Residually free 3–manifolds

HENRY WILTON

We classify those compact 3–manifolds with incompressible toral boundary whose fundamental groups are residually free. For example, if such a manifold M is prime and orientable and the fundamental group of M is nontrivial then $M \cong \Sigma \times S^1$, where Σ is a surface.

The theory of *residually free groups*, and in particular *fully residually free groups*, has been the subject of intense study in recent years, for the most part because of the important role that finitely generated fully residually free groups (also known as *limit groups*) play in the logic of free groups (see Sela [20] *et seq* and Kharlampovich and Myasnikov [11; 12] *et seq*). A group G is residually free if for each nontrivial element g there is a homomorphism f from G to a free group so that $f(g)$ is nontrivial; it is fully residually free if the same holds when the element g is replaced by a finite set of nontrivial elements. The simplest examples of residually free groups, aside from free groups and free abelian groups, are the fundamental groups of (most) compact surfaces.

The techniques of residual *finiteness* have been applied successfully in low-dimensional topology, and in particular in the study of 3–manifolds. Indeed, Haken 3–manifolds have residually finite fundamental group by Hempel [8], and attempts to prove stronger results about the profinite topology on certain 3–manifold groups are the subject of much active research. There is also interest in *large* 3–manifolds—those whose fundamental groups virtually surject nonabelian free groups. It seems natural to speculate that the theory of residually free groups might also be fruitful in the 3–manifold context. In this spirit, Leonid Potyagailo asked which 3–manifold groups are residually free [19].

At least for closed 3–manifolds, the theory of residually free groups is too powerful. Our main theorem asserts that there are very few interesting examples of prime, compact 3–manifolds with incompressible toral boundary whose fundamental groups are residually free.

Main Theorem *If M is a prime, compact 3–manifold with incompressible toral boundary then $\pi_1(M)$ is residually free and nontrivial if and only if M is one of the following.*

(1) *A trivial circle bundle over an orientable surface.*

(2) *A circle bundle with trivial monodromy over a nonorientable surface of Euler characteristic less than -1.*

(3) *The nontrivial circle bundle with trivial monodromy over the projective plane.*

Free products of residually free groups are not always themselves residually free. In Corollary 2.11 we remove the assumption of primeness and classify all compact residually free 3–manifolds with incompressible toral boundary.

The article is organized as follows. In Section 1 we recall the theory of residually free and fully residually free groups. Section 1.1 pays particular attention to their respective behaviours under free and direct products. In a nutshell, residually free groups are well behaved under direct products but not under free products, whereas fully residually free groups are well behaved under free products but not under direct products. In Section 1.2 we prove necessary conditions for residually free groups to split over abelian subgroups. Section 1.3 contains an elementary proof that residually free groups virtually retract onto cyclic subgroups.

Section 2 is concerned with applying this theory in the context of compact 3–manifolds with incompressible toral boundary. In Section 2.1, we classify all residually free circle bundles. We then examine the torus decomposition of an arbitrary (compact, irreducible, orientable) 3–manifold with incompressible toral boundary. We examine each piece and prove that, if residually free, it must be a circle bundle over a surface. The conditions from Section 1.2 are enough to deduce that any residually free graph manifold has trivial torus decomposition.

In Section 3 we discuss the case of manifolds with more general boundary. A complete classification here seems intractable without answers to some fundamental questions in the theory of fully residually free groups.

1   Residually free groups

1.1   Free products and direct products

We shall start by recalling some of the theory of residually free and fully residually free groups. Several of the results of this section are old results of Baumslag [1]. The point

of view that we emphasize is that residually free groups are well behaved under direct products, whereas fully residually free groups are well behaved under free products. Fix a nonabelian free group $\mathbb{F}$ of finite rank at least 2.

Definition 1.1 A group G is *residually free* if, for every nontrivial $g \in G$, there exists a homomorphism $f \colon G \to \mathbb{F}$ such that $f(g) \neq 1$.

It is immediate that subgroups of residually free groups are residually free, and that direct products of free groups are residually free. It is also clear that residually free groups are torsion-free.

Example 1.2 Any subgroup of a direct product of free groups is residually free.

We also have the following classification result.

Lemma 1.3 Every 2–generator residually free group is isomorphic to one of 1, $\mathbb{Z}$, $\mathbb{Z}^2$ or F_2, the nonabelian free group of rank 2.

