Angle-deformations in Coxeter groups

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The isomorphism problem for Coxeter groups has been reduced to its "reflection preserving version" by B Howlett and the second author. Thus, in order to solve it, it suffices to determine for a given Coxeter system (W, R) all Coxeter generating sets S of W which are contained in R^W, the set of reflections of (W, R). In this paper, we provide a further reduction: it suffices to determine all Coxeter generating sets $S \subseteq R^W$ which are sharp-angled with respect to R.

1 Introduction

Let W be a group and let $R \subseteq W$. We call R a *Coxeter generating set* of W if (W, R) is a Coxeter system. All Coxeter systems (W, R) considered in this paper are assumed to have finite rank, ie R is a finite set.

Let (W, R) be a Coxeter system and let $S \subseteq R^W$ be a Coxeter generating set of W. A subset J of S is called *spherical* if it generates a finite subgroup; if it is of cardinality 2, it is called an *edge* of S. Let $\{s, t\} \subseteq S$ be an edge of S. By basic results on Coxeter groups, one knows that there exist $r, r' \in R$ and $w \in W$ such that $\langle s, t \rangle^w = \langle r, r' \rangle$. If there exist $r, r' \in R$ and $w \in W$ such that $\{s, t\}^w = \{r, r'\}$, then we call the edge $\{s, t\}$ *sharp-angled* with respect to R. We call S *sharp-angled* with respect to R if all edges of S are sharp-angled with respect to R. The trivial example of the dihedral groups shows that there are examples of Coxeter systems (W, R) admitting Coxeter generating sets $S \subseteq R^W$ which are not sharp-angled with respect to R.

In [11], the second author conjectured that for any Coxeter generating set $S \subseteq R^W$, there exists an automorphism α of W such that $\alpha(S) \subseteq R^W$ and such that $\alpha(S)$ is sharp-angled [11, Conjecture 1]. This conjecture may be seen as a reduction step in order to state the main conjecture about the solution of the isomorphism problem for Coxeter groups, which is Conjecture 2 in [11] (see Remark 1 below).

It was mentioned without proof in [11] that Conjecture 1 is true if there is no subdiagram of type H_3. It turned out that this conjecture was too optimistic if there are

H_3–subdiagrams. Counterexamples have been found independently by Ratcliffe and Tschantz [12] and by Grassi [9]. This motivates the question whether it is still true that one can reduce the solution for the isomorphism problem to the main conjecture. The goal of this paper is to show that this is indeed the case.

Our first result is the following.

Theorem 1 *Let (W, R) be a Coxeter system. Let $S \subseteq R^W$ be a Coxeter generating set of W having no subsystem of type H_3. Then there exists an automorphism α of W such that $\alpha(S)$ is sharp-angled with respect to R.*

As already mentioned before, Theorem 1 was announced in [11] and it is a special case of Theorem 2 below. Its proof is given in Section 6. We prefer to present it separately since it is rather easy and provides at the same time a good overview on the kind of arguments that will yield Theorem 2.

The situation becomes considerably more complicated if H_3–subdiagrams are allowed. First of all, the counterexamples to [11, Conjecture 1] show that one cannot expect to produce sharp-angled Coxeter generating sets from S by automorphisms. So, we have to produce the desired Coxeter generating set starting from S by a sequence of operations which we call *angle-deformations*.

In order to define *angle-deformations*, we analyse the situation where we are given a Coxeter system (W, R) and a Coxeter generating set $S \subseteq R^W$ such that there is an edge J of S which is not sharp-angled with respect to R. It turns out that the Coxeter diagram of the system (W, S) has to satisfy several conditions with respect to the subset J. These conditions will be deduced in Section 8. An edge satisfying these conditions will be called a Δ–*edge*.

Let (W, S) be a Coxeter system and $J = \{r, s\}$ be a Δ–edge of S. Then we construct a mapping $\delta \colon S \to W$ such that $\delta(s) = s, \delta(r) \in \langle s, r \rangle$ and such that $S' := \{\delta(x) \mid x \in S\}$ is a Coxeter generating set with the property that all spherical 2–subsets $\{x', y'\} \neq \{\delta(r), \delta(s)\}$ are sharp-angled with respect to S. We call these mappings J–*deformations*. In the case where there are no H_3–subdiagrams, it is easy to give the definition of these J–deformations. If there are H_3–subdiagrams, the definition is given recursively. We first define J–deformations for a class of diagrams which we call *tame*. The general case will then be treated by induction on the number of "wild" vertices.

The construction of J–deformations will enable us to prove our main result, which is the following.

Theorem 2 *Let (W, R) be a Coxeter system and let $S \subseteq R^W$ be a Coxeter generating set of W. Then there exists a sequence $S = S_0, \ldots, S_k = S'$ of Coxeter generating sets S_i such that S_i is a J_i–deformation of S_{i-1} for some Δ–edge J_i of S_{i-1} for each $1 \leq i \leq k$, and such that S' is sharp-angled with respect to R.*

We remark that the proof of Theorem 2 is constructive. Hence it provides a concrete algorithm to obtain the set S' starting from S. Combining the theorem above with the fact that the isomorphism problem for Coxeter groups is reduced to its "reflection-preserving version" (as described in [11]), we obtain the following.

Corollary 1.1 *The isomorphism problem for Coxeter groups is solved as soon as the following problem is solved.*

Problem Let (W, R) be a Coxeter system. Find all Coxeter generating sets $S \subset R^W$ such that S is sharp-angled with respect to R.

Remark 1 There is a conjecture about the solution of the above problem. This is Conjecture 2 in [11] and it is a refinement of Conjecture 8.1 in Brady–McCammond–Mühlherr–Neumann [2]. It says that if R and S are as in the problem above, one can transform S into R by a sequence of twists introduced in [2]. The conjecture has been proved for various classes of Coxeter systems; the reader may refer to [11] for a survey on its status in 2005. Recently, it was shown by Ratcliffe and Tschantz in [12] that the conjecture holds for chordal Coxeter systems as well.

Remark 2 In [12], Ratcliffe and Tschantz obtained Theorem 2 for chordal Coxeter systems. Their methods are quite different from ours. Their arguments rely heavily on a very strong property of chordal Coxeter groups which is not available in the general case.

The paper is organized as follows. In Section 2, we fix notation, recall some basic facts on Coxeter groups and provide some preliminary results. In Section 3, we introduce angle-deformations and make some observations about them. In Section 4, we prepare the proof of Theorem 1. In this section we introduce Θ–edges, which are special cases of Δ–edges. Section 5 is devoted to introducing and investigating the notion of a sharp-angled set of reflections in a Coxeter group. This will enable us to give the proof of Theorem 1 in Section 6. In Section 7, we collect information about angle-deformations of Coxeter systems with subdiagrams of type H_3 and H_4. In Section 8, we define Δ–edges. Later on, these turn out to be precisely the edges of a Coxeter system for which there are nontrivial angle-deformations. This fact is a consequence

of Proposition 6.1 and Theorem 10.4, and it is indeed the key ingredient of the proof of our main result. Section 9 can be seen as a preparatory section for the proof of Theorem 10.4 which will be completed in Section 10. In Section 11, we finally give the proof of our main result, Theorem 2.

2 Preliminaries

2.1 Graphs

For a set X, denote by $P_2(X)$ the set of all subsets of X having cardinality 2. A *graph* is a pair (V, E) consisting of a set V and a set $E \subseteq P_2(V)$. The elements of V and E are called *vertices* and *edges* respectively.

Let $\Gamma = (V, E)$ be a graph. Let v, w be two vertices of Γ. They are called *adjacent* if $\{v, w\} \in E$. In this paper, a *path* from v to w is a sequence $v = v_0, v_1, \ldots, v_k = w$, where v_{i-1} is adjacent to v_i for all $1 \leq i \leq k$ and where $v_1, \ldots, v_k$ are pairwise distinct; the number k is the *length* of the path. The path is *minimal* if it is of minimal length. The *distance* between v and w (denoted by $\delta(v, w)$) is the length of a minimal path joining them; if there is no path joining v and w, we put $\delta(v, w) = \infty$.

A path $v = v_0, v_1, \ldots, v_k = w$ is said to be *chordfree* if $E \cap P_2(\{v_0, \ldots, v_k\}) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$. A path $v = v_0, v_1, \ldots, v_k = w$ is called a *circuit* if $v = w$ and $k \geq 2$.

The relation $R \subseteq V \times V$ defined by $R = \{(v, w) | \delta(v, w) \neq \infty\}$ is an equivalence relation whose equivalence classes are called the *connected components* of Γ. A graph is said to be *connected* if it has only one connected component.

2.2 Coxeter systems

Let (W, S) be a pair consisting of a group W and a set $S \subseteq W$ of involutions. For $r, s \in S$, denote by $m_{rs} \in \mathbb{N} \cup \{\infty\}$ the order of the product rs in W. Note that we will also use the notation $o(rs)$ instead of m_{rs}. Define $E(S) := \{\{r, s\} \subseteq S \mid 1 \neq m_{rs} \neq \infty\}$ to be the set of *edges* of S. Then $\Gamma(S)$ is the graph $(S, E(S))$ whose edges are labelled by the corresponding m_{rs}. Throughout this text, any graph notion (such as paths and circuits) associated to the pair (W, S) must be understood as being in $\Gamma(S)$. In particular, when we speak about the "diagram of (W, S)", we refer to $\Gamma(S)$.

The *Coxeter diagram* associated to (W, S) is the graph $(S, E'(S))$ where $E'(S) := \{\{r, s\} \subseteq S | m_{rs} \geq 3\}$ and where the edges are labelled by the corresponding m_{rs}. A subset K of S is said to be *irreducible* if the underlying Coxeter subdiagram

$(K, E'(K))$ is connected. We call K *spherical* if it generates a finite group. Finally, K is 2–*spherical* if $m_{rs} < \infty$ for all $r, s \in K$. If S is irreducible, spherical or 2–spherical, we say that (W, S) is *irreducible*, *spherical* or 2–*spherical*, respectively. Note that sometimes, we use the same notions for $\Gamma(S)$ instead of (W, S).

We say that (W, S) is a *Coxeter system* if S generates W and if the relations $((rs)^{m_{rs}})_{r,s \in S}$ form a presentation of W. We call $R \subseteq W$ a *Coxeter generating set* if (W, R) is a Coxeter system.

Let (W, R) be a Coxeter system. An element of W is called a *reflection* if it is conjugate in W to an element of R; the set of all reflections is denoted by R^W.

2.3 Conventions about figures

Here are some conventions about the figures appearing in the paper, which the reader may refer back to when needed.

Throughout this text, all figures represent diagrams of the form $\Gamma(K)$ for some Coxeter system (W, S) and $K \subseteq S$. The edges in plain have a finite label, while the edges in strips have an infinite label. An absence of edge does not imply anything. If there is a single edge with more than one label (say $m > 1$), then the figure must be understood as m different figures, one for each of these labels. If there are two or more edges with more than one label, then these edges will have the same number $m > 1$ of labels. In that case, the figure must be understood as m different figures, the i–th figure being obtained by taking the i–th label from each of these edges.

