A manifold calculus approach to link maps and the linking number

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We study the space of link maps $\mathrm{Link}(P_1, \ldots, P_k; N)$, the space of smooth maps $P_1 \sqcup \cdots \sqcup P_k \to N$ such that the images of the P_i are pairwise disjoint. We apply the manifold calculus of functors developed by Goodwillie and Weiss to study the difference between it and its linear and quadratic approximations. We identify an appropriate generalization of the linking number as the geometric object which measures the difference between the space of link maps and its linear approximation. Our analysis of the difference between link maps and its quadratic approximation resembles recent work of the author on embeddings, and is used to show that the Borromean rings are linked.

1 Introduction

Let $P_1, \ldots, P_k$ be smooth closed manifolds of dimensions $p_1, \ldots, p_k$, and let N be a smooth manifold.

Definition 1.1 The space of *link maps* of $P_1, \ldots, P_k$ in N is

$$\mathrm{Link}(P_1, \ldots, P_k; N) = \{f_i \colon P_i \to N \,|\, f_i(P_i) \cap f_j(P_j) = \varnothing \text{ for } i \neq j\},$$

topologized as a subspace of the space of smooth maps.

It is important to distinguish this from the space of embeddings of the disjoint union, since all we assume about the individual maps f_i is that they are smooth. Thus we are ignoring "self-linking". Alternately, one could study the case where the f_i are immersions, a case considered by Hatcher and Quinn [10] and Schneiderman and Teichner [23]. Our description of the linking number was inspired by Section 1.4 of Goodwillie, Klein and Weiss [4], which describes double point obstructions to embeddings, and we also rely heavily on [22] for technical details, in which the author measured secondary obstructions (to the double point obstruction) for embedding a manifold in Euclidean space.

The study of the space of link maps began with the work of Milnor [20], who worked in the classical case where the P_i are circles and N is a 3–manifold. He was mostly concerned with the fundamental group of the complement of a given link map, which in this case may be assumed to be embedded. This was further investigated by Levine [18], Habegger and Lin [9] and many others. The higher dimensional analog, which we are most concerned with here, has been extensively studied by Hatcher and Quinn [10], Koschorke [16; 15; 14; 13], Koschorke and Rolfsen [17], Habegger and Kaiser [8], Massey and Rolfsen [19], Scott [24], Skopenkov [25], Schneiderman and Teichner [23] and Klein and Williams [12].

Our analysis of "linking manifolds" in Section 5 owes much to the work of Hatcher and Quinn [10], as well as to the survey paper of Goodwillie, Klein, and Weiss [5]. It also intersects the recent work of Klein and Williams [12, Section 9]. Our discussion of the linking number also relates to work of Chernov and Rudyak [1]. But for us, a linking number depends on the choice of a path between the link in question and the unlink, while they wish to keep track of such choices.

Our goal is to understand the space of link maps from the point of view of the manifold calculus of functors developed by Weiss and Goodwillie [26; 7]. We consider the space of link maps $\mathrm{Link}(P_1, \ldots, P_k; N)$ as one value of a contravariant functor from the poset of open subsets of $P_1 \sqcup \cdots \sqcup P_k$. We describe briefly in Section 4 how this theory assigns j–th degree "polynomial" approximations to such a functor F, denoted $T_j F$. We will give a geometric description of the fiber of $\mathrm{Link}(P_1, \ldots, P_k; N) \to T_j \mathrm{Link}(P_1, \ldots, P_k; N)$ when $j = 1, 2$. This geometric description is in terms of cobordism spaces, the relevant details of which can be found in [22], and the relevant definitions and results are discussed in Section 3. Our Theorem 1.2 (resp. Theorem 1.3) states that there is a map from the homotopy fiber of the space of link maps to its linear (resp. quadratic) approximation to a cobordism space. As the difference between the space of link maps and its linear (resp. quadratic) approximation begins with quadratic (resp. cubic) information, the relevant cobordism spaces are models for the quadratic (resp. cubic) homogeneous parts of the Taylor tower for these functors (see Section 4 for explanation of this terminology). The real content of these theorems is that the maps in question have geometric content, and one of them has a connectivity that the author and Tom Goodwillie [6] have computed, and will appear soon. Choose a basepoint $(f_1, f_2) \in \mathrm{Link}(P_1, P_2; N)$.

Theorem 1.2   *There is a map*

$$l_2 \colon \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; N) \to \mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N)) \to \Omega C_2(P_1, P_2; N),$$

which can be interpreted as the "generalized linking number".

The space $C_2(P_1, P_2; N)$ is a representing space for the cobordism, twisted by a bundle described in Section 5, of

$$E_{12} = \text{holim}(P_1 \xrightarrow{f_1} N \xleftarrow{f_2} P_2),$$

which may be thought of as a homotopy theoretic model for the intersections of P_1 and P_2 in N. Indeed, a point in E_{12} is a pair (p_1, p_2) with $p_i \in P_i$ and a path between their images $f_i(p_1)$ in N, and we like to interpret this path as a plan for eliminating possible intersections of $f_1'(p_1)$ with $f_2'(p_2)$ for some link map (f_1', f_2') homotopic to (f_1, f_2) in $\text{map}(P_1, N) \times \text{map}(P_2, N)$. See Section 5.3 to see why the map l_2 deserves to be called the linking number. We will also show that a special case of this is the α invariant of Massey and Rolfsen [19], discussed in Section 5.3.

Theorem 1.3 *There is a map*

$l_3: \text{hofiber}(\text{Link}(P_1, P_2, P_3; N) \to \mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)) \to \Omega C_3(P_1, P_2, P_3; N)$

which measures when the generalized Whitney trick fails.

The space $C_3(P_1, P_2, P_3; N)$ is also a representing space for the cobordism of a space twisted by a bundle (described in Section 6). That space is the homotopy limit of the following diagram, which we call E_{123}.
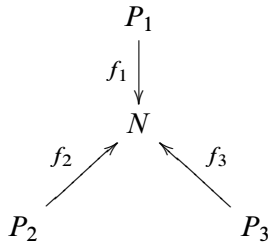
This too is a homotopy theoretic model for the intersections of the P_i in N. Its importance will become apparent in Section 6 when we discuss how $\mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)$ encodes the setup for the generalized Whitney trick: it tells one how to build generalized "Whitney disks" and the map l_3 measures intersections of the Whitney disks and intersections of P_k with a Whitney disk meant to eliminate intersections of P_i with P_j for i, j, k distinct.

One can ask whether or not these cobordism spaces describe these homotopy fibers in the sense that there is a highly connected map between them. This is indeed the case for l_2. The author and Tom Goodwillie [6] have shown:

Theorem 1.4 *The map l_2 is $(2(n-p_1-p_2)-3)$–connected.*

The work of Scott [24] and Massey and Rolfsen [19] relates to this theorem, although it goes beyond the scope of this paper to discuss these results in any depth. The connectivity of l_3 is unknown at this time, and it may be a very difficult problem. We should note that the connectivity estimates for spaces of embeddings, due to Goodwillie and Klein [3] are extremely difficult to obtain, and it is possible this will be the case for link maps as well. We will discuss some of the difficulties in Section 4.

The two applications given for Theorem 1.2 and Theorem 1.3 are to identify the map of spaces given in Theorem 1.2 with the classical linking number, and to use Theorem 1.3 to prove that the Borromean rings are linked. Theorem 1.3 requires the use of manifolds with corners, and Section 2 addresses the relevant background. Section 3 gives the necessary background for cobordism spaces. Section 4 gives the necessary background for manifold calculus and contains the models for $\mathcal{T}_1\mathrm{Link}$ and $\mathcal{T}_2\mathrm{Link}$ which we utilize, as well as a brief discussion of the problem of analyticity of the functor $\mathrm{Link}(-; N)$ and its importance. Section 5 is devoted to constructing the map of spaces appearing in Theorem 1.2, and Section 5.3 discusses why this is the same as the linking number in all of the classical cases. Section 6 is spent constructing the map of spaces in Theorem 1.3, and we use this map in Section 6.6 to prove that the Borromean rings are linked.

1.1 Conventions

We write QX for $\Omega^\infty\Sigma^\infty X$ where X is a based space, and Q_+X for $Q(X_+)$ when X is unbased and where $+$ denotes a disjoint basepoint. When we say a map is an *equivalence*, we mean it is a weak equivalence, unless otherwise noted. For a vector bundle ξ over a space X, we let $T(X; \xi)$ denote the Thom space. We will use the same capital T for tangent bundles, but this should cause no confusion. If X is a finite dimensional unbased space equipped with vector bundles ξ and η, choose a vector bundle monomorphism $\eta \to \epsilon^i$, and let $Q_+(X; \xi-\eta)$ be $\Omega^i Q T(X; \xi\oplus\epsilon^i/\eta)$. We write Spaces for the category of fibrant simplicial sets, and Top for the category of compactly generated topological spaces. Our mapping spaces are simplicial sets. For instance, $\mathrm{map}(M, N)$ is the simplicial set whose k–simplices are the fiber-preserving smooth maps of $M \times \Delta^k \to N \times \Delta^k$. By fiber-preserving we mean that $f_k(m, s) = (n, s)$. Other mapping spaces are translated to the category of simplicial sets in a similar manner.

2 Manifolds with corners

A manifold with corners is a generalization of the notion of a manifold with boundary. We will need to deal with the issue of giving a smooth structure to a manifold formed by

gluing together several manifolds with corners to construct the map l_3 of Theorem 1.3, which will be discussed in Section 6. We have two goals. The first is to explain how to make a smooth manifold by gluing together smooth manifolds with corners, at least in the case where the codimension of the corners is small. The second is to explain how to glue together maps of vector bundles, given over a collection of manifolds with corners which glue together to form a smooth manifold, to make a vector bundle map over their union. The former will be used to show that the manifold defined in Section 6 is smooth, and the latter to show that vector bundle isomorphisms, given by transversality, can be glued together to form a vector bundle isomorphism over that manifold. Our definition of a manifold with corners is a modification of the definition of a smooth manifold given by Milnor in [21]. Following this work, we begin by discussing smooth functions on half-spaces.

Let $\mathbf{R}^m_+ = [0, \infty)^m$. We can also think of $\mathbf{R}^m_+$ as a subset of $\mathbf{R}^n$ for $n > m$ by the inclusion of $\mathbf{R}^m$ in $\mathbf{R}^n$ as the first m coordinates. For $0 \le k \le m$, let $\partial_k \mathbf{R}^m_+$ denote the subspace where at least k of the coordinates are zero. We have $\partial_m \mathbf{R}^m_+ \subset \cdots \subset \partial_k \mathbf{R}^m_+ \subset \cdots \subset \partial_0 \mathbf{R}^m_+ = \mathbf{R}^m_+$. We call $\partial_k \mathbf{R}^m_+$ the k–*stratum* of $\mathbf{R}^m_+$.

Definition 2.1 We say that a map $f \colon \mathbf{R}^m_+ \to \mathbf{R}^n_+$ is *smooth* if it is the restriction of some smooth function $f' \colon \mathbf{R}^m \to \mathbf{R}^n$.

Definition 2.2 A subset $M \subset \mathbf{R}^k$ is a *smooth m–manifold with corners* if each $x \in M$ has a neighborhood U in M that is diffeomorphic to an open subset V in $\mathbf{R}^m_+$. The k–*stratum* (or *codimension k boundary*) of M is the set of all points in M that correspond to points of $\partial_k \mathbf{R}^m_+$ under such a diffeomorphism, and we denote this set by $\partial_k M$.

Let M is a smooth m–manifold with corners, and let $k \le m$ be fixed. If M requires only charts of the form $\mathbf{R}^i_+ \times \mathbf{R}^{m-i}$ for $i \le k$, then we say M *has at most k–strata*. The manifolds we will ultimately be interested in have at most 2–strata.

