lemma 10.2. □

Lemma 10.5 *Suppose $\mathbf{A}$ is a full subcategory of $\mathbf{B}$, C is a set, and*

$$\begin{tikzcd}
\mathbf{A} \times C_{\mathrm{disc}} \arrow[r, "F"] \arrow[d, hook] & \mathbf{D} \arrow[d] \\
\mathbf{B} \times C_{\mathrm{disc}} \arrow[r] & \mathbf{P}
\end{tikzcd}$$

is a pushout in $\mathbf{Cat}$*. Then the objects of* $\mathbf{P}$ *are*

$$\operatorname{Obj}\mathbf{P} = \operatorname{Obj}\mathbf{D} \coprod ((\operatorname{Obj}\mathbf{B} \setminus \operatorname{Obj}\mathbf{A}) \times C)$$

and the morphisms of $\mathbf{P}$ *have two forms:*

(1) *A morphism* $(B_0, c) \xrightarrow{f} (B_1, c)$ *with* c *in* C *and* $f = (f', c)$ *in* $(\operatorname{Mor}\mathbf{B} \setminus \operatorname{Mor}\mathbf{A}) \times C$.

(2) *A path* $X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$ *where* d *is a morphism in* $\mathbf{D}$*, and each of* f_1 *and* f_2 *is either in* $\operatorname{Mor}\mathbf{B} \setminus \operatorname{Mor}\mathbf{A} \times C$ *or an identity morphism.*

Moreover, if f_1 *or* f_2 *is not an identity morphism in (2), then the path has one of the two respective forms*

$$(B_1, c_1) \xrightarrow{(f_1', c_1)} (A_1, c_1) \xrightarrow{d} D_2 \xrightarrow{f_2} X_2$$

$$X_1 \xrightarrow{f_1} D_1 \xrightarrow{d} (A_2, c_2) \xrightarrow{(f_2', c_2)} (B_2, c_2)$$

where $c_1, c_2 \in C$, $B_1, B_2 \in \operatorname{Obj}\mathbf{B} \setminus \operatorname{Obj}\mathbf{A}$, $A_1, A_2 \in \operatorname{Obj}\mathbf{A}$, $f_1', f_2' \in \operatorname{Mor}\mathbf{B} \setminus \operatorname{Mor}\mathbf{A}$, *and* $d \in \operatorname{Mor}\mathbf{D}$.

Proof This follows from Lemma 10.2. □

Let us recall the two full inclusions $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ under consideration. The first case in which we are interested is the full inclusion of posets $c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$. Here $c\colon \mathbf{SSet} \longrightarrow \mathbf{Cat}$ denotes the fundamental category functor as described in Section 6 and $\mathrm{Sd}\colon \mathbf{SSet} \longrightarrow \mathbf{SSet}$ is the subdivision functor recalled in [39] and in Section 7.1.

The second full inclusion $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ of interest is $\{1\} \longrightarrow \mathbf{I}$. The category $\mathbf{I}$ consists of two objects 0 and 1 and four morphisms: an isomorphism and its inverse between 0 and 1, and the identity maps. The discrete subcategory $\{1\}$ is clearly full.

We can now give an explicit description of pushouts in $\mathbf{DblCat}$ along

$$i \boxtimes 1_{\mathbf{C}}\colon \mathbf{A} \boxtimes \mathbf{C} \longrightarrow \mathbf{B} \boxtimes \mathbf{C}$$

which we use immediately in Theorem 10.7 for the transfer.

