Smooth surfaces with non-simply-connected complements

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We consider two constructions of surfaces in simply-connected 4–manifolds with non simply-connected complements. One is an iteration of the twisted rim surgery introduced by the first author [8]. We also construct, for any group G satisfying some simple conditions, a simply-connected symplectic manifold containing a symplectic surface whose complement has fundamental group G. In each case, we produce infinitely many smoothly inequivalent surfaces that are equivalent up to smooth s–cobordism and hence are topologically equivalent for good groups.

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1 Introduction

In this paper, we study surfaces embedded in simply-connected 4–manifolds whose complements are not simply-connected. We investigate the possibilities for the fundamental group of the surface complement, and give constructions of smooth knottings that do not change the fundamental group (and often, the topological type of the knot). In the first part of the paper, we use a variation of the Fintushel–Stern rim surgery technique [3] introduced in the first author's thesis [8], called m*–twist* rim surgery. Starting with an embedded surface $\Sigma \subset X$, and a knot K in S^3, m–twist rim surgery, described in Section 2, produces a new surface $\Sigma_K(m) \subset X$.

Twisted rim surgery construction shares with rim surgery the property that for suitable initial pairs (SW–pairs in the terminology of [3]) the resulting surface $(X, \Sigma_K(m))$ is smoothly knotted with respect to (X, Σ). For instance, this will be the case if X is symplectic and Σ symplectically embedded with $\Sigma \cdot \Sigma \geq 0$, and the Alexander polynomial of K is nontrivial. For some choices of the parameter m, this construction preserves the fundamental group, while for others, it produces new knot groups of interest. For example, for any odd number p, we construct in Theorem 5.1 infinitely many smoothly distinct surfaces in $S^2 \times S^2$ with knot group a dihedral group D_{2p}.

If $m = 1$, we show that m–twist rim surgery does not change the surface knot group $\pi_1(X - \Sigma)$. This is also the case if $(m, d) = 1$, where $H_1(X - \Sigma) = \mathbb{Z}/d$. In both

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of these cases, we show that $(X, \Sigma_K(m))$ is topologically unknotted when the knot K is chosen carefully.

Theorem 1 (Theorem 4.4) *If J is a ribbon knot and its d–fold branched cover is an integral homology 3–sphere with $(m, d) = 1$, and G is a good group, then (X, Σ) is pairwise homeomorphic to $(X, \Sigma_J(m))$.*

This should be compared with our earlier paper [9] which dealt with the case that $\pi_1(X - \Sigma)$ is finite cyclic. On the one hand the hypothesis on the fundamental group is greatly loosened, but on the other hand we have restricted the knot J. These results rely on the 5–dimensional s–cobordism theorem, which at present holds for a restricted class of groups (Freedman–Quinn [5], Freedman–Teichner [6] and Krushkal–Quinn [12]), normally referred to as 'good' groups. Without the hypothesis that G be a good group, the conclusion of Theorem 4.4 would be that (X, Σ) is equivalent to $(X, \Sigma_J(m))$ up to s–cobordism.

In a somewhat different direction, we investigate in Section 3 the possibilities for the knot group of a symplectic surface in a simply-connected symplectic manifold. We show that the obvious topological necessary conditions on a group G are in fact sufficient to show that $G = \pi_1(X - \Sigma)$ where Σ is a symplectic surface and X is simply-connected.

Theorem 2 (Theorem 3.1) *Let G be a finitely-presented group. There is a simply-connected symplectic 4–manifold M containing a symplectically embedded surface S with $\pi_1(M - S) \cong G$ if and only if $H_1(G)$ is cyclic, and there is an element $\gamma \in G$ such that $G/\langle\gamma\rangle = \{1\}$.*

These surfaces can be further modified by twisted rim surgery to produce infinite families of smoothly knotted surfaces.

A brief outline of the paper: In Section 2 we discuss the twisted rim surgery construction, in particular its effect on the fundamental group. In Section 3 we characterize the fundamental groups that can appear as the complement of a symplectically embedded surface in a simply-connected symplectic 4–manifold. The final two sections discuss the topological (Section 4) and smooth (Section 5) classification of the surfaces constructed in the first half of the paper.

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2 Twisted rim surgery

Let X be a simply-connected 4–manifold and Σ an oriented embedded surface in X. Like the original rim surgery (Fintushel–Stern [3]), the operation of m–twist rim surgery (Kim [8]) provides a method to modify surfaces without changing the ambient 4–manifold X. The extra twist in the construction gives rise to some interesting surface knot groups. Let us briefly review the construction. Let K be any knot in S^3 and $E(K)$ be its exterior. Consider a torus T with $T \cdot T = 0$, called a *rim torus*, which is the preimage in $\partial\nu(\Sigma)$ of a closed curve α in Σ. A new surface $\Sigma_K(m)$ is obtained by taking out a neighborhood $T \times D^2$ of a rim torus from X and gluing $S^1 \times E(K)$ back using an additional twist on the boundary. An equivalent description of this construction is given in [8]. Identify the neighborhood $\nu(\alpha)$ of the curve α in X with $S^1 \times B^3$ so that the restriction of $\nu(\alpha)$ to Σ has the form $S^1 \times I$. We now consider a self diffeomorphism τ of (S^3, K) called the 'twist map' along K. Let $\partial E(K) \times I = K \times \partial D^2 \times I$ be a collar of $\partial E(K)$ in $E(K)$ under a suitable trivialization with 0–framing. The map τ is given by

$$\tau(\overline{\theta}, e^{i\varphi}, t) = (\overline{\theta}, e^{i(\varphi+2\pi t)}, t) \text{ for } (\overline{\theta}, e^{i\varphi}, t) \in K \times \partial D^2 \times I \tag{1}$$

and otherwise, $\tau(y) = y$. (Here we use $K \cong S^1 \cong \mathbb{R}/\mathbb{Z}$.) We remark, for later use, that the map on π_1 induced by τ is conjugation by the meridian of K, ie $\tau_*(\beta) = \mu_K^{-1}\beta\mu_K$ for any β in $\pi_1(S^3 - K)$.

For any integer m, we define the m–twist rim surgery on (X, Σ) by an operation producing a new pair

$$(X, \Sigma_K(m)) = (X, \Sigma) - S^1 \times (B^3, I) \cup_\partial S^1 \times_{\tau^m} (B^3, K_+). \tag{2}$$

Here, we have written $(S^3, K) = (B^3, K_+) \cup (B^3, K_-)$ where (B^3, K_-) is an unknotted ball pair. Note that $E(K)$ can be viewed as a codimension–0 submanifold of the complement $C_+(K) = B^3 - K_+$ onto which $C_+(K)$ deformation retracts. Hence we can regard τ as an automorphism of the pair (B^3, K_+), or equally as an automorphism of $C_+(K)$ that is the identity near K_+.

We record here some standing assumptions and notation that will be in use for the rest of the paper. In doing any rim surgery (twisted or otherwise) we assume that $\alpha \subset \Sigma$ is an embedded curve for which there is a framing of $\nu(\Sigma)$ along α such that the push-off of α into $\partial\nu(\Sigma)$ is null-homotopic in $X - \Sigma$. Also, we will denote by Y^d a cyclic d–fold cover of a space Y, and by $(Y, K)^d$ a d–fold cover of Y branched along a submanifold K.

2.1 Twisted rim surgery and the knot group

For a surface Σ carrying a non-trivial homology class in a simply-connected 4–manifold X, the first homology group $H_1(X-\Sigma)$ is always finite cyclic, of order that we will usually write as d. This coincides with the multiplicity of the homology class carried by Σ in $H_2(X)$. The way in which m–twist rim surgery affects the fundamental group of a surface knot depends to some degree on the relation between m and d.

In our previous paper [9], we considered the m–twist rim surgery in the case that $\pi_1(X-\Sigma)=\mathbb{Z}/d$ and $(m,d)=1$, and showed that the group $\pi_1(X-\Sigma_K(m))$ is $\mathbb{Z}/d$, no matter what K is. We take up two variations of this result. In this subsection, we consider the situation in which $\pi_1(X-\Sigma)=\mathbb{Z}/d$, but we take $m=d$, and calculate the fundamental group $\pi_1(X-\Sigma_K(m))$. Using m–twist rim surgery along appropriate knots, we will construct surfaces in X whose surface knot group are some non-abelian finite groups. In the next subsection, we will work in an opposite direction, without hypothesis on $\pi_1(X-\Sigma)$. We show, in Proposition 2.4, that when $(m,d)=1$ the fundamental group $\pi_1(X-\Sigma_K(m))$ is the same as $\pi_1(X-\Sigma)$.