2.1 Tangent space

To define the tangent space of a smooth manifold $M^m \subset \mathbf{R}^k$ at $x \in M$ we first pick a parametrization $g \colon U \to M$ of a neighborhood $g(U)$ of $x \in M$ with $g(u) = x$, and since $M \subset \mathbf{R}^k$ we may think of this as a map $g \colon \mathbf{R}^m \to \mathbf{R}^k$. We then define the tangent space $T_x M$ to be the image of dg_u. Now if M is a manifold with corners whose interior is a smooth manifold, we can define as above the tangent space to M at any point. In this case the parametrization g is a smooth map $g \colon \mathbf{R}^m_+ \to \mathbf{R}^k$, and by definition g is the restriction of a smooth map $\mathbf{R}^m \to \mathbf{R}^k$.

2.2 Gluing and smooth structures

We will only discuss gluing together manifolds with at most 2–strata, as that is all we require. The case $k = 1$ is simple and well understood, so we review the relevant details here. A manifold with corners with only 1–strata is a manifold with boundary. Suppose one has two manifolds M_1 and M_2 with boundary and a diffeomorphism $j \colon \partial M_1 \to \partial M_2$. Choose collars $\partial M_1 \times [0, 1) \subset M_1$ and $\partial M_2 \times [0, 1) \subset M_2$. We think of these as embeddings of $\partial M_i \times [0, 1)$ in M_i such that $(x, 0)$ maps to x. Then the union $M = M_1 \cup_j M_2$ has the structure of a smooth manifold, since now every point in M has a neighborhood diffeomorphic to $\mathbf{R}^m$. In particular, it is then clear what we mean by its tangent bundle TM. If we denote by ∂ the subset of M corresponding to $j(\partial M_1) = \partial M_2$, then what we have shown is that ∂ has a neighborhood in M homeomorphic to $\partial \times B^1$, where B^1 is an open 1–disk.

One can generalize the existence of collars to manifolds with corners as follows.

Proposition 1 *Let M^m be a smooth manifold with corners with k–stratum $\partial_k M$. Then there exists a neighborhood N of $\partial_k M$ in M and a diffeomorphism $N \cong \partial_k M \times [0, 1)^k$.*

One can mimic the proof using transversality Hirsch gives of the collaring theorem in [11]. We are going to use this to give a smooth structure to a closed topological manifold which is made from the union of smooth manifolds with corners. As we mentioned above, we will only discuss manifolds with at most 2–strata.

Proposition 2 *Let $M_1, \ldots, M_n$ be manifolds with corners with at most 2–strata such that $\partial_1 M_i = \partial_1^{i+1} M_i \cup \partial_1^{i-1} M_i$ and $\partial_1^{i+1} M_i \cap \partial_1^{i-1} M_i = \partial_2 M_i$ for all i (the subscripts are to be read modulo n). Let $j_{i,i+1} \partial_1^{i+1} M_i \to \partial_1^i M_{i+1}$ be diffeomorphisms, where again the subscripting integers are read modulo n. Suppose that the composition $j_{n,1} \circ \cdots \circ j_{1,2} = \mathrm{id}$ when restricted to $\partial_2 M_i$. Then the union $M = \bigcup_i M_i$ can be given the structure of a smooth closed manifold.*

The idea is to show that the image of the 2–stratum ∂_2 in M should have a neighborhood homeomorphic to $\partial_2 \times B^2$, where B^2 is the open 2–disk.

Proof Divide $\mathbf{R}^2$ up into n equal *sectors* S_i for $i = 1, \ldots n$. By a sector we mean the area between two rays meeting at the origin in angle $2\pi/n$. Denote these rays by $\partial_{i-1} S_i$ and $\partial_{i+1} S_i$, for $i = 1$ to n modulo n.

Then $\partial_2 M_i$ has a neighborhood U_i in M_i diffeomorphic to $\partial_2 M_i \times [0, 1)^2$ by Proposition 1. We then choose diffeomorphisms $\partial_2 M_i \times [0, 1)^2 \cong \partial_2 M_i \times S_i$ in compatible

with the diffeomorphisms $j_{i,i+1}$. By compatible we mean that the following diagram should commute.

$$
\begin{array}{ccc}
\partial_1^{i+1} U_i & \xrightarrow{\ j_{i,i+1}\ } & \partial_1^i U_{i+1} \\[2pt]
\cong \big\downarrow & & \big\downarrow \cong \\[2pt]
\partial_2 \times \partial_{i+1} S_i & \xrightarrow{\ =\ } & \partial_2 \times \partial_i S_{i+1}
\end{array}
$$

It follows that there is a neighborhood of ∂_2 in M homeomorphic to $\partial_2 \times B^2$. We use this homeomorphism to give M its smooth structure. This gives M a tangent bundle. Moreover, at each point in ∂_2, the restriction of the tangent bundle to each sector S_i is the tangent bundle already given to ∂_2 on that sector by the inclusion of M_i. □

2.3 Vector bundles

Proposition 3 *Let ξ and η be vector bundles over a space X. Suppose we have manifolds M_1 and M_2, continuous proper maps $f_i\colon M_i \to X$ and isomorphisms $\phi_i\colon TM_i \oplus f_i^*(\xi) \to f_i^*(\eta)$ for $i = 1, 2$, and a diffeomorphism $j\colon \partial M_1 \to \partial M_2$ satisfying $f_1 = f_2 \circ j$ on the boundary. Suppose that the restriction of ϕ_1 and $j^*\phi_2$ to ∂M_1 and ∂M_2 respectively are homotopic. Then $M = M_1 \cup_j M_2$ is a smooth manifold and there is an isomorphism $\phi\colon TM \oplus f^*(\xi) \to f^*(\eta)$ compatible with ϕ_i for $i = 1, 2$.*

Proof Using collars of ∂M_i and the diffeomorphism j we can make a smooth manifold $M = M_1 \cup_j M_2$ with a continuous map $f\colon M \to X$, as discussed in Section 2.2. The only thing left is to make a commutative diagram

$$
\begin{array}{ccc}
TM_1|_{\partial M_1} \oplus f_1^*(\xi)|_{\partial M_1} & \xrightarrow{\ \phi_1|_{\partial M_1}\ } & f_1^*(\eta)|_{\partial M_1} \\[6pt]
\big\downarrow & & \big\downarrow \\[6pt]
j^*TM_2|_{\partial M_2} \oplus j^* f_2^*(\xi)|_{\partial M_2} & \xrightarrow{\ j^*\phi_2|_{\partial M_2}\ } & j^* f_2^*(\eta)|_{\partial M_2}
\end{array}
$$

where the horizontal isomorphisms are the ϕ_i, the right vertical isomorphism is the identity, and the leftmost isomorphism is given as follows. We write $TM_i|_{\partial M_i} = T\partial M_i \oplus \epsilon$ by identifying ϵ with $\nu(\partial M_i \subset M_i)$ and letting 1 correspond to the outer unit normal via some Riemannian metric on M_1, and let 1 correspond to the inner unit normal on M_2. Then the isomorphism between $T\partial M_1 \oplus \epsilon$ and $j^*T\partial M_2 \oplus j^*\epsilon$ is the

obvious one (it is 1 on the ϵ part). If this diagram commutes, then the isomorphisms ϕ_i agree on ∂M_i, so we have produced a definite isomorphism $\phi\colon TM \oplus f^*(\xi) \to f^*(\eta)$ at all points $x \in M$. In fact, it is enough if the diagram above commutes up to homotopy. Let ϕ_t be the homotopy from ϕ_1 to ϕ_2. For notational convenience, we suppress the diffeomorphism j, and denote ∂M_i by ∂. We have $M = \mathrm{colim}(M_1 \leftarrow \partial \to M_2)$, and we have bundle isomorphisms on M_i and a homotopy between them on ∂. If we set $M' = \mathrm{hocolim}(M_1 \leftarrow \partial \to M_2)$, then we have over each point in M' a definite isomorphism $TM \oplus f^*(\xi) \cong f^*(\eta)$. It is ϕ_1 on M_1, ϕ_2 on M_2, and ϕ_t on $\partial \times \{t\}$. The canonical map $M' \to M$ is a homotopy equivalence, and we can pull back the bundle isomorphism on M' to M by a homotopy inverse to get the desired bundle isomorphism. $\qquad\square$

Remark In our case the bundle isomorphisms are going to be given by transversality. That is, they will be induced by the derivatives of certain functions which define manifolds with corners, and we will need to check that they are homotopic.

As the proof of Proposition 3 indicates, the issues of giving a smooth structure to the union $M_1 \cup M_2$ and that of gluing together the bundle isomorphisms can be dealt with separately. We next consider the process of forming a map of vector bundles over a space which is the union of other spaces by gluing together vector bundle maps given on the smaller pieces.

Our immediate goal is to describe a generalization of Proposition 3. This is the content of the next two propositions. The first describes the conditions under which one can form a vector bundle over a colimit of spaces given a vector bundle over each individual space. The second describes the conditions under which we can make a section of this bundle by gluing sections given in each space in the colimit.

Definition 2.3 Let α and β be vector bundles over a space Z. Define $L_{\max}(\alpha, \beta)$ to be the space of vector bundle maps from α to β over Z such that the linear map of fibers is of maximal rank.

Let $\mathcal{C}$ be a small category, and suppose $\mathcal{X}\colon \mathcal{C} \to \mathrm{Top}$, is a covariant functor whose values we will denote $X(c)$, and assume that $X(c)$ is compact. Let $X(\mathcal{C})$ denote the diagram of spaces obtained from $\mathcal{X}$. Let $f_c\colon X(c) \to Y$ be maps for each $c \in \mathcal{C}$ such that the diagram $X(\mathcal{C}) \to Y$ commutes. Then $X = \mathrm{hocolim}_{\mathcal{C}} \mathcal{X}$ is a space with a map $f\colon X \to Y$ given by the f_c. The model for the homotopy colimit of a functor $F\colon \mathcal{C} \to \mathrm{Top}$, we have in mind for the homotopy colimit is the coequalizer

$$\mathrm{hocolim}_{\mathcal{C}} F = \mathrm{coeq}(\coprod_{c \to c'} |c' \downarrow \mathcal{C}| \times F(c) \rightrightarrows \coprod_c |c \downarrow \mathcal{C}| \times F(c)).$$

Proposition 4 *Let $\mathcal{C}$, $\mathcal{X}$, Y, ξ, η, and $\{f_c\}_{c \in \mathcal{C}}$ be as above, and suppose ξ and η are vector bundles over Y. Consider the functor $\mathcal{E} \colon \mathcal{C} \to \mathrm{Top}$ for which $\mathcal{E}(c) = L_{\max}(f_c^*\xi, f_c^*\eta)$ as a bundle over $X(c)$ and the maps $\mathcal{E}(c) \to \mathcal{E}(c')$ are given by pullback of the map $X(c) \to X(c')$. Then $E = \mathrm{hocolim}_{\mathcal{C}}\mathcal{E}$ is a vector bundle over $X = \mathrm{hocolim}_{\mathcal{C}}\mathcal{X}$.*

Proof This follows from the fact that the maps $\mathcal{E}(c) \to \mathcal{E}(c')$ are given by the pullback by $X(c) \to X(c')$ and the coequalizer definition of the homotopy colimit. □

We wish to describe how to build a section of $E \to X$ from sections of $\mathcal{E}(c) \to \mathcal{X}(c)$. Let $\chi_{c \to c'} \colon X(c) \to X(c')$ and $\Xi_{c \to c'} \colon L_{\max}(f_c^*\xi, f_c^*\eta) \to L_{\max}(f_{c'}^*\xi, f_{c'}^*\eta)$ denote the maps given by the functors $\mathcal{X}$ and $\mathcal{E}$ respectively. Let $u_{c \to c'} \colon |c' \downarrow \mathcal{C}| \to |c \downarrow \mathcal{C}|$ be the induced map of realizations of under categories.