Theorem 10.6 *Let* $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ *be either of the following full inclusions.*

$$c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$$
$$\{1\} \longrightarrow \mathbf{I}$$

Let $\mathbf{C}$ *be a category (eg the finite ordinal* $[n]$*), and* $\mathbb{D}$ *a double category. Then the pushout*

$$\begin{tikzcd} \mathbf{A} \boxtimes \mathbf{C} \arrow[r, "F"] \arrow[d, "i \boxtimes 1_{\mathbf{C}}"'] & \mathbb{D} \arrow[d] \\ \mathbf{B} \boxtimes \mathbf{C} \arrow[r] & \mathbb{P} \end{tikzcd}$$

in **DblCat** *has the following explicit description:*

$$\mathrm{Obj}\,\mathbb{P} = \mathrm{Obj}\,\mathbb{D} \coprod ((\mathrm{Obj}\,\mathbf{B} \setminus \mathrm{Obj}\,\mathbf{A}) \times \mathrm{Obj}\,\mathbf{C}) \tag{22}$$

$$(\mathbf{H}\mathbb{P})_0 = (\mathbf{H}\mathbb{D})_0 \coprod ((\mathrm{Obj}\,\mathbf{B} \setminus \mathrm{Obj}\,\mathbf{A})_{\mathrm{disc}} \times \mathbf{C}) \tag{23}$$

$$\begin{aligned} \mathrm{Mor}(\mathbf{V}\mathbb{P})_0 = \{&\textit{paths of the form (1) and (2)} \\ &\textit{in Lemma 10.5 with } C = \mathrm{Obj}\,\mathbf{C} \\ &\textit{and } \mathbf{D} = (\mathbf{V}\mathbb{D})_0\}. \end{aligned} \tag{24}$$

Squares of $\mathbb{P}$ *have two forms:*

(1) *A square*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

in $\mathrm{Sq}(\mathbf{B} \boxtimes \mathbf{C}) \setminus \mathrm{Sq}(\mathbf{A} \boxtimes \mathbf{C})$.

(2) *A vertical path of squares*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

where δ *is a square in* $\mathbb{D}$ *and each of* β_1 *and* β_2 *is either a vertical identity square (on a horizontal morphism) in* $\mathbb{P}$ *or is in* $\mathrm{Sq}(\mathbf{B} \boxtimes \mathbf{C}) \setminus \mathrm{Sq}(\mathbf{A} \boxtimes \mathbf{C})$*. Moreover, in the case of* $c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$*, the square* β_1 *is always a vertical identity square.*

Note that $\mathrm{Sq}(\mathbf{B}\boxtimes\mathbf{C})\setminus\mathrm{Sq}(\mathbf{A}\boxtimes\mathbf{C}) =$

$$\left\{ \begin{array}{c|c} \begin{tikzcd} (B,C) \arrow[r, "(1_B{,}g)"] \arrow[d, "(f{,}1_C)"'] & (B,C') \arrow[d, "(f{,}1_{C'})"] \\ (B',C) \arrow[r, "(1_{B'}{,}g)"'] & (B',C') \end{tikzcd} & \begin{array}{c} g \in \mathrm{Mor}\,\mathbf{C}, \\ f \in \mathrm{Mor}\,\mathbf{B}\setminus \mathrm{Mor}\,\mathbf{A} \end{array} \end{array} \right\}.$$

Proof We use Theorem 4.6. First we calculate the pushout $\mathbb{S}$ of the underlying double derivation schemes. The object set $\mathrm{Obj}\,\mathbb{S} = \mathrm{Obj}\,\mathbb{P}$ is the pushout of the object sets, so (22) follows from Lemma 10.1. The horizontal and vertical 1–categories of $\mathbb{S}$ (and $\mathbb{P}$) are the pushouts of the horizontal and vertical 1–categories, so (23) follows from Lemma 10.4 and (24) follows from Lemma 10.5. By Lemma 10.1 again, the pushout of the *sets* of squares is

$$\mathrm{Sq}\,\mathbb{S} = \mathrm{Sq}\,\mathbb{D} \coprod (\mathrm{Sq}(\mathbf{B}\boxtimes\mathbf{C})\setminus\mathrm{Sq}(\mathbf{A}\boxtimes\mathbf{C})). \tag{25}$$

Thus we have calculated the pushout $\mathbb{S}$ of the underlying double derivation schemes, its horizontal and vertical 1–categories coincide with those of $\mathbb{P}$, and they have the form claimed in the theorem. It only remains to show that the squares of $\mathbb{P}$ have the form claimed in the theorem.