Proof Let G be a nonabelian residually free group generated by a and b. Since $[a, b] \neq 1$ there exists a homomorphism $f \colon G \to \mathbb{F}$ such that $f([a, b]) \neq 1$. The image of f is a 2–generator, nonabelian subgroup of $\mathbb{F}$ and so is isomorphic to F_2. Since G is a 2–generator group that surjects onto F_2 it follows that $G \cong F_2$. $\square$

We will need one consequence of this. Let G be a group and H a subgroup. Denote by $Z_G(H)$ the centralizer of H, and $Z(G) = Z_G(G)$ the centre of G. Let $[G, G]$ be the commutator subgroup of G.

Lemma 1.4 If G is residually free then

$$Z_G([G, G]) = Z(G).$$

Proof Let $z \in Z_G([G, G])$. For every $g \in G$, $[z, [z, g]] = 1$. Therefore $\langle z, g \rangle$ is not isomorphic to F_2, and so $[z, g] = 1$ by Lemma 1.3. Hence $Z_G([G, G]) \subset Z(G)$. Since the reverse inclusion is obvious, the result follows. $\square$

In order to understand residually free groups, we need to consider the special class of *fully residually free groups*.

Definition 1.5 A group G is *fully residually free* (also ω–*residually free* or, in the finitely generated case, a *limit group*) if, for any finite collection of nontrivial elements $g_1, \ldots, g_n \in G$, there exists a homomorphism $f \colon G \to \mathbb{F}$ such that $f(g_i) \neq 1$ for all $1 \leq i \leq n$.

Of course, free groups are fully residually free. It is an easy exercise to show that free abelian groups are fully residually free. The next examples of fully residually free groups are (most) surface groups.

Definition 1.6 If Σ is a compact (not necessarily closed) surface and $\pi_1(\Sigma)$ is residually free then we call Σ a *residually free surface*.

If Σ has nonempty boundary then $\pi_1(\Sigma)$ is free, and so certainly fully residually free.

Example 1.7 Let Σ be a closed surface. If Σ is one of the nonorientable surfaces of Euler characteristic -1, 0 or $+1$ then $\pi_1(\Sigma)$ is *not* residually free. Otherwise, $\pi_1(\Sigma)$ is fully residually free.

It is clear that the fundamental groups of the projective plane and the Klein bottle are not residually free. Lyndon [13] proved that the fundamental group of the surface of Euler characteristic -1 is not residually free. See Lemma 3.1 for an idea of how to start proving that the remaining surface groups are fully residually free.

The class of residually free groups is closed under taking direct products. But a free product is rarely residually free.

Lemma 1.8 (cf Baumslag [1, Theorem 6]) *If A and B are groups, A is nontrivial and B is not fully residually free then $A * B$ is not residually free.*

We prove this in the case when B is not 2–*residually free*, that is when there exist nontrivial $b_1, b_2 \in B$ such that every homomorphism $B \to \mathbb{F}$ kills one of b_1 or b_2. The general proof is similar. (In fact, every 2–residually free group is fully residually free, so this is the general case; see Remeslennikov [16].)

Proof of Lemma 1.8 Let $a \in A$ be nontrivial and let $b_1, b_2 \in B \smallsetminus 1$ be such that every homomorphism $B \to \mathbb{F}$ kills one of b_1 or b_2. Define

$$\xi = [b_1, [b_2, a]].$$

Since
$$\xi = b_1 b_2 a b_2^{-1} a^{-1} b_1^{-1} a b_2 a^{-1} b_2^{-1}$$

is a reduced word in the free product decomposition of G, it follows that $\xi \neq 1$. But $f(\xi) = 1$ for every homomorphism $f \colon A * B \to \mathbb{F}$, so G is not residually free. $\qquad \square$

By contrast, the class of *fully* residually free groups is well known to be closed under taking free products, but a direct product is rarely fully residually free.

Lemma 1.9 *If G is fully residually free then G is* commutative transitive; *that is, if $a, b, c \in G \smallsetminus 1$ with $[a, b] = [b, c] = 1$ then $[a, c] = 1$.*

Two immediate consequences are that a nonabelian fully residually free group has trivial centre, and that if A is nontrivial and B is nonabelian then $A \times B$ is not fully residually free. The proof of Lemma 1.9 is well known; however, it will be important, so we recall it here.