Proposition 5 *Suppose we are given maps $\Phi_{c \to c'} \colon |c' \downarrow \mathcal{C}| \times X(c) \to L_{\max}(f_c^*\xi, f_c^*\eta)$. If these maps satisfy $\Phi(s', x) = \Phi(u(s'), x)$ and $\Xi(\Phi(s, x)) = \phi(s, \chi(x))$, then they piece together to form a section of $E \to X$.*

Proof This follows immediately from the coequalizer definitions of E and X. □

We will be interested in the case when then $X(c)$ are manifolds with corners, $X := \mathrm{colim}_{\mathcal{C}} X$ has the structure of a smooth manifold, $X' := \mathrm{hocolim}_{\mathcal{C}} X$ and the map $X' \to X$ is a homotopy equivalence so that we can pull back the bundle map obtained over X' by a homotopy inverse to X. In our case, the maps $\Phi_{c \to c'}$ will be given by various homotopies, much as in Proposition 3. See Lemma 6.8.

3 Cobordism spaces

We begin with a very brief description of cobordism spaces which we will employ in our description of $\mathcal{T}_i \mathrm{Link}(P_1, \ldots, P_k; N)$ for $i = 1, 2$. These spaces were used extensively in [22]. Identifying a good model for these spaces is necessary to define the maps in Section 5 and Section 6.

Let X be a space, and ξ and η vector bundles on X. An element of the cobordism group $\Omega_k^{\xi - \eta}(X)$ is represented by a triple (W^k, f, ϕ) (sometimes denoted by just W) where W is a k–dimensional smooth manifold embedded in $\mathbf{R}^\infty$, $f \colon W \to X$ is continuous and proper, and ϕ is a stable isomorphism $TW \oplus f^*\xi \cong f^*\eta$. The equivalence relation for representatives is the usual one defined by $(k+1)$–dimensional manifolds with boundary. We seek a space whose homotopy groups are the cobordism groups described above, and we call such a space a cobordism space.

Definition 3.1 (Simplicial model for a cobordism space) The simplicial set $C_\bullet^{\xi-\eta}(X)$ has as its k–simplices the set $C_k = \{(W^{d+k}, f, \phi)\}$ where $d = \dim(\eta) - \dim(\xi)$, W is a smooth $(k+d)$–dimensional manifold embedded in $\mathbf{R}^\infty \times \Delta^k$, W is transverse to $\mathbf{R}^\infty \times \partial_S \Delta^k$ for all nonempty subsets $S \subset \{0, 1, \ldots, k\}$, $f: W \to X$ is continuous and proper, and $\phi: TW \oplus f^*(\xi) \to f^*(\eta)$ is a stable isomorphism.

The manifolds $W^{d+k} \subset \Delta^k \times \mathbf{R}^\infty$ should the *conditioned*. To be conditioned means that if we denote by W_t the part of W that sits over $t \in \Delta^k$, then W_t should be independent of t in a neighborhood of $\cup_i \partial_i \Delta^k$ (ie, $W_t = W'$ for a fixed W', and t near $\bigcup_i \partial_i \Delta^k$).

The face and degeneracy maps are induced by those of $\Delta^\bullet$. The i–th face map $d_i: C_k \to C_{k-1}$ is just the intersection of W^{d+k} with the i–th face of Δ^k. The i–th degeneracy map $s_i: C_k \to C_{k+1}$ takes W to the fiber product W'

$$
\begin{array}{ccc}
W' & \longrightarrow & \mathbf{R}^\infty \times \Delta^{k+1} \\
\downarrow & & \downarrow {\scriptstyle s_i} \\
W & \longrightarrow & \mathbf{R}^\infty \times \Delta^k
\end{array}
$$

where s_i is the i–th degeneracy for $\Delta^\bullet$. That $C_\bullet^{\xi-\eta}(X)$ satisfies the axioms for a simplicial set is straightforward; we are building on the usual simplicial structure on Δ^k.

The next three propositions are established in [22].

Proposition 6 $C_\bullet^{\xi-\eta}(X)$ *is a Kan complex.*

Proposition 7 *There is an equivalence* $C_{d+l}^{\xi-\eta}(X) \simeq \Omega^l C_d^{\xi-\eta}(X)$.

Remark This cobordism space is equivalent to $QT(X; \xi - \eta)$. To see the equivalence, consider the subcomplex of the total singular complex of $QT(X; \xi - \eta)$ consisting of those k–simplices $\kappa: \Delta^k \to \Omega^n \Sigma^n(T(\xi - \eta))$ that correspond to maps $\kappa': \Sigma^n(\Delta^k) \to \Sigma^n(T(\xi - \eta))$ which are transverse to the zero section of $T(\xi - \eta)$. This subcomplex is equivalent to the full complex and the map $\kappa \mapsto \kappa'^{-1}(0)$ to the cobordism model is an equivalence. See Goodwillie [2] for a similar construction.

That $C_\bullet$ is a Kan complex ensures that the homotopy groups of its realization will be the cobordism groups we want. That is, $\pi_k |C_d^{\xi-\eta}(X)| = \Omega_{d+k}^{\xi-\eta}(X)$. The second proposition, together with the next, will be useful in explicitly identifying these cobordism groups in special cases, and will be used in our identification of the linking number in Section 5.3.

Proposition 8 *If X is connected, the group $\Omega_0^{\xi-\eta}(X)$ is isomorphic with $\mathbf{Z}$ if $w(\xi) = w(\eta)$, and $\mathbf{Z}/2$ if $w(\xi) \neq w(\eta)$.*

4 Manifold calculus

Manifold calculus, developed by Goodwillie and Weiss [7; 26] studies contravariant functors $F\colon \mathcal{O}(M) \to$ Spaces, where $\mathcal{O}(M)$ is the poset of open subsets of a smooth manifold M. Examples include $U \mapsto \mathrm{Emb}(U, N)$, the space of embeddings of U in a smooth manifold N, $U \mapsto \mathrm{map}(U, X)$, the space of maps of U to a space X, and $U \mapsto \mathrm{Imm}(U, N)$, the space of immersions of U in a smooth manifold N. To such a functor F, the manifold calculus associates a tower of functors
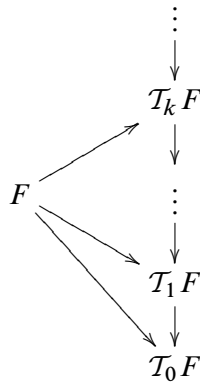
called the *Taylor tower* of F.

Definition 4.1 For a contravariant functor $F\colon \mathcal{O}(M) \to$ Spaces we define the k–th *Taylor approximation* to F, denoted $\mathcal{T}_k F\colon \mathcal{O}(M) \to$ Spaces, by

$$\mathcal{T}_k F(U) = \mathrm{holim}_{V \in \mathcal{O}_k(U)} F(V).$$

Here $\mathcal{O}_k(U)$ is the subcategory of $\mathcal{O}(U)$ consisting of those open sets $V \subset U$ which are diffeomorphic to at most k disjoint open balls.

Definition 4.2 We say that F is *polynomial of degree* $\leq k$ if given pairwise disjoint closed subsets $A_0, A_1, \ldots, A_k$ of $U \in \mathcal{O}(M)$, the map

$$F(U) \to \mathrm{holim}_{\varnothing \neq S \subset \{0,1,\ldots,k\}} F(U_S),$$

where $U_S = U - \bigcup_{i \in S} A_i$, is a weak equivalence. We say F is *homogeneous of degree k* if additionally $\mathcal{T}_{k-1} F(U)$ is contractible for all U.

For instance, the functors $U \mapsto \mathrm{map}(U, X)$ and $U \mapsto \mathrm{Imm}(U, N)$ are polynomial of degree ≤ 1, and the functor $U \mapsto \mathrm{map}(U^k, X)$ is a polynomial of degree $\leq k$. See Weiss [26] for proofs. The next two theorems state that the functors $\mathcal{T}_k F$ are polynomial and are essentially determined by their values on special open sets.

Theorem 4.3 [26, Theorem 6.1] *The cofunctor $\mathcal{T}_k F$ is polynomial of degree $\leq k$. If F is already polynomial of degree $\leq k$, then $F \to \mathcal{T}_k F$ is an equivalence.*

Remark 1 For a smooth manifold X and a vector bundle $\xi \to X$, let $\Gamma(\xi, X)$ denote the space of sections. $\Gamma(\xi, -)$ is a polynomial of degree ≤ 1 from $\mathcal{O}(X) \to$ Spaces. Hence, $\Gamma(\xi, X) \simeq \mathrm{holim}_{U \in \mathcal{O}(X)} \Gamma(\xi, U)$. In order to produce a section defined on all of X is therefore enough to produce an open cover $\mathcal{U}$ of X, an element of $\Gamma(\xi, U)$ for each $U \in \mathcal{U}$, and homotopies between these sections on their intersections. We will use this fact in Lemma 6.7.

Theorem 4.4 [26, Theorem 5.1] *Suppose that $\gamma \colon F_1 \to F_2$ is a morphism of good cofunctors, and that F_i is polynomial of degree k for $i = 1, 2$. If $\gamma \colon F_1(V) \to F_2(V)$ is a homotopy equivalence for all $V \in \mathcal{O}_k(M)$, then it is a homotopy equivalence for all $V \in \mathcal{O}(M)$.*

From its definition we see that the values of $\mathcal{T}_k F$ are completely determined by its values on $\mathcal{O}_k(M)$, so Theorem 4.4 is not too surprising.

Finally, we state the classification theorem for homogeneous functors.

Theorem 4.5 [26, Theorem 8.1] *Let F be a homogeneous cofunctor of degree k. Then there is an equivalence*

$$F(V) \to \Gamma^c \left(p; \binom{V}{k} \right)$$

where $V \in \mathcal{O}(M)$, and $\Gamma^c\left(p; \binom{V}{k}\right)$ is the space of compactly supported sections of a fibration $p \colon E \to \binom{V}{k}$.

Here "compactly supported" means equal to some fixed section, given by a choice of basepoint in $F(M)$, in some neighborhood of the fat diagonal. In particular, the functors $\mathcal{L}_k F = \mathrm{hofiber}(\mathcal{T}_k F \to \mathcal{T}_{k-1} F)$ are homogeneous, and we will make use of the classification theorem in Section 6.6.

4.1 Mapping space models for $\mathcal{T}_1\mathrm{Link}$ and $\mathcal{T}_2\mathrm{Link}$

Let $P_1, \ldots P_k$ be smooth closed compact manifolds of dimension $p_1, \ldots p_k$. We wish to apply manifold calculus to study the space $\mathrm{Link}(P_1, \ldots P_k; N)$. The functor in question is $\mathrm{Link}(-; N)\colon \mathcal{O}(\coprod_{i=1}^k P_i) \to$ Spaces. Note that there is an equivalence of categories $\mathcal{O}(\coprod_{i=1}^k P_i) \to \prod_{i=1}^k \mathcal{O}(P_i)$ given by $U \mapsto (U_1, \ldots, U_k)$, where $U_i = U \cap P_i$. We now give mapping space models for $\mathcal{T}_1\mathrm{Link}(P_1, \ldots, P_k; N)$ and $\mathcal{T}_2\mathrm{Link}(P_1, \ldots, P_k; N)$ which will be used in the proof of the main theorems.

Proposition 9 $\mathcal{T}_1\mathrm{Link}(P_1, \ldots, P_k; N) \simeq \prod_{i=1}^k \mathrm{map}(P_i, N)$.

Proof Both of the functors in question are polynomial of degree ≤ 1, and their values clearly agree when one P_i is an open ball and the others are empty. Now consider the following diagram:

$$
\begin{array}{ccc}
\mathrm{Link}(P_1, \ldots, P_k; N) & \longrightarrow & \mathcal{T}_1\mathrm{Link}(P_1, \ldots, P_k; N) \\
\downarrow & & \downarrow \\
\prod_{i=1}^k \mathrm{map}(P_i, N) & \longrightarrow & \mathcal{T}_1 \prod_{i=1}^k \mathrm{map}(P_i, N)
\end{array}
$$

The lower horizontal arrow is an equivalence by Theorem 4.3, and the result now follows from Theorem 4.4. □

Proposition 10 $\mathcal{T}_2\mathrm{Link}(P_1, P_2; N)$ *is equivalent to the homotopy pullback of:*

$$
\begin{array}{ccc}
 & & \mathrm{map}(P_1 \times P_2, N \times N - \Delta_N) \\
 & & \downarrow \\
\mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N) & \longrightarrow & \mathrm{map}(P_1 \times P_2, N \times N)
\end{array}
$$

Proof Both are polynomials of degree less than or equal to 2, and we claim that the map from $\mathcal{T}_2\mathrm{Link}(P_1, P_2; N)$ to the homotopy limit is an equivalence when $P_1 \sqcup P_2$ is a manifold consisting of at most two open balls. The cases $P_1 = \varnothing$ or $P_2 = \varnothing$ are straightforward to deal with. Without loss of generality, suppose $P_2 = \varnothing$. Then, on the one hand, P_1 is a final object in $\mathcal{O}_2(P_1)$ and so $\mathcal{T}_2\mathrm{Link}(P_1, \varnothing; N) \simeq \mathrm{Link}(P_1, \varnothing; N) = \mathrm{map}(P_1, N)$. On the other hand, the homotopy pullback diagram in question reduces to $\mathrm{map}(P_1, N)$ since $P_1 \times \varnothing = \varnothing$.