The double category $\mathbb{P}$ is the free double category on the double derivation scheme $\mathbb{S}$ modulo the smallest congruence making the natural morphisms of double derivation schemes from $\mathbf{A}\boxtimes\mathbf{C}, \mathbf{B}\boxtimes\mathbf{C}$, and $\mathbb{D}$ to $\mathbb{P}$ into double functors. Squares of $\mathbb{P}$ are represented by allowable compatible arrangements in $\mathbb{S}$. To prove that squares of $\mathbb{P}$ have the form (1) or (2), it suffices to show that any allowable compatible arrangement of squares in $\mathbb{S}$ can be transformed into (1) or (2) using the relations of the congruence and the double category associativity, identity, and interchange axioms. The congruence allows us to compose squares according to the relations in the double categories $\mathbf{A}\boxtimes\mathbf{C}, \mathbf{B}\boxtimes\mathbf{C}$, and $\mathbb{D}$.

We must treat the two inclusions i separately.

Let $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ be the full inclusion $c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$. Recall from Section 7.1 that $c\,\mathrm{Sd}^2\,\Lambda^k[m]$ and $c\,\mathrm{Sd}^2\,\Delta[m]$ are respectively the posets of nondegenerate simplices of $\mathrm{Sd}\,\Lambda^k[m]$ and $\mathrm{Sd}\,\Delta[m]$, and that there is a morphism of the form $(u_0,\dots,u_p) \longrightarrow (v_0,\dots,v_q)$ in $\mathbf{B}$ if and only if

$$\{u_0,\dots,u_p\} \subseteq \{v_0,\dots,v_q\}.$$

Also, an object $(v_0, \dots, v_q)$ of $\mathbf{B}$ is in $\mathbf{A}$ if and only *all* v_r are faces of $\Lambda^k[m]$. Thus, we see for any path of composable morphisms in $\mathbf{B}$

$$B_0 \xrightarrow{f_1} B_1 \xrightarrow{f_2} B_2 \cdots\cdots B_{n-1} \xrightarrow{f_n} B_n$$

with B_j not in $\mathbf{A}$, all f_ℓ and B_ℓ with $\ell \geq j$ are also not in $\mathbf{A}$. Thus, once a path leaves $\mathbf{A}$, it cannot return to $\mathbf{A}$. In particular, if $B \longrightarrow B'$ is a morphism in $\mathbf{B}$ and B is not in $\mathbf{A}$, then B' is also not in $\mathbf{A}$. Another useful property of $\mathbf{B}$ is that every morphism has a unique decomposition into irreducibles. These special features of the posets $\mathbf{A}$ and $\mathbf{B}$ allow us to put the squares of $\mathbb{P}$ into the desired form (1) or (2), as we do now.

Suppose R is an allowable compatible arrangement of squares in $\mathbb{S}$, that is, a representative of a square in $\mathbb{P}$. If R consists entirely of squares in $\mathbb{D}$, then it is equivalent to its composition in $\mathbb{D}$, so it has the form (2) and we are finished.