A dotted line between two vertices means that there is a path (in plain) joining these two vertices, but the other vertices in the path were omitted. (It will be always clear from the context what the omitted vertices are). For example, in Section 8, Figure 2 and Figure 3 contain a path $\{S(1), S(2), \ldots, S(n-1), S(n)\}$. We denote by X this set and we assume $n \geq 2$. Let $X_1 := X \setminus \{S(1)\}$ and $X_n := X \setminus \{S(n)\}$. We assume X has the following property:

$$o(S(i)S(j)) = \infty \text{ for all } i, j \text{ such that } 1 \leq i < j \leq n \text{ and } |i - j| \geq 2.$$

Finally, for a vertex $y \notin X_1$, we mean by $\overline{X_1 y} = \infty$ that $m_{xy} = \infty$ for all $x \in X_1$.

2.4 Coxeter generating sets and automorphisms

Lemma 2.1 Let (W, S) be a Coxeter system and let S_1, S_2 be subsets of S such that each edge of S is contained in S_1 or S_2. Put $S_0 := S_1 \cap S_2$. Let $\delta : S \to W$ be a mapping such that $\delta(S_i)$ is a Coxeter generating set of $\langle S_i \rangle$ for $i = 0, 1, 2$. Then $\delta(S)$

is a Coxeter generating set of W. Moreover, if the restrictions of δ to S_1 and S_2 extend to automorphisms of $\langle S_1 \rangle$ and $\langle S_2 \rangle$ respectively, then δ extends to an automorphism of W.

Proof This follows immediately from the fact that $W = \langle S_1 \rangle *_{\langle S_0 \rangle} \langle S_2 \rangle$. □

The following lemma follows easily by the pigeon-hole principle.

Lemma 2.2 Let G be a finite group, let α be an automorphism of G and let $g \in G$. Then $\alpha^m(g)\alpha^{m-1}(g)\ldots\alpha^2(g)\alpha(g)g = 1_G$ for some $m \geq 0$.

Using the previous lemma, one immediately obtains the following proposition.

Proposition 2.3 Let (W, S) be a Coxeter system and let $\alpha: W \to W$ be an epimorphism. Suppose that there is a subset $\mathbf{K}$ of 2^S such that the following holds:

(1) All elements of $\mathbf{K}$ are spherical.

(2) For all $K \in \mathbf{K}$, the mapping $\alpha|_{\langle K \rangle}$ is an automorphism of $\langle K \rangle$.

(3) For all $s \in S$, there exists $w_s \in \bigcup_{K \in \mathbf{K}} \langle K \rangle$ such that $\alpha(s) = w_s s w_s^{-1}$.

Then α is an automorphism of W which is of finite order.

2.5 The geometric representation of a Coxeter system

In this subsection, we collect several basic results about the geometric representation of a Coxeter system. The standard references are Bourbaki [1] and Humphreys [10].

Throughout this paper, Ω and Ω' are the following subsets of $\mathbb{R}$:

$$\Omega := \{\cos(\pi/m) \mid m \in \mathbb{N}\} \cup [1, \infty)$$

and $\Omega' := \Omega \setminus \{-1\}$. Moreover, we define a mapping $C: \mathbb{N} \cup \{\infty\} \to -\Omega$ by setting $C(m) := -\cos(\pi/m)$ if $m \in \mathbf{N}$ and $C(\infty) := -1$.

Let V be a real vector space endowed with a symmetric bilinear form $b: V \times V \to \mathbb{R}$. The set of vectors $v \in V$ with $b(v, v) = 1$ is denoted by $U(V, b)$ and for each such vector, the corresponding orthogonal reflection with respect to b is denoted by ρ_v; hence $\rho_v(x) = x - 2b(x, v)v$ for each $x \in V$.

Let (W, R) be a Coxeter system. Let $V := \mathbb{R}^R$ and $(e_r)_{r \in R}$ be the canonical basis of V. Furthermore, let $b: V \times V \to \mathbb{R}$ be the symmetric bilinear form defined by $b(e_r, e_s) := C(o(rs))$.

Theorem 2.4 *The mapping $r \mapsto \rho_{e_r}$ from R into $O(V, b)$ extends to a monomorphism from W into $O(V, b)$.*

Thus, by the above construction, we obtain a canonical faithful linear representation of the Coxeter group W which is called the *geometric representation* of (W, R). We now identify W with its image in $O(V, b)$ and we put $\Phi(W, R) := \{w(e_r) \mid w \in W, r \in R\}$. We have the following:

Lemma 2.5 *For all $r \in R$ and $w \in W$, we have $\rho_{w(e_r)} = wrw^{-1}$; in particular, $R^W = \{\rho_\alpha \mid \alpha \in \Phi(W, R)\}$. Moreover, if $\alpha, \beta \in \Phi(W, R)$ are such that $\rho_\alpha = \rho_\beta$, then $\beta = \alpha$ or $\beta = -\alpha$.*

The set $\Phi := \Phi(W, R)$ is called the *root system* of (W, R). We put

$$V^+ := \{\Sigma_{r \in R} \mu_r e_r \mid \mu_r \geq 0 \text{ for all } r \in R\}$$

and $V^- := -V^+$; furthermore, we put $\Phi^+ := V^+ \cap \Phi$ and $\Phi^- := V^- \cap \Phi$.

Lemma 2.6 $\Phi = \Phi^+ \cup \Phi^-$.

The elements of Φ^+ are called the *positive roots* of (W, R). A subset Π of Φ is called a *root-subbase* of Φ if $\Pi \subseteq \Phi^+$ and if $b(\alpha, \beta) \in -\Omega'$ for all $\alpha \neq \beta \in \Pi$.

The following theorem is a consequence of the main result in Deodhar [6] and Dyer [7].

Theorem 2.7 *Let Π be a root subbase of Φ and put $S := \{\rho_\alpha \mid \alpha \in \Pi\}$. Then $(\langle S \rangle, S)$ is a Coxeter system. Conversely, let W' be a subgroup of W which is generated by a set of reflections. Then there exists a root-subbase Π' of Φ such that $W' = \langle \rho_\alpha \mid \alpha \in \Pi' \rangle$.*

2.6 Flexibility

Let (W, S) be a Coxeter system and $J \subseteq S$. We define the following notions and notation:

- $J^\perp := \{s \in S \mid \forall\, j \in J : m_{sj} = 2\}$.
- $J^{\text{fin}} := \{s \in S \setminus J \mid m_{sj} < \infty\ \forall\, j \in J\}$.
- $J^\infty := \{s \in S \setminus J \mid \exists\, j \in J : m_{sj} = \infty\}$.
- $\mathcal{G}_J := (J^\infty, \{\{a, b\} \subseteq J^\infty \mid m_{ab} < \infty\})$.
- A *J–component* is a connected component of $\mathcal{G}_J$.

- Let L be a J–component. We shall say that $j \in J$ is *L–free* if $m_{jl} = \infty$ for all $l \in L$.

- An element j of J is *J^∞–free* if it is L–free for every J–component L.

- The J–component L is said to be *flexible* if there exists $j \in J$ such that j is L–free.

- Finally, we will say that J is *flexible* if all J–components are.

Here is a first observation.

Lemma 2.8 *Let (W, S) be a Coxeter system and let $J = \{r, s\}$ be an edge of S. Then J is flexible if and only if there is no chordfree circuit in $\Gamma(S)$ of length at least 4 containing J.*

Proof Suppose first J is not flexible. Then there exists a J–component L and $x, y \in L$ such that $m_{xr} < \infty$ and $m_{ys} < \infty$. Let $x = x_0, x_1, \ldots, x_k = y$ be a minimal path in L joining x to y. Define

$$M := \min \{i \mid 0 < i \le k;\ m_{x_i s} < \infty\}$$

and
$$m := \max \{i \mid 0 \le i < M;\ m_{x_i r} < \infty\}.$$

Then the subpath $x_m, x_{m+1}, \ldots, x_M$ from x_m to x_M is still minimal, hence chordfree, and possesses the following properties:

(1) $m_{x_i s} = \infty$ for all i such that $m \le i < M$ (by definition of M).

(2) $m_{x_i r} = \infty$ for all i such that $m < i \le M$ (by definition of m).

Moreover, $m_{x_M s} < \infty$ and $m_{x_m r} < \infty$. We then obtain a chordfree circuit $r, x_m, x_{m+1}, \ldots, x_M, s, r$, as required. The situation is illustrated on Figure 1.

The converse is obvious.   □

3 Angle-deformations

Definition 3.1 Let (W, S) be a Coxeter system, let $J = \{r, s\}$ be an edge of S and let $\omega \in \langle J \rangle$ be such that $\omega r \omega^{-1}$ and s generate $\langle J \rangle$. An (r, s, ω)–*deformation* of S is a mapping $\delta \colon S \to W$ satisfying the following properties:

(AD1) $\delta(x) \in S^W$ for all $x \in S$.

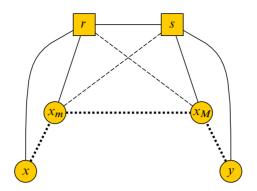(AD2) $\delta(r) = \omega r \omega^{-1}$ and $\delta(s) = s$.

Figure 1. Lemma 2.8

(AD3) $\delta(S)$ is a Coxeter generating set of W.

(AD4)   There exists a bijection Δ from the set of edges of S onto the set of edges of $\delta(S)$ such that $\Delta(J) = \{\omega r \omega^{-1}, s\}$ and such that for each edge $K \neq J$ of S, there exists $w_K \in W$ with $\Delta(K) = K^{w_K}$.

Definition 3.2   Let (W, S) be a Coxeter system and let $J = \{r, s\}$ be an edge of S. A *J–deformation* of S is an (r, s, ω)–deformation of S for some $\omega \in \langle J \rangle$. An *angle-deformation* of S is a *J*–deformation for some edge J of S.

The following proposition is a consequence of Lemma 2.1.

Proposition 3.3   *Let (W, S) be a Coxeter system and let S_1, S_2 be subsets of S such that each edge of S is contained in S_1 or S_2 and put $S_0 := S_1 \cap S_2$. Let J be an edge contained in S_0 and assume that $\delta_i \colon S_i \to \langle S_i \rangle$ are J–deformations of S_i for $i = 0, 1, 2$ and that $\delta_0 = \delta_i|_{S_0}$ for $i = 1, 2$. Define $\delta \colon S_1 \cup S_2 \to \langle S_1 \cup S_2 \rangle$ by setting $\delta|_{S_i} := \delta_i$ for $i = 1, 2$. Then δ is a J–deformation of $S_1 \cup S_2$.*

Proposition 3.4   *Let (W, S) be a Coxeter system, $J := \{r, s\}$ be an edge of S and let $\omega \in \langle J \rangle$ be such that $\omega r \omega^{-1}$ and s generate $\langle J \rangle$. Let **K** be a set of spherical subsets of S such that each element of **K** contains J and let $\delta \colon S \to W$ be a mapping with the following properties:*

a) $\delta(r) = \omega r \omega^{-1}$ and $\delta(s) = s$.

b) $\langle \delta(S) \rangle = W$.

c)   *For all $x \in S$, there exists an element w_x in $\bigcup_{K \in \mathbf{K}} \langle K \rangle$ such that $\delta(x) = w_x x w_x^{-1}$.*

    d) *For each edge E of S different from J, there exists an element $w_E \in W$ such that $\delta(E) = E^{w_E}$.*

Then δ is an (r, s, ω)–deformation of S which extends to an automorphism of W.