Lemma 2.1 *Suppose $\pi_1(X-\Sigma)\cong\mathbb{Z}/d$. Then $\pi_1(X-\Sigma_K(d))$ is a semi-direct product of $\pi_1((S^3,K)^d)$ and $\mathbb{Z}/d$, where the action of $\mathbb{Z}/d$ is by the covering transformations of the branched cover.*

Proof Write $H=\pi_1((S^3,K)^d)$. It is sufficient to show that the fundamental group of the d–fold unbranched cover $(X-\Sigma_K(d))^d$ is H, and that the exact sequence

$$0\to H\to\pi_1(X-\Sigma_K(d))\xrightarrow{\text{hurew.}}\mathbb{Z}/d\to 0 \tag{3}$$

splits; the identification of the action of $\mathbb{Z}/d$ should be clear by the end of the argument. Considering (2) and the choice of the curve α, we decompose $(X-\Sigma_K(d))$ as

$$X-\Sigma_K(d)=X-\Sigma-(S^1\times(B^3-I))\cup_\partial S^1\times_{\tau^d}C_+(K) \tag{4}$$

with a corresponding decomposition for the d–fold cover:

$$(X-\Sigma_K(d))^d=(X-\Sigma)^d-(S^1\times(B^3-I))\cup_\partial S^1\times_{\tilde{\tau}^d}C_+(K)^d. \tag{5}$$

Referring to the decomposition (4), note that the inclusion of $X-\Sigma-(S^1\times(B^3-I))$ into $X-\Sigma$ induces an isomorphism on π_1, so the meridian μ_Σ has order d in $\pi_1(X-\Sigma_K(d))$. It follows that the sequence (3) splits, as asserted, and that the action of $\mathbb{Z}/d$ on the kernel of the Hurewicz map is given by conjugation by μ_Σ.

Applying van Kampen's Theorem to the decomposition (5) gives the following diagram:

$$\begin{tikzcd}[column sep=small]
 & \pi_1((X-\Sigma)^d - S^1\times(B^3-I)) \arrow[dr, "\psi_1"] & \\
\pi_1(S^1\times(\partial B^3-\{\text{two points}\})) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & \pi_1((X-\Sigma_K(d))^d) \\
 & \pi_1(S^1\times_{\widetilde{\tau}^d} C_+(K)^d) \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

Note that $\pi_1((X-\Sigma)^d - S^1\times(B^3-I))$ is isomorphic to $\pi_1((X-\Sigma)^d)$ which is trivial because $\pi_1((X-\Sigma)) = \mathbb{Z}/d$. So, the diagram shows

$$\pi_1((X-\Sigma_K(d))^d) = \langle \pi_1(E(K)^d,*) \mid \mu_{\widetilde{K}} = 1, \beta = \widetilde{\tau}^d_*(\beta), \forall \beta\in\pi_1(E(K)^d,*)\rangle \tag{6}$$

where $\mu_{\widetilde{K}}$ is a meridian of the lifted knot $\widetilde{K}$.

Recall that the lift $\widetilde{\tau}$ is given in [8];

$$\widetilde{\tau}(x) = \begin{cases} \phi(x) & \text{if } x\in E(K)^d - \partial E(K)^d\times I \\ (\overline{\theta}, e^{i((s/d)\cdot 2\pi+\varphi)}, s) & \text{if } x = (\overline{\theta}, e^{i\varphi}, s)\in \partial E(K)^d\times I \\ x & \text{otherwise} \end{cases} \tag{7}$$

where ϕ is the canonical generator of the group $\mathbb{Z}/d$ of covering transformations.

We observe that the lifted map $\widetilde{\tau}^d$ is the same as a twist map $\tau_{\widetilde{K}}$ along the lifted knot of K as in (1) and so in the presentation (6), we have that $\widetilde{\tau}^d_*(\beta) = \mu_{\widetilde{K}}^{-1}\beta\mu_{\widetilde{K}}$ for any $\beta\in\pi_1(S^3,K,*)^d$. Since $\mu_{\widetilde{K}} = 1$, $\widetilde{\tau}^d_*(\beta) = \beta$. This implies

$$\pi_1((X-\Sigma_K(d))^d) = \langle \pi_1(E(K)^d,*) \mid \mu_{\widetilde{K}} = 1\rangle = \pi_1((S^3,K)^d) = H. \qquad \square$$

Corollary 2.2 *If $\pi_1(X-\Sigma)\cong\mathbb{Z}/d$ and $\pi_1((S^3,K)^d)$ is finite, then so is $\pi_1(X-\Sigma_K(d))$.*

Since $\pi_1(X-\Sigma_K(d))$ is a semi-direct product of H and $\mathbb{Z}/d$, we denote $\pi_1(X-\Sigma_K(d))$ by G.

Corollary 2.2 suggests the question: what finite groups can be obtained by twisted rim surgery, starting with a surface whose knot group is cyclic? From Perelman's work on geometrization (Perelman [17], Lott [10], Cao–Zhu [2] and Morgan–Tian [14; 15]) it suffices to know which spherical space forms arise as cyclic branched covers of knots in S^3. The possibilities for the the fundamental groups of such branched covers, as well as the action of $\mathbb{Z}/d$, were determined by Plotnick and Suciu [19, Section 5]. The full list is a little complicated, but the following are worth noting:

(1) Taking $d = 3$ and K a trefoil knot then H is a quaternion group Q_8, with the action of $\mathbb{Z}/3$ permuting the unit quaternions $\imath$, $\jmath$ and k. So for example, starting with a degree–3 curve in $\mathbf{CP}^2$, we obtain an embedded torus in $\mathbf{CP}^2$ with group $G = Q(8) \rtimes \mathbb{Z}/3$.

(2) Taking $d = 2$ and K to be a 2–bridge knot $K_{p,q}$ with $(p,q) = 1$ then H is a cyclic group $\mathbb{Z}/p$, where $\mathbb{Z}/2$ acts by multiplication by -1, so that G is a dihedral group D_{2p}. This group can be realized as a surface knot group in $\mathbf{CP}^2$, by taking Σ to be a degree-2 curve (a sphere) with a handle added to create a torus. In the next section, we will want to perform a further twisted rim surgery on the resulting surface $\Sigma_{K_{p,q}}(2)$, but $(\mathbf{CP}^2, \Sigma_{K_{p,q}}(2))$ is not an SW–pair in the sense of [3], and so is not a good starting point for rim surgery constructions. One could instead choose Σ to be a curve in $S^2 \times S^2$ of bidegree $(2,2)$.

(3) The Poincare homology sphere is the p fold cover of the (q,r) torus knot for $\{p,q,r\} = \{2,3,5\}$, giving three different extensions G with subgroup $H = I^* = \pi_1(PHS)$. For $d = 3, 5$ one obtains interesting surfaces in $\mathbf{CP}^2$, while for $d = 2$ we would work with surfaces in $S^2 \times S^2$.

A further interesting aspect of the second family of surfaces is that the group doesn't depend on q, but the knots $\Sigma_{K_{p,q}}(2)$ and $\Sigma_{K_{p,q'}}(2)$ will be different if $\Delta_{K_{p,q}}(t) \neq \Delta_{K_{p,q'}}(t)$ and (X, Σ) is an SW–pair. So we can in principle obtain many knotted surfaces with a given dihedral knot group. However, in Section 5, we will do better than this, and obtain infinitely many such surfaces.

2.2 Rim surgeries that preserve π_1

We have constructed surfaces whose surface knot group is no longer abelian by d–twist rim surgery, starting with a surface whose complement has $\pi_1 = \mathbb{Z}/d$. Now, we seek to modify these surfaces using twist rim surgery without changing the fundamental group. With the correct choice of knot K, this will produce surfaces that are smoothly knotted but topologically standard.

Our first result is that 1–twist rim surgery always preserves the surface knot group.