So suppose that R contains at least one square in $\mathrm{Sq}\,\mathbf{B} \boxtimes \mathbf{C} \setminus \mathrm{Sq}\,\mathbf{A} \boxtimes \mathbf{C}$. Then R has at least one vertex (B, C) in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, that is, B is in $\mathbf{B}$ but not in $\mathbf{A}$. Any horizontal morphism in R with source (respectively target) (B, C) is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, as (B, C) is not in $\mathbf{A} \boxtimes \mathbf{C}$. Thus the target (respectively source) of such a morphism has the form (B, C') and is also in $\mathbf{B} \boxtimes \mathbf{C}$ but not $\mathbf{A} \boxtimes \mathbf{C}$. Any vertical morphism in R with *source* (B, C) is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, as (B, C) is not in $\mathbf{A} \boxtimes \mathbf{C}$. Thus the target of such a vertical morphism is of the form (B', C) with B' not in $\mathbf{A}$ by the special feature of the posets $\mathbf{A}$ and $\mathbf{B}$ described in the preceding paragraph. From the original vertex (B, C) we traverse down a vertical morphism with source (B, C) if there is one, otherwise we traverse to the right along a horizontal morphism with source (B, C). In either case, we arrive at another vertex (B_1, C_1) which is in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$. From this vertex we repeat the procedure, moving either to the right or down. We continue in this way until we reach the bottom edge of the allowable compatible arrangement R. We conclude that the entire bottom edge of the diagram consists of objects and horizontal morphisms in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, and hence not in $\mathbb{D}$.

Each of these horizontal morphisms on the bottom edge is the bottom edge of a square in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$, since squares of $\mathbb{D}$ only have vertices in $\mathbb{D}$ (some objects of $\mathbb{D}$ are identified with objects of $\mathbf{A} \boxtimes \mathbf{C}$). Thus, the bottom portion of R looks like Figure 6 with all squares in $\mathbf{B} \boxtimes \mathbf{C}$ but not in $\mathbf{A} \boxtimes \mathbf{C}$.

Next we factor the vertical morphisms of Figure 6 into irreducibles, which we can do since these vertical morphisms are of the form (f, C) where f is a morphism in $\mathbf{B}$ and C is an object of $\mathbf{C}$. By the uniqueness of the factorization and the form of squares in $\mathbf{B} \boxtimes \mathbf{C}$, we can factor these squares at the height of the shortest one as illustrated in

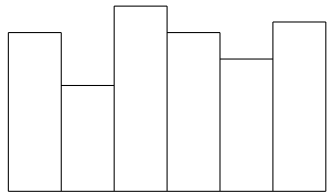


Figure 6: The bottom portion of R

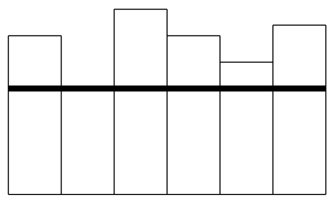


Figure 7: A factorization of the squares in Figure 6

Figure 7. We include these new horizontal morphisms into the allowable compatible arrangement R, and obtain a new compatible arrangement R_1. The compatible arrangement R_1 is also allowable, since the same cuts that make R allowable also make R_1 allowable.

The bold horizontal line in Figure 7 is a full length cut on an allowable compatible arrangement R_1, hence it divides R_1 into two allowable compatible arrangements by Proposition 3.7. We denote the upper allowable compatible arrangement by $R_{1,1}$ and the lower allowable compatible arrangement by $R_{1,2}$. Then $R_{1,1}$ has at least one square less than R, since we cut off at the height of the shortest square whose bottom edge is on the bottom edge of R. If we argue by induction on the number of squares in an allowable compatible arrangement, we may assume that $R_{1,1}$ is equivalent to a square of the form (1) or (2). The allowable compatible arrangement $R_{1,2}$ is equivalent to a square of the form (1), as it can be composed horizontally. Finally, we compose $R_{1,1}$ with $R_{1,2}$ to conclude that R is also equivalent to a compatible arrangement of the form (1) or (2).

We only need an argument for the triviality of β_1 whenever a compatible arrangement is equivalent to one of the form (2). Suppose β_1 is in $\mathrm{Sq}(\mathbf{B} \boxtimes \mathbf{C}) \setminus \mathrm{Sq}(\mathbf{A} \boxtimes \mathbf{C})$. Then its lower two vertices cannot be in $\mathbf{A} \boxtimes \mathbf{C}$ (for if they were, the upper two vertices must also be in $\mathbf{A} \boxtimes \mathbf{C}$, and the square β_1 would be in $\mathbf{A} \boxtimes \mathbf{C}$). Thus, the upper two vertices of the square δ are not in $\mathbf{A} \boxtimes \mathbf{C}$, a contradiction. Thus β_1 must be a vertical identity. This completes the proof of Theorem 10.6 for the case $c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$.