Proof By the universal property of (W, S) and Property d), δ extends to an endomorphism α of W which is in fact an epimorphism because of Property b). By Proposition 2.3, it follows now from Property c) that α is an automorphism. Hence $\delta(S)$ is a Coxeter generating set of W and the mapping $E \mapsto \delta(E)$ is a bijection as required in Condition (AD4). As (AD1) is a consequence of Property c), and as (AD2) is precisely Property a), the proposition is proved.   □

4 Angle-deformations involving dihedral groups

Throughout this section, (W, S) is a Coxeter system and $J = \{r, s\}$ is an edge of S such that $o(rs) \geq 3$.

4.1 Condition (TWa)

Definition 4.1 Let $a \in J$. We say that J is an *a–special* subset of S if the following condition (TWa) holds.

(TWa) For all $x \in S \setminus J$ we have $o(xa) \in \{2, \infty\}$, and if $o(xa) = 2$ then $x \in J^\perp$.

The following observation is immediate.

Lemma 4.2 *Let $a \in J$ be such that J is a–special. Then the following holds.*

    a) $\{J, J^\infty, J^\perp\}$ *is a partition of S.*

    b) *a is J^∞–free; in particular, J is flexible.*

For the remainder of this subsection, we assume that $a \in J$ is such that J is a–special, and $\omega \in \langle J \rangle$ is such that $\omega r \omega^{-1}$ and s generate $\langle J \rangle$. We put $\pi := 1_W$ if $a = r$ and $\pi := \omega$ if $a = s$. Moreover, we let $\delta \colon S \to W$ be the mapping defined by $\delta(r) = \omega r \omega^{-1}$, $\delta(y) = y$ for $y \in \{s\} \cup J^\perp$ and $\delta(x) := \pi x \pi^{-1}$ if $x \in J^\infty$.

Lemma 4.3 *Let $E = \{x, y\}$ be an edge of S different from J. Then there exists w_E such that $\delta(E) = E^{w_E}$.*

Proof Note first that each $y \in J^\perp$ commutes with ω and π. Hence, if $E \subseteq \{s\} \cup J^\perp$, then we may choose $w_E = 1_W$; if $E \subseteq \{r\} \cup J^\perp$, then we may choose $w_E = \omega$; and if $E \subseteq J^\infty \cup J^\perp$, then we may choose $w_E = \pi$.

By the previous lemma, we are left with the case where $E \subseteq J \cup J^\infty$. As a is J^∞–free and $E \neq J$, we are now left with the case where $E = \{b, x\}$ for some $x \in J^\infty$ and where b is the element of J distinct from a. If $a = r$, we may choose $w_E = 1_W$ and if $a = s$, we may choose $w_E = \omega$. $\qquad\square$

Proposition 4.4 *The mapping δ is an (r, s, ω)–deformation of S which extends to an automorphism of W.*

Proof Setting $\mathbf{K} := \{J\}$, Properties a), b) and c) required in [Proposition 3.4](#) are clear from the definition of δ and Property d) is settled by the previous lemma. $\qquad\square$

4.2 Θ–edges

Definition 4.5 We say that J is a Θ–*edge* of S if J is flexible and if there is no 2–spherical and irreducible subset of S containing J properly.

Remark If J is a Θ–edge, then $\{J, J^\infty, J^\perp\}$ is a partition of S.

For the remainder of this subsection, we suppose that J is a Θ–edge of S. Moreover, we assume that $\omega \in \langle J \rangle$ is such that $\omega r \omega^{-1}$ and s generate $\langle J \rangle$.

Let L be a J–component. We denote the set of L–free vertices in J by $\Pi(L)$. It is nonempty because J is assumed to be flexible. If $r \in \Pi(L)$, we put $a_L := r$ and $\gamma_L := 1_W$; if this is not the case, we set $a_L := s$ and $\gamma_L := \omega$. We set $K_L := J \cup L \cup J^\perp$. We define $\delta_L \colon K_L \to \langle K_L \rangle$ by $\delta_L(r) := \omega r \omega^{-1}$, $\delta_L(y) := y$ for all $y \in \{s\} \cup J^\perp$ and $\delta_L(x) := \gamma_L x \gamma_L^{-1}$ for all $x \in L$.

Proposition 4.6 *Let $\delta \colon S \to W$ be the unique mapping such that $\delta|_{K_L} = \delta_L$ for every J–component L. Then δ is an (r, s, ω)–deformation of S which extends to an automorphism of W.*

Proof Let L be a J–component. The edge J is an a_L–special subset of K_L and hence it follows by [Proposition 4.4](#) that δ_L is an (r, s, ω)–deformation of K_L. An obvious induction on the number of J–components using [Proposition 3.3](#) and [Lemma 2.1](#) yields the claim. $\qquad\square$

5   Sharp-angled sets of reflections

Throughout this section, (W, R) denotes a Coxeter system, where W is identified with its image in $O(V, b)$ by its geometric representation and $\Phi \subseteq U(V, b)$ is its root system.

Lemma 5.1   Let $\alpha, \beta \in \Phi$.

   a)   If $|b(\alpha, \beta)| < 1$, then $o(\rho_\alpha \rho_\beta)$ is finite and $b(\alpha, \beta) = -\cos(p\pi/q)$ for some integers p and q.

   b)   If $\rho_\alpha \neq \rho_\beta$ and $|b(\alpha, \beta)| \geq 1$, then $\rho_\alpha \rho_\beta$ has infinite order.

   c)   If $\rho_\alpha \neq \rho_\beta$, then $o(\rho_\alpha \rho_\beta)$ is finite if and only if $|b(\alpha, \beta)| < 1$.

Proof   Assertion a) is Proposition 1.4 in Brink and Howlett [3], whereas Assertion b) is an easy exercise in linear algebra. Assertion c) is an immediate consequence of a) and b).   □

Definition 5.2   Let $s \neq t \in R^W$ be such that $o(st)$ is finite. Let $\alpha, \beta \in \Phi$ be such that $s = \rho_\alpha$ and $t = \rho_\beta$. Then we call the 2-set $\{s, t\}$ *sharp-angled* if $|b(\alpha, \beta)| \in \Omega$.

Remark   Note that this definition does not depend on the choice of α and β in view of the last statement of Lemma 2.5.

The following two lemmas are easy.

Lemma 5.3   Let $s \neq r \in R^W$ be such that $o(rs)$ is finite. If $\{r, s\}$ is not sharp-angled, then $o(rs) \geq 5$.

Lemma 5.4   Let $s \neq r \in R^W$ be such that $o(rs)$ is finite and suppose $\{r, s\}$ is not sharp-angled. Then there exists an element $w \in \langle s, r \rangle$ such that the set $\{s, wrw^{-1}\}$ is sharp-angled. Moreover, if $o(rs) = 5$, we may choose w to be srs.

Definition 5.5   A set $S \subset R^W$ is called *sharp-angled* if each edge of S is sharp-angled.

The following lemma follows from the fact that W is a subgroup of $O(V, b)$ and from the first statement of Lemma 2.5.

Lemma 5.6   Let S be a set of reflections and let $w \in W$. Then S^w is sharp-angled if and only if S is sharp-angled.

The following fact follows from the definition of a root-subbase:

Lemma 5.7   Let Π be a root-subbase of Φ and $S := \{\rho_\alpha \mid \alpha \in \Pi\}$. Then S is sharp-angled.

5.1 Fundamental sets of reflections

Definition 5.8 A subset S of R^W is called *fundamental* if $(\langle S \rangle, S)$ is a Coxeter system.

Theorem 5.9 *Let $S \subset R^W$ be a fundamental set of reflections and suppose that one of the following holds:*

A) *The Coxeter system $(\langle S \rangle, S)$ is 2–spherical, irreducible and nonspherical.*

B) *$\Gamma(S)$ is a chordfree circuit of length at least 4.*

Then S is sharp-angled.

Proof As $W' := \langle S \rangle$ is generated by a set of reflections, we may apply the second part of Theorem 2.7 to see that there is a root-subbase Π of Φ such that the set $S' := \{\rho_\alpha \mid \alpha \in \Pi\}$ is a Coxeter generating set of W'. It is known by Caprace–Mühlherr [4] and Charney–Davis [5] that the Coxeter system $(\langle S \rangle, S)$ is strongly reflection rigid and hence S and S' are conjugate in W' and the claim follows from Lemma 5.6 and Lemma 5.7. □

6 Proof of Theorem 1

Throughout this section, (W, R) is a Coxeter system and $S \subseteq R^W$ is a fundamental set of reflections. Moreover, we assume that S contains no subset of type H_3.

Proposition 6.1 *Suppose that J is an edge of S which is not sharp-angled. Then J is a Θ–edge of S.*

Proof Put $J = \{r, s\}$. By Lemma 5.3, we have $o(rs) \geq 5$. Let $t \in S$ be such that $o(rt)$ and $o(st)$ are finite. By Theorem 5.9 and our hypothesis that there are no subsets of type H_3, we have that $t \in J^\perp$. Hence there is no irreducible 2–spherical subset of S containing J properly. Furthermore, again by Theorem 5.9, there is no chordfree circuit of length at least 4 containing J. By Lemma 2.8, it follows that J is flexible. Hence J is indeed a Θ–edge of S. □

Corollary 6.2 *Suppose that J is an edge of S which is not sharp-angled. Then there exists a J–deformation δ of S such that $\delta(J)$ is sharp-angled and such that δ is the restriction of an automorphism of $\langle S \rangle$.*

Proof Put $J = \{r, s\}$. By Lemma 5.4, we can find an element $\omega \in \langle J \rangle$ such that $\omega r \omega^{-1}$ and s generate $\omega \in \langle J \rangle$ and such that $\{\omega r \omega^{-1}, s\}$ is sharp-angled. By the previous proposition, we know that J is a Θ–edge of S and hence, by Proposition 4.6, we can find an (r, s, ω)–deformation of S which extends to an automorphism of $\langle S \rangle$ and we are done. □

Conclusion of the proof of Theorem 1 Let $S \subset R^W$ be a Coxeter generating set which is not sharp-angled. Suppose S contains $n \geq 1$ edges which are not sharp-angled and choose one of them. Call it J. By the previous corollary, there exists a J–deformation δ of S which extends to an automorphism of W (because $\langle S \rangle = W$) and such that $\delta(J)$ is sharp-angled. Let J' be an edge of S different from J. Then $\delta(J')$ is W–conjugate to J' by Property (AD4) of δ; in particular, $\delta(J')$ is sharp-angled if and only if J' is sharp-angled. Hence the number of edges in $\delta(S)$ which are not sharp-angled is $n - 1$. Thus the statement follows by an obvious induction on the number of edges of S which are not sharp-angled. □

7 Angle-deformations involving H_k

7.1 Coxeter systems of type H_3

Lemma 7.1 Let (W, S) be a Coxeter system of type H_3, where $S = \{r, s, t\}$ and $o(rs) = 5$, $o(st) = 3$. Set $\omega := tsrtst$, $\pi := trs$ and define $\delta: S \to W$ by $\delta(r) := rsr$, $\delta(s) := s$ and $\delta(t) := \omega t \omega^{-1}$. Then we have the following:

(1) $\omega s \omega^{-1} = s$, $\omega t \omega^{-1} = \pi r \pi^{-1}$, $\pi t \pi^{-1} = rsr$.

