Poincaré duality complexes in dimension four

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Generalising Hendriks' fundamental triples of PD^3–complexes, we introduce fundamental triples for PD^n–complexes and show that two PD^n–complexes are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic. As applications we establish a conjecture of Turaev and obtain a criterion for the existence of degree 1 maps between n–dimensional manifolds. Another main result describes chain complexes with additional algebraic structure which classify homotopy types of PD^4–complexes. Up to 2–torsion, homotopy types of PD^4–complexes are classified by homotopy types of chain complexes with a homotopy commutative diagonal.

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Introduction

In order to study the homotopy types of closed manifolds, Browder and Wall introduced the notion of Poincaré duality complexes. A Poincaré duality complex, or PD^n–complex, is a CW–complex X whose cohomology satisfies a certain algebraic condition. Equivalently, the chain complex $\widehat{C}(X)$ of the universal cover of X must satisfy a corresponding algebraic condition. Thus Poincaré complexes form a mixture of topological and algebraic data and it is an old quest to provide purely algebraic data determining the homotopy type of PD^n–complexes. This has been achieved for $n=3$, but, for $n=4$, only partial results are available in the literature.

Homotopy types of 3–manifolds and PD^3–complexes were considered by Thomas [17], Swarup [15] and Hendriks [9]. The homotopy type of a PD^3–complex X is determined by its *fundamental triple*, consisting of the fundamental group $\pi=\pi_1(X)$, the orientation character ω and the image in $\mathrm{H}_3(\pi,\mathbb{Z}^\omega)$ of the fundamental class $[X]$. Turaev [18] provided an algebraic condition for a triple to be realizable by a PD^3–complex. Thus, in dimension 3, there are purely algebraic invariants which provide a complete classification.

Using primary cohomological invariants like the fundamental group, characteristic classes and intersection pairings, partial results were obtained for $n=4$ by imposing

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conditions on the fundamental group. For example, Hambleton, Kreck and Teichner classified PD^4–complexes with finite fundamental group having periodic cohomology of dimension 4 (see Hambleton and Kreck [6], Teichner [16] and Hambleton, Kreck and Teichner [7]). Cavicchioli and Hegenbarth [4] and Hegenbarth and Piccarreta [8] studied PD^4–complexes with free fundamental group, as did Hillman [10], who also considered PD^4–complexes with fundamental group a PD^2–group [11]. Recently, Hillman [12] considered homotopy types of PD^4–complexes whose fundamental group has cohomological dimension 2 and one end.

It is doubtful whether primary invariants are sufficient for the homotopy classification of PD^4–complexes in general and we thus follow Ranicki's approach [13; 14] who assigned to each PD^n–complex X an *algebraic Poincaré duality complex* given by the chain complex $\widehat{C}(X)$, together with a *symmetric structure*. However, Ranicki considered neither the realizability of such algebraic Poincaré duality complexes nor whether the homotopy type of a PD^n–complex is determined by the homotopy type of its algebraic Poincaré duality complex.

This paper presents a structure on chain complexes which completely classifies PD^4–complexes up to homotopy. The classification uses *fundamental triples* of PD^4–complexes, and, in fact, the chain complex model yields algebraic conditions for the realizability of fundamental triples.

A fundamental triple of formal dimension $n \geq 3$ comprises an $(n-2)$–type T, a homomorphism $\omega\colon \pi_1(T) \to \mathbb{Z}/2\mathbb{Z}$ and a homology class $t \in \mathrm{H}_n(T, \mathbb{Z}^\omega)$. There is a functor,

$$\tau_+\colon \mathbf{PD}^n_+ \longrightarrow \mathbf{Trp}^n_+,$$

from the category $\mathbf{PD}^n_+$ of PD^n–complexes and maps of degree one to the category $\mathbf{Trp}^n_+$ of triples and morphisms inducing surjections on fundamental groups. Our first main result is:

Theorem 3.1 *The functor τ_+ reflects isomorphisms and is full for $n \geq 3$.*

Corollary 3.2 *Take $n \geq 3$. Two closed n–dimensional manifolds or two PD^n–complexes, respectively, are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic.*

Corollary 3.2 extends results of Thomas [17], Swarup [15] and Hendriks [9] for dimension 3 to arbitrary dimension and establishes Turaev's conjecture [18] on PD^n–complexes whose $(n-2)$–type is an Eilenberg–Mac Lane space $K(\pi_1 X, 1)$. Corollary 3.2 is even of interest in the case of simply connected or highly connected manifolds.