Now we turn to the squares in the second case. Let $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ be the full inclusion $\{1\} \longrightarrow \mathbf{I}$ where $\mathbf{I}$ is the category with two objects 0 and 1 and an isomorphism between them. We will again argue by induction on the number of squares in the allowable compatible arrangement, but the special features of the inclusion $\{1\} \longrightarrow \mathbf{I}$ are different from those of the previous case. Note that $\mathbf{B} \boxtimes \mathbf{C}$ only has the four types of squares listed in Figure 8. The only vertical morphisms in $\mathbf{B} \boxtimes \mathbf{C}$ that are identified

$$\begin{tikzcd}
(0,C_1) \arrow[r] \arrow[d, equal] & (0,C_2) \arrow[d, equal] & (1,C_1) \arrow[r] \arrow[d, equal] & (1,C_2) \arrow[d, equal] \\
(0,C_1) \arrow[r] & (0,C_2) & (1,C_1) \arrow[r] & (1,C_2)
\end{tikzcd}$$

$$\begin{tikzcd}
(0,C_1) \arrow[r] \arrow[d] & (0,C_2) \arrow[d] & (1,C_1) \arrow[r] \arrow[d] & (1,C_2) \arrow[d] \\
(1,C_1) \arrow[r] & (1,C_2) & (0,C_1) \arrow[r] & (0,C_2)
\end{tikzcd}$$

Figure 8: The four types of squares in $\mathbf{B} \boxtimes \mathbf{C}$

with a morphism in $\mathbb{D}$ are the trivial vertical morphisms $\mathrm{id}^v_{(1,C)}\colon (1,C) \longrightarrow (1,C)$.

Suppose that any allowable compatible arrangement of squares in $\mathbb{S}$ with fewer than n squares is equivalent in $\mathbb{P}$ to one of the form (1) or (2). Let R be an allowable compatible arrangement of n squares in $\mathbb{S}$. Since R is allowable, it admits a full length cut $\mathcal{C}$ which divides R into two allowable compatible arrangements each with fewer than n squares. We now recombine these two smaller allowable compatible arrangements to show that R is equivalent to a compatible arrangement of the form (1) or (2), but the argument is slightly different depending on whether $\mathcal{C}$ is horizontal or vertical.

Suppose the full length cut $\mathcal{C}$ is horizontal. Let R_1 and R_2 be the allowable compatible arrangements above and below $\mathcal{C}$ respectively. Since R_1 and R_2 have fewer than n squares, they must be equivalent to compatible arrangements of the form (1) or (2). If R_1 and R_2 both are equivalent to compatible arrangements of the form (2), then by the fullness of $\mathbf{A}$ in $\mathbf{B}$ their vertical composite is also of the form (2), and hence R is equivalent to a compatible arrangement of the form (2). If one or both of R_1 and R_2

has the form (1), then one can similarly conclude that R is equivalent to a compatible arrangement of form (1) or (2).

Suppose the full length cut $\mathcal{C}$ is vertical. Let Q^ℓ and Q^r be the allowable compatible arrangements to the left and to the right of $\mathcal{C}$ respectively. Since Q^ℓ and Q^r have fewer than n squares, they must be equivalent to compatible arrangements of the form (1) or (2). There are several cases to consider.

If both Q^ℓ and Q^r are equivalent to compatible arrangements of the form (1), then their horizontal composite R is clearly in $\mathbf{B} \boxtimes \mathbf{C}$, and hence also equivalent to a compatible arrangement of the form (1) or (2).