Proof of Lemma 1.9   Suppose $[a, c] \neq 1$. Then there exists a homomorphism $f \colon G \to \mathbb{F}$ so that $f(b) \neq 1$ and $f([a, c]) \neq 1$. But

$$[f(a), f(b)] = [f(b), f(c)] = 1$$

so $f(a)$, $f(b)$ and $f(c)$ all lie in the same cyclic subgroup of $\mathbb{F}$; so $f([a, c]) = 1$, a contradiction.   □

In short: residually free groups are well behaved under direct products, but not under free products; by contrast, fully residually free groups are well behaved under free products, but not direct products.

From this point of view, the simplest residually free group that is not fully residually free is $\mathbb{F} \times \mathbb{Z}$. Indeed, it is a theorem of B Baumslag that this is the only obstruction to being fully residually free.

Theorem 1.10   [1, Theorem 1]   *Every finitely generated residually free group is either fully residually free or contains $\mathbb{F} \times \mathbb{Z}$.*

1.2   Amalgamated products and HNN extensions

We have seen that fully residually free groups can be nontrivial free products, whereas residually free groups that are not fully residually free cannot be. We now turn our attention to splittings (as amalgamated products or HNN extensions) over abelian subgroups.

Theorem 3.2 of [20] asserts that every finitely generated fully residually free group splits nontrivially over an abelian (possibly trivial) subgroup. The following is a straightforward consequence.

Theorem 1.11   [20]   *Every finitely generated noncyclic fully residually free group has infinitely many outer automorphisms.*

We now address splittings of residually free groups over abelian subgroups. The idea is to generalize the proof of Lemma 1.8. For any subgroup $H \subset G$, define the *normal core* of H in G to be the largest normal subgroup of G contained in H; that is,

$$\mathrm{Core}_G(H) = \bigcap_{g \in G} H^g.$$

Of course, $\mathrm{Core}_G(H) = H$ if and only if H is normal in G.

Proposition 1.12 *Let $G = A *_C B$ be an amalgamated product over an abelian subgroup, and suppose that G is residually free. If A is nonabelian then*

$$Z(A) \cap C \subset \mathrm{Core}_B(C).$$

Proof Fix $z \in Z(A) \cap C$. Since A is nonabelian, A contains a nonabelian free subgroup by Lemma 1.3. The edge group C intersects this free subgroup in a cyclic subgroup, and so there exist $a_1, a_2 \in A$ such that $[a_1, a_2]$ is not contained in C. Let $b \in B \smallsetminus C$. Consider

$$\xi = [[a_1, a_2], [z, b]].$$

For every $f: G \to \mathbb{F}$, either $f(z) = 1$ or $f([a_1, a_2]) = 1$, (to see this, apply the proof of Lemma 1.9 to the triple a_1, z, a_2) and so $f(\xi) = 1$. Since G is residually free, $\xi = 1$. Therefore

$$\xi = [a_1, a_2][z, b][a_2, a_1][b, z]$$

is not a reduced word in the amalgamated product decomposition of G, so $[z, b] \in C$. Therefore every $b \in B$ conjugates z into C. Hence $z \in \mathrm{Core}_B(C)$. □

In the case of an HNN extension we get a stronger result, that holds for *any* edge group.

Proposition 1.13 *Let $G = A *_C$ be an HNN extension and suppose G is residually free. Let t be a stable letter. Then*

$$(Z(A) \cap C)^t \subset Z(A).$$

Proof Fix $z \in Z(A) \cap C$, and let $a_1, a_2 \in A$. Consider

$$\xi = [[a_1, a_2], [z, t]].$$

For every $f: G \to \mathbb{F}$, either $f(z) = 1$ or $f([a_1, a_2]) = 1$ (again by the proof of Lemma 1.9) so $f(\xi) = 1$. Since G is residually free, $\xi = 1$. If $z^t = a$ then ξ can be written as

$$\xi = [[a_1, a_2], za^{-1}] = [[a_1, a_2], a^{-1}].$$

So a commutes with every commutator in A; that is, $a \in Z_A([A, A])$ which equals $Z(A)$ by Lemma 1.4. □

1.3 Virtual retractions onto cyclic subgroups

A subgroup H of a group G is a *retract* if the inclusion map $H \hookrightarrow G$ has a left inverse. Likewise H is a *virtual retract* of G if it is contained as a retract in a finite-index subgroup of G. In Bridson and Wilton [5] it is shown that residually free groups virtually retract onto many subgroups. Here we give an elementary argument to prove the special case that cyclic subgroups of residually free groups are virtual retracts. We shall apply this in Section 2.5 to prove that residually free torus bundles are 3–tori.