Proposition 2.3 *For any knot K, the surface $\Sigma_K(1)$ obtained by 1–twist surgery along the rim torus parallel to α has $\pi_1(X - \Sigma_K(1)) \cong \pi_1(X - \Sigma)$.*

Proof The van Kampen theorem for the decomposition $X - \Sigma_K(1)$ shows the following diagram;

$$\begin{tikzcd}
& \pi_1(X - \Sigma - S^1 \times (B^3 - I)) \arrow[dr, "\psi_1"] & \\
\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\})) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & \pi_1(X - \Sigma_K(1)) \\
& \pi_1(S^1 \times_\tau (C_+(K)) \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

In the diagram, consider the generators $[S^1]$ and $[\mu]$ in $\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\}))$. Note that $\varphi_1[\mu]$ is the meridian $[\mu_\Sigma]$ in $\pi_1(X - \Sigma - S^1 \times (B^3 - I)) \cong \pi_1(X - \Sigma)$ and $\varphi_1[S^1]$ is $[\alpha]$ which is trivial. So, the presentation for $\pi_1(X - \Sigma_K(1))$ is

$$\langle \pi_1(X - \Sigma) * \pi_1(S^1 \times_\tau (C_+(K))) \mid [S^1] = 1,$$
$$\mu_\Sigma = \mu_K, \mu_K^{-1} \beta \mu_K = \beta \forall \beta \in \pi_1(C_+(K)) \rangle,$$

that is isomorphic to $\pi_1(X - \Sigma)$. □

In Section 4.1, we will show that if K is a slice knot, then 1–twist rim surgery does not change the s–cobordism class of the knot Σ, and hence the resulting surface is topologically unknotted for good fundamental groups. In that section, we will make use of the observation, referring to the decomposition above, that the image of $\pi_1(S^1 \times_\tau (C_+(K))$ in $\pi_1(X - \Sigma_K(1))$ is the cyclic subgroup generated by the meridian of $\Sigma_K(1)$. In Section 5 we will use this result to get infinitely many smoothly knotted surfaces with a given fundamental group.

More generally, one can ask when an m–twist rim surgery preserves the fundamental group. For any integer m and knot J, consider the m–twist rim surgery along a rim torus $\alpha \times \mu_\Sigma$. Then in certain cases, the knot group of the surface is preserved.

Proposition 2.4 *Suppose that the surface $\Sigma \subset X$ has $H_1(X - \Sigma) = \mathbb{Z}/d$, and that the meridian μ_Σ has order d in $G = \pi_1(X - \Sigma)$. If $(m, d) = 1$ then the knot group of $\Sigma_J(m)$ is isomorphic to $G = \pi_1(X - \Sigma)$.*

We remark that the hypothesis about the meridian of Σ holds for the surfaces constructed above in Section 2.1.

Proof of Proposition 2.4 We first note that the Hurewicz homomorphism gives

$$\pi_1(X - \Sigma_J(m)) \to H_1(X - \Sigma_J(m)) = H_1(X - \Sigma) = \mathbb{Z}/d.$$

We shall show that the fundamental group of the d–fold cover $(X-\Sigma_J(m))^d$ of $X-\Sigma_J(m)$ is isomorphic to $\pi_1((X-\Sigma)^d)$. Identifying a lift $\widetilde{\alpha}$ of α in d–fold cover of X branched along Σ as S^1, we have a decomposition similar to that in (5):

$$(X-\Sigma_J(m))^d = (X-\Sigma)^d - S^1\times(B^3-I)\cup_\partial S^1\times_{\widetilde{\tau}^m} C_+(J)^d.$$

The van Kampen Theorem for this decomposition gives the following diagram:

(8)
$$\begin{tikzcd}[column sep=small]
 & \pi_1((X-\Sigma)^d - S^1\times(B^3-I)) \arrow[dr, "\psi_1"] & \\
\pi_1(S^1\times(\partial B^3-\{\text{two points}\})) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & \pi_1((X-\Sigma_J(m))^d) \\
 & \pi_1(S^1\times_{\widetilde{\tau}^m} C_+(J)^d) \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

We claim that φ_1 is a trivial map. Since $\pi_1(S^1\times(\partial B^3-\{\text{two points}\}))$ is $\mathbb{Z}^2$ generated by $[S^1]$ and $[\mu]$, it is sufficient to check that their images under φ_1 are trivial. Note that $\varphi_1([\mu])\in\pi_1((X-\Sigma)^d)$ is the meridian $[\mu_{\widetilde{\Sigma}}]$ of the lifted surface of Σ and so it projects to $[\mu_\Sigma^d]\in\pi_1(X-\Sigma)$, which is trivial. Similarly, $\varphi_1([S^1])$ is sent to $[\alpha]\in\pi_1(X-\Sigma)$ and so it is also trivial.

Now, we note that

(9)
$$\pi_1(S^1\times_{\widetilde{\tau}^m} C_+(J)^d)/im(\varphi_2)\cong\langle\pi_1(S^1\times_{\widetilde{\tau}^m}(S^3,J)^d)\mid[S^1]=1\rangle.$$

In [18], Plotnick constructs a knotted 2–sphere $A(J)$ in a homotopy 4–sphere that depends on a knot J in S^3 and 3×3 matrix A. In many cases, the knot $A(J)$ is fibred and the homotopy 4–sphere is smoothly S^4. If m and d are relatively prime then with an appropriate choice of matrix A, Plotnick's construction yields a knotted 2–sphere $A(J)$ in S^4 whose fiber is the punctured d–fold branched cover of J and the monodromy is $\widetilde{\tau}^m$ described in (7). For details, see [18, Theorem 5.6].

So, we observe that the presentation (9) is the knot group of $A(J)$ in S^4 with the relation $[S^1]=1$, which is indeed the meridian of $A(J)$ in its construction and hence $\pi_1(S^1\times_{\widetilde{\tau}^m} C_+(J)^d)/im(\varphi_2)$ is trivial. This shows that $\pi_1((X-\Sigma_J(m))^d)$ is isomorphic to $\pi_1((X-\Sigma)^d-S^1\times(B^3-I))\cong\pi_1((X-\Sigma)^d)$.

In order to see that $\pi_1(X-\Sigma_J(m))$ is isomorphic to G, we consider a short exact sequence

(10)
$$0\to H\to\pi_1(X-\Sigma_J(m))\overset{\text{hurew.}}{\to}\mathbb{Z}/d\to 0,$$

where H is $\pi_1((X-\Sigma)^d)$. We are assuming that the meridian of Σ has order d in $\pi_1(X-\Sigma)$. It is easy to check that this still holds when we remove the rim torus $\alpha \times \mu_\Sigma$, so that the meridian $\mu_{\Sigma_J(m)}$ has order d as well. Thus the sequence (10) splits, and the action of $\mathbb{Z}/d$ is again by conjugation by $\mu_{\Sigma_J(m)}$ on H. Keeping track of this conjugation in the isomorphism described above shows that the sequences (3) and (10) yield the same extension. □

Given a sequence $K_1,\dots,K_n$ of knots and integers $m_1,\dots,m_n$, and a surface $\Sigma\subset X$, we can do a sequence of twisted rim surgeries. The result of this iterated rim surgery will be denoted $(X,\Sigma_{K_1,\dots,K_n}(m_1,\dots,m_n))$. We will assume that the curves $\alpha_i\subset\Sigma$ that determine the rim tori are all parallel on Σ, and recall our standing assumption that this curve α has a pushoff that is homotopically trivial in $X-\Sigma$. It is easy to see that this condition is preserved after each surgery. Using this iterated construction, we obtain the following corollary.

Corollary 2.5 *Consider a surface $\Sigma\subset X$ with $\pi_1(X-\Sigma)\cong\mathbb{Z}/m_1$, and a sequence of integers $m_2,\dots,m_n$ such that $(m_1,m_i)=1$ for all $i>1$. Then the knot group of $\Sigma_{K_1,\dots,K_n}(m_1,\dots,m_n)$ is isomorphic to that of $\Sigma_{K_1}(m_1)$.*

We remark that iterated twisted rim surgery can be done in a single operation, as follows. Suppose that $K_1,\dots,K_n$ are knots in S^3, and that integers $m_1,\dots,m_n$ are given. Then the exterior $E(K_1\#\cdots\#K_n)$ is contains the exteriors $E(K_i)$ in a standard way, bounded by incompressible tori. Performing the twist maps τ^{m_i} along these tori gives a diffeomorphism $\mathcal{T}$ of $E(K_1\#\cdots\#K_n)$ which gives rise to a new surface knot $(X,\Sigma_{K_1,\dots,K_n}(\mathcal{T}))$ as in (2). This is the same as doing m_i–twist rim surgeries along the knots K_i in any sequence.

3 Symplectic tori with arbitrary knot group

In this section, we discuss the question of when a given finitely presented group G is the fundamental group of $X-S$, where X is a simply-connected symplectic 4–manifold, and S is a symplectic surface. Note that S being symplectic implies that $[S]$ is non-trivial in $H_2(X;\mathbb{R})$, which in turn implies that $H_1(X-S;\mathbb{Z})$ is finite, in fact isomorphic to $\mathbb{Z}/d$ where d is the divisibility of $[S]\in H_2(X;\mathbb{Z})$. Thus, the following conditions are necessary for G to be isomorphic to $\pi_1(X-S)$:

(K_d) $\quad H_1(G)=\mathbb{Z}/d$ for some d, and $\exists\,\gamma\in G$ such that $G/\langle\gamma\rangle=\{1\}$.