If each of the P_i is an open ball, then once again $P_1 \sqcup P_2$ is a final object in $\mathcal{O}_2(P_1 \sqcup P_2)$, and so $\mathcal{T}_2\mathrm{Link}(P_1, P_2; N) \simeq \mathrm{Link}(P_1, P_2; N) \simeq \mathrm{Link}(*, *; N) \simeq N \times N - \Delta$. As for the homotopy pullback in question, clearly

$$
\mathrm{map}(P_1 \times P_2, N \times N - \Delta_N) \simeq \mathrm{map}(* \times *, N \times N - \Delta_N) = N \times N - \Delta_N,
$$

and so the diagram reduces to $\mathrm{holim}(N \times N \to N \times N \leftarrow N \times N - \Delta_N)$. The result follows from Theorem 4.4. Compare Theorem 1.3 of [5] in the case $k = 2$. $\square$

Corollary 1 *When $N = \mathbf{R}^n$, $\mathcal{T}_2\mathrm{Link}(P_1, P_2; \mathbf{R}^n) \simeq \mathrm{map}(P_1 \times P_2, S^{n-1})$.*

See Section 5.3 for applications of this to the classical linking number.

Corollary 2 *The functor $\mathcal{T}_2\mathrm{Link}(P_1, \ldots, P_k; N)$ is equivalent to product over $i < j$ of the homotopy pullbacks of:*

$$\mathrm{map}(P_i \times P_j, N \times N - \Delta_N)$$
$$\downarrow$$
$$\mathrm{map}(P_i, N) \times \mathrm{map}(P_j, N) \longrightarrow \mathrm{map}(P_i \times P_j, N \times N)$$

Proof This follows from Proposition 10 along with the fact that $\mathrm{map}(X \sqcup Y, Z) = \mathrm{map}(X, Z) \times \mathrm{map}(Y, Z)$, by induction on k. $\square$

Hence, we may think of an element of $\mathcal{T}_2\mathrm{Link}(P_1, \ldots, P_k; N)$ as a tuple $(\mathbf{f}, \mathbf{H}, \mathbf{F})$, where $\mathbf{f} = (\mathbf{f_1}, \ldots, \mathbf{f_k})$ satisfies $f_i \in \mathrm{map}(P_i, N)$, $\mathbf{F} = (\mathbf{F_{12}}, \ldots, \mathbf{F_{(k-1)k}})$ satisfies $F_{ij} \in \mathrm{map}(P_i \times P_j, N \times N - \Delta_N)$, and H_{ij} is a homotopy between F_{ij} and $f_i \times f_j$.

4.2 Analyticity

One can ask whether these polynomial approximations actually approximate the functor, and whether or not $\mathrm{Link}(P_1, \ldots, P_k; N) \to \mathrm{holim}_j \mathcal{T}_j\mathrm{Link}(P_1 \ldots, P_k; N)$ is an equivalence. If so, we would say that the polynomial approximations *converge* to the original functor and that the functor is *analytic*. For the functor $\mathrm{Link}(-; N)$, this appears to be a very difficult problem.

For smooth manifolds M^m and N^n and the functor $U \mapsto \mathrm{Emb}(U, N)$ for $U \subset M$, Goodwillie and Klein [3] prove that if $n - m - 2 > 0$, then the maps $\mathrm{Emb}(M, N) \to \mathcal{T}_k\mathrm{Emb}(M, N)$ have connectivity increasing with k, and so the Taylor tower converges to this functor, and $\mathrm{Emb}(-, N)$ is analytic with *radius of convergence* $n - m - 2$. The proof of these connectivity estimates is very difficult, and involves, among other things, diffeomorphisms and their relationship (via the isotopy extension theorem) to embeddings. There is no analog for link maps of the isotopy extension theorem, because the restriction map $\mathrm{Link}(K_2; N) \to \mathrm{Link}(K_1; N)$ for compact sets $K_1 \subset K_2 \subset P_1 \sqcup P_2$ is not in general a fibration. For example, let $P_1 = \{p\}$ be a single point, $P_2 = \{q_1, q_2\}$, and let $K_1 = P_2$ and $K_2 = P_1 \sqcup P_2$. This makes it difficult to analyze homotopy fibers of such restrictions, unlike the case for embeddings, where the isotopy extension theorem says that the restriction map is a fibration.

5 Linking manifolds and quadratic obstructions

In this section we are going to examine the difference between $\mathrm{Link}(P_1, P_2; N)$ and $\mathcal{T}_1\mathrm{Link}(P_1, P_2; N)$. It should be clear from what follows that for $\mathrm{Link}(P_1, \ldots, P_k; N)$ we make $\binom{k}{2}$ linking manifolds, one for each pair (i, j) with $i < j$. Recall from Proposition 9 that $\mathcal{T}_1\mathrm{Link}(P_1, P_2; N) \simeq \mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N)$.

Let $(f_1, f_2) \in \mathrm{Link}(P_1, P_2; N)$ be the basepoint, and let

$$\alpha \in \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; N) \to \mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N)).$$

This is a map $\alpha\colon I \to \mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N)$ such that $\alpha(0) = (f_1, f_2)$, and $\alpha(1) \in \mathrm{Link}(P_1, P_2; N)$. We will write $\alpha(t) = (f_{1,t}, f_{2,t})$ so that $f_{1,0} = f_1$ and $f_{2,0} = f_2$.

5.1 Quadratic cobordism space model

The constructions in this section were inspired by a very similar construction in Section 1.4 of Goodwillie, Klein and Weiss [5]. Compare also Klein and Williams [12].

Definition 5.1 $E_{12} = \mathrm{holim}(P_1 \xrightarrow{f_1} N \xleftarrow{f_2} P_2)$.

Hence E_{12} is the set of all (x_1, x_2, ω) where $(x_1, x_2) \in P_1 \times P_2$, $\omega\colon [-1, 1] \to N$, $\omega(-1) = f_1(x_1)$ and $\omega(1) = f_2(x_2)$.

E_{12} has maps to $P_1 \times P_2$ and N given by projection and evaluation of ω at 0. Hence we may pull back $TP_1 \times TP_2$ and TN to E_{12}. From this data we can make a cobordism space $C_{\bullet}^{TN-TP_1 \times TP_2}(E_{12})$. For us, $d = p_1 + p_2 - n$, so that a 0–simplex in this space is a manifold of dimension $(p_1 + p_2 - n)$. We will abbreviate this space by $C_2(P_1, P_2; N)$. It is equivalent to $QT(E_{12}; TN - TP_1 \times TP_2)$ by the Pontryagin–Thom construction.

We are now going to produce a path in $C_2(P_1, P_2; N)$ from our choice of $\alpha \in \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; N) \to \mathrm{map}(P_1, N) \times \mathrm{map}(P_2, N))$. Recall from above the notation $\alpha_t = (f_{1,t}, f_{2,t})$. We may regard $f_{1,t} \times f_{2,t}\colon P_1 \times P_2 \to N \times N$ as a map $F\colon P_1 \times P_2 \times I \to N \times N$. We may assume that F is transverse to the diagonal $\Delta_N \subset N \times N$.

Definition 5.2 $L_\alpha = F^{-1}(\Delta_N)$.

Proposition 11 *L is a smooth closed compact (p_1+p_2+1-n)–dimensional manifold with stable normal bundle $TN - TP_1 \times TP_2$. Also, L defines an element of $\Omega QT(E_{12}; TN - TP_1 \times TP_2)$.*

Proof This follows from transversality. $\qquad\square$

$L_\alpha \subset E_{12} \times [0,1]$ is the subset of all (x_1, x_2, ω, t) such that $f_{1,t}(x_1) = f_{2,t}(x_2)$, where ω is the path

$$\omega(s) = \begin{cases} f_{1,(1+s)t} & \text{if } -1 \le s \le 0 \\ f_{2,(1-s)t} & \text{if } 0 \le s \le 1. \end{cases}$$

Thus L_α determines a point in $\Omega C_2(P_1, P_2; N)$ the space whose 0–simplices are cobordisms of $\varnothing$ with itself. This works for arbitrary maps of a k–simplex into hofiber(Link($P_1, P_2; N$) $\to$ map(P_1, N) $\times$ map(P_2, N)), producing a k–simplex in $\Omega C_2(P_1, P_2; N)$, or equivalently a k–simplex in $\Omega QT(E_{12}; TN - TP_1 \times TP_2)$. We need to note that the subcomplex of maps which satisfy the necessary transversality condition to define L_α are homotopy equivalent to the full complex (see, for instance, Hypothesis 3.18 of [2]). This proves Theorem 1.2, restated here for convenience.

Theorem 5.3 *There is a map of spaces*

$$l\colon \text{hofiber}(\text{Link}(P_1, P_2; N) \to \text{map}(P_1, N) \times \text{map}(P_2, N)) \to \Omega C_2(P_1, P_2; N)$$

which sends α to L_α, and can be described as the "generalized linking number".

It is not necessary to assume that the basepoint (f_1, f_2) is in the image of the space of link maps. In this case the above would be modified only slightly. Instead of cobordisms of the empty set with itself, we would have nullcobordisms of the intersection of $f_1(P_1)$ with $f_2(P_2)$. As mentioned in the introduction, Theorem 1.4, to appear in [6], tells us the connectivity of this map.

5.2 Properties of the linking manifold

We have a map

$$\text{hofiber}(\text{Link}(P_1, P_2; N) \to \text{map}(P_1, N) \times \text{map}(P_2, N)) \to \Omega C_2(P_1, P_2; N).$$

As above, for each $\alpha \in$ hofiber(Link($P_1, P_2; N$) $\to$ map(P_1, N) $\times$ map(P_2, N)), we write $\alpha_t = (f_{1,t}, f_{2,t})$ and from this produce an element of $\Omega C_2(P_1, P_2; N)$, which is represented by a smooth closed compact manifold L_α of dimension $(p_1 + p_2 - n + 1)$, called the *linking manifold* of α, and it plays the role of the linking number of (f_1, f_2).

Clearly the linking manifold depends both on the choice of basepoint (f_1, f_2) and the choice of $\alpha \in$ hofiber(Link($P_1, P_2; N$) $\to$ map(P_1, N) $\times$ map(P_2, N)). The dependence on the basepoint is unavoidable, but in many cases, the linking manifold does not depend on the choice of path α, only its values at 0 and 1. See Section 5.2.1 for more.

5.2.1 Dependence on element of the homotopy fiber Let us now examine the extent to which the linking manifold depends on the path α. These results are simple but useful corollaries of Theorem 1.2.