If Q^ℓ is equivalent to a compatible arrangement of the form (1) and Q^r is equivalent to a compatible arrangement of the form (2), then β_1^r and β_2^r must be in $\mathbf{B} \boxtimes \mathbf{C}$ as in Figure 9. Further, the vertical morphism $k^r \colon (B_2^r, C) \longrightarrow (B_3^r, C)$ must be the

$$\begin{tikzcd}
Q^\ell & (B_1, C_1^\ell) \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta^\ell"] & (B_1, C) \arrow[d, "p^\ell"] \\
& (B_4, C_1^\ell) \arrow[r] & (B_4, C)
\end{tikzcd}
\qquad
\begin{tikzcd}
(B_1, C) \arrow[r] \arrow[d, "j^r"'] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] & \\
(B_2^r, C) \arrow[r] \arrow[d, "k^r"'] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] & Q^r \\
(B_3^r, C) \arrow[r] \arrow[d, "m^r"'] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] & \\
(B_4, C) \arrow[r] & {} &
\end{tikzcd}$$

Figure 9: Q^ℓ of form (1) and Q^r of form (2)

vertical identity $\mathrm{id}^v_{(1,C)}$ on the object $(1, C)$, since $k^r = (m^r)^{-1} p^\ell (j^r)^{-1}$ lies in both $\mathbf{B} \boxtimes \mathbf{C}$ and $\mathbb{D}$, and the only vertical morphisms in both $\mathbf{B} \boxtimes \mathbf{C}$ and $\mathbb{D}$ are such vertical identities. Then we can subdivide β^ℓ in $\mathbf{B} \boxtimes \mathbf{C}$ as in Figure 10. The middle square of Q^ℓ is now an identity square on a horizontal morphism in $\mathbb{D}$, and hence is also a square in $\mathbb{D}$. Finally, we horizontally compose Q^ℓ and Q^r and use the interchange law to obtain a compatible arrangement of the form (1) or (2). Hence R is equivalent to a compatible arrangement of the form (1) or (2).

Next we consider the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (2) *and* the squares $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ are in $\mathbf{B} \boxtimes \mathbf{C}$ as in

$$
Q^\ell \qquad
\begin{tikzcd}
(B_1, C_1^\ell) \arrow[r] \arrow[d] & (B_1, C) \arrow[d] \\
(1, C_1^\ell) \arrow[r] \arrow[d, equal] & (1, C) \arrow[d, equal] \\
(1, C_1^\ell) \arrow[r] \arrow[d] & (1, C) \arrow[d] \\
(B_4, C_1^\ell) \arrow[r] & (B_4, C)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(B_1, C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] \\
(1, C) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] \\
(1, C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] \\
(B_4, C) \arrow[r] & {}
\end{tikzcd}
\qquad Q^r
$$

Figure 10: β^ℓ subdivided and $k^r = \mathrm{id}^v_{(1,C)}$

$$
Q^\ell \qquad
\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1^\ell"] & (B_1, C_1) \arrow[d, "j^\ell"] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta^\ell"] & (B_2^\ell, C_1) \arrow[d, "k^\ell"] \\
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2^\ell"] & (B_3^\ell, C_2) \arrow[d, "m^\ell"] \\
{} \arrow[r] & (B_4, C_2)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(B_1, C_1) \arrow[r] \arrow[d, "j^r"'] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] \\
(B_2^r, C_1) \arrow[r] \arrow[d, "k^r"'] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] \\
(B_3^r, C_2) \arrow[r] \arrow[d, "m^r"'] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] \\
(B_4, C_2) \arrow[r] & {}
\end{tikzcd}
\qquad Q^r
$$

Figure 11: Q^ℓ and Q^r both of form (2) *and* $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ in $\mathbf{B} \boxtimes \mathbf{C}$

Figure 11. Then $B_2^\ell = 1 = B_2^r$ and $B_3^\ell = 1 = B_3^r$, since the only objects of $\mathbf{B} \boxtimes \mathbf{C}$ that are identified with an object of $\mathbb{D}$ are of the form $(1, C)$. Thus $j^\ell = j^r$ and $m^\ell = m^r$, as there is a unique vertical morphism from any object of $\mathbf{B} \boxtimes \mathbf{C}$ to another. Since j^ℓ and m^ℓ are invertible and $m^\ell k^\ell j^\ell = m^\ell k^r j^\ell$, we see also that $k^\ell = k^r$. Hence Q^ℓ and Q^r can be horizontally composed to obtain a compatible arrangement equivalent to (1) or (2).