Theorem 3.1 also yields a criterion for the existence of a map of degree one between PD^n–complexes, recovering Swarup's result for maps between 3–manifolds and Hendriks' result for maps between PD^3–complexes.

In the oriented case, special cases of Corollary 3.2 were proved by Hambleton and Kreck [6] and Cavicchioli and Spaggiari [5]. In fact, in [6], Corollary 3.2 is obtained under the condition that either the fundamental group is finite or the second rational homology of the 2–type is nonzero. Corresponding conditions were used in [5] for oriented PD^{2n}–complexes with $(n-1)$–connected universal covers, and Teichner extended the approach of [6] to the nonoriented case in his thesis [16]. Our result shows that the conditions on finiteness and rational homology used in these papers are not necessary.

It follows directly from Poincaré duality and Whitehead's Theorem that the functor τ_+ reflects isomorphisms. To show that τ_+ is full requires work. Given PD^n–complexes Y and X, $n \geq 3$, and a morphism $f\colon \tau_+ Y \to \tau_+ X$ in $\mathbf{Trp}^n_+$, we first construct a chain map $\xi\colon \widehat{C}(Y) \to \widehat{C}(X)$ preserving fundamental classes, that is, $\xi_*[Y] = [X]$. Then we use the category $\mathbf{H}^c_{k+1}$ of homotopy systems of order $(k+1)$ introduced by the first author in [1] to realize ξ by a map $\overline{f}\colon Y \to X$ with $\tau_+(\overline{f}) = f$.

Our second main result describes algebraic models of homotopy types of PD^4–complexes. We introduce the notion of PD^n*–chain complex* and show that PD^3–chain complexes are equivalent to PD^3–complexes up to homotopy. In Section 5 we show that PD^4–chain complexes classify homotopy types of PD^4–complexes up to 2–torsion. In particular, we obtain:

Theorem 5.3 *The functor $\widehat{C}$ induces a 1–1 correspondence between homotopy types of PD^4–complexes with finite fundamental group of odd order and homotopy types of PD^4–chain complexes with homotopy commutative diagonal and finite fundamental group of odd order.*

This result is a consequence of the following.

Theorem *Let C be a PD^4–chain complex with homotopy commutative diagonal, fundamental group π and homology module $H_2 = \mathrm{H}_2(C)$. If $\mathrm{H}_0(\pi, \Lambda^2 H_2^{\omega})$ has no 2–torsion, then C is realizable by a PD^4–complex, and the 2–torsion group $\ker H_*$ in Theorem 5.1 acts transitively and effectively on the set of realizations.*

To obtain a complete homotopy classification of PD^4–complexes, we study the chain complex of a 2–type in Section 6. We compute this chain complex up to dimension 4 in terms of Peiffer commutators in pre-crossed modules. This allows us to introduce PD^4–chain complexes together with a β–invariant, and we prove:

Corollary 7.4 *The functor $\widehat{C}$ induces a 1–1 correspondence between homotopy types of PD^4–complexes and homotopy types of β–PD^4–chain complexes.*

Corollary 7.4 highlights the crucial rôle of Peiffer commutators for the homotopy classification of 4–manifolds.

The proofs of our results rely on the obstruction theory in [1] for the realizability of chain maps which we recall in Section 8.

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1 Chain complexes

Let X^n denote the n–skeleton of the CW–complex X. We call X reduced if $X^0 = *$ is the base point. The objects of the category $\mathbf{CW_0}$ are reduced CW–complexes X with universal covering $p\colon \widehat{X} \to X$, such that $p(\widehat{*}) = *$, where $\widehat{*} \in \widehat{X}^0$ is the base point of $\widehat{X}$. Here the n–skeleton of $\widehat{X}$ is $\widehat{X}^n = p^{-1}(X^n)$. Morphisms in $\mathbf{CW_0}$ are cellular maps $f\colon X \to Y$ and homotopies in $\mathbf{CW_0}$ are base point preserving. A map $f\colon X \to Y$ in $\mathbf{CW_0}$ induces a unique covering map $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ with $\widehat{f}(\widehat{*}) = \widehat{*}$, which is equivariant with respect to $\varphi = \pi_1(f)$.

