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Large scale geometry of commutator subgroups

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Let G be a finitely presented group, and G' its commutator subgroup. Let C be the Cayley graph of G' with *all commutators in* G as generators. Then C is large scale simply connected. Furthermore, if G is a torsion-free nonelementary word-hyperbolic group, C is one-ended. Hence (in this case), the asymptotic dimension of C is at least 2.

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1 Introduction

Let G be a group and let $G' := [G, G]$ denote the commutator subgroup of G. The group G' has a canonical generating set S, which consists precisely of the set of commutators of pairs of elements in G. In other words,

$$S = \{[g, h] \text{ such that } g, h \in G\}.$$

Let $C_S(G')$ denote the Cayley graph of G' with respect to the generating set S. This graph can be given the structure of a (path) metric space in the usual way, where edges have length 1 by fiat.

By now it is standard to expect that the large scale geometry of a Cayley graph will reveal useful information about a group. However, one usually studies finitely generated groups G and the geometry of a Cayley graph $C_T(G)$ associated to a finite generating set T. For typical infinite groups G, the set of commutators S will be infinite, and the Cayley graph $C_S(G')$ will not be locally compact. This is a significant complication. Nevertheless, $C_S(G')$ has several distinctive properties which invite careful study:

(1) The set of commutators of a group is *characteristic* (ie invariant under any automorphism of G), and therefore the semidirect product $G' \rtimes \mathrm{Aut}(G)$ acts on $C_S(G')$ by isometries.

(2) The metric on G' inherited as a subspace of $C_S(G')$ is both left- and right-invariant (unlike the typical Cayley graph, whose metric is merely left-invariant).

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(3) Bounded cohomology in G is reflected in the geometry of G'; for instance, the translation length $\tau(g)$ of an element $g \in G'$ is the stable commutator length $\mathrm{scl}(g)$ of g in G.

(4) Simplicial loops in $C_S(G')$ through the origin correspond to (marked) homotopy classes of maps of closed surfaces to a $K(G,1)$.

These properties are straightforward to establish; for details, see Section 2.

This paper concerns the connectivity of $C_S(G')$ in the large for various groups G. Recall that a *thickening* Y of a metric space X is an isometric inclusion $X \to Y$ into a bigger metric space, such that the Hausdorff distance in Y between X and Y is finite. A metric space X is said to be *large scale k–connected* if for any thickening Y of X there is another thickening Z of Y which is k–connected (ie $\pi_i(Z) = 0$ for $i \leq k$; also see the definitions in Section 3). Our first main theorem, proved in Section 3, concerns the large scale connectivity of $C_S(G')$ where G is finitely presented:

Theorem A *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

As well as large scale connectivity, one can study connectivity *at infinity*. In Section 4 we specialize to word-hyperbolic G and prove our second main theorem, concerning the connectivity of G' at infinity:

Theorem B *Let G be a torsion-free nonelementary word-hyperbolic group. Then $C_S(G')$ is one-ended; ie for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least R from* id *can be joined by a path which does not come closer than distance r to* id.

Combined with a theorem of Fujiwara–Whyte [7], Theorem A and Theorem B together imply that for G a torsion-free nonelementary word-hyperbolic group, $C_S(G')$ has asymptotic dimension at least 2 (see Section 5 for the definition of asymptotic dimension).

2 Definitions and basic properties

Throughout the rest of this paper, G will denote a group, G' will denote its commutator subgroup, and S will denote the set of (nonzero) commutators in G, thought of as a generating set for G'. Let $C_S(G')$ denote the Cayley graph of G' with respect to the generating set S. As a graph, $C_S(G')$ has one vertex for every element of G', and two elements $g, h \in G'$ are joined by an edge if and only if $g^{-1}h \in S$. Let d denote distance in $C_S(G')$ restricted to G'.

Definition 2.1 Let $g \in G'$. The *commutator length* of g, denoted $\mathrm{cl}(g)$, is the smallest number of commutators in G whose product is equal to g.

From the definition, it follows that $\mathrm{cl}(g) = d(\mathrm{id}, g)$ and $d(g,h) = \mathrm{cl}(g^{-1}h)$ for $g, h \in G'$.