Proposition 12 *Let $\alpha, \beta \in \text{hofiber}(\text{Link}(P_1, P_2; N) \to \text{map}(P_1, N) \times \text{map}(P_2, N))$ and that $\alpha(1) = \beta(1)$. If α and β are homotopic, then the linking manifolds L_α and L_β are cobordant.*

Corollary 3 *If $\text{map}(P_1, N)$ and $\text{map}(P_2, N)$ are simply connected, then the cobordism class of the linking manifold of $(f_{1,1}, f_{2,1})$ only depends on the choice of basepoint (f_1, f_2).*

Proof If $\text{map}(X, Y)$ is simply connected, then any two paths in $\text{map}(X, Y)$ sharing the same endpoints are homotopic through paths fixed at their endpoints. The result follows from Proposition 12. $\qquad\square$

Of course, $\text{map}(X, \mathbf{R}^k)$ is contractible. In general, it is straightforward to show that $\text{map}(X, Y)$ is $(\text{conn}(Y) - \dim(X))$–connected by considering it as the space of sections of a trivial bundle. If we choose another basepoint $(f_1', f_2') \in \text{Link}(P_1, P_2; N)$ in the same path component as (f_1, f_2), then the linking manifold remains unchanged. In particular, this is the case if N is connected and (f_1, f_2) and (f_1', f_2') send P_1 and P_2 to a distinct pair of points in N.

Remark In the work of Chernov and Rudyak [1], the authors have a way to produce a linking number for a given link from a choice of the unlink in the case where $p_1 + p_2 + 1 = n$. It is an element of a cobordism group modulo a group of indeterminacies. Clearly our construction depends also on a choice of path in the space $\text{map}(P_1, N) \times \text{map}(P_2, N)$. In the case where $\text{map}(P_1, N) \times \text{map}(P_2, N)$ is simply connected, this group of indeterminacies vanishes. This group of indeterminacies also vanishes in other special situations. See especially Section 6 of [1] for details.

5.3 The classical linking number

Suppose $f_1 \colon S^{p_1} \to \mathbf{R}^{p_1+p_2+1}$ and $f_2 \colon S^{p_2} \to \mathbf{R}^{p_1+p_2+1}$ are smooth maps with disjoint images. This produces a map

$$f_1 \times f_2 \colon S^{p_1} \times S^{p_2} \to \mathbf{R}^{p_1+p_2+1} \times \mathbf{R}^{p_1+p_2+1} - \Delta_{\mathbf{R}^{p_1+p_2+1}},$$

and by subtraction and retraction, a map $F \colon S^{p_1} \times S^{p_2} \to S^{p_1+p_2}$. One classical definition of the *linking number* of f_1 and f_2 is the degree of this map, which may

be computed by computing the cardinality of the inverse image of a regular value $x \in S^{p_1+p_2}$ with signs. The sign convention is as follows. Give an orientation to S^{p_1}, S^{p_2}, and $S^{p_1+p_2+1}$. Each point (x_1, x_2) in $F^{-1}(x) \subset S^{p_1} \times S^{p_2}$ is assigned the value $+1$ or -1 according to whether the derivative $DF: T_{x_1}S^{p_1} \times T_{x_2}S^{p_2} \to T_x S^{p_1+p_2}$ is orientation preserving or reversing respectively. We will examine this is a slightly more general context where the p_1–sphere and the p_2–sphere are replaced by smooth closed compact manifolds P_1 and P_2 of dimensions p_1 and p_2 respectively. If these manifolds are oriented, then the sign convention is the same as that above, and if either of them is not oriented, then the linking number is an integer mod 2.

Let us first deal with the right way to count the linking number from our perspective. By Proposition 8, $\Omega_0^{T\mathbf{R}^{p_1+p_2+1}-TP_1 \times TP_2}(E_{12})$ is $\mathbf{Z}$ if P_1 and P_2 are oriented, and $\mathbf{Z}/2$ otherwise.

Consider the map $l_2: \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; \mathbf{R}^{p_1+p_2+1}) \to \mathrm{map}(P_1, \mathbf{R}^{p-1+p_1+1}) \times \mathrm{map}(P_2, \mathbf{R}^{p_1+p_2+1})) \to \Omega C_2(P_1, P_2; \mathbf{R}^{p_1+p_2+1})$. Hence for each

$$\alpha \in \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; \mathbf{R}^{p_1+p_2+1}) \to \mathrm{map}(P_1, \mathbf{R}^{p_1+p_2+1}) \times \mathrm{map}(P_2, \mathbf{R}^{p_1+p_2+1})$$

we get a manifold L of dimension 0. Taking the induced map on π_0 and using Proposition 7, we see that $l(\alpha)$ determines a class $[L_\alpha]$ in $\Omega_0^{TN-TP_1 \times TP_2}(E_{12})$.

Next we will show that our definition agrees with the classical one when the basepoint is chosen so that the images of f_1 and f_2 lie inside disjoint open balls, and that the choice of lift α is immaterial.

Let

$$\alpha, \beta \in \mathrm{hofiber}(\mathrm{Link}(P_1, P_2; \mathbf{R}^{p_1+p_2+1}) \to \mathrm{map}(S^p, \mathbf{R}^{p+q+1}) \times \mathrm{map}(S^q, \mathbf{R}^{p+q+1}))$$

satisfy $\alpha(1) = \beta(1)$. We write $\alpha(t) = (f_{1,t}, f_{2,t})$ as before, and $\beta(t) = (f'_{1,t}, f'_{2,t})$. Let H be the straight line homotopy between them. That is,

$$H(s, t) = (sf_{1,t} + (1-s)f'_{1,t}, sf_{2,t} + (1-s)f'_{2,t}).$$

Then H gives rise to a cobordism between L_α and L_β.

Consider the composite map

$$P_1 \times P_2 \xrightarrow{f_1 \times f_2} \mathbf{R}^{p_1+p_2+1} \times \mathbf{R}^{p_1+p_2+1} - \Delta_{\mathbf{R}^{p_1+p_2+1}} \xrightarrow{d} \mathbf{R}^{p_1+p_2+1} - \{0\} \xrightarrow{r} S^{p_1+p_2}.$$

As we mentioned, the linking number is the degree of the composed map. As a manifold, it is the inverse image of a regular value $x \in S^{p_1+p_2}$, counted with signs (equivalently, framed) as described above. By inspection, this is the solutions to the equation $f_1(x) - tz = f_2(y)$, where $t > 0$ and z is a regular value of the composite above. By

compactness we and reparametrization of the time coordinate we can assume all these solutions occur for $t \in [0, 1]$. Now we can interpret $(f_1(x) - tz, f_2(y))$ as an element of hofiber(Link$(P_1, P_2; \mathbf{R}^{p_1+p_2+1}) \to \text{map}(S^p, \mathbf{R}^{p+q+1}) \times \text{map}(S^q, \mathbf{R}^{p+q+1}))$, and clearly $(f_1(x) - z, f_2(y))$ is the standard unlink: the images of the two spheres lie in disjoint open balls.

This is clearly related to the α invariant of Massey and Rolfsen [19]. From maps $f_1 \colon S^{p_1} \to \mathbf{R}^{p_1+p_2+1}$ and $f_2 \colon S^{p_2} \to \mathbf{R}^m$ with disjoint images we can construct a map $F \colon S^{p_1} \times S^{p_2} \to S^{m-1}$ as above. If $p_1, p_2 \le m-2$, then the homotopy classes of maps $S^{p_1} \times S^{p_2} \to S^{m-1}$ are in bijective correspondence with elements of $\pi_{p_1+p_2}(S^{m-1})$, and Massey and Rolfsen define $\alpha(F)$ to be the element of $\pi_{p_1+p_2}(S^{m-1})$ corresponding to the homotopy class of F.

5.3.1 Linking in $S^{p_1+p_2+1}$ By imposing some mild dimensional assumptions we can argue by using our work in the previous section that the classical linking number for linking in spheres agrees with ours. The key observation is that made in the comments following Corollary 3: map(X, Y) is $(\text{conn}(Y) - \dim(X))$–connected. Since $S^{p_1+p_2+1}$ is (p_1+p_2)–connected, map$(P_1, S^{p_1+p_2+1})$ and map$(P_2, S^{p_1+p_2+1})$ are simply connected provided $p_1, p_2 \ge 1$. In order to use the straight line homotopy as we did before, we need to make sure that every 1–simplex (homotopy of maps) in map$(P_1, S^{p_1+p_2+1})$ and map$(P_2, S^{p_1+p_2+1})$ misses a point (so that they may be considered as maps to $\mathbf{R}^{p_1+p_2+1}$ and thus we can use the straight line homotopy between them). This will happen provided $p_1+1 < p_1+p_2+1$ and $p_2+1 < p_1+p_2+1$, or $p_1, p_2 \ge 1$.

6 Cubic obstructions

In this section we seek to describe the difference between Link$(P_1, \ldots, P_k; N)$ and T_2Link$(P_1, \ldots, P_k; N)$ in a similar fashion to the way we described the difference between Link$(P_1, \ldots, P_k; N)$ and T_1Link$(P_1, \ldots, P_k; N)$ in the previous section. That is, we will give a map from hofiber(Link$(P_1, \ldots, P_k; N) \to T_2$Link$(P_1, \ldots, P_k; N))$ to a cobordism space. As an example, we will use this map to show that the Borromean rings are linked in Section 6.6. We begin by giving a cobordism space model for hofiber(Link$(P_1, P_2, P_3; N) \to T_2$Link$(P_1, P_2, P_3; N))$, and then go about constructing a manifold in a similar manner to that above.

6.1 Cubic cobordism space model

Choose a basepoint $(\mathbf{f}, \mathbf{F}, \mathbf{H}) \in T_2$Link$(P_1, P_2, P_3; N)$. As before $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{F} = (F_{12}, F_{23}, F_{31})$, and $\mathbf{H} = (H_{12}, H_{23}, H_{31})$. $H_{ij} \colon P_i \times P_j \times I \to N \times N$ is a homotopy between $H_{ij}(0) = (f_i, f_j)$ and $H_{ij}(1) = F_{ij}$.

Consider the following diagram $\mathcal{D}$:

$$
\begin{array}{ccc}
 & P_1 & \\
 & \downarrow f_1 & \\
 & N & \\
 f_2 \nearrow & & \nwarrow f_3 \\
 P_2 & & P_3
\end{array}
$$
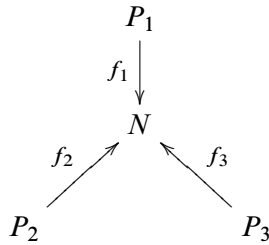
Definition 6.1 Define $E_{123} = \text{holim}(\mathcal{D})$.

Thus a point in E_{123} is a tuple $(x_1, x_2, x_3, \omega_1, \omega_2, \omega_3)$, where $x_i \in P_i$, and $\omega_i \colon I \to N$ is a path such that $\omega_i(0) = f_i(x_i)$, and $\omega_1(1) = \omega_2(1) = \omega_3(1)$ for $i = 1, 2, 3$. Evidently E_{123} has a map to $P_1 \times P_2 \times P_3$ given by projection, which we use to pull back $TP_1 \times TP_2 \times TP_3$ to E_{123}. We also have a map $E_{123} \to N \times N$, defined as follows. There is a map

$$
E_{123} \to \prod_{i \neq j} \text{holim}(P_i \xrightarrow{f_i} N \xleftarrow{f_j} P_j)
$$

given by $(\omega_1, \omega_2, \omega_3) \mapsto (\omega_1 \cdot \omega_2^{-1}, \omega_2 \cdot \omega_3^{-1}, \omega_3 \cdot \omega_1^{-1})$, where $\cdot$ indicates path multiplication. If we reparametrize the resulting paths to be defined on $[-1, 1]$ then the map to $E_{123} \to N \times N$ can be defined as the composition of this map with the evaluation $(\omega_1 \cdot \omega_2^{-1}, \omega_2 \cdot \omega_3^{-1}, \omega_3 \cdot \omega_1^{-1}) \mapsto (\omega_1 \cdot \omega_2^{-1}(0), \omega_2 \cdot \omega_3^{-1}(0))$, and we use this map to pull back $TN \times TN$ to E_{123}.

Definition 6.2 Define $C_3(P_1, P_2, P_3; N) = C_\bullet^{TN \times TN - TP_1 \times TP_2 \times TP_3}(E_{123})$.

In this case, $d = p_1 + p_2 + p_3 - 2n$, but for indexing purposes, it is more convenient to shift this down by 2 (this could be done honestly by subtracting twice the trivial 1–dimensional bundle, but that is rather cumbersome notationally). Thus, a 0–simplex in this space will be represented by a manifold of dimension $p_1 + p_2 + p_3 - 2n + 2$. Note that C_3 is equivalent to $QT(E_{123}; TN \times TN - TP_1 \times TP_2 \times TP_3)$ by the Pontryagin–Thom construction.