We consider pairs (π, C), where π is a group and C a chain complex of left modules over the group ring $\mathbb{Z}[\pi]$. We write $\Lambda = \mathbb{Z}[\pi]$ and C for (π, C), whenever π is understood. We call (π, C) *free* if each C_n, $n \in \mathbb{Z}$, is a free Λ–module. Let $\mathrm{aug}\colon \Lambda \to \mathbb{Z}$ be the augmentation homomorphism, defined by $\mathrm{aug}(g) = 1$ for all $g \in \pi$. Every group homomorphism, $\varphi\colon \pi \to \pi'$, induces a ring homomorphism $\varphi_\sharp\colon \Lambda \to \Lambda'$, where $\Lambda' = \mathbb{Z}[\pi']$. A *chain map* is a pair $(\varphi, F)\colon (\pi, C) \to (\pi', C')$, where φ is a group homomorphism and $F\colon C \to C'$ a φ–equivariant chain map, that is a chain map of the underlying abelian chain complexes, such that $F(\lambda c) = \varphi_\sharp(\lambda) F(c)$ for $\lambda \in \Lambda$ and $c \in C$. Two such chain maps are *homotopic*, $(\varphi, F) \simeq (\psi, G)$ if $\varphi = \psi$ and if there is a φ–equivariant map $\alpha\colon C \to C'$ of degree $+1$ such that $G - F = d\alpha + \alpha d$.

A pair (π, C) is a *reduced* chain complex if $C_0 = \Lambda$ with generator $*$, $C_i = 0$ for $i < 0$ and $\mathrm{H}_0 C = \mathbb{Z}$ such that $C_0 = \Lambda \to \mathrm{H}_0 C = \mathbb{Z}$ is the augmentation of Λ. A chain map,

$(\varphi, f)\colon (\pi, C) \to (\pi', C')$, of reduced chain complexes, is *reduced* if f_0 is induced by $\varphi_\sharp$, and a chain homotopy α of reduced chain maps is *reduced* if $\alpha_0 = 0$. The objects of the category $\mathbf{H}_0$ are reduced chain complexes and the morphisms are reduced chain maps. Homotopies in $\mathbf{H}_0$ are reduced chain homotopies. Every chain complex (π, C) in $\mathbf{H}_0$ is equipped with an augmentation $\varepsilon\colon C \to \mathbb{Z}$ in $\mathbf{H}_0$. The ring homomorphism $\mathbb{Z} \to \Lambda$ yields the co–augmentation $\iota\colon \mathbb{Z} \to C$, where we view $\mathbb{Z} = (0, \mathbb{Z})$ as chain complex with trivial group $\pi = 0$ concentrated in degree 0. Note that $\varepsilon\iota = \mathrm{id}_\mathbb{Z}$, and the composite $\iota\varepsilon\colon C \to C$ is the *trivial* map.

For an object X in $\mathbf{CW}_0$, the cellular chain complex $C(\widehat{X})$ of the universal cover $\widehat{X}$ is given by $C_n(\widehat{X}) = \mathrm{H}_n(\widehat{X}^n, \widehat{X}^{n+1})$, the n–th relative singular homology of the pair $(\widehat{X}^n, \widehat{X}^{n-1})$. The fundamental group $\pi = \pi_1(X)$ acts on $C(\widehat{X})$, and viewing $C(\widehat{X})$ as a complex of left Λ–modules, we obtain the object $\widehat{C}(X) = (\pi, C(\widehat{X}))$ in $\mathbf{H}_0$. Moreover, a morphism $f\colon X \to Y$ in $\mathbf{CW}_0$ induces the homomorphism $\pi_1(f)$ on the fundamental groups and the $\pi_1(f)$–equivariant map $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ which, in turn, induces the $\pi_1(f)$–equivariant chain map $\widehat{f}_*\colon C(\widehat{X}) \to C(\widehat{Y})$ in $\mathbf{H}_0$. As $\widehat{f}$ preserves base points, $\widehat{C}(f) = (\pi_1(f), \widehat{f}_*)$ is a reduced chain map. We obtain the functor

$$\widehat{C}\colon \mathbf{CW}_0 \longrightarrow \mathbf{H}_0. \tag{1–1}$$

The chain complex C in $\mathbf{H}_0$ is 2*–realizable* if there is an object X in $\mathbf{CW}_0$ such that $\widehat{C}(X^2) \cong C_{\leq 2}$, that is, $\widehat{C}(X^2)$ is isomorphic to C in degree ≤ 2.