(2) There is an automorphism α of W which extends δ.

(3) δ is an (r, s, srs)–deformation of (W, S).

Proof Part (1) is a straightforward calculation. Moreover, it is clear that $\delta(S)$ is contained in S^W and that it generates W. It follows from (1) that $\{\delta(s), \delta(t)\} = \{s, t\}^\omega$ and $\{\delta(r), \delta(t)\} = \{r, t\}^\pi$. Furthermore, we have $o(\delta(r)\delta(s)) = o(rsrs) = 5$. By the universal property of Coxeter systems, it follows that δ extends to an endomorphism α of W. Since $\delta(S)$ generates W, α is surjective and hence an automorphism because W is finite. This finishes (2) and shows in particular that $\delta(S)$ is a Coxeter generating set. Assertion (3) is now a consequence of the information collected so far. □

Corollary 7.2 Let (W, S), ω, π and δ be as in the previous lemma and set $c := rsrs$, $\omega_1 := c\omega$, $\pi_1 := c\pi$ and $\delta_1 := \gamma_c \circ \delta$, where γ_c is the inner automorphism $w \mapsto cwc^{-1}$ of W. Then we have the following:

   (1) $\omega_1 = rsrtsrst$, $\pi_1 = rsrsrts$.

   (2) $\omega_1 s \omega_1^{-1} = srs$, $\omega_1 t \omega_1^{-1} = \pi_1 r \pi_1^{-1}$ and $\pi_1 t \pi_1^{-1} = r$.

   (3) There is an automorphism α_1 of W which extends δ_1.

   (4) δ_1 is an (s, r, rsr)–deformation of (W, S).

Proof Assertions (1) and (2) are straightforward calculations. Since γ_c is a reflection-preserving automorphism of W, Assertions (3) and (4) follow from Assertions (2) and (3) of the previous lemma, respectively. $\qquad\square$

Corollary 7.3 Let (W, S) be a Coxeter system of type H_3 where $S = \{r, s, t\}$ and $o(rs) = 5$, $o(rt) = 3$. Set $\omega := srstrsrt$, $\pi := srsrstr$ and define $\delta \colon S \to W$ by $\delta(r) := rsr$, $\delta(s) := s$ and $\delta(t) := \omega t \omega^{-1}$. Then we have the following:

   (1) $\omega r \omega^{-1} = rsr$, $\omega t \omega^{-1} = \pi s \pi^{-1}$ and $\pi t \pi^{-1} = s$.

   (2) There is an automorphism α of W which extends δ.

   (3) δ is an (r, s, srs)–deformation of (W, S).

Proof This follows by exchanging the roles of r and s in the previous corollary. $\quad\square$

Remark Corollary 7.3 is obtained from Lemma 7.1 by conjugating by $rsrs$ and then relabelling. We refer to this technique again in Section 7.7 without giving further details.

7.2 Coxeter systems of type H_4

Throughout this subsection, (W, S) is a Coxeter system of type H_4, where $S = \{r, s, t, u\}$ and $o(rs) = 5$, $o(st) = 3$. Set $J := \{r, s\}$, $\omega_1 := rsturstrsrstusrsrts$, $\omega_2 := tsrsrutsrsrtsrsutsrsr$, $\omega_3 := srsrutsrsrtsrsutsrsrtsr$, $\omega := rsrsr\omega_2$, $\pi := \omega\omega_1 utu$, $\tau := trs\omega_3 \omega^{-1}$ and define $\delta \colon S \to W$ by $\delta(r) := rsr$, $\delta(s) := s$, $\delta(t) := \omega t \omega^{-1}$ and $\delta(u) = u$.

Lemma 7.4 We have the following:

   a) $\pi r \pi^{-1} = rsr$, $\omega s \omega^{-1} = s$, $\omega t \omega^{-1} = \pi t \pi^{-1}$ and $\omega u \omega^{-1} = u = \pi u \pi^{-1}$.

   b) $\{\delta(r), \delta(t)\} = \{r, t\}^{\pi}$, $\{\delta(r), \delta(u)\} = \{r, u\}^{srs}$, $\{\delta(s), \delta(t)\} = \{s, t\}^{\omega}$, $\{\delta(s), \delta(u)\} = \{s, u\}^{1_W}$ and $\{\delta(t), \delta(u)\} = \{t, u\}^{\omega}$.

c) $\tau r s r \tau^{-1} = r s r$, $\tau s \tau^{-1} = s$ and $\tau \omega t \omega^{-1} \tau^{-1} = (tsrtst)t(tsrtst)^{-1}$.

Proof The relations in a) and c) are easily deduced from relations given in Franzsen and Howlett [8, page 333], and b) is an immediate consequence of a). □

Note The relations for τ will only be needed in Section 10.

Proposition 7.5 δ is an (r, s, srs)–deformation of S which extends to an automorphism of W.

Proof Clearly, $rsr = (srs)r(srs)$ and s generate $\langle J \rangle$ and $\delta(S)$ generates W. Setting $K := \{S\}$, it follows that δ has Properties a), b) and c) of Proposition 3.4, while Property d) is a consequence of the previous lemma. This proves the claim. □

7.3 Conditions (TWa)–(TWt)

Throughout this subsection, (W, S) is a Coxeter system and K is a subset of S of type H_k, where $k \in \{3, 4\}$ and where $r, s, t \in K$ are such that $o(rs) = 5$ and $o(st) = 3$; if $k = 4$, the unique element in $K \setminus \{r, s, t\}$ is denoted by u. Furthermore, we put $J := \{r, s\}$ and $\omega := tsrtst$ if $k = 3$, $\omega := rsrsr\omega_2$ if $k = 4$, $\pi := trs$ if $k = 3$ and $\pi := rsrsr\omega_2\omega_1 utu$ if $k = 4$, where ω_1 and ω_2 are as in Section 7.2.

Definition 7.6 Let $a \in J$. We say that K is an *a–special* subset of S or that K is *a–special* in S if the following two Conditions (TWa) and (TWt) hold.

(TWa) For all $x \in S \setminus K$ we have $o(xa) \in \{2, \infty\}$, and if $o(xa) = 2$ then $x \in J^{\perp}$.

(TWt) If $y \in J^{\perp} \setminus K$ is such that $o(xy) < \infty$ for some $x \in J^{\infty} \cup \{t\}$, then $y \in K^{\perp}$.

Lemma 7.7 Let $a \in J$ be such that K is a–special in S. Then we have the following.

a) $\{K, J^{\infty}, J^{\perp} \setminus K\}$ is a partition of S; if $k = 3$ then $K \cap J^{\perp} = \varnothing$ and if $k = 4$ then $K \cap J^{\perp} = \{u\}$.

b) If $y \in J^{\perp}$ is such that $o(xy) < \infty$ for some $x \in J^{\infty} \cup \{t\}$, then y commutes with ω and with π.

Proof Part a) is immediate and Part b) is a consequence of (TWt) and Lemma 7.4 a). □

7.4 Angle-deformations for a–special subsets of S

We adopt the hypotheses of the previous subsection. Furthermore, we assume that $a \in J$ is such that K is a–special in S.

We define the mapping $\delta \colon S \to W$ as follows. We put $\delta(r) := rsr$, $\delta(y) := y$ for all $y \in \{s\} \cup J^\perp$ and $\delta(t) := \omega t \omega^{-1}$. Let $x \in J^\infty$. Then we put $\delta(x) := \omega x \omega^{-1}$ if $a = r$ and $\delta(x) := \pi x \pi^{-1}$ if $a = s$.

Lemma 7.8 *The mapping δ has the following properties.*

    a) $\delta(r) = (srs)r(srs)$ *and* $\delta(s) = s$ *generate* $\langle J \rangle$.

    b) $\delta(S)$ *generates* W.

    c) $\delta \mid_K$ *is an* (r, s, srs)–*deformation of* K *which extends to an automorphism of* $\langle K \rangle$.

    d) *For each* $x \in S$, *there exists an element* $w_x \in \langle K \rangle$ *such that* $\delta(x) = w_x x w_x^{-1}$.

Proof Assertion a) is obvious. Assertions b) and d) are immediate consequences of the definition of δ. Finally, Assertion c) is a consequence of Lemma 7.1 if $k = 3$ and of Proposition 7.5 if $k = 4$. □

Lemma 7.9 *Let E be an edge of S different from J. If $k = 3$ and $a = s$, suppose in addition that E is not of the form $\{z, x\}$ with $z \in \{r, t\}$ and $x \in J^\infty$. Then there exists an element $w_E \in W$ with $\delta(E) = E^{w_E}$.*

Proof Let $E = \{x, y\}$ be such an edge of S.

If E is contained in K, the assertion follows from Lemma 7.1 for $k = 3$ and Lemma 7.4 for $k = 4$.

If E is contained in J^∞, then we may choose $w_E = \omega$ if $a = r$ and $w_E = \pi$ if $a = s$.

If E is contained in $\{s\} \cup J^\perp$, we may choose $w_E = 1_W$.

If E is contained in $\{r\} \cup J^\perp$, we may choose $w_E = srs$.

Suppose E is contained in $\{t\} \cup J^\perp$. As the case $E \subseteq J^\perp$ is already covered by the above, we may assume that $E = \{t, y\}$ for some $y \in J^\perp$. Since $o(yt)$ is finite, it follows from Lemma 7.7 b) that y commutes with ω. Hence we may choose $w_E = \omega$.

Suppose now that $x \in J^\infty$ and $y \in J^\perp$. Again by Lemma 7.7 b), we know that y commutes with ω and with π. Hence, we may choose $w_E = \omega$ if $a = r$ and $w_E = \pi$ if $a = s$.

Up to renaming the elements of E, we are now left with the case where $x \in \{r, s, t\}$ and $y \in J^\infty$.

Suppose first that $a = r$. Then the case $x = r$ is not possible and hence E is contained in $\{s, t\} \cup J^\infty$. As s commutes with ω (by Lemma 7.4 a)), we may thus choose $w_E = \omega$.

Suppose now that $a = s$. Then the case $x = s$ is not possible and by hypothesis, we only have to consider the case $k = 4$. In view of the relations given in Lemma 7.4 b), we may choose $w_E = \pi$ in this case, and we are done.   □

Proposition 7.10 *If $(a, k) \neq (s, 3)$, then δ is an (r, s, srs)–deformation of S which is the restriction of an automorphism of W.*

Proof Setting $\mathbf{K} = \{K\}$ in Proposition 3.4, the two previous lemmas show that δ has the required properties and we are done.   □

Lemma 7.11 *Suppose $(a, k) = (s, 3)$ and let $x \in J^\infty$. Then $\delta(\{r, x\}) = \{t, x\}^\pi$ and $\delta(\{t, x\}) = \{r, x\}^\pi$.*

Proof This is an immediate consequence of the relations given in Lemma 7.1 and the definition of δ.   □

7.5 K–Mirrors

Throughout this subsection, let (W, S) be a Coxeter system and let $K = \{r, s, t\} \subseteq S$ be of type H_3 such that $o(rs) = 5$ and $o(st) = 3$.