6.2 Whitney circles and Whitney disks

Proposition 10 tells us that we may think of an element of $\mathcal{T}_2\text{Link}(P_1, P_2; N)$ as a triple $((f_1, f_2), F_{12}, H_{12})$ where $f_i \colon P_i \to N$, $F_{12} \colon P_1 \times P_2 \to N \times N - \Delta_N$, and $H_{12} \colon P_1 \times P_2 \times I \to N \times N$ is a homotopy from f_1 to f_2. Suppose H_{12} is transverse

to Δ_N. Then $D_{12} = H_{12}^{-1}(\Delta_N)$ is a (p_1+p_2+1-n)–dimensional submanifold of $P_1 \times P_2 \times I$ which we call the *Whitney circle*. The reason for this terminology is that in case D_{12} is 1–dimensional, the images under f_i of the projections of D_{12} to P_i for $i = 1, 2$ form a bent circle in N. This circle bounds a disk in N as follows. For each $(x_1, x_2, t_0) \in D_{12}$ we give a path in N from $f_1(x_1)$ to $f_2(x_2)$ by going from $f_1(x_1) = p_1 H_{12}(x_1, x_2, 0)$ to $p_1 H_{12}(x_1, x_2, t_0) = p_2 H_{12}(x_1, x_2, t_0)$ by letting t run from 0 to t_0, and then to $f_2(x_2)$ by letting t run from t_0 back down to 0. We call this disk (and its higher-dimensional counterpart) the *Whitney disk*. Our obstruction manifold measures the intersections of the Whitney circles, and the intersections of the manifolds themselves with the Whitney disks. Compare an identical description in Schneiderman and Teichner [23] in the case of 2–spheres in a 4–manifold.

6.3   Construction of the obstruction manifold

We will construct the obstruction manifold in the case $k = 3$, so the space under consideration is $\text{Link}(P_1, P_2, P_3; N)$. For higher k, one constructs $\binom{k}{3}$ manifolds in the manner described below. There are two types of intersections which form the obstruction manifold. The first is the intersections of P_j with the generalized Whitney disks for the pair (P_h, P_i), where h, i, j are distinct. The second is the intersections of the generalized Whitney circles of the pair (P_h, P_i) with that for (P_i, P_j), again for h, i, j distinct.

6.3.1   Intersection of the Whitney disk with the P_i Let $S = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Let $T_1 = \{(s, t) \mid 0 \le s \le t \le 1\}$. For $(h, i, j) \in S$ consider the maps $\Phi_{1,(hi)j} \colon P_1 \times P_2 \times P_3 \times T \to N^4$ given by

$$\Phi_{1,(hi)j}(x_1, x_2, x_3, s, t) = (f_j(x_j), p_1 H_{hi}(x_h, x_i, s), H_{hi}(x_h, x_i, t)).$$

Let $T_2 = \{(s, t) \mid 0 \le t \le s \le 1\}$. For $(h, i, j) \in S$ consider the maps $\Phi_{2,(hi)j} \colon P_1 \times P_2 \times P_3 \times T \to N^4$ given by

$$\Phi_{2,(hi)j}(x_1, x_2, x_3, s, t) = (H_{hi}(x_h, x_i, s), p_2 H_{hi}(x_h, x_i, t), f_j(x_j)).$$

Definition 6.3   For $a = 1, 2$, define $X_{a,(hi)j} = \Phi_{a,(hi)j}^{-1}(\Delta_N \times \Delta_N)$.

Proposition 13   *For $a = 1, 2$, if $\Phi_{a,(hi)j}$ and its restrictions to $\partial_1(P_1 \times P_2 \times P_3 \times T_a)$ and $\partial_2(P_1 \times P_2 \times P_3 \times T_a)$ are transverse to $\Delta_N \times \Delta_N$, then $X_{a,(hi)j}$ is a compact $(p_1+p_2+p_3+2-2n)$–dimensional manifold with at most 2–strata, and stable normal bundle $TN \times TN - TP_1 \times TP_2 \times TP_3$.*

Proof Transversality gives an isomorphism

$$TX_{a,(hi)j} \oplus TN \oplus TN \to TP_1 \oplus TP_2 \oplus TP_3 \oplus \epsilon^2. \qquad \square$$

The boundary of $X_{a,(hi)j}$ naturally decomposes in to three parts according to whether $t = 0$, $s = 0$ or $t = s$. $\partial X_{a,(hi)j} = \partial_t X_{a,(hi)j} \cup \partial_s X_{a,(hi)j} \cup \partial_{t=s} X_{a,(hi)j}$. Also, $\partial_{t=s} X_{1,(hi)j} = \partial_{t=s} X_{2,(hi)j}$. The 2–stratum of $X_{a,(hi)j}$ consists of the points $(p_1, p_2, p_3, 0, 0)$ such that $f_1(p_1) = f_2(p_2) = f_3(p_3)$, a compact $(p_1 + p_2 + p_3 - 2n)$–dimensional manifold we will call T.

6.3.2 Intersections of the Whitney circles For $(h, i, j) \in S$, consider the maps $\Phi_{hij} \colon P_1 \times P_2 \times P_3 \times I \times I \to N^4$ given by

$$\Phi_{hij}(p_1, p_2, p_3, s, t) = (H_{hi}(p_h, p_i, s), H_{ij}(p_i, p_j, t)).$$

Definition 6.4 $W_{hij} = \Phi_{hij}^{-1}(\Delta_N \times \Delta_N)$.

Proposition 14 *Suppose Φ_{hij} and its restrictions to $\partial_1(P_1 \times P_2 \times P_3 \times I \times I)$ and $\partial_2(P_1 \times P_2 \times P_3 \times I \times I)$ are transverse to $\Delta_N \times \Delta_N$. Then W_{hij} is a compact $(p_1 + p_2 + p_3 + 2 - 2n)$–dimensional manifold with boundary, and stable normal bundle $TN \times TN - TP_1 \times TP_2 \times TP_3$.*
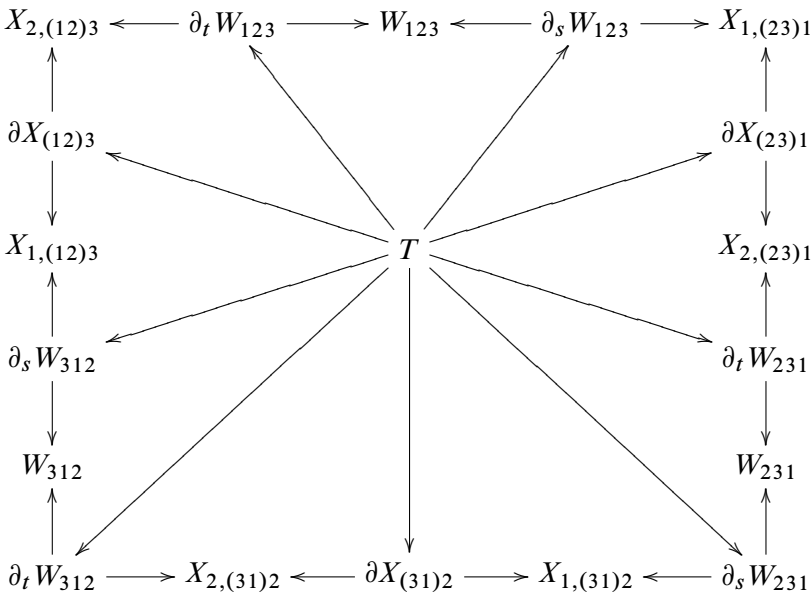
Proof Transversality gives an isomorphism

$$TW_{hij} \oplus TN \oplus TN \to TP_1 \oplus TP_2 \oplus TP_3 \oplus \epsilon^2. \qquad \square$$

The boundary ∂W_{hij} occurs when either $s = 0$ or $t = 0$, and gives a decomposition $\partial W_{hij} = \partial_s W_{hij} \cup \partial_t W_{hij}$. The 2–stratum of W_{hij} is the set T described above.

6.3.3 Forming the obstruction manifold The following lemma tells us how the boundaries of the W_{hij} and $X_{a,(hi)j}$ fit together, and can be verified by noting that the equations which define these manifolds are the same when the appropriate value of t or s is zero.

Lemma 6.5 *For $(h, i, j) \in S$, there are diffeomorphisms of manifolds with boundary $\partial_t W_{hij} \cong \partial_t X_{2,(hi)j}$, $\partial_s W_{hij} \cong \partial_s X_{1,(ij)h}$, and $\partial_{s=t} X_{1,(hi)j} \cong \partial_{s=t} X_{2,(hi)j}$.*

We define $\partial X_{(hi)j} = \partial_{s=t} X_{1,(hi)j}$, and arrange this into the following diagram:

$$
\begin{array}{ccccccc}
X_{2,(12)3} & \longleftarrow \partial_t W_{123} & \longrightarrow W_{123} & \longleftarrow \partial_s W_{123} & \longrightarrow & X_{1,(23)1} \\
\uparrow & & & & & \uparrow \\
\partial X_{(12)3} & & & & & \partial X_{(23)1} \\
\downarrow & & & & & \downarrow \\
X_{1,(12)3} & & T & & & X_{2,(23)1} \\
\uparrow & & & & & \uparrow \\
\partial_s W_{312} & & & & & \partial_t W_{231} \\
\downarrow & & & & & \downarrow \\
W_{312} & & & & & W_{231} \\
\uparrow & & & & & \uparrow \\
\partial_t W_{312} & \longrightarrow X_{2,(31)2} & \longleftarrow \partial X_{(31)2} & \longrightarrow X_{1,(31)2} & \longleftarrow & \partial_s W_{231}
\end{array}
$$

Let $\mathcal{I}$ denote the underlying category for this diagram, and let $\mathcal{Z}\colon \mathcal{I} \to$ Spaces be the functor which gives the diagram above.

Definition 6.6 Define $Z = \mathrm{colim}_{\mathcal{I}}\mathcal{Z}$, and let $Z' = \mathrm{hocolim}_{\mathcal{I}}\mathcal{Z}$.

Note that the canonical map $a\colon Z' \to Z$ is a homotopy equivalence. We wish to show how to piece together the bundle isomorphisms given by transversality in Proposition 13 and Proposition 14 to make a bundle isomorphism over Z'. In order to do this, we need to compare the isomorphisms along the common boundary they define and invoke Proposition 4 and Proposition 5. In order to apply these propositions, we need to first describe the map $Z \to E_{123}$.