We show that these topological conditions are sufficient for the existence of a symplectic surface. The technique is a relative version of Gompf's construction of symplectic manifolds with arbitrary fundamental group. We have not made any effort to be efficient in this construction, with regard to keeping $\chi(M)$ or $\sigma(M)$ (or some combination thereof) small. It seems likely that this could be achieved for specific groups, following Baldridge–Kirk [1].

Theorem 3.1 *If G satisfies conditions (K_d), then there is a simply-connected symplectic 4–manifold M containing a symplectically embedded surface S with $\pi_1(M - S) \cong G$.*

Proof Take a finite presentation of G:

$$G = \langle x_1, \dots, x_l \mid r_1, \dots r_m \rangle.$$

Since $H_1(G) \cong \mathbb{Z}/d$, there are elements $a_i,\ b_i \in G$ with $\gamma^d = \prod_i^n [a_i, b_i]$, where γ is the group element in (K_d). Write a_i and b_i as words in the generators x_j, symbolically $a_i = v_i(x_1, \dots, x_l)$ and $b_i = w_i(x_1, \dots, x_l)$. Similarly, write γ as a word $w(x_1, \dots, x_l)$. By construction, the equation $w^d = \prod_i^n [v_i, w_i]$ in the free group generated by the $\{x_i\}$ is a consequence of the relations $\{r_j\}$.

Consider a surface Σ_1 of genus $l + n$ with one boundary component η', containing a standard symplectic basis of curves $\{x_1, y_1, \dots, x_l, y_l, a_1, b_1, \dots, a_n, b_n\}$. Let P denote a d–punctured disc $D^2 - (\delta_1 \cup \cdots \cup \delta_d)$, with boundary $\partial P = \eta \bigcup \cup_i \partial(\delta_i)$. Write $\Sigma_d = \Sigma_1 \cup_{\eta=\eta'} P$, and Σ for Σ_d with the disks δ_i glued back in. This is illustrated, for $d = 2$, in Figure 1 below.

The disks δ_j should be cyclically arranged around a circle in $\mathrm{int}(D)$ as shown below in Figure 2. The boundaries of the δ_j, oriented counterclockwise, together with the indicated base paths, will be denoted $\gamma_j \in \pi_1(P, 1)$. With the given orientations and base paths, $\eta = \gamma_d \cdot \gamma_{d-1} \cdots \gamma_1$. Consider a diffeomorphism $\rho\colon D \to D$ that is the identity on ∂D, and permutes the δ_j cyclically. The effect of ρ on $\pi_1(P, 1)$ is given by $\rho_*(\gamma_1) = \gamma_2, \dots,\ \rho_*(\gamma_{d-1}) = \gamma_d$, and $\rho_*(\gamma_d) = \eta\gamma_1\eta^{-1}$. Extend ρ by the identity on Σ_1 so that it becomes a diffeomorphism on Σ_d.

Form the manifold $X = S^1 \times S^1 \times \Sigma$, with a product symplectic structure. We will call the first circle factor α and the second one β and write $T = \alpha \times \beta$. Note that the surface $T^2 \times 0$ where 0 is the center of D^2 is a symplectic submanifold of X. In the solid torus $\beta \times D^2$ there is a braid that forms a $(d, 1)$ torus knot meeting D in the centers of the disks δ_j. Let S_d be the product of α with this braid; it is straightforward to check that the symplectic structure on X can be arranged so that S_d is symplectic.

The complement X_d of S_d is equal to the product of the circle α with the mapping torus $S^1 \times_\rho \Sigma_d$; we will identify the circles transverse to $\Sigma_1 \subset \Sigma_d$ with the β circles.

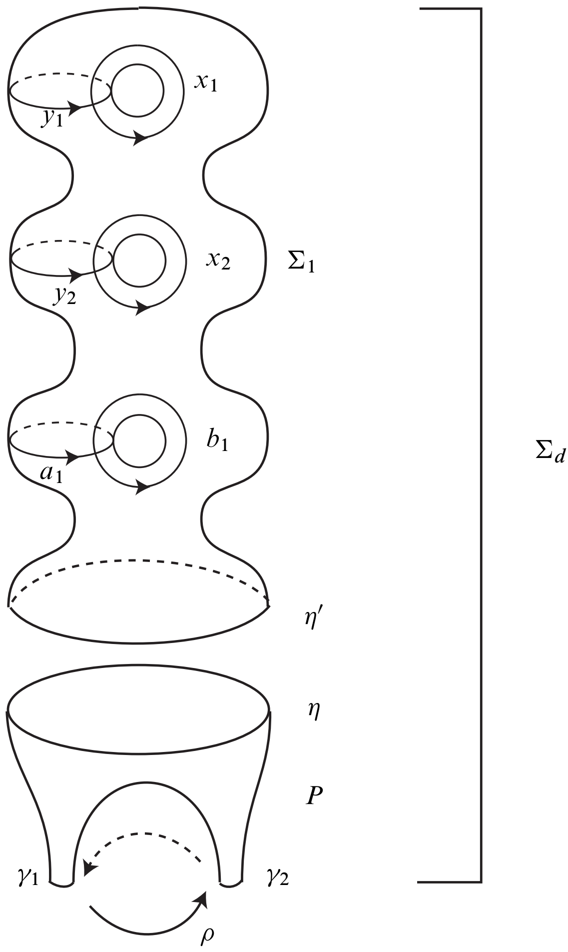

Figure 1

Another way to observe the embedding of X_d in X is to note that if ρ is extended over the disks δ_j, then it is actually isotopic to the identity map of Σ, and so the mapping torus becomes a product manifold.

The fundamental group of X_d is generated by α, β and

$$\{x_i, y_i, a_j, b_j, \gamma_k\} \text{ for } \quad i = 1 \dots l, \quad j = 1 \dots n, \quad k = 1 \dots d,$$

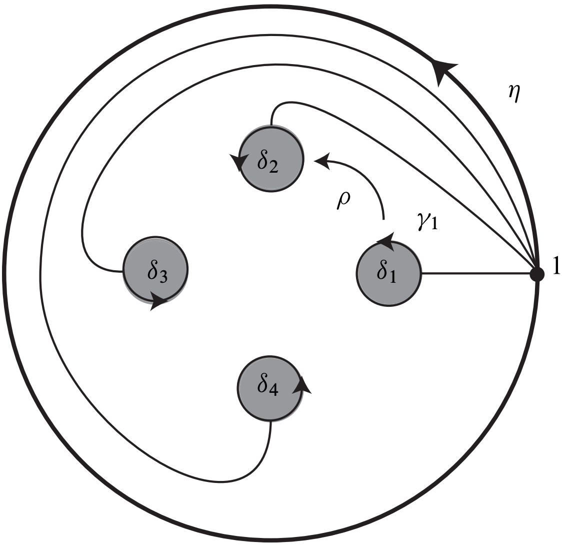

Figure 2

with relations that α commutes with everything, β commutes with x_i, y_i, a_j and b_j,

$$\gamma_k^\beta = \gamma_{k+1} \text{ for } k = 1 \dots d-1 \tag{11}$$

$$\gamma_d^\beta = \gamma_1^\eta, \text{ and} \tag{12}$$

$$\prod_{i=1}^{l} [x_i, y_i] \prod_{j=1}^{n} [a_j, b_j] = \gamma_d \cdot \gamma_{d-1} \cdots \gamma_1. \tag{13}$$

Choose, for $j \geq 1$, immersed curves $\epsilon_j \subset \Sigma_1$ representing the homotopy classes (in Σ_1) of

$$y_1, \dots, y_l, r_1, \dots, r_m, \text{ and } a_1^{-1} v_1, \dots, a_n^{-1} v_n, b_1^{-1} w_1, b_n^{-1} w_n \tag{14}$$

and a curve ϵ_0 in Σ_d representing $\gamma_1^{-1} w$. Following Gompf [7], replace Σ_d by its connected sum with many copies of T^2 and the ϵ_j by their connected sum with curves running over these tori so that $\alpha \times \epsilon_j$ can be arranged to be embedded and symplectic. (The collection of curves $\{\epsilon_j\}$ has been enlarged in this process to include the generators π_1 of each torus added on.) Note that the connected sums can all be arranged to take place in $\Sigma_1 \subset \Sigma_d$, and so the diffeomorphism ρ extends to the new Σ_d.

Now do the symplectic sum of X with copies of the elliptic surface $E(1)$, where a fiber F of $E(1)$ is identified with the each of the tori $\alpha \times \epsilon_j$. Do one further symplectic sum where F is identified with a copy of the torus $\alpha \times \beta$. Write M for the result of all of these fiber sums with X, with M_d, the complement of S_d, being the same fiber sums with X_d.