Lemma 2.2 *The group $G' \rtimes \mathrm{Aut}(G)$ acts on $C_S(G')$ by isometries.*

Proof $\mathrm{Aut}(G)$ acts as permutations of S, and therefore the natural action on G extends to $C_S(G')$. Further, G' acts on $C_S(G')$ by left multiplication. □

Lemma 2.3 *The metric on $C_S(G')$ restricted to G' is left- and right-invariant.*

Proof Since the inverse of a commutator is a commutator, we have $\mathrm{cl}(g^{-1}h) = \mathrm{cl}(h^{-1}g)$. Since the conjugate of a commutator by any element is a commutator, we have $\mathrm{cl}(h^{-1}g) = \mathrm{cl}(gh^{-1})$. This completes the proof. □

Definition 2.4 Given a metric space X and an isometry h of X, the *translation length* of h on X, denoted $\tau(h)$, is defined by the formula

$$\tau(h) = \lim_{n\to\infty} \frac{d(p, h^n(p))}{n}$$

where $p \in X$ is arbitrary.

By the triangle inequality, the limit does not depend on the choice of p.

For $g \in G'$ acting on $C_S(G')$ by left multiplication, we can take $p = \mathrm{id}$. Then $d(\mathrm{id}, g^n(\mathrm{id})) = \mathrm{cl}(g^n)$.

Definition 2.5 Let G be a group, and $g \in G'$. The *stable commutator length* of g is the limit

$$\mathrm{scl}(g) = \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}.$$

Hence we have the following:

Lemma 2.6 *Let $g \in G'$ act on $C_S(G')$ by left multiplication. There is an equality $\tau(g) = \mathrm{scl}(g)$.*

Proof This is immediate from the definitions. □

Stable commutator length is related to two-dimensional (bounded) cohomology. For an introduction to stable commutator length, see the book by the first author [3]; for an introduction to bounded cohomology, see Gromov [8].

If X is a metric space, and g is an isometry of X, one can obtain lower bounds on $\tau(g)$ by constructing a Lipschitz function on X which grows linearly on the orbit of a point under powers of g. One important class of Lipschitz functions on $C_S(G')$ are *quasimorphisms*:

Definition 2.7 Let G be a group. A function $\phi\colon G \to \mathbb{R}$ is a *quasimorphism* if there is a least positive real number $D(\phi)$ called the *defect*, such that for all $g, h \in G$ there is an inequality

$$|\phi(g) + \phi(h) - \phi(gh)| \le D(\phi).$$

From the defining property of a quasimorphism, $|\phi(\mathrm{id})| \le D(\phi)$ and therefore by repeated application of the triangle inequality, one can estimate

$$|\phi(f[g,h]) - \phi(f)| \le 7D(\phi)$$

for any $f, g, h \in G$. In other words:

Lemma 2.8 *Let G be a group, and let $\phi\colon G \to \mathbb{R}$ be a quasimorphism with defect $D(\phi)$. Then ϕ restricted to G' is $7D(\phi)$–Lipschitz in the metric inherited from $C_S(G')$.*

Word-hyperbolic groups admit a rich family of quasimorphisms. We will exploit this fact in Section 4.

3 Large scale simple connectivity

The following definitions are taken from Gromov [10, pages 23–24].

Definition 3.1 A *thickening* Y of a metric space X is an isometric inclusion $X \to Y$ with the property that there is a constant C so that every point in Y is within distance C of some point in X.

Definition 3.2 A metric space X is *large scale k–connected* if for every thickening $X \subset Y$ there is a thickening $Y \subset Z$ which is k–connected in the usual sense; ie Z is path-connected, and $\pi_i(Z) = 0$ for $i \le k$.

For G a finitely generated group with generating set T, Gromov outlines a proof [10, $1.C_2$] that the Cayley graph $C_T(G)$ is large scale 1–connected if and only if G is finitely presented, and $C_T(G)$ is large scale k–connected if and only if there exists a proper simplicial action of G on a $(k+1)$–dimensional k–connected simplicial complex X with compact quotient X/G.

For T an infinite generating set, large scale simple connectivity is equivalent to the assertion that G admits a presentation $G = \langle T \mid R \rangle$ where all elements in R have *uniformly bounded length* as words in T; ie all relations in G are consequences of relations of bounded length.

To show that $C_S(G')$ is large scale 1–connected, it suffices to show that there is a constant K so that for every simplicial loop γ in $C_S(G')$ there are a sequence of loops $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_n$ where γ_n is the trivial loop, and each γ_i is obtained from γ_{i-1} by cutting out a subpath $\sigma_{i-1} \subset \gamma_{i-1}$ and replacing it by a subpath $\sigma_i \subset \gamma_i$ with the same endpoints, so that $|\sigma_{i-1}| + |\sigma_i| \le K$.