6.4 The map $z\colon Z \to E_{123}$

The map $z\colon Z \to E_{123}$ comes in several pieces, because Z itself is defined in several pieces. Note that, by composition, this induces a map $z'\colon Z' \to E_{123}$. A point in Z is a tuple (x_1, x_2, x_3, s, t) which is in one of the W_{hij} or $X_{a,(hi)j}$. If $(x_1, x_2, x_3, s, t) \in W_{hij}$, then the corresponding point in E_{123} has

$$
\omega_h(r_h) = \begin{cases} p_1 H_{hi}(x_h, x_i, 2r_h s) & \text{if } 0 \leq r_h \leq 1/2 \\ p_2 H_{hi}(x_h, x_i, 2(1-r_h)s) & \text{if } 1/2 \leq r_h \leq 1, \end{cases}
$$

$\omega_i(r_i) = f_i(x_i)$, and

$$\omega_j(r_j) = \begin{cases} p_2 H_{ij}(x_i, x_j, 2r_j t) & \text{if } 0 \le r_j \le 1/2 \\ p_1 H_{ij}(x_i, x_j, 2(1-r_j)t) & \text{if } 1/2 \le r_j \le 1, \end{cases}$$

for all $r_i \in [0, 1]$. If $(x_1, x_2, x_3, s, t) \in X_{1,(hi)j}$, then

$$\omega_h(r_h) = p_1 H_{hi}(x_h, x_i, r_h s),$$

$$\omega_i(r_i) = \begin{cases} p_2 H_{hi}(x_h, x_i, 2r_i t) & \text{if } 0 \le r_i \le 1/2 \\ p_1 H_{hi}(x_h, x_i, 2(1-r_i)t + 2(r_i - 1/2)s) & \text{if } 1/2 \le r_j \le 1, \end{cases}$$

and $w_j(r_j) = f_j(x_j)$ for all $r_j \in [0, 1]$. Finally, if $(x_1, x_2, x_3, s, t) \in X_{2,(hi)j}$, then

$$\omega_h(r_h) = \begin{cases} p_1 H_{hi}(x_h, x_i, 2r_h s) & \text{if } 0 \le r_j \le 1/2 \\ p_2 H_{hi}(x_h, x_i, 2(1-r_h)s + 2(r_h - 1/2)t) & \text{if } 1/2 \le r_j \le 1, \end{cases}$$

$\omega_i(r_i) = p_2 H_{hi}(x_h, x_i, r_i t)$, and $\omega_j(r_j) = f_j(x_i)$ for all $r_j \in [0, 1]$. By inspection, these maps agree on the intersections of the W_{hij} with the $X_{a,(hi)j}$. Hence we have a map $Z \to E_{123}$. Now we are going to produce a path in $C_3(P_1, P_2, P_3; N)$ from an element α of

$$\text{hofiber}(\text{Link}(P_1, P_2, P_3; N) \to \mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)).$$

Write $\alpha(u) = (H'_{12}(u), H'_{23}(u), H'_{31}(u))$, where $u \in I$, $\alpha(0) = (H_{12}, H_{23}, H_{31}) \in \mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)$, and $\alpha(1) = (H'_{12}, H'_{23}, H'_{31})$. Here $H'_{ij}: P_i \times P_j \times I \to N \times N$ satisfies $H'_{ij} = f'_i \times f'_j$ for some $(f'_1, f'_2, f'_3) \in \text{Link}(P_1, P_2, P_3; N)$. Since we can think of α as a parametrized family of maps in $\mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)$, we can apply the construction of Z to this family. What remains is to give the map to E_{123}, which means associating three paths with the point. For the sake of brevity, let us do this in the specific case where $(x_1, x_1, x_3, s, t, u) \in W_{123}$. This means that $H_{12}(x_1, x_2, s, u) \in \Delta_N$ and $H_{23}(x_2, x_3, t, u) \in \Delta_N$. Then, for instance, we define

$$\omega_1(r_1) = \begin{cases} p_1 H_{12}(x_1, x_2, 2r_1 s, 2r_1 u) & \text{if } 0 \le r_j \le 1/2 \\ p_2 H_{12}(x_1, x_2, 2(1-r_1)s, 2(1-r_1)u) & \text{if } 1/2 \le r_j \le 1. \end{cases}$$

The others are defined similarly.

6.5 Gluing the bundle data

For vector spaces V, W, let $L_{\max}(V, W)$ denote the space of linear maps $V \to W$ of maximal rank. There is an isomorphism $L_{\max}(V, W) \to L_{\max}(W, V)$ given by sending a linear transformation to its transpose. When $\dim(V) \le \dim(W)$, the space $L_{\max}(V, W)$ is the Stiefel manifold, which is well-known to be

$(\dim(W) - \dim(V) - 1)$–connected. We consider bundles of spaces $L_{\max}(V, W)$, as introduced in Definition 2.3.

Lemma 6.7 *The derivatives $D\Phi_{hij}$ and $D\Phi_{a,(hi)j)}$ are homotopic where it makes sense to compare them.*

Proof Consider the space $L_{\max}(TP \times \epsilon^2, TN \times TN)$ as a space over $\partial_t W_{hij} \cong \partial_t X_{2,(hi)j}$. The maps $D\Phi_{hij}$ and $D\Phi_{2,(hi)j)}$ define sections of these bundles, and we wish to show they are homotopic.

Take an open cover $\mathcal{U}$ of $\partial_t W_{hij}$ consisting of contractible open sets U so the bundle $L_{\max}(TP \times \epsilon^2, TN \times TN)$ is trivial over each $U \in \mathcal{U}$. Now $L_{\max}(TP \times \epsilon^2, TN \times TN)$ is $(p_1 + p_2 + p_3 - 2n + 1)$–connected, and if $p_1 + p_2 + p_3 + 2 - 2n \leq 0$, there is nothing to prove. Otherwise, since $(p_1 + p_2 + p_3 - 2n + 1) \geq 0$, two such maps are always homotopic. It follows that $D\Phi_{hij}$ and $D\Phi_{2,(hi)j}$ are homotopic, by Remark 1. The same proof shows that the restrictions of $D\Phi_{hij}$ and $D\Phi_{1,(ij)h}$ and $D\Phi_{1,(ij)h}$ and $D\Phi_{2,(ij)h}$ are homotopic where it makes sense to compare them. As for the 2–strata T, we must compare these chosen homotopies on T and ensure that the composed homotopy is homotopic to a constant homotopy. If $p_1 + p_2 + p_3 + 2 - 2n \leq 1$ there is nothing to prove, since the 2–strata is empty. Otherwise $(p_1 + p_2 + p_3 - 2n + 1) \geq 1$ and the space $L_{\max}(TP \times \epsilon^2, TN \times TN)$ is ≥ 1–connected. $\square$

Lemma 6.8 *There is an isomorphism $TZ \oplus TN \oplus TN \to TP_1 \oplus TP_2 \oplus TP_3 \oplus \epsilon^2$.*

Proof Let $P = P_1 \times P_2 \times P_3$, and identify $\nu(\Delta_N \times \Delta_N \subset N^4)$ with $TN \times TN$. For $(h, i, j) \in S$, the derivatives $D\Phi_{hij} \colon TP \times \epsilon^2 \to TN^4$ give sections of the bundles $L_{\max}(TP \times \epsilon^2, TN \times TN)$ over W_{hij}. The same is true of $D\Phi_{a,(hi)j} \colon TP \times \epsilon^2 \to TN \times TN$ and the manifolds $X_{a,(hi)j}$. These maps are homotopic by Lemma 6.7. By Propositions 4 and 5, they combine to give an element of the space $\operatorname{hocolim}_{\mathcal{I}} L_{\max}(z'^* TP \times \epsilon^2, z'^* TN \times TN)$, which is a space over Z'. Pulling back by a homotopy inverse to $a \colon Z' \to Z$, we obtain $\Phi \in L_{\max}(z^* TP \times \epsilon^2, z^* TN \times TN)$. The fiberwise kernel of Φ is, by construction, TZ (see Proposition 2), and this gives the desired isomorphism. $\square$

Theorem 6.9 *Z can be given the structure of a smooth closed compact manifold of dimension $p_1 + p_2 + p_3 + 2 - 2n$, and TZ is stably isomorphic to $TP_1 \times TP_2 \times TP_3 - TN \times TN$.*

Proof That Z is smooth follows from Proposition 2, since the diffeomorphisms between the 1–strata of the pieces which form Z all restrict to the identity on T. Note

that the canonical map $a\colon Z' \to Z$ is a homotopy equivalence. By Lemma 6.8, we pull back the bundle isomorphism over Z' by a homotopy inverse to z to give the desired bundle isomorphism. $\qquad\square$

What we have shown is that there is a map from the subcomplex of 0–simplices of

$$\text{hofiber}(\text{Link}(P_1, P_2, P_3; N) \to \mathcal{T}_2\text{Link}(P_1, P_2, P_3; N))$$

such that the above transversality conditions are met, to the 0–simplices of the space $\Omega C_3(P_1, P_2, P_3; N)$. Getting a map of simplicial sets is easy, since the above works for families of maps too, and since the subcomplex of transverse maps is homotopy equivalent to the full complex. Thus we have proven Theorem 1.3, restated below.

Theorem 6.10 *For a basepoint in the image of* $\text{Link}(P_1, P_2, P_3; N)$, *there is a map of spaces*

$$l_3\colon \text{hofiber}(\text{Link}(P_1, P_2, P_3; N) \to \mathcal{T}_2\text{Link}(P_1, P_2, P_3; N)) \to \Omega C_3(P_1, P_2, P_3; N).$$

It would be interesting to know the connectivity of the map l_3.

6.6 Example: The Borromean rings

As an example, we show that the Borromean rings are not homotopic to the unlink. We proceed by finding a path from image of the Borromean rings to the unlink in $\mathcal{T}_1\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$, lifting this path to a path in $\mathcal{T}_2\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$, and then constructing the obstruction manifold for this family, which turns out to be a single point.

We need a lemma that ensures this construction really distinguishes the Borromean rings from the unlink, and it concerns the connecting map in the fibration sequence $\mathcal{L}_3\text{Link}(S^1, S^1, S^1; \mathbf{R}^3) \to \mathcal{T}_3\text{Link}(S^1, S^1, S^1; \mathbf{R}^3) \to \mathcal{T}_2\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$.

Lemma 6.11 $\pi_1\mathcal{T}_2\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ *acts trivially on* $\pi_0\mathcal{L}_3\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$.

Proof Let $\mathcal{T}_2 = \mathcal{T}_2\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$, $\mathcal{L}_3 = \mathcal{L}_3\text{Link}(S^1, S^1, S^1; \mathbf{R}^3)$, $S^{(3)} = S^1 \times S^1 \times S^1$, $\Phi_3 = \text{tfiber}(S \mapsto \text{Emb}(S, \mathbf{R}^3))$, and $\widehat{\Phi}_3 = \text{holim}_S \text{Emb}(S, \mathbf{R}^3)$, where S ranges over proper subsets of $\{1, 2, 3\}$.

Consider the following commutative diagram.

$$
\begin{array}{ccccccc}
\pi_1\mathcal{T}_2 & \xrightarrow{D_1} & \pi_1\text{map}(S^{(3)}, \widehat{\Phi}_3) & \xleftarrow{C_1} & \pi_1\text{map}_*(S^{(3)}, \widehat{\Phi}_3) & \xleftarrow{B_1} & \pi_1\text{map}_*(S^3, \widehat{\Phi}_3) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle A} \\
\pi_0\mathcal{L}_3 & \xrightarrow{D_2} & \pi_0\text{map}(S^{(3)}, \Phi_3) & \xleftarrow{C_2} & \pi_0\text{map}_*(S^{(3)}, \Phi_3) & \xleftarrow{B_2} & \pi_0\text{map}_*(S^3, \Phi_3)
\end{array}
$$

We wish to show the left-most vertical map is zero. It is enough to show that A is the zero map, that B_1 and B_2 are injective, that C_1 and C_2 are isomorphisms, that D_2 is an isomorphism, and that the images D_1 and $C_1 \circ B_1$ coincide.

That D_2 is an isomorphism follows from Theorem 4.5 by inspection, as the underlying spaces are the same. C_2 is an isomorphism since Φ_3 is simply connected. C_1 is an isomorphism because we may regard $\pi_1 \mathrm{map}_*(S^{(3)}, \widehat{\Phi}_3) = \pi_0 \mathrm{map}_*(S^{(3)}, \Omega\widehat{\Phi}_3)$, and since $\Omega\widehat{\Phi}_3$ is an H–space, its fundamental group acts trivially on the based mapping space. For $i = 1, 2$, B_i is induced by the cofiber sequence $X \to S^{(3)} \to S^3$, where $X = S^1 \times S^1 \times * \cup S^1 \times * \times S^1 \cup * \times S^1 \times S^1$. If we suspend this cofiber sequence, we see that the map $\Sigma X \to \Sigma S^{(3)}$ has a section since ΣX is an H–cospace. It follows from the long exact sequence in homotopy obtained by applying the functor $\mathrm{map}_*(-, \Phi_3)$ that B_2 is injective. The same argument also shows that B_1 is injective.