Since the fiber in $E(1)$ has simply-connected complement, van Kampen's theorem implies that each fiber sum kills the precisely elements of the fundamental group of the torus in X or X_d. Let us compute the fundamental group of X_d. Note that after relations killing the fundamental group elements α, β and those listed in (14) are imposed, then only the generators $\{\gamma_1, x_1, \ldots, x_l\}$ are needed, and the relation (13) reduces to

$$\prod_{j=1}^{n} [v_j, w_j] = \gamma_1^d.$$

Doing the final fiber sum along $\alpha \times \epsilon_0$ makes γ_1 a word in the x_i, with this relation automatically satisfied. Thus $\pi_1(M_d)$ is generated by the $\{x_i\}$ with relations $\{r\}$, and is thus isomorphic to G. The fundamental group of M is trivial, because we kill the element $\gamma = \gamma_1$, which by hypothesis normally generates G. □

There are two special properties of the surface constructed in the above proof: it is a torus and has trivial self-intersection. It is straightforward, when $d = 1$, to modify the construction to produce surfaces of arbitrary positive genus g. In that case, instead of taking a product with a torus to form X_1, simply take the product with a surface F_g of genus g. To kill the extra fundamental group introduced in this way, one needs to take a symplectic sum of X_1 along F_g with a symplectic manifold Y containing a copy of F_g with simply-connected complement; such are easily found. It seems a little more difficult to find such surfaces for $d > 1$.

Finding a symplectic sphere has a rather different character. The fundamental group of a sphere with non-zero self-intersection m in a simply-connected 4–manifold satisfies extra relations, because the element γ in conditions (K_d) satisfies $\gamma^m = 1$. There are certainly many groups that satisfy conditions (K_d) but have no elements of finite order (for example d–framed surgery on most knots in S^3). So to get a group satisfying conditions (K_d) we would want the sphere to have trivial normal bundle. However, smoothly embedded spheres with trivial normal bundles are relatively rare in symplectic manifolds, and so we conjecture that there are some groups that simply cannot be realized.

In general, if a surface Σ of genus g is embedded in a 4–manifold with self-intersection k, note that its divisibility d must divide k. By considering a pushoff of Σ, one sees that the group of Σ satisfies conditions (K_d) with the extra proviso that the element γ may be chosen so that γ^k is a product of at most g commutators. In principle, this places some extra restriction on the group G, but this seems hard to work with because there may be many choices for γ.

If we do not care whether the ambient manifold is symplectic, then it is easy to find embedded surfaces of any genus with arbitrary group satisfying conditions (K_d).

Proposition 3.2 *Let G be a finitely presented group satisfying (K_d). Then $G = \pi_1(X - S)$ for an embedded 2–sphere S in a smooth simply-connected 4–manifold X.*

Note that by adding on handles to S, we get surfaces of arbitrary genus.

Proof Construct a handlebody Y with 1–handles and 2–handles corresponding to the generators and relations of a presentation of G. Represent γ by an embedded circle in ∂Y, and let Z be Y together with a 2–handle attached along γ (with arbitrary framing). Then Z is simply-connected, and contains a properly embedded disc Δ (the cocore of this handle) such that $Z - \Delta$ deformation retracts onto Y. In particular, $\pi_1(Z - \Delta) \cong G$. Let X be the double of Z, and take S to be the double of the disk Δ. Note that $\pi_1 \partial Y \to \pi_1 Y$ is surjective, which implies that $\pi_1(X - S) \cong G$. □

Finally, we remark that by repeatedly crossing with S^2, Theorem 3.1 implies a similar result for codimension–2 symplectic submanifolds in arbitrary dimensions.

Theorem 3.3 *If G satisfies conditions (K_d), then (for $n \geq 2$) there is a simply-connected symplectic $2n$–manifold M containing a symplectically embedded $(2n-2)$-submanifold S with $\pi_1(M - S) \cong G$.*

4 Topological classification

The iterated twisted rim surgery construction of Section 2.2 and the construction of symplectic surfaces in Section 3 (combined with 1–twist rim surgery as in Proposition 2.3) give large families of surface knots with the same knot group. This section will treat the topological classification of these knots, with the smooth classification considered in the next section.

First we discuss the iterated twist rim surgery construction, starting with a surface (X, Σ). We make the same hypotheses as in Section 2.2 on the group G of Σ and the curve α that determines the rim torus. We perform a twisted rim surgery (with twisting m such that $(m, d) = 1$) to obtain a new knot $(X, \Sigma_J(m))$, with the same group as Σ. In the case that the knot groups were cyclic, we used in our earlier paper [9] a computation in surgery theory to show that all such knots are pairwise homeomorphic. This relied on the vanishing of the Wall groups $L_5^h(\mathbb{Z}[\mathbb{Z}_d])$ and $L_5^s(\mathbb{Z}[\mathbb{Z}_d])$, which does not hold for arbitrary fundamental groups. So we return to the method of [8], and work under the assumption that J is a ribbon knot. To compare the surfaces Σ and $\Sigma_J(m)$ topologically in X, we will construct a relative h–cobordism between their exteriors as in [8]; a further condition on the Alexander polynomial of J will ensure the vanishing of the torsion. Note that for simplicity of notation, our computations in the first 3 sections had to do with knot and surface *complements*; for this section we will work with the *exteriors* so we can use the relative s–cobordism theorem. To this end, we will write $\overset{\circ}{\nu}(J)$ for an open tubular neighborhood of J.

Let us briefly review the construction from [8], to which we refer for further details. If J is a ribbon knot ie $(S^3, J) = \partial(B^4, \Delta)$ for some ribbon disc Δ in B^4 then there is a concordance A in $S^3 \times I$ between J and an unknot O, such that the map $\pi_1(S^3 - J) \to \pi_1(S^3 \times I - A)$ is a surjection. The twist map τ on (S^3, J) extends to a self diffeomorphism with the same name on $(S^3 \times I, A)$ as follows. On the collar of $\partial\nu(A) \cong A \times \partial D^2 \times I$,

$$\tau(x \times e^{i\theta} \times t) = x \times e^{i(\theta+2\pi t)} \times t \text{ for } x \times e^{i\theta} \times t \in A \times \partial D^2 \times I$$

and otherwise, τ is the identity. Note that the restrictions τ to $S^3 \times \{0\}$ and $S^3 \times \{1\}$ are the twist maps τ_O and τ_J generated by O and J.

Write $S^3 = B^3_+ \cup B^3_-$ and let $J = J_+ \cup J_-$ where $J_+ = B^3_+ \cap J$ and J_- is an unknotted arc in B^3_-. We obtain a restricted concordance between the arcs J_+ and O_+ by taking out $B^3_- \times I$ from $(S^3 \times I, A)$ and then denote the concordance by A_+ in $B^3_+ \times I$. Using τ restricted to $(B^3_+ \times I, A_+)$, we obtain a new pair $(X \times I, (\Sigma \times I)_A(m))$ by taking out the neighborhood of the curve $\alpha \subset \Sigma$ in $X \times I$ and gluing back $(B^3_+ \times I, A_+)$ along τ^m. Explicitly, we write

$$(X \times I, (\Sigma \times I)_A(m)) = X \times I - S^1 \times (B^3 \times I, I \times I) \cup S^1 \times_{\tau^m} (B^3 \times I, A_+).$$

Note that in this construction, $X \times 1 = (X, \Sigma_J(m))$ and $X \times 0 = (X, \Sigma)$. Consider the exterior $X \times I - \overset{\circ}{\nu}((\Sigma \times I)_A(m))$, denoted by W, which provides a homology cobordism between $X - \overset{\circ}{\nu}(\Sigma)$ and $X - \overset{\circ}{\nu}(\Sigma_J(m))$ (see the proof of [8, Proposition 4.3]). Like all of the cobordisms we will consider in this section, W is a product along the boundary. Let $M_0 = X - \overset{\circ}{\nu}(\Sigma)$ and $M_1 = X - \overset{\circ}{\nu}(\Sigma_J(m))$. From the decomposition of $(X \times I, (\Sigma \times I)_A(m))$, we write W as

$$W = (X - (S^1 \times B^3) - \overset{\circ}{\nu}(\Sigma)) \times I \cup S^1 \times_{\tau^m} (B^3 \times I - \overset{\circ}{\nu}(A_+)). \tag{15}$$

We assert that W is an h–cobordism; the first step is to show that $\pi_1(W) = G$. Then we show, for the universal covers $\widetilde{W}$ and $\widetilde{M_1}$ of W and M_1 respectively, that $H_*(\widetilde{W}, \widetilde{M_1})$ is trivial. The Whitehead Theorem shows that the inclusion, $i\colon M_1 \to W$ is a homotopy equivalence.