More generally, we call the operation of cutting out a subpath σ and replacing it by a subpath σ' with the same endpoints where $|\sigma| + |\sigma'| \le K$ a *K–move*.

Definition 3.3 Two loops γ and γ' are *K–equivalent* if there is a finite sequence of K–moves which begins at γ, and ends at γ'.

K–equivalence is (as the name suggests) an equivalence relation. The statement that $C_S(G')$ is large scale 1–connected is equivalent to the statement that there is a constant K such that every two loops in $C_S(G')$ are K–equivalent.

First we establish large scale simple connectivity in the case of a free group.

Lemma 3.4 *Let F be a finitely generated free group. Then $C_S(F')$ is large scale simply connected.*

Proof Let γ be a loop in $C_S(F')$. After acting on γ by left translation, we may assume that γ passes through id, so we may think of γ as a simplicial path in $C_S(F')$ which starts and ends at id. If $s_i \in S$ corresponds to the i–th segment of γ, we obtain an expression

$$s_1 s_2 \cdots s_n = \mathrm{id}$$

in F, where each s_i is a commutator. For each i, let $a_i, b_i \in F$ be elements with $[a_i, b_i] = s_i$ (note that a_i, b_i with this property are not necessarily unique). Let Σ be a surface of genus n, and let α_i, β_i for $i \le n$ be a standard basis for $\pi_1(\Sigma)$; see Figure 1.

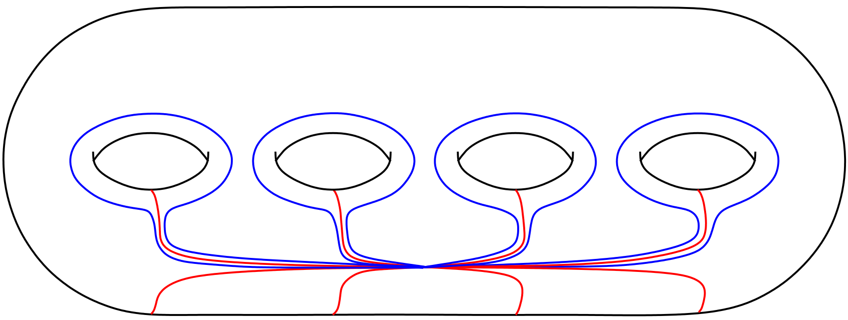

Figure 1: A standard basis for $\pi_1(\Sigma)$ where Σ has genus 4. The α_i curves are in red, and the β_i curves are in blue.

Let X be a wedge of circles corresponding to free generators for F, so that $\pi_1(X) = F$. We can construct a basepoint preserving map $f\colon \Sigma \to X$ with $f_*(\alpha_i) = a_i$ and $f_*(\beta_i) = b_i$ for each i. Since X is a $K(F,1)$, the homotopy class of f is uniquely determined by the a_i, b_i. Informally, we could say that loops in $C_S(F')$ correspond to based homotopy classes of maps of marked oriented surfaces into X (up to the ambiguity indicated above).

Let ϕ be a (basepoint preserving) self-homeomorphism of Σ. The map $f\circ\phi\colon \Sigma \to X$ determines a new loop in $C_S(F')$ (also passing through id) which we denote $\phi_*(\gamma)$ (despite the notation, this image does not depend only on γ, but on the choice of elements a_i, b_i as above).

Sublemma 3.5 *There is a universal constant K independent of γ or of ϕ (or even of F) so that after composing ϕ by an inner automorphism of $\pi_1(\Sigma)$ if necessary, γ and $\phi_*(\gamma)$ as above are K–equivalent.*

Proof Suppose we can express ϕ as a product of (basepoint preserving) automorphisms

$$\phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$$

such that if α_i^j, β_i^j denote the images of α_i, β_i under $\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1$, then ϕ_{j+1} fixes all but K consecutive pairs α_i^j, β_i^j up to (basepoint preserving) homotopy. Let $s_i^j = [f_*\alpha_i^j, f_*\beta_i^j]$, and let γ^j be the loop in $C_S(F')$ corresponding to the identity $s_1^j s_2^j \cdots s_n^j = \mathrm{id}$ in F.