Remark 1.1 A chain complex C in $\mathbf{H}_0$ is 2–realizable if and only if C is realizable, up to isomorphism, by an object in the category $\mathbf{H}_3^c$ (compare Section 3.2 in [1]). Hence the condition of 2–realizability is needed to apply the obstruction theory in Section 8.

Given two objects X and Y in $\mathbf{CW}_0$, their product again carries a cellular structure and we obtain the object $X \times Y$ in $\mathbf{CW}_0$ with base point $(*, *)$ and universal cover $(X \times Y)\widehat{\ } = \widehat{X} \times \widehat{Y}$, so that

$$\widehat{C}(X \times Y) = (\pi \times \pi, C(\widehat{X}) \otimes_\mathbb{Z} C(\widehat{Y})). \tag{1–2}$$

For $i = 1, 2$, let $p_i\colon X \times X \to X$ be the projection onto the i–th factor. A *diagonal* $\Delta\colon X \to X \times X$ in $\mathbf{CW}_0$ is a cellular map with $p_i \Delta \simeq \mathrm{id}_X$ in $\mathbf{CW}_0$ for $i = 1, 2$. A *diagonal* on (π, C) in $\mathbf{H}_0$ is a chain map $(\delta, \Delta)\colon (\pi, C) \to (\pi \times \pi, C \otimes_\mathbb{Z} C)$ in $\mathbf{H}_0$ with $\delta\colon \pi \to \pi \times \pi, g \mapsto (g, g)$, such that $p_i \Delta \simeq \mathrm{id}_C$ for $i = 1, 2$, where $p_1 = \mathrm{id} \otimes \varepsilon$ and $p_2 = \varepsilon \otimes \mathrm{id}$.

The diagonal (δ, Δ) in $\mathbf{H}_0$ is homotopy associative if the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes_{\mathbb{Z}} C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes_{\mathbb{Z}} C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes_{\mathbb{Z}} C \otimes_{\mathbb{Z}} C \end{tikzcd}$$

commutes up to chain homotopy in $\mathbf{H}_0$. The diagonal (δ, Δ) in $\mathbf{H}_0$ is homotopy commutative if the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & C \otimes_{\mathbb{Z}} C \arrow[d, "T"] \\ & C \otimes_{\mathbb{Z}} C \end{tikzcd}$$

commutes up to chain homotopy in $\mathbf{H}_0$, where T is given by $T(c\otimes d)=(-1)^{|c||d|}d\otimes c$.

By the cellular approximation theorem, every object, X, in $\mathbf{CW}_0$ has a diagonal $\Delta\colon X \to X \times X$ in $\mathbf{CW}_0$. Applying the functor $\widehat{C}$ to such a diagonal, we obtain the diagonal $\widehat{C}(\Delta)$ in $\mathbf{H}_0$. This raises the question of realizability, that is, given a diagonal $(\delta, \Delta)\colon \widehat{C}(X) \to \widehat{C}(X) \otimes_{\mathbb{Z}} \widehat{C}(X)$ in $\mathbf{H}_0$, is there a diagonal Δ in $\mathbf{CW}_0$ with $\widehat{C}(\Delta) = (\delta, \Delta)$? As $\widehat{C}(\Delta)$ is homotopy associative and homotopy commutative for any diagonal Δ in $\mathbf{CW}_0$, homotopy associativity and homotopy commutativity of (δ, Δ) are necessary conditions for realizability.