The proof that W is an h–cobordism uses an observation from [8] which we quote for later use.

Lemma 4.1 [8, Lemma 4.2] *If J is a ribbon knot whose d–fold branched cover is an integral homology 3–sphere, then the d–fold cover $(B^4 - \Delta)^d$ of $B^4 - \Delta$ is a homology circle.*

Lemma 4.2 *The inclusion $i\colon M_1 \to W$ induces an isomorphism of $\pi_1(W) \to G$.*

Proof We shall show first that the d–fold covers of W and M_1 have the same fundamental group and then deduce that $\pi_1(M_1) \cong \pi_1(W)$.

Since the homology class of α is trivial in W and M_1, we decompose the d–fold covers W^d and M_1^d from (15) as follows.

$$M_1^d = (X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d \cup S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}(J_+))^d \tag{16}$$

$$W^d = (X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d \times I \cup S^1 \times_{\tilde{\tau}^m} (B^3 \times I - \mathring{\nu}(A_+))^d. \tag{17}$$

Applying the van Kampen theorem to these decompositions, we can compare the two diagrams:

$$\begin{tikzcd}
 & \pi_1((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d) \arrow[dr, "\psi_1"] \arrow[dd, dash, "i_2"] & \\
\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\})) \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"] \arrow[ddd, "i_1"] & & \pi_1(M_1^d) \arrow[ddd, "i_4"] \\
 & \pi_1(S^1 \times_{\tilde{\tau}_J^m} C_+(J)^d) \arrow[ur, "\psi_2"] \arrow[d] \arrow[dd, "i_3"] & \\
 & \pi_1((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d \times I) \arrow[dr, "\psi_1'"] & \\
\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\} \times I)) \arrow[ur, "\varphi_1'"] \arrow[dr, "\varphi_2'"] & & \pi_1(W^d) \\
 & \pi_1(S^1 \times_{\tilde{\tau}_A^m} (B^3 \times I - \mathring{\nu}(A_+))^d) \arrow[ur, "\psi_2'"] &
\end{tikzcd}$$

Here, i_1, i_2, i_3 and i_4 are the maps induced by inclusions. Obviously, i_1 and i_2 are isomorphisms. Moreover, i_3 is onto since J is a ribbon knot and so the induced map i_4 is onto. Recall from Proposition 2.4 that $\pi_1(M_1^d)$ is isomorphic to $\pi_1((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d)$. Our claim is that $\pi_1(W^d)$ is isomorphic to $\pi_1((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d \times I)$ and so it is isomorphic to $\pi_1(M_1^d)$. This will then imply that $\pi_1(W)$ is isomorphic to $\pi_1(M_1)$.

If we consider the argument of Proposition 2.4, φ_1 is the zero map and the quotient $\pi_1(S^1 \times_{\tilde{\tau}_J^m} C_+(J)^d)/im(\varphi_2)$ is trivial. In the diagram of $\pi_1(W^d)$, similarly φ_1' is the zero map and the quotient $\pi_1(S^1 \times_{\tilde{\tau}_A^m} (B^3 \times I - \mathring{\nu}(A_+))^d)/im(\varphi_2')$ is also trivial by the commutativity and surjectivity of i_3. This shows that $\pi_1(W^d)$ is isomorphic to $\pi_1((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d \times I)$.

Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(M_1^d) & \longrightarrow & \pi_1(M_1) & \longrightarrow & H_1(M_1) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & \pi_1(W^d) & \longrightarrow & \pi_1(W) & \longrightarrow & H_1(W) & \longrightarrow & 0
\end{array}$$

Note that $\pi_1(M_1)$ maps to $\pi_1(W)$ surjectively and some simple diagram-chasing shows that it is an isomorphism. □

Proposition 4.3 *If J is a ribbon knot and its d–fold branched cover is an integral homology 3–sphere with $(m,d) = 1$ then there exists an h–cobordism W between $M_0 = X - \mathring{\nu}(\Sigma)$ and $M_1 = X - \mathring{\nu}(\Sigma_J(m))$ rel ∂.*

Proof We will show that for the universal coverings of W and M_1 denoted by $\widetilde{W}$ and $\widetilde{M_1}$ respectively, $H_*(\widetilde{W}, \widetilde{M_1})$ is trivial. It follows by the Whitehead theorem that the inclusion $i\colon M_1 \to W$ is a homotopy equivalence.

We first consider the d–fold covers of W and M_1 associated to $\pi_1(W) \to H_1(W) = \mathbb{Z}/d$ and $\pi_1(M_1) \to H_1(M_1) = \mathbb{Z}/d$. As before, denote by H the groups $\pi_1(W^d) \cong \pi_1(M_1^d)$. Note that the universal covers of W^d and M_1^d are the universal covers of W and M_1. We will denote the preimage, under the universal cover, of a subset S of W^d or M_1^d by S^H, and refer to this as the H–cover of S. Then the universal covers of W and M decompose into the preimages of the pieces in the decompositions (17) and (16) of their d–fold covers W^d and M^d:

$$\widetilde{W} = ((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d)^H \times I \cup (S^1 \times_{\tilde{\tau}^m} (B^3 \times I - \mathring{\nu}(A_+))^d)^H \tag{18}$$

and

$$\widetilde{M}_1 = ((X - S^1 \times B^3 - \mathring{\nu}(\Sigma))^d)^H \cup (S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}(J_+))^d)^H. \tag{19}$$

In order to describe the H–cover of $S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}\,(J_+))^d$, we consider the inclusion-induced map $\pi_1(S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}\,(J_+))^d) \to \pi_1(M_1^d)$. In the diagram (8) induced by the van Kampen theorem to the decomposition (16) of M_1^d, the argument of Proposition 2.4 shows that $\psi_2\colon \pi_1(S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}\,(J_+))^d) \to \pi_1(M_1^d)$ is trivial.

This means that the H–cover of $S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}\,(J_+))^d$ is the disjoint union of copies of $S^1 \times_{\tilde{\tau}_J^m} (B^3 - \mathring{\nu}\,(J_+))^d$, indexed by elements of H. A similar argument shows that the H–cover of $S^1 \times_{\tilde{\tau}^m} (B^3 \times I - \mathring{\nu}\,(A_+))^d$ is the disjoint union of copies of $S^1 \times_{\tilde{\tau}^m} (B^3 \times I - \mathring{\nu}\,(A_+))^d$ indexed by elements of H as well. So, by excision, the relative homology for the pair $(\widetilde{W}, \widetilde{M_1})$ takes the following simple form:

$$H_*(\widetilde{W}, \widetilde{M_1}) \cong \overset{H}{\oplus} H_*((B^3 \times I - \mathring{\nu}\,(A_+))^d, (B^3 - \mathring{\nu}\,(J_+))^d))$$

where $\overset{H}{\oplus}$ means a direct sum indexed by the elements of H.

Lemma 4.1 of [8] implies that $H_*(\widetilde{W}, \widetilde{M_1})$ is trivial. □

The homotopy equivalence $i\colon M_1 \to W$ induces a well-defined Whitehead torsion $\tau(W, M_1) \in Wh(G)$; this is the torsion of the (based) acyclic chain complex $C(\widetilde{W}, \widetilde{M_1})$ over $\mathbb{Z}[G]$. If this is zero then we would obtain an s–cobordism W between $M_0 = X - \mathring{\nu}\,(\Sigma)$ and $M_1 = X - \mathring{\nu}\,(\Sigma_J(m))$ rel ∂ that is topologically trivial if G is a good group.

Our strategy to compute $\tau(W, M_1)$ is to apply the multiplicative property of torsion (Turaev [20]) to the decomposition (15) of W. This induces a decomposition of the (W, M_1) of the form

$$\begin{aligned} &((X - S^1 \times B^3 - \mathring{\nu}\,(\Sigma)) \times I, X - S^1 \times B^3 - \mathring{\nu}\,(\Sigma)) \cup \\ &\quad (S^1 \times_{\tau^m} (B^3 \times I - \mathring{\nu}\,(A_+)), S^1 \times_{\tau_J^m} (B^3 - \mathring{\nu}\,(J_+))). \end{aligned} \tag{20}$$

For a pair $i\colon (U, V) \hookrightarrow (W, M_1)$, suppose that the chain complex $C_*(\widetilde{U}, \widetilde{V})$ becomes acyclic when tensored with $\mathbb{Z}[G]$ via i_*. Even if V is not a deformation retract of U, the torsion of this chain complex is defined torsion, and will be denoted $\tau^i(U, V) \in Wh(G)$. The first pair in (20) is a product, and so its torsion τ^{i_1} vanishes. However, the second pair need not be a product up to homotopy. With the same assumptions as in Lemma 4.1, we will show that the chain complex for the universal cover of $(S^1 \times_{\tau^m} (B^3 \times I - \mathring{\nu}\,(A_+)), S^1 \times_{\tau_J^m} (B^3 - \mathring{\nu}\,(J_+)))$ is acyclic when tensored with $\mathbb{Z}[G]$ and so it has a torsion $\tau^{i_2} \in Wh(G)$. We will show that τ^{i_2} also vanishes, implying that (W, M_1) is an s–cobordism.