For each j, let supp_{j+1} denote the *support* of ϕ_{j+1}; ie the set of indices i such that $\phi_{j+1}(\alpha_i^j) \neq \alpha_i^j$ or $\phi_{j+1}(\beta_i^j) \neq \beta_i^j$. By hypothesis, supp_{j+1} consists of at most K indices for each j.

Because it is just the marking on Σ which has been changed and not the map f, if $k \le i \le k+K-1$ is a maximal consecutive string of indices in supp_{j+1}, then there is an equality of products

$$s_k^j s_{k+1}^j \cdots s_{k+K-1}^j = s_k^{j+1} s_{k+1}^{j+1} \cdots s_{k+K-1}^{j+1}$$

as elements of F. This can be seen geometrically as follows. The expression on the left is the image under f_* of an element represented by a certain embedded based loop in Σ, while the expression on the right is its image under $f_* \circ \phi_{j+1}$. The automorphism ϕ_{j+1} is represented by a homeomorphism of Σ whose support is contained in regions bounded by such loops. Hence the expressions are equal. It follows that γ^j and γ^{j+1} are $2K$–equivalent.

So to prove the sublemma it suffices to show that any automorphism of S can be expressed (up to inner automorphism) as a product of automorphisms ϕ_i with the property above.

The hypothesis that we may compose ϕ by an inner automorphism means that we need only consider the image of ϕ in the mapping class group of Σ. It is well-known since Dehn [5] that the mapping class group of a closed oriented surface Σ of genus g is generated by twists in a finite standard set of curves, each of which intersects at most two of the α_i, β_i essentially; see Figure 2.

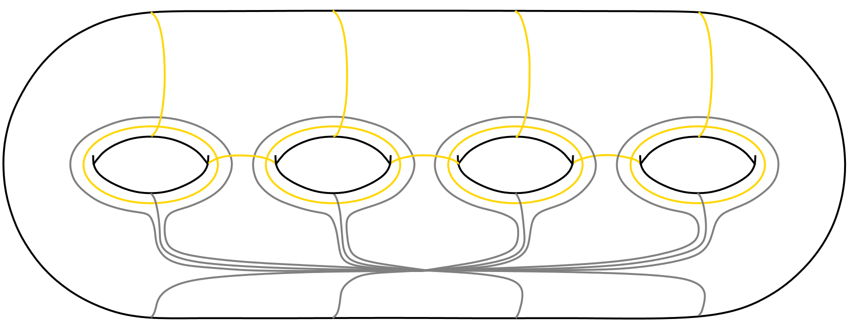

Figure 2: A standard set of $3g-1$ simple curves, in yellow. Dehn twists in these curves generate the mapping class group of Σ.

So write $\phi = \tau_1 \tau_2 \cdots \tau_m$ where the τ_i are all standard generators. Now define

$$\phi_j = \tau_1 \tau_2 \cdots \tau_{j-1} \tau_j \tau_{j-1}^{-1} \cdots \tau_1^{-1}.$$

We have

$$\phi_j \phi_{j-1} \cdots \phi_1 = \tau_1 \tau_2 \cdots \tau_j.$$

Moreover, each ϕ_j is a Dehn twist in a curve which is the image of a standard curve under $\phi_{j-1}\cdots\phi_1$, and therefore intersects α_i^{j-1} and β_i^{j-1} essentially for at most 2 (consecutive) indices i. This completes the proof of the sublemma (and shows, in fact, that we can take $K=4$). □

We now complete the proof of the lemma. As observed by Stallings (in eg [14]), a nontrivial map $f\colon \Sigma \to X$ from a closed, oriented surface to a wedge of circles factors (up to homotopy) through a *pinch* in the following sense. Make f transverse to some edge e of X, and look at the preimage Γ of a regular value of f in e. After homotoping inessential loops of Γ off e, we may assume that for some edge e and some regular value, the preimage Γ contains an embedded essential loop δ.

There are two cases to consider. In the first case, δ is nonseparating. In this case, let ϕ be an automorphism which takes α_1 to the free homotopy class of δ. Then γ and $\phi_*(\gamma)$ are K–equivalent by the sublemma. However, since $f(\delta)$ is homotopically trivial in X, there is an identity $[\phi_*\alpha_1,\phi_*\beta_1]=\mathrm{id}$ and therefore $\phi_*(\gamma)$ has length 1 shorter than γ.