To discuss questions of realizability for a functor $\lambda\colon \mathbf{A} \to \mathbf{B}$, we consider pairs (A, b), where $b\colon \lambda A \cong B$ is an equivalence in $\mathbf{B}$. Two such pairs are equivalent, written $(A, b) \sim (A', b')$, if and only if there is an equivalence $g\colon A' \cong A$ in $\mathbf{A}$ with $\lambda g = b^{-1}b'$. The classes of this equivalence relation form the classes of λ–realizations of B:

$$\mathrm{Real}_\lambda(B) = \{(A, b) \mid b\colon \lambda A \cong B\}/\sim. \tag{1–3}$$

We say that B is λ–realizable if $\mathrm{Real}_\lambda(B)$ is nonempty. The functor $\lambda\colon \mathbf{A} \to \mathbf{B}$ is *representative* if all objects B in $\mathbf{B}$ are λ–realizable. Further, we say that λ *reflects isomorphisms* if a morphism f in $\mathbf{A}$ is an equivalence whenever $\lambda(f)$ is an equivalence in $\mathbf{B}$. The functor λ is *full* if, for every morphism $\overline{f}\colon \lambda(A) \to \lambda(A')$ in $\mathbf{B}$, there is a morphism $f\colon A \to A'$ in $\mathbf{A}$, such that $\lambda(f) = \overline{f}$. We then say $\overline{f}$ is *λ–realizable*.

2 PD–chain complexes and PD–complexes

We begin with a description of the cap product on chain complexes. We fix a homomorphism $\omega\colon \pi \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ which gives rise to the anti-isomorphism of rings, $\bar{\ }\colon \Lambda \to \Lambda$, defined by $\overline{g} = (-1)^{\omega(g)} g^{-1}$ for $g \in \pi$. With the left Λ–module M we associate the right Λ–module M^ω having the same underlying abelian group and action given by $\lambda.m = m.\bar{\lambda}$ for $m \in M$ and $\lambda \in \Lambda$. Proceeding analogously for a right Λ–module N, we obtain a left Λ–module ${}^\omega N$. We put

$$\mathrm{H}_n(C, M^\omega) = \mathrm{H}_n(M^\omega \otimes_\Lambda C), \quad \mathrm{H}^k(C, M) = \mathrm{H}_{-k}(\mathrm{Hom}_\Lambda(C, M)).$$

To define the ω–twisted cap product $\cap$ for a chain complex C in $\mathbf{H}_0$ with diagonal (δ, Δ), write $\Delta(c) = \sum_{i+j=n,\alpha} c'_{i,\alpha} \otimes c''_{j,\alpha}$ for $c \in C$. Then

$$\begin{aligned} \cap\colon \mathrm{Hom}_\Lambda(C, M)_{-k} \otimes_{\mathbb{Z}} (\mathbb{Z}^\omega \otimes_\Lambda C)_n &\to (M^\omega \otimes_\Lambda C)_{n-k} \\ \psi \otimes (z \otimes c) &\mapsto \sum_\alpha z\psi(c'_{k,\alpha}) \otimes c''_{n-k,\alpha} \end{aligned}$$

for every left Λ–module M. Passing to homology and composing with

$$\begin{gathered} \mathrm{H}^*(C, M) \otimes_{\mathbb{Z}} \mathrm{H}_*(C \otimes_{\mathbb{Z}} C, \mathbb{Z}^\omega) \to \mathrm{H}_*(\mathrm{Hom}_\Lambda(C, M)\big) \otimes_{\mathbb{Z}} \big(\mathbb{Z}^\omega \otimes_\Lambda (C \otimes_{\mathbb{Z}} C))), \\ [\psi] \otimes [y] \mapsto [\psi \otimes y], \end{gathered}$$

we obtain

$$\cap\colon \mathrm{H}^k(C, M) \otimes_{\mathbb{Z}} \mathrm{H}_n(C, \mathbb{Z}^\omega) \to \mathrm{H}_{n-k}(C, M^\omega). \tag{2–1}$$