Theorem 4.4 *If J is a ribbon knot and its d–fold branched cover is an integral homology 3–sphere with $(m,d)=1$, and G is a good group, then (X,Σ) is pairwise homeomorphic to $(X,\Sigma_J(m))$.*

Proof According to Proposition 4.3 and the above argument, it is sufficient to show that the torsion $\tau(W,M_1)$ is trivial. We denote by $K_1\cup K_2$ the union of each component of the decomposition (15) of W, and likewise $L_1\cup L_2$ for M_1. Note that from (18) and (19) in Proposition 4.3, the universal cover of W is the union of H–covers of the d–fold covers of each component and the same form works for M_1. Extending the notation, we write $\widetilde{W}=\widetilde{K}_1\cup\widetilde{K}_2$ and $\widetilde{M}_1=\widetilde{L}_1\cup\widetilde{L}_2$, where for each $\alpha=1,2$, $\widetilde{K}_\alpha$ is the preimage of K_α under the universal cover of W associated to the inclusion induced map $i_{\alpha*}\colon \pi_1K_\alpha\to\pi_1W=G$ and likewise, $\widetilde{L}_\alpha$ for L_α.

Then we have the Mayer–Vietoris sequence for the pair $(\widetilde{W},\widetilde{M}_1)=(\widetilde{K}_1,\widetilde{L}_1)\cup(\widetilde{K}_2,\widetilde{L}_2)$ and $(\widetilde{K}_0,\widetilde{L}_0)=(\widetilde{K}_1,\widetilde{L}_1)\cap(\widetilde{K}_2,\widetilde{L}_2)$. If the torsion of each component in the decomposition of (W,M_1) associated to the inclusion induced morphism $i_{\alpha*}\colon \mathbb{Z}[\pi_1(K_\alpha)]\to\mathbb{Z}[\pi_1(W)]$ is well defined then the Mayer–Vietoris sequence and the multiplicativity of torsion shows

$$\tau(W,M_1)\cdot\tau^{i_0}(K_0,L_0)=\tau^{i_1}(K_1,L_1)\cdot\tau^{i_2}(K_2,L_2).$$

Since the chain complexes $C_*(\widetilde{K}_0,\widetilde{L}_0)$ and $C_*(\widetilde{K}_1,\widetilde{L}_1)$ are obviously acyclic, their torsion $\tau^{i_0}(K_0,L_0)$ and $\tau^{i_1}(K_1,L_1)$ are well defined and moreover they are trivial. So, we only need to compute $\tau^{i_2}(K_2,L_2)=\tau^{i_2}(S^1\times_{\tau^m}(B^3\times I-\mathring{\nu}(A_+)),S^1\times_{\tau_J^m}(B^3-\mathring{\nu}(J_+)))$ to get $\tau(W,M_1)$. In order to define the torsion $\tau^{i_2}(K_2,L_2)$, we need to check if the chain complex $C_*(\widetilde{K}_2,\widetilde{L}_2)$ is acyclic. In Proposition 4.3, $\widetilde{K}_2$ is the disjoint union of H–copies of $S^1\times_{\tilde{\tau}^m}(B^3\times I-\mathring{\nu}(A_+))^d$. Similarly, $\widetilde{L}_2$ is the disjoint union of H–copies of $S^1\times_{\tilde{\tau}_J^m}(B^3-\mathring{\nu}(J_+))^d$ and so we write

$$(\widetilde{K}_2,\widetilde{L}_2)=(\coprod^H S^1\times_{\tilde{\tau}^m}(B^3\times I-\mathring{\nu}(A_+))^d,\coprod^H S^1\times_{\tilde{\tau}_J^m}(B^3-\mathring{\nu}(J_+))^d).$$

So, the chain complex is the form of

$$C_*(\widetilde{K}_2,\widetilde{L}_2)\cong\overset{H}{\oplus}C_*(S^1\times_{\tilde{\tau}^m}(B^3\times I-\mathring{\nu}(A_+))^d,S^1\times_{\tilde{\tau}_J^m}(B^3-\mathring{\nu}(J_+))^d).$$

Since the d–fold branched cover of J is an integral homology 3–sphere, Lemma 4.1 of [8] shows that $C_*(\widetilde{K}_2,\widetilde{L}_2)$ is acyclic. Now, in the pair $(K_2,L_2)=(S^1\times_{\tau^m}(B^3\times I-\mathring{\nu}(A_+)),S^1\times_{\tau_J^m}(B^3-\mathring{\nu}(J_+)))$, we consider it as a relative smooth fiber bundle over

S^1 with the fiber $(B^3 \times I - \mathring{\nu}(A_+), B^3 - \mathring{\nu}(J_+))$:

$$(B^3 \times I - \mathring{\nu}(A_+), B^3 - \mathring{\nu}(J_+)) \hookrightarrow \\ (S^1 \times_{\tau^m} (B^3 \times I - \mathring{\nu}(A_+)), S^1 \times_{\tau_J^m} (B^3 - \mathring{\nu}(J_+))) \longrightarrow S^1.$$

For simplicity, we denote its relative fiber by (F, F_0) and so we write the pair of covers $(\widetilde{K}_2, \widetilde{L}_2)$ as $(\coprod^H S^1 \times_{\tilde{\tau}^m} \widetilde{F}, \coprod^H S^1 \times_{\tilde{\tau}^m} \widetilde{F}_0)$, where $(\widetilde{F}, \widetilde{F}_0)$ is the pair of the d–fold covers associated to the inclusions. Using the same techniques as in [8, Proposition 4.4], we have the following exact sequence for $(\coprod^H S^1 \times_{\tilde{\tau}^m} \widetilde{F}, \coprod^H S^1 \times_{\tilde{\tau}^m} \widetilde{F}_0)$;

$$\begin{aligned} 0 &\longrightarrow \overset{H}{\oplus}(C_*(\widetilde{F}, \widetilde{F}_0) \oplus C_*(\widetilde{F}, \widetilde{F}_0)) \\ &\longrightarrow \overset{H}{\oplus}(C_*([0,1/2] \times \widetilde{F}, [0,1/2] \times \widetilde{F}_0) \oplus C_*([1/2,1] \times \widetilde{F}, [1/2,1] \times \widetilde{F}_0)) \\ &\longrightarrow \overset{H}{\oplus}(C_*(S^1 \times_{\tilde{\tau}^m} \widetilde{F}, S^1 \times_{\tilde{\tau}^m} \widetilde{F}_0)) \longrightarrow 0. \end{aligned}$$

By the assumption that $(\widetilde{F}, \widetilde{F}_0)$ is homologically trivial, it follows that if

$$j\colon \mathbb{Z}[\pi_1(F)] \longrightarrow \mathbb{Z}[\pi_1(W)]$$

denotes the morphism induced by inclusion then the torsion $\tau^j(F, F_0)$ is defined. From the above short exact sequence and the multiplicativity of the torsion we obtain $\tau^{i_2}(S^1 \times_{\tau^m} F, S^1 \times_{\tau^m} F_0) = 1$, which implies that $\tau^{i_2}(K_2, L_2)$ is also trivial and thus the torsion $\tau(W, M_1) \in Wh(G)$ is trivial. □

The homology condition on a knot J can be expressed in terms of its Alexander polynomial. Fox [4] proved that

$$|H_1((S^3, J)^d)| = \prod_{i=0}^{d-1} \Delta_J(\zeta^i)$$

where ζ is a primitive dth root of unity. So the d–fold branched cover of a knot is an integral homology 3–sphere if and only if $\prod_{i=0}^{d-1} \Delta_J(\zeta^i) = 1$. It is easy to verify this condition for (say) the d–fold cover of a (p, q) torus knot when d is relatively prime to both p and q.

4.1 Topological triviality for 1–twist rim surgery

In Proposition 2.3, we showed that a 1–twist rim surgery does not change the fundamental group. In this section, we show that the surface $\Sigma_K(1)$ produced by such a surgery is standard up to s–cobordism. The construction is similar to that in the previous section, but we allow K to be slice (rather than ribbon), and impose no hypothesis on the cyclic coverings of $S^3 - K$.