Definition 7.12 The K–*mirror* of (W, S) is the Coxeter system $(\overline{W}, \overline{S})$ with the property that there exists a bijection $x \mapsto \overline{x}$ from S onto $\overline{S}$ such that $o(\overline{r}\,\overline{x}) = o(tx)$ and $o(\overline{t}\,\overline{x}) = o(rx)$ if $x \in J^\infty$, and $o(\overline{x}\,\overline{y}) = o(xy)$ in the remaining cases.

Remark Let $(\overline{W}, \overline{S})$ be the K–mirror of (W, S) and for each $X \subseteq S$, put $\overline{X} := \{\overline{x} \mid x \in X\}$. Then $\overline{K}$ is a subset of $\overline{S}$ of type H_3 and (W, S) is the $\overline{K}$–mirror of $(\overline{W}, \overline{S})$.

Remark Let $(\overline{W}, \overline{S})$ be the K–mirror of (W, S). Then we have an obvious bijection between the edges of S and the edges of $\overline{S}$ which we will call the *canonical bijection* and which will be denoted by θ.

The following lemma is obvious.

Lemma 7.13 *Let $(\overline{W}, \overline{S})$ be the K–mirror of (W, S). Then K is s–special in S if and only if $\overline{K}$ is $\overline{s}$–special in $\overline{S}$.*

7.6 The case $(a, k) = (s, 3)$

Throughout this subsection, let (W, S) be a Coxeter system and let $K = \{r, s, t\} \subseteq S$ be of type H_3 such that $o(rs) = 5$ and $o(st) = 3$. We put $\omega := tsrtst$, $\pi := trs$ and $J := \{r, s\}$. Moreover, $(\overline{W}, \overline{S})$ denotes the K–mirror of (W, S). We assume furthermore that K is s–special in S. Note that this implies that $\overline{K}$ is $\overline{s}$–special in $\overline{S}$. We define the mapping $\delta \colon S \to W$ by $\delta(x) := x$ if $x \in J^\perp \cup \{s\}$, $\delta(x) := \pi x \pi^{-1}$ if $x \in J^\infty$, $\delta(r) := rsr$ and $\delta(t) := \omega t \omega^{-1}$.

Lemma 7.14 *Let $\{x, y\}$ be an edge of S. Then $o(\delta(x)\delta(y)) = o(\overline{x}\,\overline{y})$.*

Proof This is a consequence of Lemmas 7.9 and 7.11. □

Lemma 7.15 *$\delta(S)$ is a Coxeter generating set of W. Moreover, there exists a bijection Δ from the set of edges of S onto the set of edges of $\delta(S)$ such that $\Delta(J) = \{rsr, s\}$ and such that for each edge $E \neq J$ of S, there exists $w_E \in W$ with $\Delta(E) = E^{w_E}$.*

Proof By the universal property of $(\overline{W}, \overline{S})$ and Lemma 7.14 and Lemma 7.8, there is an epimorphism $\overline{\beta} \colon \overline{W} \to W : \overline{x} \mapsto \delta(x)$ with the following properties:

  a) $\overline{\beta}|_{\langle \overline{K} \rangle}$ is an isomorphism from $\langle \overline{K} \rangle$ onto $\langle K \rangle$.

  b) For each $x \in S$, there exists an element $w_x \in \langle K \rangle$ such that $\overline{\beta}(\overline{x}) = w_x x w_x^{-1}$.

By Lemma 7.13, $\overline{K}$ is $\overline{s}$–special in $\overline{S}$. Hence, by defining $\overline{\omega}, \overline{\pi} \in \overline{W}$ and $\overline{\delta} \colon \overline{S} \to \overline{W}$ for $(\overline{W}, \overline{S})$, we obtain also an epimorphism $\beta \colon W \to \overline{W}$ with the following properties:

  a) $\beta|_{\langle K \rangle}$ is an isomorphism from $\langle K \rangle$ onto $\langle \overline{K} \rangle$.

  b) For each $x \in S$, there exists an element $\overline{w}_x \in \langle \overline{K} \rangle$ such that $\beta(x) = \overline{w}_x \overline{x}\, \overline{w}_x^{-1}$.

We put $\alpha := \overline{\beta} \circ \beta$ and for each $x \in S$, we set $v_x := \overline{\beta}(\overline{w}_x)w_x$. Then $\alpha \colon W \to W$ is an epimorphism with the following properties:

  a) $\alpha|_{\langle K \rangle}$ is an automorphism of $\langle K \rangle$.

  b) For each $x \in S$, we have $v_x \in \langle K \rangle$ and $\alpha(x) = v_x x v_x^{-1}$.

Now, it follows from Proposition 2.3 (with $\mathbf{K} = \{K\}$) that α is an automorphism of W. In particular, $\overline{\beta}$ is an isomorphism. As $\delta(S) = \overline{\beta}(\overline{S})$, the set $\delta(S)$ is a Coxeter generating set of W.

It remains to find an appropriate Δ. As $\overline{\beta}$ is an isomorphism, we have a canonical bijection Δ_1 from the set of edges of $\overline{S}$ onto the set of edges of $\delta(S)$. Let θ be the canonical bijection from the set of edges of S onto the set of edges of $\overline{S}$. It is then readily verified, using Lemmas 7.9 and 7.11, that $\Delta := \Delta_1 \circ \theta$ is the required bijection. This finishes the proof of the lemma. □

Proposition 7.16 δ *is an* (r, s, srs)*–deformation of* (W, S).

Proof This is a consequence of the two previous lemmas. □

7.7 The relabeled version

Throughout this subsection, (W, S) is a Coxeter system and K is a subset of S of type H_k, where $k \in \{3, 4\}$ and where $r, s, t \in K$ are such that $o(rs) = 5$ and $o(rt) = 3$; if $k = 4$, the unique element in $K \setminus \{r, s, t\}$ is denoted by u. Define $\overline{\omega}_i$ for $i \in \{1, 2\}$ by exchanging r and s in the expression of ω_i given in [Section 7.2](#), where t and u are as above. Also, let $c := rsrs$ and $\overline{c} := srsr$. We put $J := \{r, s\}$, $\omega := \overline{c}trstrt = srstrsrt$ if $k = 3$, $\omega := \overline{c}srsrs\overline{\omega}_2 = r\overline{\omega}_2$ if $k = 4$, $\pi := \overline{c}tsr = srsrstr$ if $k = 3$ and $\pi := \overline{c}srsrs\overline{\omega}_2\overline{\omega}_1utu = r\overline{\omega}_2\overline{\omega}_1utu$ if $k = 4$. We assume that $a \in J$ is such that K is a–special in S and we define $\delta: S \to W$ as follows. We put $\delta(r) := rsr$, $\delta(y) := y$ for all $y \in \{s\} \cup J^\perp$ and $\delta(t) := \omega t\omega^{-1}$. Let $x \in J^\infty$. Then we put $\delta(x) := \omega x\omega^{-1}$ if $a = s$ and $\delta(x) := \pi x\pi^{-1}$ if $a = r$.

The following proposition is obtained from [Proposition 7.10](#) and [Proposition 7.16](#) by relabelling.

Proposition 7.17 *The mapping* δ *is an* (r, s, srs)*–deformation of* (W, S). *Moreover, if* $(k, a) \neq (3, r)$, *it is the restriction of an automorphism of* W.

8 Δ–edges

8.1 Some particular diagrams

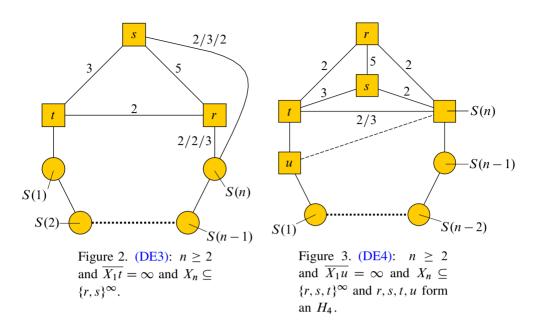Throughout this subsection, we put $\lambda := 2\cos(\pi/5)$.

Let (W_1, R_1) be a Coxeter system whose diagram is as in [Figure 2](#) and let (W_2, R_2) be a Coxeter system whose diagram is as in [Figure 3](#). Hence, we have $R_1 = \{r, s, t\} \cup X$ and $R_2 = \{r, s, t, u\} \cup X$ where $X = \{S(i) \mid 1 \leq i \leq n\}$.

For $k = 1, 2$, we consider the geometric representation of (W_k, R_k) and its root system Φ_k; in particular we identify W_k with its image in $O(V_k, b_k)$.

We put $\alpha_1 := rs(e_r) = \lambda e_r + \lambda e_s$, $\Pi_1 := \{\alpha_1, e_t\} \cup \{e_{S(i)} \mid 1 \leq i \leq n\}$, $S_1 := \{\rho_\alpha \mid \alpha \in \Pi_1\}$ and $\omega_1 := rst$.

We put $\alpha_2 := srstrs(e_r) = (\lambda + 1)e_r + 2\lambda e_s + \lambda e_t$, $\Pi_2 := \{\alpha_2, e_u\} \cup \{e_{S(i)} \mid 1 \leq i \leq n\}$, $S_2 := \{\rho_\alpha \mid \alpha \in \Pi_2\}$ and $\omega_2 := srstrsut$.

Figure 2. (DE3): $n \geq 2$ and $\overline{X_1 t} = \infty$ and $X_n \subseteq \{r,s\}^\infty$.

Figure 3. (DE4): $n \geq 2$ and $\overline{X_1 u} = \infty$ and $X_n \subseteq \{r,s,t\}^\infty$ and r,s,t,u form an H_4.

The following facts are easily verified for $k = 1, 2$:

a) $\rho_{\alpha_k} = \omega_k r \omega_k^{-1}$.

b) Π_k is a root subbase of Φ_k; in particular S_k is a fundamental set.

c) $\Gamma(S_k)$ is a chordfree circuit.

d) $\omega_1 s \omega_1^{-1} = t$ and $\omega_2 s \omega_2^{-1} = u$.

8.2   Coxeter systems containing some particular subsystems

Throughout this subsection, (W, R) is a Coxeter system and W is identified with its image in $O(V, b)$ via its geometric representation.

Proposition 8.1   *For $k = 1, 2$, let $R_k \subseteq R^W$ be a fundamental set of reflections and put $W_k := \langle R_k \rangle$. Suppose that (W_k, R_k) is a Coxeter system whose diagram is as in Figure 2 if $k = 1$ and as in Figure 3 if $k = 2$. Then $\{r, s\}$ is sharp-angled.*

Proof   For $k = 1, 2$, we define $\omega_k \in W_k$ as in the previous subsection. We put $S_1 := (R_1 \setminus \{r, s\}) \cup \{\omega_1 r \omega_1^{-1}\}$ and $S_2 := (R_2 \setminus \{r, s, t\}) \cup \{\omega_2 r \omega_2^{-1}\}$. By the considerations above, we know that the set S_k is a fundamental set of reflections. Moreover, $\Gamma(S_k)$ is a chordfree circuit. By Theorem 5.9, it follows that S_k is sharp-angled. Hence $\{\omega_1 r \omega_1^{-1}, t\}$ and $\{\omega_2 r \omega_2^{-1}, u\}$ are sharp-angled. As ω_k is an element of W which conjugates $\{r, s\}$ onto $\{\omega_1 r \omega_1^{-1}, t\}$ for $k = 1$, and onto $\{\omega_k r \omega_k^{-1}, u\}$ for $k = 2$, it follows that $\{r, s\}$ is sharp-angled as well. $\square$

Corollary 8.2 Let $S \subseteq R^W$ be a fundamental set of reflections and let $J = \{r, s\}$ be an edge of S such that $o(rs) = 5$, and which is not sharp-angled. Then there is no subset K of S as in Figure 2 or Figure 3.