Next we consider the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (2), but the squares $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ may be vertical identity squares in $\mathbb{P}$, that is, not necessarily in $\mathbf{B} \boxtimes \mathbf{C}$, as in Figure 12. Suppose β_1^ℓ is a vertical

$$\begin{tikzcd}
 & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_1^\ell"] & P_1 \arrow[d, "j^\ell"] & P_1 \arrow[r, "f^r"] \arrow[d, "j^r"'] \arrow[dr, phantom, "\beta_1^r"] & {} \arrow[d] & \\
Q^\ell & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta^\ell"] & D_1^\ell \arrow[d, "k^\ell"] & D_1^r \arrow[r] \arrow[d, "k^r"'] \arrow[dr, phantom, "\delta^r"] & {} \arrow[d] & Q^r \\
 & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\beta_2^\ell"] & D_2^\ell \arrow[d, "m^\ell"] & D_2^r \arrow[r] \arrow[d, "m^r"'] \arrow[dr, phantom, "\beta_2^r"] & {} \arrow[d] & \\
 & {} \arrow[r] & P_2 & P_2 \arrow[r] & {} &
\end{tikzcd}$$

Figure 12: Q^ℓ and Q^r both of form (2) *and* one of $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ not in $\mathbf{B} \boxtimes \mathbf{C}$

identity square. Then $P_1 = D_1^\ell$. We claim that β_1^r is also a vertical identity square; there are two cases to prove. If D_1^ℓ is an object of $\mathbb{D}$ that is not of the form $(1, C)$, then f^r cannot be in $\mathbf{B} \boxtimes \mathbf{C}$ (as its source is not in $\mathbf{B} \boxtimes \mathbf{C}$). Hence β_1^r is a vertical identity square. For the second case, if D_1^ℓ is of the form $(1, C)$, then j^r is a vertical arrow in $\mathbf{B} \boxtimes \mathbf{C}$ with source and target $(1, C)$. By the special form of squares in $\mathbf{B} \boxtimes \mathbf{C}$ in Figure 8, we see that β_1^r is also a vertical identity square. Thus, we have proved, if β_1^ℓ is a vertical identity square, then β_1^r is also a vertical identity square. One can similarly show that if any one of $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ is a vertical identity square, then the square next to it is also.

Let us continue the case where Q^ℓ and Q^r are both equivalent to compatible arrangements of the form (2) as in Figure 12, and suppose again that β_1^ℓ is a vertical identity square. Then β_1^r is also a vertical identity square. If either of β_2^ℓ or β_2^r is a vertical identity square, then so is the other, in which case Q^ℓ and Q^r can be horizontally composed to give a compatible arrangement equivalent to one of the form (1) or (2). If neither β_2^ℓ nor β_2^r is a vertical identity square, then they are both in $\mathbf{B} \boxtimes \mathbf{C}$, and we can argue as in Figure 11 to conclude $D_2^\ell = D_2^r$, $m^\ell = m^r$, and $k^\ell = k^r$, in which case Q^ℓ and Q^r can be horizontally composed to give a compatible arrangement equivalent to one of the form (1) or (2).

The other cases of Figure 12, where one or more of $\beta_1^\ell, \beta_2^\ell, \beta_1^r, \beta_2^r$ is a vertical identity square in $\mathbb{P}$, are similar.