Lemma 1.14 *Let G be a residually free group and Z a cyclic subgroup. Then G has a finite-index subgroup G', containing Z, such that Z is a retract of G'.*

Proof Let $f\colon G \to \mathbb{F}$ be a homomorphism that is injective on Z. By Marshall Hall's Theorem [7] there exists a finite-index subgroup F' of $\mathbb{F}$ that retracts onto $f(Z)$. Let $G' = f^{-1}(F')$. Then f composed with the retraction $F' \to f(Z)$ defines a retraction $G' \to Z$, as required. $\square$

2 3–manifolds

2.1 Circle bundles

The observations of Section 1 give us some easy examples of 3–manifolds with residually free fundamental groups.

Example 2.1 If Σ is a residually free surface then $\Sigma \times S^1$ has residually free fundamental group.

In this subsection we shall examine the question of which other circle bundles over closed surfaces have residually free fundamental group. Any circle bundle M over a surface Σ is determined by its monodromy and by the Euler class. If the fundamental group of M is infinite then it fits into a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \xrightarrow{\eta} \pi_1(\Sigma) \to 1.$$

The action of $\pi_1(\Sigma)$ on $\mathbb{Z}$ by conjugation is the *monodromy* of the bundle. The Euler class is an element of $H^2(\Sigma, \mathbb{Z})$, where the coefficients are twisted by the monodromy, and the Euler class itself can be interpreted as the obstruction to the sequence splitting. (See pages 434–5 of Scott [18] for more details.)

If the fundamental group of a circle bundle is residually free then it follows from Lemma 1.3 that the monodromy is trivial.

Example 2.2 (Bundles over orientable surfaces)   Suppose Σ is closed and orientable and $\pi_1(M)$ is nontrivial and residually free. Then the fundamental group of M has a presentation of the form

$$\left\langle a_1, \ldots, a_g, b_1, \ldots, b_g, z \mid \prod_i [a_i, b_i] = z^e, [a_1, z] = \ldots = [b_g, z] = 1 \right\rangle$$

where e times the fundamental class of Σ can be identified with the Euler class (up to sign). Consider a homomorphism $f \colon \pi_1(M) \to \mathbb{F}$. If $f(z)$ is nontrivial then it commutes with everything else in the image, so the image is abelian and f factors through $H_1(M, \mathbb{Z})$. If $e \neq 0$ then the homology class of z is torsion, so z dies in every map to $\mathbb{F}$. Therefore only trivial circle bundles over orientable surfaces are residually free.

Because every other case has nontrivial centre but is not abelian, $\pi_1(M)$ is fully residually free only if Σ is the 2–sphere or the 2–torus.

We now consider the case in which Σ is nonorientable.

Example 2.3 (Bundles over nonorientable surfaces)   Suppose Σ is closed and non-orientable and $\pi_1(M)$ is residually free. In this case $H^2(\Sigma, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$, so there are only two choices for the Euler class, and M has a presentation of the form

$$\left\langle a_1, \ldots, a_m, z \mid \prod_i a_i^2 = z^e, [a_1, z] = \ldots = [b_m, z] = 1 \right\rangle$$

where e can be taken to be 0 or 1, depending on the Euler class. In either case, it is clear that the infinite cyclic subgroup generated by z, which corresponds to the circle fibre, injects under the map to $H_1(M, \mathbb{Z})$.

As $\langle z \rangle$ is precisely the kernel of the map $\eta \colon \pi_1(M) \to \pi_1(\Sigma)$, it follows that if Σ is residually free (that is, if $\chi(\Sigma) < -1$) then M has residually free fundamental group.