A PD^{n}*–chain complex* $C = ((\pi, C), \omega, [C], \Delta)$ consists of a free chain complex (π, C) in $\mathbf{H}_0$ with π finitely presented and $\mathrm{H}_1 C = 0$, a group homomorphism $\omega\colon \pi \to \mathbb{Z}/2\mathbb{Z}$, a fundamental class $[C] \in \mathrm{H}_n(C, \mathbb{Z}^\omega)$ and a diagonal $\Delta\colon C \to C \otimes C$ in $\mathbf{H}_0$, such that

$$\cap [C]\colon \mathrm{H}^r(C, M) \to \mathrm{H}_{n-r}(C, M^\omega); \quad \alpha \mapsto \alpha \cap [C] \tag{2–2}$$

is an isomorphism of abelian groups for every $r \in \mathbb{Z}$ and every left Λ–module M. A morphism of PD^{n}–chain complexes $f\colon ((\pi, C), \omega, [C], \Delta) \to ((\pi', C'), \omega', [C'], \Delta')$ is a morphism $(\varphi, f)\colon (\pi, C) \to (\pi', C')$ in $\mathbf{H}_0$ such that $\omega = \omega'\varphi$ and $(f \otimes f)\Delta \simeq \Delta' f$. The category $\mathbf{PD}^n_*$ is the category of PD^{n}–chain complexes and morphisms between them. Homotopies in $\mathbf{PD}^n_*$ are reduced chain homotopies. The subcategory $\mathbf{PD}^n_{*+}$ of $\mathbf{PD}^n_*$ is the category consisting of PD^n–chain complexes and *oriented* or *degree* 1 morphisms of PD^n–chain complexes, that is, morphisms $f\colon C \to D$ with $f_*[C] = [D]$.

Wall [20] showed that it is enough to demand that (2–2) be an isomorphism for $M = \Lambda$. If $1 \otimes x \in \mathbb{Z}^\omega \otimes_\Lambda C_n$ represents the fundamental class $[C]$, where C_i is finitely generated for $i \in \mathbb{Z}$, then $\cap [C]$ in (2–2) is an isomorphism if and only if

$$\cap 1 \otimes x\colon C^* = {}^\omega\mathrm{Hom}_\Lambda(C, {}^\omega\Lambda) \to \Lambda \otimes_\Lambda C = C \tag{2–3}$$

is a homotopy equivalence of chain complexes of degree n. Here finite generation implies that C^* is a free chain complex.

Lemma 2.1 *Every* PD^n*–chain complex is homotopy equivalent in* $\mathbf{PD}^n_*$ *to a* 2*–realizable* PD^n*–chain complex.*

Proof This follows from Theorem III 2.9, Proposition III 2.13 and Theorem III 2.12 in [1]. □

A PD^{n}*–complex* $X = (X, \omega, [X], \Delta)$ consists of an object X in $\mathbf{CW}_0$ with finitely presented fundamental group $\pi_1(X)$, a group homomorphism $\omega\colon \pi_1 X \to \mathbb{Z}/2\mathbb{Z}$, a fundamental class $[X] \in \mathrm{H}_n(X, \mathbb{Z}^\omega)$ and a diagonal $\Delta\colon X \to X \times X$ in $\mathbf{CW}_0$, such that $(\widehat{C}X, \omega, [X], \widehat{C}\Delta)$ is a PD^{n}–chain complex. A morphism of PD^{n}–complexes $f\colon (X, \omega, [X], \Delta) \to (X', \omega', [X'], \Delta')$ is a morphism $f\colon X \to X'$ in $\mathbf{CW}_0$ such that $\omega = \omega'\pi_1(f)$. The category $\mathbf{PD}^n$ is the category of PD^{n}–complexes and morphisms between them. Homotopies in $\mathbf{PD}^n$ are homotopies in $\mathbf{CW}_0$. The subcategory $\mathbf{PD}^n_+$ of $\mathbf{PD}^n$ is the category consisting of PD^n–complexes and *oriented* or *degree* 1 morphisms of PD^n–complexes, that is, morphisms $f\colon X \to Y$ with $f_*[X] = [Y]$.

Remark 2.2 Our PD^n–complexes have finitely presented fundamental groups by definition and are thus finitely dominated by Proposition 1.1 in [21].

Let X be a PD^n–complex with $n \geq 3$. We say that X is *standard*, if X is an n–dimensional CW–complex with exactly one n–cell e^n. We say that X is *weakly standard*, if X has a subcomplex X' with $X = X' \cup e^n$, where X' is n–dimensional and satisfies $\mathrm{H}^n(X', B) = 0$ for all coefficient modules B. In this sense X' is homologically $(n-1)$–dimensional. Of course standard implies weakly standard with $X' = X^{n-1}$.