8.3 Definition of Δ–edges

Definition 8.3 Let W be a group and S a subset of involutions of W. Let $J = \{r, s\}$ be an edge of S. We call J a Δ–*edge* of S if there is no subset K of S containing J having one of the following properties:

(DE1) $\Gamma(K)$ is nonspherical, 2–spherical and irreducible.

(DE2) $\Gamma(K)$ is a chordfree circuit of length at least 4.

(DE3) $\Gamma(K)$ is a diagram as shown in Figure 2.

(DE4) $\Gamma(K)$ is a diagram as shown in Figure 3.

Remark Note that if $o(rs) \neq 5$, then J is a Δ–edge if and only if (DE1) and (DE2) hold; if $o(rs) = 5$, the same remains true if there is no subset of type H_3 containing J.

The definition of Δ–edges is motivated by the following proposition, which is a consequence of Theorem 5.9 and Corollary 8.2.

Proposition 8.4 Let (W, R) be a Coxeter system, let $S \subseteq R^W$ be a fundamental set of reflections and suppose that J is an edge of S which is not sharp-angled with respect to R. Then J is a Δ–edge of S.

9 Δ–edges of type H_2

Throughout this section, (W, S) is a Coxeter system and $J = \{r, s\} \subseteq S$ is a Δ–edge of (W, S) with $o(rs) = 5$. Moreover, we define several subsets of S as follows.

- $T := \{t \in S \mid \text{type}(\{r, s, t\}) = H_3\} = T_r \amalg T_s$, where $T_r := \{t \in T \mid m_{rt} = 3\}$ and $T_s := \{t \in T \mid m_{st} = 3\}$.
- For a J–component L, put $T_L := \{t \in T \mid \exists x \in L \ : \ m_{xt} < \infty\}$.
- $U := \{u \in S \mid \exists t \in T \text{ such that type}(\{r, s, t, u\}) = H_4\}$.
- For $t \in T$, set $U_t := \{u \in U \mid \text{type}(\{r, s, t, u\}) = H_4\}$.
- For $t \in T$ and L a J_t–component, let $U_L := \{u \in U_t \mid \exists x \in L \ : \ m_{xu} < \infty\}$.
- For $t \in T$ and $u \in U_t$, let $J_t := J \cup \{t\}$ and $J_{t,u} := J \cup \{t\} \cup \{u\}$.
- For $u \in U$, set $T_u := \{t \in T \mid \text{type}(\{r, s, t, u\}) = H_4\}$.
- $T^3 := \{t \in T \mid U_t = \varnothing\}$.
- $T^4 := T \setminus T^3$.
- For $a \in J$ and $k \in \{3, 4\}$, put $T_a^k := T_a \cap T^k$.

9.1 Some preliminary observations

Lemma 9.1 *J is flexible.*

Proof This is Lemma 2.8. □

Lemma 9.2 *There are no edges in T and for each $t \in T$, there are no edges in U_t.*

Proof This follows from (DE1). □

9.2 Flexibility of J_t and consequences

Proposition 9.3 *For all $t \in T$, the set J_t is flexible.*

Proof Let $t \in T$ and let L be a J_t–component. If L is also a J–component, then L is flexible by Lemma 9.1 and we are done. So, we may assume there exists an $x \in L$ such that $x \in J^{\text{fin}}$ (thus $m_{xt} = \infty$). Suppose by contradiction there exists $y \in L$ such that $m_{yt} < \infty$. Then $m_{yr} = \infty$ or $m_{ys} = \infty$.

Let $x = x_0, x_1, \ldots, x_k = y$ be a minimal path in L joining x to y. Define

$$M := \min\{i \mid 0 < i \leq k; \ m_{x_i t} < \infty\}$$

and
$$m := \max\{i \mid 0 \leq i < M; \ x_i \in J^{\text{fin}}\}.$$

Then the subpath $x_m, x_{m+1}, \ldots, x_M$ from x_m to x_M is still minimal, hence chordfree, and possesses the following properties:

(1) $(m_{x_m r}, m_{x_m s}) \in \{(2, 2), (2, 3), (3, 2)\}$ (by (DE1)).

(2) $m_{x_i t} = \infty$ for all i such that $m \leq i < M$ (by definition of M).

(3) $x_i \in J^\infty$ for all i such that $m < i \leq M$ (by definition of m).

Moreover, $m_{x_M t} < \infty$. Thus, we obtain a subgraph $\{r, x_m, x_{m+1}, \ldots, x_M, t, s\}$ as pictured in Figure 4, contradicting (DE3). □

Corollary 9.4 *Let $t \in T$ and let L be a J–component such that there exists $z \in L$ with $o(zt) < \infty$. If $y \in J^{\text{fin}} \setminus \{t\}$ is such that there exists an $x \in L$ with $o(xy) < \infty$, then $y \in J_t^{\text{fin}}$.*

Proof Let L' be the J_t–component containing L. If $o(yt) = \infty$, we get $y \in L'$ because $o(xy) < \infty$. But then z and y belong to L', contradicting the fact that J_t is flexible. Hence $o(yt) < \infty$ and so $y \in J_t^{\text{fin}}$ because $y \in J^{\text{fin}}$ by assumption. □
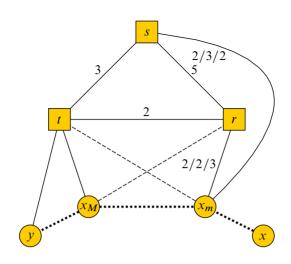
Figure 4. Contradicts (DE3)

Corollary 9.5 Let L be a J–component, then $|T_L| \leq 1$.

Proof This follows from the previous corollary and Lemma 9.2.   □

Definition 9.6 Let L be a J–component. If T_L is nonempty, then $t(L)$ denotes its unique element; if T_L is empty, we put $t(L) := \infty$.

9.3 Flexibility of $J_{t,u}$ and consequences

Proposition 9.7 Let $t \in T$ and $u \in U_t$. Then $J_{t,u}$ is flexible.

Proof Let L be a $J_{t,u}$–component. If L is also a J_t–component, then it is free by Proposition 9.3 and we are done. So, we may assume there exists an $x \in L$ such that $x \in J_t^{\text{fin}}$ (thus $m_{xu} = \infty$). Suppose by contradiction there exists $y \in L$ such that $m_{yu} < \infty$. Then $y \in J_t^{\infty}$.

Let $x = x_0, x_1, \dots, x_k = y$ be a minimal path in L joining x to y. Define

$$M := \min\{i \mid 0 < i \leq k; \, m_{x_i u} < \infty\}$$

and
$$m := \max\{i \mid 0 \leq i < M; \, x_i \in J_t^{\text{fin}}\}.$$

Then the subpath $x_m, x_{m+1}, \dots, x_M$ from x_m to x_M is still minimal, hence chordfree, and possesses the following properties:

(1) $(m_{x_m r}, m_{x_m s}, m_{x_m t}) \in \{(2,2,2), (2,2,3)\}$ (by (DE1)).

(2) $m_{x_i u} = \infty$ for all i such that $m \le i < M$ (by definition of M).

(3) $x_i \in J_t^\infty$ for all i such that $m < i \le M$ (by definition of m).

Moreover, $m_{x_M u} < \infty$. Thus, we obtain a subgraph $\{r, x_m, x_{m+1}, \ldots, x_M, u, t, s\}$ as pictured in Figure 5, contradicting (DE4). □
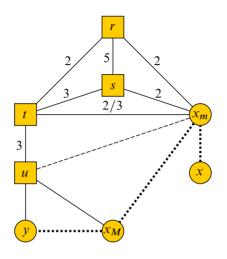

Figure 5. Contradicts (DE4)

Corollary 9.8 *Let* $t \in T$, $u \in U_t$ *and* L *be a* J_t–*component containing an element* z *with* $o(zu) < \infty$. *Suppose that* $y \in J_t^{\text{fin}}$ *is such that there exists* $x \in L$ *with* $o(xy) < \infty$. *Then* $y \in J_{t,u}^{\text{fin}} \cup \{u\}$; *in particular, if* $y \ne u$, *then* $y \in J_{t,u}^\perp$.

Proof Let L' be the $J_{t,u}$–component containing L and suppose $y \ne u$. If $o(yu) = \infty$, we get $y \in L'$ since $o(xy) < \infty$. But then z and y belong to L', contradicting the flexibility of $J_{t,u}$. Hence $o(yu) < \infty$ and so $y \in J_{t,u}^{\text{fin}}$ because $y \in J_t^{\text{fin}}$ by assumption. Now, (DE1) implies that $J_{t,u}^{\text{fin}} = J_{t,u}^\perp$, so we are done. □

Corollary 9.9 *Let* $t \in T$ *and let* L *be a* J_t–*component. Then* $|U_L| \le 1$.

Proof This follows from the previous corollary. □

Definition 9.10 *Let* $t \in T$ *and let* L *be a* J_t–*component. If* U_L *is nonempty, then* $u(L)$ *denotes its unique element; if* U_L *is empty, we put* $u(L) := \infty$.

Remark Let $t \neq t' \in T$. By Lemma 9.2, we can talk about the "J_t–component containing t'" as we will do in the following proposition.

Proposition 9.11 Let $t \neq t' \in T$, let L be the J_t–component containing t' and put $K := J_t \cup U_L$. Then $J_{t'}^{\text{fin}}$ is contained in $K^{\text{fin}} \cup L \cup U_L$.

Proof Let $y \in J_{t'}^{\text{fin}}$. Then we have in particular $o(yt') < \infty$. Hence, if $o(yt) = \infty$, we have $y \in L$. Thus we are left with the case where $o(yt) < \infty$. As $y \in J_{t'}^{\text{fin}}$, we get that $y \in J_t^{\text{fin}}$. In particular, we are already done if $u(L) = \infty$.

Let us now assume that $U_L \neq \varnothing$ and put $u := u(L)$. Then there exists an element $z \in L$ such that $o(uz) < \infty$ and there exists an element $x \in L$ (namely t') such that $o(xy) < \infty$. As $y \in J_t^{\text{fin}}$, the claim follows from Corollary 9.8. $\square$

9.4 Tameness

Definition 9.12 Let $t \in T$ and let K be a subset of S containing J_t. Then t is called *tame* in K if there is no subset K' of K containing J_t such that $\Gamma(K')$ is as in Figure 6. We call t *tame*, if it is tame in S. Otherwise, we call it *wild*.
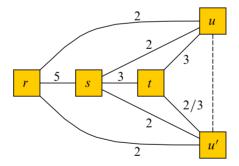

Figure 6. Tameness

Here are some basic observations. The first two of them are obvious whereas the third one is a consequence of Lemma 9.2.