If Σ has Euler characteristic 1, 0 or -1 then $\pi_1(\Sigma)$ is not residually free. Indeed, in these cases it follows from [13] that every homomorphism $\pi_1(\Sigma) \to \mathbb{F}$ factors through $H_1(\Sigma, \mathbb{Z})$.[1] Just as in the orientable case, every homomorphism $\pi_1(M) \to \mathbb{F}$ either factors through $H_1(M, \mathbb{Z})$ or $\pi_1(\Sigma)$, so in fact every homomorphism to $\mathbb{F}$ has abelian image. If $\chi(\Sigma)$ is 0 or -1, in which case $\pi_1(M)$ is not abelian, it follows that $\pi_1(M)$ is not residually free. However, in the case when Σ is the projective plane and $e = 1$, we see that $\pi_1(M)$ is infinite cyclic, so residually free.

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[1]It is an entertaining exercise to prove this topologically. Consider a cellular map from Σ to a graph whose fundamental group is $\mathbb{F}$, and pull back the midpoints of the edges. The resulting simple closed curves cut Σ into simpler pieces.

This last example is the only one that is fully residually free—all the others have a nontrivial centre but are not abelian.

We summarize these examples in the following proposition.

Proposition 2.4 *If M is a circle bundle with incompressible boundary over a compact surface Σ then $\pi_1(M)$ is residually free and nontrivial if and only if M is of one of the following forms.*

(1) *The base Σ is orientable and residually free and M is the trivial bundle.*

(2) *The base Σ is nonorientable and residually free and M is any circle bundle with trivial monodromy.*

(3) *The base Σ is the projective plane and M is the nontrivial bundle with trivial monodromy.*

Of these, only $S^2 \times S^1$, the 3–torus and the nontrivial bundle over the projective plane have fully residually free fundamental groups.

Note that the third example can also be thought of as the twisted sphere bundle over the circle.

2.2 The torus decomposition

We shall apply the theory of residually free groups to a compact, irreducible, orientable 3–manifold M with (possibly empty) toral boundary. Our principal tool is the torus decomposition of Jaco–Shalen [9] and Johannson [10].

Theorem 2.5 *Let M be a compact, irreducible, orientable 3–manifold with incompressible boundary. Then M can be cut along a canonical embedded collection of disjoint incompressible 2–sided tori*

$$\mathcal{T} = T_1 \sqcup \ldots \sqcup T_k$$

such that each component of the complement $M \smallsetminus \mathcal{T}$ is either atoroidal or Seifert fibred.

See Neumann and Swarup [15] for an excellent exposition of this result. For (sketchy) definitions of atoroidal and Seifert fibred manifolds, see the next two subsections.

Since being residually free passes to subgroups, we start by classifying the possible pieces of the torus decomposition of M. If $\pi_1(M)$ is residually free and nontrivial

then it clearly has positive first Betti number. It follows that M is Haken and so satisfies Thurston's geometrization conjecture, which asserts that each piece of the torus decomposition carries one of eight model geometries. See Scott [18] for a detailed account. In particular, we assume that the atoroidal pieces are either Seifert fibred or carry hyperbolic structures. Under the additional hypothesis that the boundary of M is toral, these hyperbolic structures are of finite volume. Although we give a unified treatment for the Seifert fibred pieces in Section 2.4, the reader may find it instructive to consider which of the fibred geometries can be supported by a residually free 3–manifold.

2.3 Hyperbolic pieces

First, consider an *atoroidal* piece. This means that every essential embedded torus is boundary parallel. As mentioned above, we assume that such pieces are Seifert fibred or admit hyperbolic geometries of finite volume. Seifert fibred pieces are dealt with in the next subsection. The following proposition rules out hyperbolic pieces.

Proposition 2.6 *If M is a hyperbolic 3–manifold of finite volume then $\pi_1(M)$ is not residually free.*

Proof Suppose $\pi_1(M)$ is residually free. By Theorem 1.10, either $\pi_1(M)$ contains $\mathbb{F} \times \mathbb{Z}$ or $\pi_1(M)$ is fully residually free. As M is hyperbolic any $\mathbb{Z}^2$ subgroup of $\pi_1(M)$ is malnormal, and so $\pi_1(M)$ cannot contain $\mathbb{F} \times \mathbb{Z}$—therefore, it is fully residually free, and by Theorem 1.11 has infinitely many outer automorphisms. But it is a well known consequence of Mostow Rigidity that the fundamental groups of finite-volume hyperbolic 3–manifolds have finite outer automorphism group.   □

2.4 Seifert fibred pieces

As none of the pieces of the torus decomposition can be hyperbolic, every compact 3–manifold with incompressible toral boundary and nontrivial, residually free fundamental group is a *graph manifold*—that is, every piece of the torus decomposition is Seifert fibred. (For us, torus bundles over circles are graph manifolds.) Furthermore, the Seifert fibred pieces must be of a very restricted type.