Theorem 4.5 *Suppose that $\alpha \subset \Sigma$ is an embedded curve that has a null-homotopic pushoff into $X - \Sigma$. Then for any slice knot K, the surface $\Sigma_K(1)$ obtained by 1–twist rim surgery along α is s–cobordant to Σ.*

The homology computation requires a preliminary lemma. Suppose that $\pi\colon \widetilde{Y} \to Y$ is an infinite cyclic cover, and that $T\colon \widetilde{Y} \to \widetilde{Y}$ generates the group of covering translations. Note that for any k, the quotient $Y^k = \widetilde{Y}/\langle T^k\rangle$ is a k–fold cyclic cover of Y, and that T descends to a generator of the covering translations of Y^k.

Lemma 4.6 *The mapping torus $S^1 \times_T \widetilde{Y}$ is homeomorphic to $\mathbb{R} \times Y$. Moreover, $S^1 \times_T Y^k$ is homeomorphic to $S^1 \times Y$.*

Proof Our convention is that the mapping torus is given by $\mathbb{R}\times\widetilde{Y}$, modulo the relation $(x, y) \sim (x-1, Ty)$. The map $(x, y) \mapsto \pi(y)$ descends to an $\mathbb{R}$ bundle over Y, and the same map descends to an S^1 bundle $S^1 \times_T Y^k \to Y$. To see that these bundles are trivial, note that the covers $\widetilde{Y} \to Y$ and $Y^k \to Y$ are induced from the standard infinite and finite cyclic covers $\mathbb{R} \to S^1$ and $S^1 \to S^1$ by a map $f\colon Y \to S^1$. This implies that the $\mathbb{R}$–bundle $S^1 \times_T \widetilde{Y} \to Y$ is induced from $S^1 \times_T \mathbb{R} \to S^1$ by the same map, and likewise for the circle bundles.

Hence, it suffices to consider the case where $Y = S^1 = \mathbb{R}/\mathbb{Z}$, with $\widetilde{Y} = \mathbb{R}$. Then the first bundle is trivialized by the isomorphism $\mathbb{R} \times S^1 \to S^1 \times_T \mathbb{R}$ that takes $(r, [y])$ to $[y + r, -y]$, where the brackets $[\,]$ denote equivalence classes. This trivialization descends to a trivialization of the circle bundle as well. $\square$

Proof of Theorem 4.5 The proof uses the same technique as in Proposition 4.3 and Theorem 4.4, so we will be brief. A concordance of K to the unknot produces a cobordism between $X - \mathring{\nu}(\Sigma)$ and $X - \mathring{\nu}(\Sigma_K(1))$, as described at the beginning of this section. The isomorphism $G = \pi_1(X - \mathring{\nu}(\Sigma)) \cong \pi_1(X - \mathring{\nu}(\Sigma_K(1)))$ established in Proposition 2.3 works as well to calculate that the fundamental group of $W = X \times I - \mathring{\nu}((\Sigma \times I)_A(m))$ is also G. So the remaining points are to show the vanishing of the relative homology groups of the universal covers of $(W, X - \mathring{\nu}(\Sigma))$, and the Whitehead torsion.

As observed just after the proof of Proposition 2.3, the image of $\pi_1(S^1 \times_\tau (B^3 - \mathring{\nu}(K_+)))$ in $\pi_1(X - \mathring{\nu}(\Sigma_K(1)))$ is the cyclic subgroup generated by the meridian of $\Sigma_K(1)$; the same is true for the image of $\pi_1(S^1 \times_\tau (B^3 \times I - \mathring{\nu}(A_+))$ in G. It follows that in the universal cover, the preimage of $S^1 \times_\tau (B^3 - \mathring{\nu}\,(K_+))$ is a union of its finite or infinite cyclic covers, the order of the meridian in $\pi_1(X - \Sigma_K(1))$ determining the order of the covering. The same is true for the preimage of $B^3 \times I - \mathring{\nu}\,(A_+)$. Note that these coverings are mapping tori, as in Lemma 4.6, where in place of the covering transformation T, we have a lift of the twist map τ. But the lift $\tilde{\tau}$ defined in (7) is isotopic to the covering transformation, so we can apply Lemma 4.6 to compute the homology of these covering spaces. It follows that each $(B^3 \times I - \mathring{\nu}\,(A_+), B^3 - \mathring{\nu}\,(K_+))$ lifts to a relative homology cobordism, and hence (by the Mayer–Vietoris argument in Proposition 4.3) that W is a relative h–cobordism.

The torsion calculation in Theorem 4.4 depends only on the vanishing of the relative homology of $(B^3 \times I - \mathring{\nu}\,(A_+), B^3 - \mathring{\nu}\,(K_+))$, with coefficients in the group ring $\mathbb{Z}[G]$ induced by the inclusion of these spaces into W. But the argument in the preceding paragraph implies this vanishing, so that the torsion is trivial. □

Corollary 4.7 *With the hypotheses of Theorem 4.5, if the group G is good, then the knots $\Sigma_K(1)$ and Σ are topologically equivalent.*

5 Smooth classification

To distinguish, in the smooth category, the knots that we have constructed, we make use of the results of Fintushel and Stern [3]. They start with a surface Σ such that $\Sigma \cdot \Sigma = 0$ and (X, Σ) is an SW–pair, and show that for knots K_1, K_2, the equality $(X, \Sigma_{K_1}) \cong (X, \Sigma_{K_2})$ implies that the coefficients of Δ_{K_1} coincide with those of Δ_{K_2} (including multiplicities). This result (described in the addendum to the original paper) is proved using the gluing theory in Kronheimer–Mrowka [11]. Note that by blowing up, one can convert a surface with positive self-intersection into one with 0 self-intersection; symplectic surfaces of *negative* self-intersection are treated in a recent preprint of T Mark [13] using the Ozsváth–Szabó 4–manifold invariants [16] in place of Seiberg-Witten invariants. Mark's results require that $\Sigma \cdot \Sigma \geq 2 - 2g(\Sigma)$ and that the map $H^1(X - \mathring{\nu}\,(\Sigma)) \to H^1(\partial\, \mathring{\nu}\,(\Sigma))$ be trivial. See [13, Remark 1.3] for a discussion of how his work applies in our non-simply-connected setting.

The extra m–twist does not affect the gluing theorem, and so (with hypotheses as above) the surfaces $(X, \Sigma_{K_1}(m))$ and $(X, \Sigma_{K_2}(m))$ are distinguished smoothly if the coefficients of their Alexander polynomials form distinct sets. Here is a sample result that one gets by combining these observations with the constructions of Section 2.2 and the topological classification results in Section 4.

Theorem 5.1 *For any odd number p, there are infinitely many topologically equivalent but smoothly inequivalent knots in $S^2 \times S^2$ with dihedral knot group D_{2p} with homology class $(2,2)$ in the obvious basis for $H_2(S^2 \times S^2)$.*

Proof Let $X = S^2 \times S^2$. Choose a complex curve Σ in the homology class $(2,2)$; this will be a torus of square 8, and have group $\mathbb{Z}/2$. For any odd p and q relatively prime to p, the surface $(X, \Sigma_{K_{p,q}}(2))$ has group D_{2p}, by Lemma 2.1. Let J be any knot with non-trivial Alexander polynomial but with determinant 1. For any positive n, let J_n be the ribbon knot $\#_n(J\# -J)$. By Theorem 4.4, the knots $(X, \Sigma_{K_{p,q},J_n}(2,3))$ are all topologically equivalent. On the other hand, these surfaces are all distinguished smoothly because the coefficient lists for the Alexander polynomials of the J_n are distinct. □

In a different direction, the construction of symplectic surfaces also gives rise to families of smoothly distinct surfaces.

Theorem 5.2 *Let G be a group satisfying condition (K_d). Then there is a simply-connected symplectic 4–manifold M containing a symplectically embedded surface S, and infinitely many smoothly embedded surfaces S_n in the same homology class with $\pi_1(M - S_n) \cong G$. If G is a good group then these surfaces can be taken to be topologically equivalent.*

Proof Start with the symplectic surface S with group G provided by Theorem 3.1; note that since S is symplectic and has 0 self-intersection, (M, S) is an SW–pair. In the construction of the surface S, we performed fiber sums multiple times, including a fiber sum to kill the generators α and β of the fundamental group of the torus T. It follows readily that α, pushed into the complement of S, is null-homotopic in the complement of S. Choose a sequence of knots J_n as in the previous theorem, and do 1–twist rim surgeries to create new surfaces $(M, S_{J_n}(1))$, all of which have group G. These are smoothly distinct, as before.

By Theorem 4.5, since the knots J_n are slice, the knots $(M, S_{J_n}(1))$ are all s–cobordant. If the group G is good, then the knots are topologically equivalent. □

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