Lemma 9.13 Let $t \in T$ and $K_1 \subseteq K$ be subsets of S containing J_t. If t is tame in K, then it is tame in K_1.

Lemma 9.14 If $t \in T^3$, then t is tame.

Lemma 9.15 *If t is tame, then $|U_t| \leq 1$.*

Let $t \in T$ be tame. Then we put $K_t := J_t \cup U_t$.

Lemma 9.16 *Let $t \in T$ be tame. Then $J_t^\perp = K_t^\perp$ and $J_t^{\mathrm{fin}} \cup J_t = K_t \cup K_t^\perp$.*

Proof We start with the first equality which is trivial if U_t is empty. Suppose U_t is nonempty and let u denote its unique element. Obviously, we have $K_t^\perp \subset J_t^\perp$. Let now $y \in J_t^\perp$. If $o(yu) = \infty$, we get a contradiction to the tameness of t (using (DE1)) and if $2 < o(yu) < \infty$, we get a contradiction to (DE1). Hence $o(uy) = 2$ and the first equality holds.

The second equality follows now from the fact that $J_t^{\mathrm{fin}} = J_t^\perp \cup U_t$ (because of (DE1)), the definition of K_t and the first equality. $\qquad\square$

Lemma 9.17 *Let $t \in T$ be tame, $K := K_t$, let L be a J–component with $t = t(L)$ and let $a \in J$ be L–free. Then K is an a–special subset of $S' := K \cup L \cup J^\perp$.*

Proof Note first that $S' \setminus K \subseteq L \cup J^\perp$. Thus, as a is L–free, Condition (TWa) is obviously satisfied.

We now show that Condition (TWt) holds as well. Note first that $J^\infty \cap S' = L$. Let $y \in J^\perp \setminus K$ such that $o(yx) < \infty$ for some $x \in (J^\infty \cap S') \cup \{t\} = L \cup \{t\}$. We first show that $y \in J_t^{\mathrm{fin}}$, which is obvious if $x = t$. Hence we may assume $x \in L$. As $t = t(L)$, there exists $z \in L$ such that $o(tz) < \infty$. Therefore, $y \in J^\perp \subseteq J^{\mathrm{fin}} \setminus \{t\}$ and we can apply Corollary 9.4 to see that $y \in J_t^{\mathrm{fin}}$.

Now, as t is tame and y is not in K, we have $y \in J_t^\perp$ and we are done if $U_t = \varnothing$. Suppose $U_t \neq \varnothing$ and let u be the unique element of U_t. If $o(yu) = \infty$, we get a contradiction to the tameness of t and if $2 < o(yu) < \infty$, we get a contradiction to (DE1). Hence $o(yu) = 2$ and $y \in K^\perp$ because $K = J_t \cup \{u\}$ and $y \in J_t^\perp$. $\qquad\square$

9.5 The degree of a subset containing J

Definition 9.18 Let K be a subset of S containing J. The *degree* of K is the number of elements in $K \cap T$ which are wild in K. It is denoted by $\deg(K)$.

Here is a preliminary observation.

Lemma 9.19 *Let $J \subseteq K_1 \subseteq K \subseteq S$. Then $\deg(K_1) \leq \deg(K)$.*

Definition 9.20 Let $t \in T$. For each $u \in \widehat{U}_t := U_t \cup \{\infty\}$, we define the sets V_u, W_u, X_u, Y_u and Z_u as follows.

- $V_\infty := J_t$ and $V_u := J_{t,u}$ for $u \in U_t$.
- $W_u := V_u \cup V_u^\perp$.
- X_u is the union of all J_t–components L such that $u(L) = u$.
- $Y_u := W_u \cup X_u$.
- $Z_u := Y_u \cup Y_\infty$.

Lemma 9.21 Let $t \in T$ and $u \in \widehat{U}_t$. Then t is tame in Y_u. In particular, if t is wild then $\deg(Y_u) < \deg(S)$.

The following is a consequence of [Proposition 9.11](#).

Lemma 9.22 Let $t \neq t' \in T$ and $u \in \widehat{U}_t$. If t' is contained in X_u, then $J_{t'}^{\text{fin}} \subseteq Y_u$.

Lemma 9.23 Let $u \in U_t$. Then $Y_u \cap Y_\infty = J_t \cup J_{t,u}^\perp$ and if E is an edge of Z_u, then $E \subseteq Y_u$ or $E \subseteq Y_\infty$.

Proof The first statement follows from the definition of the sets Y_u and Y_∞.

Let $E = \{x, y\}$ be an edge of Z_u and suppose that $x \in Y_u$ and $y \in Y_\infty$.

Suppose first that $x \in X_u$. Then y cannot be in X_∞ since in that case x and y would be in different J_t–components. Hence, $y \in J_t \cup J_t^\perp$. If $y \in J_t \cup J_{t,u}^\perp$, then y is in Y_u and we are done. Suppose by contradiction that $y \in J_t^\perp \setminus J_{t,u}^\perp$. Then we have $o(yu) = \infty$ by [(DE1)](#). Let L be the J_t–component containing x. Then there is an element z in L such that $o(uz)$ is finite. Let L' be the $J_{t,u}$–component containing L. Then x, y and z are contained in L', contradicting the flexibility of $J_{t,u}$.

Thus we may assume that $x \in J_{t,u} \cup J_{t,u}^\perp$. If $x \neq u$, we have $x \in Y_\infty$ and we are done. Suppose that $x = u$. Then the case $y \in X_\infty$ is not possible, because otherwise we would have $u = u(L)$ for the J_t–component L containing y. Thus we may assume that $y \in J_t \cup J_t^\perp$. By [(DE1)](#), we then get $y \in J_t \cup J_{t,u}^\perp$ and hence $E \subseteq Y_u$ in this case. $\qquad\square$

10 Existence of angle-deformations

Throughout this section, (W, S) is a Coxeter system and $J = \{r, s\} \subseteq S$ is a Δ–edge of (W, S) with $o(rs) = 5$.

We adopt the notation of the previous section.

10.1 Conventions for tame elements and standard deformations

If $t \in T$ is tame, we fix the following notation:

- By Lemma 9.15, there exists precisely one element in U_t for each $t \in T^4$, which we will denote by u_t.
- If $t \in T_s^3$, we put $\omega_t := tsrtst$ and $\pi_t := trs$.
- If $t \in T_r^3$, we put $\omega_t := srstrsrt$ and $\pi_t := srsrstr$.
- If $t \in T_s^4$, we put $\omega_t := rsrsr\omega_2$ and $\pi_t := rsrsr\omega_2\omega_1 tut$, where $u := u_t$ and ω_1, ω_2 are as in Section 7.2.
- If $t \in T_r^4$, we put $\omega_t := r\bar\omega_2$ and $\pi_t := r\bar\omega_2\bar\omega_1 utu$, where $u := u_t$ and $\bar\omega_1$, $\bar\omega_2$ are as in Section 7.7.
- For $t \in T^3$, we put $K_t := J_t$ and for $t \in T^4$, we put $K_t := J_t \cup \{u_t\}$.
- We put $\widehat{T} := T \cup \{\infty\}$, $K_\infty = J_\infty := J$, $\omega_\infty := 1_W$ and $\pi_\infty := srs$.
- Finally, for $t \in \widehat{T}$, we put $K_t^{\mathrm{def}} = K_t \cup K_t^\perp$.

Let $t \in \widehat{T}$ and if $t \neq \infty$, suppose it is tame. We define $\delta_t \colon K_t^{\mathrm{def}} \to \langle K_t^{\mathrm{def}}\rangle$ by $\delta_t(r) = rsr, \delta_t(s) = s$, $\delta_t(t) = \omega_t t\omega_t^{-1}$ (for $t \neq \infty$), $\delta_t(u_t) := u_t$ for $t \in T^4$ and $\delta_t(x) := x$ for all $x \in K_t^\perp$.

Proposition 10.1 δ_t *is an* (r, s, srs)*–deformation of* K_t^{def}.

Proof This is a consequence of Lemma 7.1, Corollary 7.3 and Proposition 7.5 together with its relabeled version. $\qquad\square$

Definition 10.2 We call δ_t the *standard deformation* of K_t^{def}.

10.2 Tame angle-deformations

Definition 10.3 Let K be a subset of S containing J and let $\delta \colon K \to \langle K\rangle$ be an (r, s, srs)–deformation of K. Then we call δ *tame* if for each $t \in T \cap K$ which is tame in K, there exists an element $w_t \in \langle K\rangle$ such that $\delta(x) = w_t\delta_t(x)w_t^{-1}$ for all $x \in K_t^{\mathrm{def}}$.

The goal of this section is to prove the following result.

Theorem 10.4 *There exists a tame* (r, s, srs)*–deformation of* S.

10.3　The tame case

Throughout this subsection, we assume the following.

(TAME)　*All elements in T are tame.*

For each $t \in \widehat{T}$, let $\delta_t \colon K_t^{\mathrm{def}} \to \langle K_t^{\mathrm{def}} \rangle$ be the standard deformation.

We put $\widehat{J} := J \cup T \cup J^\perp$ and we define $\widehat{\delta} \colon \widehat{J} \to \langle \widehat{J} \rangle$ by $\widehat{\delta}|_{K_t^{\mathrm{def}}} := \delta_t$ for each $t \in \widehat{T}$ and $\widehat{\delta}|_{J^\perp} := \mathrm{id}_{J^\perp}$. Note that $\widehat{\delta}$ is well-defined.

Our first goal is to prove the following proposition.

Proposition 10.5　$\widehat{\delta}$ *is a tame (r, s, srs)–deformation of $(\langle \widehat{J} \rangle, \widehat{J})$.*

Lemma 10.6　*Let $t \in T$ and $y \in J^\perp$ such that $o(ty) < \infty$. Then, y commutes with ω_t and π_t.*

Proof　If $t \in T^3$ or if $y \neq u_t$, then the tameness of t and (DE1) imply that $y \in J_t^\perp$, and hence $y \in K_t^\perp$ by Lemma 9.16 and we are done. If $y = u_t$, then the result follows from Lemma 7.4 a). □

Lemma 10.7　*Let $E := \{x, y\}$ be an edge of $\widehat{J}$ different from J. Then, there is an element $w_E \in \bigcup_{t \in \widehat{T}} \langle J_t \rangle$ such that $\widehat{\delta}(E) = E^{w_E}$.*

Proof　If E is contained in J_t for some $t \in T$, then there exists an element $w_E \in \langle J_t \rangle$ such that $\widehat{\delta}(E) = E^{w_E}$. This follows from Lemma 7.1 and Corollary 7.3.

If E is contained in $J^\perp \cup \{s\}$, then $\widehat{\delta}(E) = E^{1_W}$. Hence, the case $s \in E$ is settled completely.

Suppose now $x = r$. In this case, we may assume $y \in J^\perp$ because the case $y \in T$ is already covered above. For all $y \in J^\perp$, we have $ysrs = srsy$ and therefore $\widehat{\delta}(E) = E^{srs}$.