For details of the theory of Seifert fibred manifolds see, for example, Scott [18]. We will not define Seifert fibred manifolds here, but we recall an elementary fact. If M is a Seifert fibred manifold then $\pi_1(M)$ fits into a short exact sequence

$$1 \to Z \to \pi_1(M) \xrightarrow{\eta} \pi_1(\Sigma) \to 1$$

where Z is cyclic. The following lemma reduces the Seifert fibred pieces to the cases studied in Section 2.1.

Lemma 2.7 *Let M be a Seifert fibred manifold with incompressible boundary. If $\pi_1(M)$ is nontrivial and residually free then M has no singular fibres and so is a circle bundle over a surface.*

Proof For a contradiction, assume that the base orbifold Σ has nontrivial singular locus. Because the fundamental group of the Klein bottle is not residually free, it follows that Σ has no reflector lines and therefore the singular fibres of M are isolated. By Lemma 1.3, Z is central in $\pi_1(M)$. Let $g \in \pi_1(M)$ be an element corresponding to a singular fibre. The image curve in Σ can be taken to be the generator corresponding to a cone point, so $\eta(g)^p = 1$ for some integer p.

It follows from the fact that $\pi_1(M)$ is infinite that the base orbifold Σ has a manifold cover (see Scott [18, Lemma 3.1]). Because the boundary of M is incompressible, Σ is not a disc with a single cone point and hence $\eta(g)$ is not central in $\pi_1(\Sigma)$. So there exists $h \in \pi_1(M)$ such that $\eta(g)$ and $\eta(h)$ do not commute. But $[g^p, h] = 1$, so $\langle g, h \rangle$ is abelian by Lemma 1.3. This contradicts the fact that $\eta(g)$ and $\eta(h)$ do not commute. $\qquad\square$

2.5 Torus bundles

Torus bundles over circles are important special cases of graph manifolds. If M is such a torus bundle then its fundamental group fits into a short exact sequence

$$1 \to A \to \pi_1(M) \to \mathbb{Z} \to 1$$

where $A \cong \mathbb{Z}^2$. We will use Lemma 1.14 to show that any residually free torus bundle is actually Seifert fibred. The proof avoids the deep results of [5].

Lemma 2.8 *If M is a torus bundle over the circle and $\pi_1(M)$ is residually free then M is homeomorphic to a 3–torus.*

Proof First, we will show that M is Seifert fibred. It suffices to show that the action of $\mathbb{Z}$ on A fixes some nontrivial element. Consider an arbitrary nontrivial element $a \in A$. By Lemma 1.14, there exists a finite-sheeted cover M' of M with a retraction $\rho \colon \pi_1(M') \to \langle a \rangle$. Let b generate $A \cap \pi_1(M') \cap \ker \rho$, so $A \cap \pi_1(M') = \langle a \rangle \times \langle b \rangle$. For any $g \in \pi_1(M)$ there is an m such that $g^m \in \pi_1(M')$. Then

$$\rho(g^m b g^{-m}) = \rho(g^m)\rho(b)\rho(g^{-m}) = 1$$

so $g^m b g^{-m} \in A \cap \pi_1(M') \cap \ker \rho$ and hence equals b^n for some n. Hence the subgroup of $\pi_1(M)$ generated by g and b is abelian by Lemma 1.3. It follows that the action of $\mathbb{Z}$ on A fixes b.

Therefore M is Seifert fibred. So, by Lemma 2.7, M is a circle bundle over a surface and hence by Proposition 2.4 a 3–torus. $\qquad\square$

2.6 Graph manifolds

We have shown that every (closed, orientable, irreducible) residually free 3–manifold is a graph manifold, and that every Seifert fibred piece is a circle bundle over a surface.