In the second case, ϕ is separating, and we can let ϕ be an automorphism which takes the free homotopy class of $[\alpha_1,\beta_1]\cdots[\alpha_j,\beta_j]$ to δ. Again, by the sublemma, γ and $\phi_*(\gamma)$ are K–equivalent. But now $\phi_*(\gamma)$ contains a subarc of length j with both endpoints at id, so we may write it as a product of two loops at id, each of length shorter than that of γ.

By induction, γ is K–equivalent to the trivial loop, and we are done. □

We are now in a position to prove our first main theorem.

Theorem A *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

Proof Let W be a smooth 4–manifold (with boundary) satisfying $\pi_1(W)=G$. If $G=\langle T \mid R\rangle$ is a finite presentation, we can build W as a handlebody, with one 0–handle, one 1–handle for every generator in T, and one 2–handle for every relation in R. If $r_i\in R$ is a relation, let D_i be the cocore of the corresponding 2–handle, so that D_i is a properly embedded disk in W. Let $V\subset W$ be the union of the 0–handle and the 1–handles. Topologically, V is homotopy equivalent to a wedge of circles. By the definition of cocores, the complement of $\bigcup_i D_i$ in W deformation retracts to V. See eg Kirby [12, Chapter 1] for an introduction to handle decompositions of 4–manifolds.

Given γ a loop in $C_S(G')$, translate it by left multiplication so that it passes through id. As before, let Σ be a closed oriented marked surface, and $f\colon \Sigma \to W$ a map representing γ.

Since G is finitely presented, $H_2(G;\mathbb{Z})$ is finitely generated. Choose finitely many closed oriented surfaces $S_1, \cdots, S_r$ in W which generate $H_2(G;\mathbb{Z})$. Let K' be the supremum of the genus of the S_i. We can choose a basepoint on each S_i, and maps to W which are basepoint preserving. By tubing Σ repeatedly to copies of the S_i with either orientation, we obtain a new surface and map $f'\colon \Sigma' \to W$ representing a loop γ' such that $f'(\Sigma')$ is null-homologous in W, and γ' is K'–equivalent to γ (note that K' depends on G but not on γ).

Put f' in general position with respect to the D_i by a homotopy. Since $f'(\Sigma')$ is null-homologous, for each proper disk D_i, the signed intersection number vanishes: $D_i \cap f'(\Sigma') = 0$. Hence $f'(\Sigma) \cap D_i = P_i$ is a finite, even number of points which can be partitioned into two sets of equal size corresponding to the local intersection number of $f'(\Sigma')$ with D_i at $p \in P_i$.

Let $p, q \in P_i$ have opposite signs, and let μ be an embedded path in D_i from $f'(p)$ to $f'(q)$. Identifying p and q implicitly with their preimages in Σ', let α and β be arcs in Σ' from the basepoint to $(f')^{-1}p$ and $(f')^{-1}q$. Since μ is contractible, there is a neighborhood of μ in D_i on which the normal bundle is trivializable. Hence, since $f'(\Sigma')$ and D_i are transverse, we can find a neighborhood U of μ in W disjoint from the other D_j, and coordinates on U satisfying:

(1) $D_i \cap U$ is the plane $(x, y, 0, 0)$.

(2) $\mu \cap U$ is the interval $(t, 0, 0, 0)$ for $t \in [0, 1]$.

(3) $f'(\Sigma') \cap U$ is the union of the planes $(0, 0, z, w)$ and $(1, 0, z, w)$.

Let A be the annulus consisting of points $(t, 0, \cos(\theta), \sin(\theta))$ where $t \in [0, 1]$. Then A is disjoint from D_i and all the other D_j, and we can tube $f'(\Sigma')$ with A to reduce the number of intersection points of $f'(\Sigma')$ with $\bigcup_i D_i$, at the cost of raising the genus by 1. Technically, we remove the disks $(f')^{-1}(0, 0, s\cos(\theta), s\sin(\theta))$ and $(f')^{-1}(1, 0, s\cos(\theta), s\sin(\theta))$ for $s \in [0, 1]$ from Σ', and sew in a new annulus which we map homeomorphically to A. The result is $f''\colon \Sigma'' \to W$ with two fewer intersection points with $\bigcup_i D_i$. This has the effect of adding a new (trivial) edge to the start of γ', which is the commutator of the elements represented by the core of A and the loop $f'(\alpha) * \mu * f'(\beta)$. Let γ'' denote this resulting loop, and observe that γ'' is 1–equivalent to γ'. After finitely many operations of this kind, we obtain $f'''\colon \Sigma''' \to W$ corresponding to a loop γ''' which is $\max(1, K')$–equivalent to γ, such that $f'''(\Sigma''')$ is disjoint from $\bigcup_i D_i$.