By Lemma 9.2, it remains to consider the case where $x \in T$ and $y \in J^\perp$. Set $x = t$. As $\{x, y\}$ is an edge, it follows from Lemma 10.6 that y commutes with ω_t. Hence we have $\widehat{\delta}(E) = E^{\omega_t}$ in this case. □

Proof of Proposition 10.5　It is readily verified that $\widehat{\delta}(\widehat{J})$ generates $\langle \widehat{J} \rangle$ and by Lemma 10.7 and Proposition 2.3 (with $\mathbf{K} = \{J_t \mid t \in \widehat{T}\}$), it follows that $\widehat{\delta}$ extends to an automorphism $\widehat{\alpha}$ of $\langle \widehat{J} \rangle$, which implies in particular that $\widehat{\delta}(\widehat{J})$ is a Coxeter generating set of $\langle \widehat{J} \rangle$. Using Lemma 10.7, it is now straightforward to check that $\widehat{\delta}$ satisfies Properties (AD1)–(AD4). The tameness of $\widehat{\delta}$ is a consequence of its definition. This concludes the proof of Proposition 10.5. □

Let L be a J–component and define the set $T_L \subseteq T$ as before. Since we assume that J is a Δ–edge, we know by Corollary 9.5 that $|T_L| \leq 1$. We define $t(L)$ as in Section 9.2. Moreover, we put $J_L := K_{t(L)}$, $K_L := J_L \cup J^\perp \cup L$ and $M_L := K_L \cup T$.

Let $\Pi(L)$ be the set of L–free vertices of J; since J is flexible (by Lemma 9.1), we know that $\Pi(L) \neq \varnothing$.

For each J–component L, we define $\gamma_L \in \langle J_L \rangle$ as follows.

If $t(L) = \infty$ and $r \in \Pi(L)$, we put $\gamma_L := 1_W$.

If $t(L) = \infty$ and $\Pi(L) = \{s\}$, we put $\gamma_L := srs$.

If $t(L) \in T_s$ and $r \in \Pi(L)$, we put $a_L := r$ and $\gamma_L := \omega_t$.

If $t(L) \in T_s$ and $\Pi(L) = \{s\}$, we put $a_L := s$ and $\gamma_L := \pi_t$.

If $t(L) \in T_r$ and $s \in \Pi(L)$, we put $a_L := s$ and $\gamma_L := \omega_t$.

If $t(L) \in T_r$ and $\Pi(L) = \{r\}$, we put $a_L := r$ and $\gamma_L := \pi_t$.

Finally, we define $\delta_L \colon K_L \to \langle K_L \rangle$ by $\delta_L|_{J_L} := \delta_{t(L)}|_{J_L}$, $\delta_L|_{J^\perp} = \mathrm{id}_{J^\perp}$ and $\delta_L(x) := \gamma_L x \gamma_L^{-1}$ for all $x \in L$. Note that δ_L is well-defined.

Lemma 10.8 *Let L be a J–component with $t := t(L) \neq \infty$. Then K_t is an a_L– special subset of K_L.*

Proof This is a consequence of Lemma 9.17. □

Lemma 10.9 *Let L be a J–component. Then δ_L is an (r, s, srs)–deformation of K_L.*

Proof This is a consequence of the previous lemma and Propositions 7.10, 7.16 and 7.17 applied to the Coxeter system $(\langle K_L \rangle, K_L)$ if $t(L) \neq \infty$, and of Proposition 4.4 applied to the same Coxeter system otherwise. □

Proposition 10.10 *Let L be a J–component. Define $\widehat{\delta}_L \colon M_L \to \langle M_L \rangle$ by $\widehat{\delta}_L|_{K_L} := \delta_L$ and $\widehat{\delta}_L|_{\widehat{J}} := \widehat{\delta}$. Then $\widehat{\delta}_L$ is an (r, s, srs)–deformation of M_L.*

Proof Note first that $\widehat{\delta}_L$ is well-defined. By the previous lemma, δ_L is an (r, s, srs)– deformation of K_L and by Proposition 10.5, $\widehat{\delta}$ is an (r, s, srs)–deformation of $\widehat{J}$. As $K_L \setminus \widehat{J} = L$ and $\widehat{J} \setminus K_L = T \setminus T_L$, all edges of M_L are contained in at least one of the two sets. Now, as δ restricted to $M_L \cap K_L = J_L \cup J^\perp$ is an (r, s, rsr)–deformation of $M_L \cap K_L$, Proposition 3.3 finishes the proof. □

Theorem 10.11 *Let* $\delta\colon S \to W$ *be the mapping defined by* $\delta|_{M_L} := \widehat{\delta}_L$ *for each* J*–component* L. *Then* δ *is a tame* (r, s, srs)*–deformation of* (W, S).

Proof Note that for two different J–components L and L', we have $M_L \cap M_{L'} = T \cup J^{\perp} \cup J$, which is independent of L and L'. Moreover, δ restricted to $T \cup J^{\perp} \cup J$ is an (r, s, rsr)–deformation of $T \cup J^{\perp} \cup J$. The claim now follows by induction on the number of J–components using Propositions 3.3 and 10.10, the tameness being a consequence of Proposition 10.5. $\qquad\square$

10.4 Proof of Theorem 10.4

The theorem will be proved by induction on $\deg(S)$. If $\deg(S) = 0$, all elements in T are tame and we are done by Theorem 10.11. Suppose now that the degree of S is at least 1. Then there exists a wild $t \in T$, which we fix throughout this subsection.

For each $u \in \widehat{U}_t$, we define the sets V_u, W_u, X_u, Y_u and Z_u as in Section 9.5. For $u \in U_t$, we put $\tau_u := \tau$ where τ is defined as in Section 7.2 and $\tau_{\infty} := 1_W$.

Let $u \in \widehat{U}_t$. By Lemma 9.21, we know that $\deg(Y_u) < \deg(S)$. Thus, we know by induction that there is a tame (r, s, srs)–deformation θ_u of Y_u. Again by Lemma 9.21, t is tame in Y_u and if we define K_t^{def} as in Section 10.1 with respect to Y_u, we have $W_u = K_t^{\mathrm{def}}$. Hence, there is an $w_u \in \langle Y_u \rangle$ such that $\mathrm{Int}(w_u) \circ \theta_u|_{W_u}$ is the standard deformation of W_u. We put $\Theta_u := \mathrm{Int}(w_u) \circ \theta_u$. The discussion above yields the following.

Lemma 10.12 *For each $u \in \widehat{U}_t$, there exists a tame (r, s, srs) deformation Θ_u of Y_u such that $\Theta_u|_{W_u}$ is the standard deformation of W_u.*

For each $u \in \widehat{U}_t$, let Θ_u be as in the previous lemma and put $\delta_u := \mathrm{Int}(\tau_u) \circ \Theta_u$.

Lemma 10.13 *For each $u \in \widehat{U}_t$ the mapping $\delta_u\colon Y_u \to \langle Y_u \rangle$ is a tame (r, s, srs)–deformation of Y_u. Moreover, we have $\delta_u|_{Y_u \cap Y_{\infty}} = \delta_{\infty}|_{Y_u \cap Y_{\infty}}$. In particular, there exists an (r, s, srs)–deformation $\widehat{\delta}_u$ of Z_u such that $\widehat{\delta}_u|_{Y_u} = \delta_u$ and $\widehat{\delta}_u|_{Y_{\infty}} = \delta_{\infty}$.*

Proof The first assertion of the lemma is clear, because Θ_u is tame and $\tau_u \in \langle Y_u \rangle$.

The second assertion is trivial for $u = \infty$, so we may assume $u \in U_t$. First remark that $Y_u \cap Y_{\infty} = J_t \cup J_{t,u}^{\perp}$ by Lemma 9.23. Since $\Theta_u|_{W_u}$ is the standard deformation and as $\tau_u \in \langle J_{t,u} \rangle$ with all elements in $J_{t,u}^{\perp}$ and with rsr and s (by Lemma 7.4 c)), it follows that $\delta_u|_{J \cup J_{t,u}^{\perp}} = \delta_{\infty}|_{J \cup J_{t,u}^{\perp}}$. Thus, it remains only to check whether $\delta_u(t) = \delta_{\infty}(t)$; but this is also a consequence of Lemma 7.4 c). This concludes the proof of the second assertion.

The last assertion is a consequence of the second, Lemma 9.23 and Proposition 3.3. $\quad\square$

Lemma 10.14 *There exists an (r, s, srs)–deformation δ of S such that $\delta|_{Y_u} = \delta_u$ for each $u \in \widehat{U}_t$.*

Proof As t is assumed to be wild, we have $|U_t| \geq 1$. We prove the lemma by induction on $|U_t|$. If $|U_t| = 1$ and if u denotes the unique element in U_t, then $S = Z_u$ and we are done by the previous lemma.

Suppose now $|U_t| > 1$ and let $u \in U_t$. Put $C_u := \bigcup_{u \neq u' \in U_t} Z_{u'}$. Note first that $C_u \cap Z_u = Y_\infty$ and that each edge in S is contained in C_u or in Z_u. By induction, there exists an (r, s, srs)–deformation δ'_u of C_u such that $\delta'_u|_{Y_{u'}} = \delta_{u'}$ for each $u' \in \widehat{U}_t$ different from u. By the previous lemma, there exists an (r, s, srs)–deformation $\widehat{\delta}_u$ of Z_u such that $\widehat{\delta}_u|_{Y_a} = \delta_a$ for $a \in \{u, \infty\}$. Now Proposition 3.3 yields the existence of δ. □

Conclusion of the proof of Theorem 10.4 The previous lemma yields the existence of an (r, s, srs)–deformation δ of S such that $\delta|_{Y_u} = \delta_u$ for each $u \in \widehat{U}_t$. It remains to show that δ is tame. Let $t' \in T$ be tame in S. Since t is assumed to be wild, we have $t' \neq t$. By Lemmas 9.22 and 9.16, there is an $u \in \widehat{U}_t$ such that $K^{\text{def}}_{t'}$ is contained in Y_u. By Lemma 10.13, we know that δ_u is a tame (r, s, srs)–deformation. Hence there exists an element $v \in \langle Y_u \rangle$ such that $\text{Int}(v) \circ \delta_u|_{K^{\text{def}}_{t'}}$ is the standard deformation of $K^{\text{def}}_{t'}$. As $\delta|_{Y_u} = \delta_u$, it follows that $\text{Int}(v) \circ \delta|_{K^{\text{def}}_{t'}}$ is the standard deformation of $K^{\text{def}}_{t'}$. Hence δ is tame. □

11 Proof of Theorem 2

Let (W, R) be a Coxeter system and let $S \subseteq R^W$ be a Coxeter generating set which is not sharp-angled. Suppose S contains $k \geq 1$ edges which are not sharp-angled and choose one of them. Call it J. By Theorem 1, we can assume that $J = \{r, s\}$ with $o(rs) = 5$. By Proposition 8.4, J is a Δ–edge. Hence, by Theorem 10.4, there exists a J–deformation δ of S sending J onto $\{rsr, s\}$. Hence, by Lemma 5.4, $\delta(J)$ is a sharp-angled edge of $\delta(S)$. Let now J' be an edge of S different from J. Then $\delta(J')$ is W–conjugate to J' by Property (AD4) of δ; in particular, $\delta(J')$ is sharp-angled if and only if J' is sharp-angled. Hence the number of edges in $\delta(S)$ which are not sharp-angled is $k - 1$. Thus the statement follows by an obvious induction on the number of edges of S which are not sharp-angled. □

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