Proposition 2.9 *Let M be a compact, irreducible, orientable graph manifold with incompressible boundary. If $\pi_1(M)$ is nontrivial and residually free then M is Seifert fibred.*

Proof Consider an essential torus T in the torus decomposition of M, and let $M' = M \smallsetminus T$. There are two inclusions $T \hookrightarrow M'$ and each induces a foliation of T by circles, inherited from the Seifert fibred submanifolds of M' that contain T. We shall show that these two foliations coincide—hence the Seifert structures extend over T. There are two cases.

Suppose T is contained in disjoint Seifert fibred submanifolds M_1 and M_2. Both M_1 and M_2 are products, by Lemma 2.7: for $i = 1$ or 2, let $M_i = \Sigma_i \times S_i^1$, where Σ_i is a surface with nontrivial boundary and $\chi(\Sigma_i) < 0$, and S_i^1 is the circle factor of M_i. The centre of $\pi_1(M_i)$ is $\pi_1(S_i^1)$, and also

$$\mathrm{Core}_{\pi_1(M_i)}(\pi_1(T)) = \pi_1(S_i^1).$$

It follows by Proposition 1.12 that the gluing along T sends circle factors to circle factors.

Similarly, suppose each inclusion of T is contained in the same Seifert fibred submanifold M'. Then $M' = \Sigma' \times S^1$, where Σ' is a surface with nontrivial boundary and nonpositive Euler characteristic. Let t be a stable letter of the corresponding decomposition of $\pi_1(M)$ as an HNN extension. If Σ' is an annulus then M is a torus bundle over a circle and the result follows from Lemma 2.8. Otherwise, $\chi(\Sigma') < 0$ and the centre of $\pi_1(M')$ is $\pi_1(S^1)$, so $(\pi_1(S^1) \cap \pi_1(T))^t \subset \pi_1(S^1)$ by Proposition 1.13. Therefore, the gluing along T sends the circle factor of M' to itself.

As the Seifert structure extends over T, it follows that T was not really in the torus decomposition of M. Hence M has trivial torus decomposition, so is Seifert fibred. $\square$

2.7 Conclusion

Combining these results we achieve a complete classification of prime, compact 3–manifolds (with incompressible toral boundary) with residually free fundamental group.

Theorem 2.10 *If M is a prime, compact 3–manifold with incompressible toral boundary and $\pi_1(M)$ is residually free and nontrivial then M is a circle bundle over a surface, and hence of one of the forms listed in Proposition 2.4.*

Proof Let $\widehat{M}$ be a finite-sheeted orientable cover of M. We first argue that $\widehat{M}$ is Seifert fibred. If $\widehat{M}$ is not irreducible then it is $S^2 \times S^1$. Otherwise $\widehat{M}$ has a well defined torus decomposition, and because $b_1(\widehat{M})$ is positive we can take the pieces that are not Seifert fibred to admit hyperbolic structures of finite volume. By Proposition 2.6 there are no hyperbolic pieces, so $\widehat{M}$ is a graph manifold and so, by Proposition 2.9, is in fact Seifert fibred. Therefore M is Seifert fibred and so, by Lemma 2.7, is a circle bundle over a surface. □

This completes the proof of the main theorem. We saw above that, of all residually free groups, only the *fully* residually free ones are well behaved under taking free products. It is therefore easy to see when connected sums are residually free. Note that, for the following result, we need to assume the Poincaré Conjecture.

Corollary 2.11 *If M is any compact 3–manifold with incompressible toral boundary and $\pi_1(M)$ is residually free and nontrivial then one of the following holds:*

(1) *$\pi_1(M)$ is fully residually free and M is a connected sum of finitely many copies of $S^2 \times S^1$, the 3–torus, and the nontrivial circle bundle over the projective plane.*

(2) *$\pi_1(M)$ is not fully residually free and M is one of the non–fully residually free circle bundles listed in Proposition 2.4.*

Proof Consider the Kneser–Milnor prime decomposition

$$M = M_1 \# \ldots \# M_n$$

of M, where no M_i is a 3–sphere. The fundamental group of M is a free product of the fundamental groups of the prime pieces $\{M_i\}$. Assuming the Poincaré Conjecture, every prime piece has nontrivial fundamental group.

If one of the M_i is not fully residually free then, by Lemma 1.8, $M = M_i$ and so we are in case 2.