After composing with a deformation retraction, we may assume f''' maps Σ''' into V. Let $F = \pi_1(V)$, and let $\rho\colon F \to G$ be the homomorphism induced by the inclusion $V \to W$. There is a loop γ^F in $C_S(F')$ corresponding to f''' such that $\rho_*(\gamma^F) = \gamma'''$ under the obvious simplicial map $\rho_*\colon C_S(F') \to C_S(G')$. By Lemma 3.4, the loop γ^F is K–equivalent to a trivial loop in $C_S(F')$. Pushing forward the sequence of intermediate loops by ρ_* shows that γ''' is K–equivalent to a trivial loop in $C_S(G')$. Since γ was arbitrary, we are done. □

Remark 3.6 A similar, though perhaps more combinatorial argument could be made working directly with 2–complexes in place of 4–manifolds.

In words, Theorem A says that for G a finitely presented group, all relations amongst the commutators of G are consequences of relations involving only boundedly many commutators.

The next example shows that the size of this bound depends on G:

Example 3.7 Let Σ be a closed surface of genus g, and $G = \pi_1(\Sigma)$. If γ is a loop in $C_S(G)$ through the origin, and $f\colon \Sigma' \to \Sigma$ is a corresponding map of a closed surface, then the homology class of Σ' is trivial unless the genus of Σ' is at least as big as that of Σ. Hence the loop in $C_S(G)$ of length g corresponding to the relation in the "standard" presentation of $\pi_1(\Sigma)$ is not K–equivalent to the trivial loop whenever $K < g$.

In light of Theorem A, it is natural to ask the following question:

Question 3.8 Let G be a finitely presented group. Is $C_S(G')$ large scale k–connected for all k?

Remark 3.9 Laurent Bartholdi has pointed out that for F a finitely generated free group, there is a confluent, Noetherian rewriting system for F', with rules of bounded length, which puts every word in F' over generators S into normal form (with respect to a "standard" free generating set for F'). By results of Groves [11] this should imply that $C_S(F')$ is large scale k–connected for all k, but we have not verified this implication carefully. In any case, it gives another more algebraic proof of Lemma 3.4.

4 Word-hyperbolic groups

In this section we specialize to the class of *word-hyperbolic groups*. See Gromov [9] for more details.

Definition 4.1 A path metric space X is *δ–hyperbolic* for some $\delta \geq 0$ if for every geodesic triangle abc, and every point p on the edge ab, there is $q \in ac \cup bc$ with $d_X(p,q) \leq \delta$. In other words, the δ neighborhood of any two sides of a geodesic triangle contains the third side.

Definition 4.2 A group G is *word-hyperbolic* if there is a finite generating set T for G such that $C_T(G)$ is δ–hyperbolic as a path metric space, for some δ.

Example 4.3 Finitely generated free groups are word-hyperbolic. The fundamental group of a closed surface with negative Euler characteristic is word-hyperbolic. Discrete cocompact groups of isometries of hyperbolic n–space are word-hyperbolic.

To rule out some trivial examples, one makes the following:

Definition 4.4 A word-hyperbolic group is *elementary* if it has a cyclic subgroup of finite index, and *nonelementary* otherwise.

The main theorem we prove in this section concerns the geometry of $C_S(G')$ at infinity, where G is a nonelementary word-hyperbolic group. For the sake of brevity we restrict attention to torsion-free G, though this restriction is not logically necessary; see Remark 4.9.

Theorem B *Let G be a torsion-free nonelementary word-hyperbolic group. Then $C_S(G')$ is one-ended; ie for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least R from* id *can be joined by a path which does not come closer than distance r to* id.

We will estimate distance to id in $C_S(G')$ using quasimorphisms, as indicated in Section 2. Hyperbolic groups admit a rich family of quasimorphisms. Of particular interest to us are the *Epstein–Fujiwara counting quasimorphisms*, introduced in [6], generalizing a construction due to Brooks [2] for free groups.