Therefore every square of $\mathbb{P}$ is equivalent to a compatible arrangement of the form (1) or (2), for both inclusions i under consideration. This completes the proof of Theorem 10.6. □

The two inclusions of Theorem 10.6 have some features in common, and the theorem holds for an entire class of inclusions $i\colon \mathbf{A} \longrightarrow \mathbf{B}$. We will return to the this topic and its interaction with Dawson–Paré–Pronk [24] in the future. Theorem 10.6 allows us to characterize the behavior of the horizontal nerve on such pushouts in Theorem 10.7, which we need to transfer the model structures from $\mathbf{Cat}^{\Delta^{op}}$ in Section 7.

Theorem 10.7 *Let $i\colon \mathbf{A} \longrightarrow \mathbf{B}$ be either of the following full inclusions.*

$$c\,\mathrm{Sd}^2\,\Lambda^k[m] \longrightarrow c\,\mathrm{Sd}^2\,\Delta[m]$$

$$\{1\} \longrightarrow \mathbf{I}$$

Let $\mathbf{C}$ be a finite ordinal $[n]$ viewed as a category, $\mathbb{D}$ a double category, and $\mathbb{P}$ the pushout

$$\begin{tikzcd} \mathbf{A} \boxtimes \mathbf{C} \arrow[r] \arrow[d, "i \boxtimes 1_{\mathbf{C}}"'] & \mathbb{D} \arrow[d] \\ \mathbf{B} \boxtimes \mathbf{C} \arrow[r] & \mathbb{P} \end{tikzcd}$$

in **DblCat**. *Then the induced map*

$$N_h(\mathbb{D}) \coprod_{N_h(\mathbf{A}\boxtimes\mathbf{C})} N_h(\mathbf{B} \boxtimes \mathbf{C}) \longrightarrow N_h(\mathbb{P})$$

is an isomorphism of simplicial objects in **Cat**.

Proof We calculate the pushout

$$N_h(\mathbb{D}) \coprod_{N_h(\mathbf{A}\boxtimes\mathbf{C})} N_h(\mathbf{B} \boxtimes \mathbf{C}) \tag{26}$$

levelwise and compare it with $N_h(\mathbb{P})$, whose form is known from Theorem 10.6. The horizontal nerve of an external product of categories is known from Proposition 5.8.

In level 0, the pushout (26) is

$$\mathbb{D}_0 \coprod_{\mathbf{A}\times(\mathrm{Obj}\,\mathbf{C})_{\mathrm{disc}}} \mathbf{B} \times (\mathrm{Obj}\,\mathbf{C})_{\mathrm{disc}}$$

which is the same as the vertical 1–category of $\mathbb{P}$ and thus $(N_h\mathbb{P})_0$.

In level $k \geq 1$ the pushout (26) is

$$N_h(\mathbb{D})_k \coprod_{\mathbf{A}\times N\mathbf{C}_k} \mathbf{B} \times N\mathbf{C}_k$$

by Proposition 5.8. An application of Lemma 10.5 to level k together with Theorem 10.6 give immediately that

$$N_h(\mathbb{D})_k \coprod_{\mathbf{A}\times N\mathbf{C}_k} (\mathbf{B} \times N\mathbf{C}_k) \longrightarrow (N_h\mathbb{P})_k$$

is full (and is the identity on objects). To see that this functor is also faithful, we only need to concern ourselves with the squares of the form (2) in Theorem 10.6. For the Thomason structure these squares always have a representative where β_2 is of the form

$$\begin{tikzcd} (A,C) \arrow[r] \arrow[d] & (A,C') \arrow[d] \\ (B,C) \arrow[r] & (B,C') \end{tikzcd}$$

such that A is a maximal element of $c\,\mathrm{Sd}^2\,\Lambda^k[n]$. This determines β_2 and δ uniquely. For the categorical structure all nonidentity β_2 squares are of the form

$$\begin{tikzcd} (1,C) \arrow[r] \arrow[d] & (1,C') \arrow[d] \\ (0,C) \arrow[r] & (0,C'). \end{tikzcd}$$

A similar argument for β_1 shows that β_1, δ, and β_2 are determined uniquely. □

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