Remark Every compact connected manifold M of dimension n has the homotopy type of a finite standard PD^{n}–complex.

Remark 2.3 Wall's Theorem 2.4 in [20] and Theorem E in [19] imply that, for $n \geq 4$, every PD^n–complex is homotopy equivalent to a standard PD^n–complex and, for $n = 3$, every PD^3–complex is homotopy equivalent to a weakly standard PD^3–complex.

Let C be a PD^n–chain complex with $n \geq 3$. We say that C is *standard*, if C is 2–realizable, $C_i = 0$ for $i > n$, and $C_n = \Lambda[e_n]$, where $[e_n] \in C_n$. We say that C is *weakly standard*, if C is 2–realizable and has a subcomplex C' with $C = C' \oplus \Lambda[e_n]$, where C' is n–dimensional and satisfies $\mathrm{H}^n(C', B) = 0$ for all coefficient modules B.

Remark 2.4 A PD^n–complex, X, is homotopy equivalent to a finite standard, standard or weakly standard PD^n–complex if and only if the PD^n–chain complex $\widehat{C}X$ is homotopy equivalent to a finite standard, standard or weakly standard PD^n–chain complex, respectively.

3 Fundamental triples

Homotopy types of 3–manifolds and PD^3–complexes were considered by Thomas [17], Swarup [15] and Hendriks [9]. In particular, Hendriks and Swarup provided a criterion for the existence of degree 1 maps between 3–manifolds and PD^3–complexes, respectively. In this section we generalize these results to manifolds and Poincaré duality complexes of arbitrary dimension.

Let k–**types** be the full subcategory of $\mathbf{CW}_0/\simeq$ consisting of CW–complexes X in $\mathbf{CW}_0$ with $\pi_i(X)=0$ for $i>k$. We define the k–th *Postnikov functor*

$$P_k\colon \mathbf{CW}_0 \to k\text{–}\mathbf{types}.$$

For X in $\mathbf{CW}_0$ we obtain P_kX by "killing homotopy groups", that is, we choose a CW–complex P_kX with $(k+1)$–skeleton $(P_kX)^{k+1}=X^{k+1}$ and $\pi_i(P_kX)=0$ for $i>k$. For a morphism $f\colon X\to Y$ in $\mathbf{CW}_0$ we may choose a map $Pf\colon P_kX\to P_kY$ which extends the restriction $f^{k+1}\colon X^{k+1}\to Y^{k+1}$ as $\pi_i(P_kY)=0$ for $i>k$. Then the functor P_k assigns P_kX to X and the homotopy class of Pf to f. Different choices of P_kX yield canonically isomorphic functors P_k. The CW–complex $P_1X=K(\pi_1X,1)$ is an Eilenberg–Mac Lane space and, as a functor, P_1 is equivalent to the fundamental group functor π_1. There are natural maps

$$p_k\colon X \longrightarrow P_kX \tag{3–1}$$

in $\mathbf{CW}_0/\simeq$ extending the inclusion $X^{k+1}\subseteq P_kX$.

For $n\geq 3$, a *fundamental triple* $T=(X,\omega,t)$ *of formal dimension* n consists of an $(n-2)$–type X, a homomorphism $\omega\colon \pi_1X\to \mathbb{Z}/2\mathbb{Z}$ and an element $t\in \mathrm{H}_n(X,\mathbb{Z}^{\omega})$. A morphism $(X,\omega_X,t_X)\to(Y,\omega_Y,t_Y)$ between fundamental triples is a homotopy class $\{f\}\colon X\to Y$ of maps of the $(n-2)$–types, such that $\omega_X=\omega_Y\pi_1(f)$ and $f_*(t_X)=t_Y$. We obtain the category $\mathbf{Trp}^n$ of fundamental triples T of formal dimension n and the functor

$$\tau\colon \mathbf{PD}^n_+/\simeq \longrightarrow \mathbf{Trp}^n, \quad X\longmapsto (P_{n-2}X,\omega_X,p_{n-2*}[X]).$$