Otherwise, $\pi_1(M_i)$ is fully residually free for each i. By Theorem 2.10 each M_i is a circle bundle over a surface, and by Proposition 2.4 each M_i is of one of the three types listed. $\qquad\square$

In summary, there are no interesting examples of closed 3–manifolds with residually free fundamental group. What is the next best thing that one might hope for? As far as residual properties go, the class of torsion-free nilpotent groups seems a natural intermediate class between free groups and finite groups.

Question 2.12 Which (closed) 3–manifolds have fundamental groups that are residually torsion-free nilpotent?

It is a theorem of Magnus (see Lyndon and Schupp [14, Proposition 10.2]) that free groups are residually torsion-free nilpotent. Furthermore, the classes of graph manifolds and right angled Artin groups intersect in the class of *right angled tree groups*; see Behrstock and Neumann [2]. In [6], Duchamp and Krob proved that all right angled Artin groups are residually torsion-free nilpotent. So the class of residually torsion-free nilpotent 3–manifolds is certainly substantially larger than the class of residually free 3–manifolds.

2.8   An alternative argument

Martin Bridson has pointed out an alternative proof of the main theorem, which we outline here. Let M be an irreducible 3–manifold with nontrivial residually free fundamental group. By the main theorem of [4], $\pi_1(M)$ is virtually a direct product of fully residually free groups. If there were more than one nonabelian factor then, by Lemma 1.3, $\pi_1(M)$ would contain a subgroup isomorphic to $\mathbb{F} \times \mathbb{F}$, which is well known to contain subgroups that are finitely generated but not finitely presentable—this contradicts the fact that 3–manifolds are coherent [17], so at most one of the direct factors can be nonabelian.

There are now two cases to consider. If there is a nontrivial abelian factor, then $\pi_1(M)$ has an infinite cyclic normal subgroup and so is Seifert fibred; one then argues as in Section 2.4. On the other hand, if there is no nontrivial abelian factor then the fundamental group is fully residually free and hence, by Theorem 1.11, has infinite outer automorphism group. As in Section 2.3, this leads to a contradiction. Bridson's argument is shorter than the one given here, but uses considerably more sophisticated machinery.

3 More general boundary

In this section we discuss compact 3–manifolds with arbitrary boundary. We may assume that the boundary is still incompressible—otherwise the fundamental group is a free product, and so is easily dealt with by the results of Section 1.

With arbitrary boundary, one can no longer guarantee that the atoroidal pieces of the torus decomposition admit hyperbolic structures of finite volume, so one can no longer apply Theorem 1.11 and Mostow Rigidity to rule them out. There are certainly more examples of both residually free 3–manifolds in this context. The following folklore lemma is useful for constructing fully residually free groups.

Lemma 3.1 *Let L be a fully residually free group and let $A \subset L$ be a maximal abelian subgroup. The double $D = L *_A L$ is also fully residually free.*

Note that it follows from Lemma 1.9 that if A is not maximal in L then D is not fully residually free.

The idea of the proof of Lemma 3.1 is as follows. Let $\rho\colon D \to L$ be the natural retraction that identifies the two copies of L and let δ be an automorphism of D given by a Dehn twist in some nontrivial element of A. Then, for any element $g \in D$, $\rho \circ \delta^n(g)$ is nontrivial for all sufficiently large n. (See Bestvina and Feighn [3] for some details.)

This enables us to construct examples of fully residually free 3–manifolds by gluing interval bundles along annuli.

Example 3.2 Let N be an interval bundle over a closed, (fully) residually free, hyperbolic surface Σ, and let A be an essential annulus embedded in the boundary of N whose core curve does not have a proper root in $\pi_1(N)$. Let M be the double of N along A. Then $\pi_1(M)$ is fully residually free.

Using Lemma 3.1 one can iterate the doubling construction to build still more complicated (fully) residually free 3–manifolds. Of course, there are also negative examples, such as an interval bundle over the surface of Euler characteristic -1. It seems hard to classify residually free manifolds in this context without a complete solution to the following challenge of Potyagailo and Sela.

Question 3.3 [19] Give sufficient or necessary conditions for a group of the form $\mathbb{F} *_{u=v} \mathbb{F}$ to be fully residually free.

Note that, by Theorem 1.10, such a group is residually free if and only if it is fully residually free.

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Department of Mathematics, University of Texas
1 University Station C1200, Austin, TX 78712-0257, USA

henry.wilton@math.utexas.edu