Fix a word-hyperbolic group G and a finite generating set T. Let $C_T(G)$ denote the Cayley graph of G with respect to T. Let σ be an oriented simplicial path in $C_T(G)$. A *copy* of σ is a translate $g \cdot \sigma$ for some $g \in G$. If γ is an oriented simplicial path in $C_T(G)$, let $|\gamma|_\sigma$ denote the maximal number of disjoint copies of σ contained in γ. For $g \in G$, define

$$c_\sigma(g) = d(\mathrm{id}, g) - \inf_\gamma(\mathrm{length}(\gamma) - |\gamma|_\sigma)$$

where the infimum is taken over all directed paths γ in $C_T(G)$ from id to g, and $d(\cdot,\cdot)$ denotes distance in $C_T(G)$.

Definition 4.5 (Epstein–Fujiwara) A *counting quasimorphism* on G is a function of the form

$$h_\sigma(g) := c_\sigma(g) - c_{\sigma^{-1}}(g)$$

where σ^{-1} denotes the same simplicial path as σ with the opposite orientation.

Since $|\gamma|_\sigma$ takes discrete values, the infimum is realized in the definition of c_σ. A path γ for which

$$c_\sigma(g) = d(\mathrm{id}, g) - \mathrm{length}(\gamma) + |\gamma|_\sigma$$

is called a *realizing path* for g. Realizing paths exist, and satisfy the following geometric property:

Lemma 4.6 (Epstein–Fujiwara [6, Proposition 2.2]) *Any realizing path for g is a (K,ϵ)–quasigeodesic in $C_T(G)$, where*

$$K = \frac{\mathrm{length}(\sigma)}{\mathrm{length}(\sigma)-1} \quad \text{and} \quad \epsilon = \frac{2\cdot \mathrm{length}(\sigma)}{\mathrm{length}(\sigma)-1}.$$

Moreover, the following holds:

Lemma 4.7 (Epstein–Fujiwara [6, Proposition 2.13]) *Let σ be a path in $C_T(G)$ of length at least 2. Then there is a constant $K(\delta)$ (where T is such that $C_T(G)$ is δ–hyperbolic as a metric space) such that $D(h_\sigma) \le K(\delta)$.*

Counting quasimorphisms are very versatile, as the following lemma shows:

Lemma 4.8 *Let G be a torsion-free, nonelementary word-hyperbolic group. Let g_i be a finite collection of elements of G. There is a commutator $s \in G'$ and a quasimorphism ϕ on G with the following properties:*

(1) $|\phi(g_i)| = 0$ *for all i.*

(2) $|\phi(s^n) - n| \le K_1$ *for all n, where K_1 is a constant which depends only on G.*

(3) $D(\phi) \le K_2$ *where K_2 is a constant which depends only on G.*

Proof Fix a finite generating set T so that $C_T(G)$ is δ–hyperbolic. There is a constant N such that for any nonzero $g \in G$, the power g^N fixes an axis L_g [9]. Since G is nonelementary, it contains quasigeodesically embedded copies of free groups, of any fixed rank. So we can find a commutator s whose translation length (in $C_T(G)$) is as big as desired. In particular, given $g_1, \cdots, g_j$ we choose s with $\tau(s) \gg \tau(g_i)$ for all i. Let L be a geodesic axis for s^N, and let σ be a fundamental domain for the action of s^N on L. Since $|\sigma| = N\tau(s) \gg \tau(g_i)$, Lemma 4.6 implies that there are no copies of σ or σ^{-1} in a realizing path for any g_i. Hence $h_\sigma(g_i)=0$ for all i. By Lemma 4.7, $D(h_\sigma)\le K(\delta)$. It remains to estimate $h_\sigma(s^n)$.

In fact, the argument of [4] Theorem A′ (which establishes explicitly an estimate that is implicit in [6]) shows that for N sufficiently large (depending only on G and not on s) no copies of σ^{-1} are contained in any realizing path for s^n with n positive, and therefore $|h_\sigma(s^n)-\lfloor n/N\rfloor|$ is bounded by a constant depending only on G. The quasimorphism $\phi = N\cdot h_\sigma$ has the desired properties. □

Remark 4.9 The hypothesis that G is torsion-free is included only to ensure that s is not conjugate to s^{-1}. It is possible to remove this hypothesis by taking slightly more care in the definition of s, using the methods of the proof of Proposition 2 from [1]. We are grateful to the referee for pointing this out.