To show that A is the zero map, consider the following fibration sequence $\Phi_3 \to \mathrm{Emb}(\underline{3}, \mathbf{R}^3) \to \widehat{\Phi}_3$. We claim it has connecting map zero in homotopy. Consider the following diagram.

$$
\begin{array}{ccccc}
\Phi_3 & \longrightarrow & S^2 \vee S^2 & \longrightarrow & S^2 \times S^2 \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
\Phi_3 & \longrightarrow & \mathrm{Emb}(\underline{3}, \mathbf{R}^3) & \longrightarrow & S^2 \times S^2 \times S^2 \\
& & \Big\downarrow & & \Big\downarrow \\
& & S^2 & \longrightarrow & S^2
\end{array}
$$

Everything in sight is a fibration sequence. The connecting map in the long exact sequence in homotopy of the top row is zero because, for spaces X and Y, the map $\Omega(X \vee Y) \to \Omega(X \times Y)$ has a section. It follows that the connecting map for in the long exact sequence in homotopy of the middle row is zero as well.

Now consider the map of spaces $\mathcal{T}_2 \to \mathrm{map}(S^{(3)}, \widehat{\Phi}_3)$ which induces D_1. Note that $\widehat{\Phi}_3 \simeq \prod_3 S^2$, $\mathcal{T}_2 = \prod_3 \mathrm{map}(S^1 \times S^1, S^2)$, and the map which induces D_1 is compatible with these equivalences and may therefore be interpreted as the map $\prod_3 \mathrm{map}(S^1 \times S^1, S^2) \to \prod_3 \mathrm{map}(S^{(3)}, S^2)$ which sends a triple (f, g, h) to the map $F(x, y, z) = (f(x, y), g(y, z), h(z, x))$. On the one hand, the component of $\prod_3 \mathrm{map}(S^1 \times S^1, S^2)$ in which we choose the basepoint is the degree zero maps (the image of the unlink is the basepoint), and thus consists of those maps $F = (f, g, h)$ such that the components are independent of one of the three variables and are null homotopic. On the other hand, $\mathrm{map}_*(S^3, \widehat{\Phi}_3)$ is the fiber of the fibration $\mathrm{map}_*(S^{(3)}, \widehat{\Phi}_3) \to \mathrm{map}_*(X, \widehat{\Phi}_3)$, and thus consists of those maps $S^{(3)} \to \widehat{\Phi}_3$ which

are constant on X. It is homotopy equivalent to the homotopy fiber of this map, which consists of those maps $S^{(3)} \to \widehat{\Phi}_3$ which are homotopic to a map constant on X. This is precisely the image of $\mathcal{T}_2$ in $\prod_3 \mathrm{map}(S^{(3)}, S^2)$.   □

Let $B = (f_1, f_2, f_3) \in \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ be the Borromean rings, where $f_1(x) = (0, 2\cos x, \sin x)$, $f_2(y) = (\cos y, 0, 2\sin y)$, and $f_3(z) = (2\cos z, \sin z, 0)$. Let the unlink be represented by the triple $U = (f_1, e_2, f_3) \in \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$, where $e_2(y) = (\cos y, 4, 2\sin y)$. The images of B and U in $\mathcal{T}_1 \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ are clearly homotopic by the path $L(t) = (f_1, H_1(y, t), f_3)$, where $H_1(y, t) = (\cos y, t, 2\sin y)$ for $t \in [0, 4]$. We lift this path to the path in $\mathcal{T}_2 \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ given by the triple $(f_1 \times f_3, H_1 \times f_1, f_3 \times H_3)$, where $H_3(y, t)$ is given by $(\cos y, t, 2\sin y - (t - 2)^2 + 4)$ for $t \in [0, 4]$ (we have altered the order in which we represent tuples in $\mathcal{T}_2$ that we used in Section 6.1, and the $\mathcal{T}_2$ we mean here is the image of an inner automorphism of $\mathcal{T}_2$ induced by a permutation of $\{1, 2, 3\}$).

In order to construct the obstruction from this family, we must make the double point construction, and form the Whitney disks. The double points of the family $L(t)$ come from the intersection of f_3 with H_1, which is clearly a pair of points $(y, z, t) = (\pi/2, \pi/3, \sqrt{3}/2), (3\pi/2, 2\pi/3, \sqrt{3}/2)$, and this intersection is transverse. To make a nullcobordism of this manifold, we define a function $G: S^1 \times S^1 \times [0, 4] \times [0, 1] \to \mathbf{R}^3 \times \mathbf{R}^3$ given by $G(y, z, t, s) = (f_3(z), sH_3(y, t) + (1 - s)H_1(y, t))$.

Proposition 15 G is transverse to $\Delta_{\mathbf{R}^3}$ and $G^{-1}(\Delta_{\mathbf{R}^3}) \subset S^1 \times S^1 \times [0, 4] \times [0, 1]$ is an arc which embeds in each copy of S^1 under the projections to the S^1 factors.

Proof   Let $H(y, t, s) = sH_3(y, t) + (1 - s)H_1(y, t)$. To see that G is transverse to the diagonal, it is equivalent to check that the images of f_3 and H intersect transversely in $\mathbf{R}^3$. The derivative

$$DH = \begin{pmatrix} -\sin y & 0 & 0 \\ 0 & 1 & 0 \\ 2\cos y & -2s(t - 2) & -(t - 2)^2 + 4 \end{pmatrix}$$

has determinant $(-(t - 2)^2 + 4)\sin y$, which vanishes if and only if either $t = 0, 4$ or $y = 0, \pi$. Since there are no solutions to $f_3(z) = H(y, 0, s)$, we only have to deal with the possibility of $y = 0, \pi$. In this case, $z = \pi/3, 2\pi/3$, and by inspection we find that for these values of z, Df_3 and the last two columns of DH span $\mathbf{R}^3$.

From transversality it follows that $D = G^{-1}(\Delta_{\mathbf{R}^3})$ is a 1–dimensional manifold with boundary. It is the solutions to the following equations:

$$2\cos z = \cos y, \quad \sin z = t, \quad 0 = 2\sin y + s(-(t - 2)^2 + 4)$$

The first two equations imply that $z \in [\pi/3, 2\pi/3]$ and $t \in [\sqrt{3}/2, 1]$. Moreover, for each such (z, t) there is a unique (y, s) which solves all three, since the third equation implies that $2 \sin y \le 0$, and we can solve for s. Let $p_i \colon S^1 \times S^1 \times [0, 4] \times [0, 1] \to S^1$ for $i = 1, 2$ be the projections. We have shown that $p_2(D) = [\pi/3, 2\pi/3]$ and that the restriction of p_2 to D is an embedding. It is also clear that $p_1(D) = [\pi, 2\pi]$ and that the projection p of D onto $S^1 \times I$, given by $p(y, z, t) = (y, t)$, is an embedding (in fact, the restriction of p_1 itself is an embedding, though we will not need this). $\qquad \square$

The Whitney circle $C = f_3(p_2 D) \cup H_1(p D)$ is a bent circle in $\mathbf{R}^3$. Let $\pi_i \colon \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ be the projections for $i = 1, 2$. C bounds a disk W given by G as follows: For each $(y, z, t, s) \in D$, G gives rise to a path in $\mathbf{R}^3$ from $f_3(z) = \pi_1 G(y, z, t, 0)$ to $H_1(y, t) = \pi_2 G(y, z, t, 0)$ because $\pi_1 G(y, z, t, s) = \pi_2 G(y, z, t, s)$. The path is defined by

$$\gamma(\sigma) = \begin{cases} \pi_1 G(y, z, t, 2\sigma s) & \text{if } \sigma \in [0, 1/2] \\ \pi_2 G(y, z, t, (2 - 2\sigma)s) & \text{if } \sigma \in [1/2, 1]. \end{cases}$$

The union of these paths forms the disk W bounding C in $\mathbf{R}^3$, and it is clearly disjoint from $f_3(S^1)$ and $H_1(S^1) \times [0, 4]$ except along its boundary. It is clear that $f_2(S^1)$ intersects W transversely in a single point, since W is the union of straight lines between z and (y, t) for each $(y, z, t, s) \in D$ (and for each y there is a unique z and vice-versa, since the projections of D to the S^1 factors are embeddings). It follows that B cannot be link homotopic to U, since $\pi_1 \mathcal{T}_2 \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ acts trivially on $\pi_0 \mathcal{L}_3 \mathrm{Link}(S^1, S^1, S^1; \mathbf{R}^3)$ by Lemma 6.11.

References

[1] **V V Chernov**, **Y B Rudyak**, *Toward a general theory of linking invariants*, Geom. Topol. 9 (2005) 1881–1913   MR2175159

[2] **T G Goodwillie**, *Calculus. I. The first derivative of pseudoisotopy theory*, K–Theory 4 (1990) 1–27   MR1076523

[3] **T G Goodwillie**, **J R Klein**, *Excision estimates for spaces of Poincaré embeddings*, In preparation

[4] **T G Goodwillie**, **J R Klein**, **M S Weiss**, *Spaces of smooth embeddings, disjunction and surgery*, from: "Surveys on surgery theory, Vol. 2", Ann. of Math. Stud. 149, Princeton Univ. Press (2001) 221–284   MR1818775

[5] **T G Goodwillie**, **J R Klein**, **M S Weiss**, *A Haefliger style description of the embedding calculus tower*, Topology 42 (2003) 509–524   MR1953238

[6] **T G Goodwillie**, **B Munson**, *Connectivity estimates for spaces of link maps*, In preparation

[7] **T G Goodwillie**, **M Weiss**, *Embeddings from the point of view of immersion theory: Part II*, Geom. Topol. 3 (1999) 103–118   MR1694808

[8] **N Habegger**, **U Kaiser**, *Link homotopy in the 2–metastable range*, Topology 37 (1998) 75–94   MR1480878

[9] **N Habegger**, **X-S Lin**, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. 3 (1990) 389–419   MR1026062

[10] **A Hatcher**, **F Quinn**, *Bordism invariants of intersections of submanifolds*, Trans. Amer. Math. Soc. 200 (1974) 327–344   MR0353322

[11] **M W Hirsch**, *Differential topology*, Graduate Texts in Math. 33, Springer, New York (1976)   MR0448362

[12] **J Klein**, **B Williams**, *Homotopical intersection theory I*   arXiv:math.AT/0512479

[13] **U Koschorke**, *A generalization of Milnor's μ–invariants to higher-dimensional link maps*, Topology 36 (1997) 301–324   MR1415590

[14] **U Koschorke**, *Link maps in arbitrary manifolds and their homotopy invariants*, J. Knot Theory Ramifications 12 (2003) 79–104   MR1953625

[15] **U Koschorke**, *Link homotopy in $S^n \times \mathbb{R}^{m-n}$ and higher order μ–invariants*, J. Knot Theory Ramifications 13 (2004) 917–938   MR2101235

[16] **U Koschorke**, *Linking and coincidence invariants*, Fund. Math. 184 (2004) 187–203   MR2128050

[17] **U Koschorke**, **D Rolfsen**, *Higher-dimensional link operations and stable homotopy*, Pacific J. Math. 139 (1989) 87–106   MR1010788

[18] **J P Levine**, *An approach to homotopy classification of links*, Trans. Amer. Math. Soc. 306 (1988) 361–387   MR927695

[19] **W S Massey**, **D Rolfsen**, *Homotopy classification of higher-dimensional links*, Indiana Univ. Math. J. 34 (1985) 375–391   MR783921

[20] **J Milnor**, *Link groups*, Ann. of Math. (2) 59 (1954) 177–195   MR0071020

[21] **J Milnor**, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville, VA (1965)   MR0226651   Based on notes by D W Weaver

[22] **B A Munson**, *Embeddings in the $3/4$ range*, Topology 44 (2005) 1133–1157   MR2168572

[23] **R Schneiderman**, **P Teichner**, *Higher order intersection numbers of 2–spheres in 4–manifolds*, Algebr. Geom. Topol. 1 (2001) 1–29 MR1790501

[24] **G P Scott**, *Homotopy links*, Abh. Math. Sem. Univ. Hamburg 32 (1968) 186–190 MR0236912

[25] **A Skopenkov**, *On the generalized Massey–Rolfsen invariant for link maps*, Fund. Math. 165 (2000) 1–15 MR1804997

[26] **M Weiss**, *Embeddings from the point of view of immersion theory: Part I*, Geom. Topol. 3 (1999) 67–101 MR1694812

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