We now give the proof of Theorem B:

Proof Let $g,h\in G'$ have commutator length at least R. Let $g=s_1s_2\cdots s_n$ and $h=t_1t_2\cdots t_m$ where $n,m\ge R$ are equal to the commutator lengths of g and h respectively, and each s_i,t_i is a commutator in G. Let s be a commutator with the properties described in Lemma 4.8 with respect to the elements g,h; that is, we want s for which there is a quasimorphism ϕ with $\phi(g)=\phi(h)=0$, with $|\phi(s^n)-n|\le K_1$ for all n, and with $D(\phi)\le K_2$. Let $N\gg R$ be very large. We build a path in $C_S(G')$ from g to h out of four segments, none of which come too close to id.

The first segment is
$$g, gs, gs^2, gs^3, \cdots, gs^N.$$
Since s is a commutator, $d(gs^i,\mathrm{id})\ge R-i$ for any i. On the other hand,
$$\phi(gs^i)\ge\phi(g)+\phi(s^i)-D(\phi)\ge i-K_2-K_1$$
where K_1,K_2 are as in Lemma 4.8 (and do not depend on g,h,s). From Lemma 2.8 we can estimate
$$d(gs^i,\mathrm{id})\ge\frac{\phi(gs^i)}{7D(\phi)}\ge\frac{i-K_2-K_1}{7K_2}.$$
Hence $d(gs^i,\mathrm{id})\ge R/14K_2-(K_1+K_2)/7K_2$ for all i, so providing $R\gg K_1,K_2$, the path gs^i never gets too close to id.

The second segment is
$$gs^N=s_1s_2\cdots s_ns^N, s_2\cdots s_ns^N,\cdots, s^N.$$

Note that consecutive elements in this segment are distance 1 apart in $C_S(G')$, by Lemma 2.3. Since $d(gs^N, \mathrm{id}) \geq (N - K_2 - K_1)/7K_2 \gg R$ for N sufficiently large, we have that for all i,

$$d(s_i \cdots s_n s^N, \mathrm{id}) \gg R.$$

The third segment is

$$s^N, t_m s^N, t_{m-1} t_m s^N, \cdots, t_1 t_2 \cdots t_m s^N = hs^N$$

and the fourth is

$$hs^N, hs^{N-1}, \cdots, hs, h.$$

For the same reason as above, neither of these segments gets too close to id. This completes the proof of the theorem, taking $r = R/14K_2 - (K_1 + K_2)/7K_2$. □

5 Asymptotic dimension

The main point of this section is to make the observation that G' for G as above is not a quasitree, and to restate this observation in terms of asymptotic dimension. We think it is worth making this restatement explicitly. The notion of asymptotic dimension was introduced by Gromov [10, page 32].

Definition 5.1 Let X be a metric space, and $X = \bigcup_i U_i$ a covering by subsets. For given $D \geq 0$, the *D–multiplicity* of the covering is at most n if for any $x \in X$, the closed D–ball centered at x intersects at most n of the U_i.

A metric space X has *asymptotic dimension at most n* if for every $D \geq 0$ there is a covering $X = \bigcup_i U_i$ for which the diameters of the U_i are uniformly bounded, and the D–multiplicity of the covering is at most $n+1$. The least such n is the *asymptotic dimension* of X, and we write

$$\mathrm{asdim}(X) = n.$$

If X is a metric space, we say $H_1(X)$ is *uniformly generated* if there is a constant L such that $H_1(X)$ is generated by loops of length at most L. It is clear that if X is large scale 1–connected, then $H_1(X)$ is uniformly generated. Fujiwara–Whyte [7] prove the following theorem:

Theorem 5.2 (Fujiwara–Whyte [7, Theorem 0.1]) *Let X be a geodesic metric space with $H_1(X)$ uniformly generated. X has $\mathrm{asdim}(X) = 1$ if and only if X is quasi-isometric to an unbounded tree.*

A group whose Cayley graph is quasi-isometric to an unbounded tree has more than one end (see eg Manning [13], especially Sections 2.1 and 2.2). Hence Theorem A and Theorem B together imply the following:

Corollary 5.3 *Let G be a nonelementary torsion-free word-hyperbolic group. Then*

$$\mathrm{asdim}(C_S(G')) \geq 2.$$

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