Every degree 1 morphism $Y\to X$ in $\mathbf{PD}^n_+$ induces a surjection $\pi_1Y\to\pi_1X$ on fundamental groups (see for example Browder [3]) and hence we introduce the subcategory $\mathbf{Trp}^n_+\subset\mathbf{Trp}^n$ consisting of all morphisms inducing surjections on fundamental groups. Then the functor τ yields the functor

$$\tau_+\colon \mathbf{PD}^n_+/\simeq \longrightarrow \mathbf{Trp}^n_+. \tag{3–2}$$

As a main result in this section we show:

Theorem 3.1 *The functor τ_+ reflects isomorphisms and is full for $n\geq 3$.*

As corollaries we mention:

Corollary 3.2 *Take $n \geq 3$. Two n–dimensional manifolds, respectively two* PD^n*–complexes, are orientedly homotopy equivalent if and only if their fundamental triples are isomorphic.*

Remark For $n = 3$, Corollary 3.2 yields the results by Thomas [17], Swarup [15] and Hendriks [9]. Turaev reproves Hendriks' result in the appendix of [18], although the proof needs further explanation. We reprove the result in a more algebraic way.

Remark Turaev conjectures in [18] that his proof for $n = 3$ has a generalization to PD^n–complexes whose $(n-2)$–type is an Eilenberg–Mac Lane space $K(\pi, 1)$. Corollary 3.2 proves this conjecture.

Take PD^n–complexes X and Y and a diagram:

(3–3)
$$\begin{tikzcd} Y \arrow[r, "p_{n-2}"] \arrow[d, dotted, "\bar{f}"'] & P_{n-2}Y \arrow[d, "f"] \\ X \arrow[r, "p_{n-2}"] & P_{n-2}X. \end{tikzcd}$$

Corollary 3.3 *For $n \geq 3$, there is a degree 1 map $\bar{f}$ rendering Diagram (3–3) homotopy commutative if and only if f induces a surjection on fundamental groups, is compatible with the orientations ω_X and ω_Y, that is, $\omega_X \pi_1(f) = \omega_Y$, and*

$$f_* p_{n-2*}[Y] = p_{n-2*}[X].$$

Remark Swarup [15] and Hendriks [9] prove Corollary 3.3 for 3–manifolds and PD^3–complexes, respectively.

Remark For a homotopy equivalence f between oriented PD^4–complexes, the map $\bar{f}$ corresponds to the map h in Hambleton and Kreck [6, Lemma 1.3]. The reader is invited to compare our proof with that of [6, Lemma 1.3] which shows the existence of h but not the fact that h is of degree 1.

By Remark 2.3, Theorem 3.1 is a consequence of Lemma 3.4 and Lemma 3.5 below.

Lemma 3.4 *The functor τ_+ reflects isomorphisms.*

Proof This is a consequence of Poincaré duality and Whitehead's Theorem. □

Remark For $n \geq 3$, let $[n/2]$ be the integer part of $n/2$. Associating with a PD^n–complex, X, the *pre-fundamental triple* $(P_{[n/2]}X, \omega_X, p_{[n/2]*}[X])$, there is an analogue of Lemma 3.4, namely, an orientation preserving map between PD^n–complexes is a homotopy equivalence if and only if the induced map between pre-fundamental triples is an isomorphism. However, pre-fundamental triples do not determine the homotopy type of a PD^n–complex as in Corollary 3.2, as is demonstrated by the fake products $X = (S^n \vee S^n) \cup_\alpha e^{2n}$, where α is the sum of the Whitehead product $[\iota_1, \iota_2]$ and an element $\iota_1\beta$ with $\beta \in \pi_{2n-1}(S^n)$ having trivial Hopf invariant. Pre-fundamental triples coincide with the fundamental triple for $n = 3$ and $n = 4$. It remains an open problem to enrich the structure of a pre-fundamental triple to obtain an analogue of Corollary 3.2.

Lemma 3.5 *Let X and Y be standard PD^n–complexes for $n \geq 4$ and weakly standard for $n = 3$ and let $f\colon \tau_+Y \to \tau_+X$ be a morphism in $\mathbf{Trp}^n_+$. Then f is τ_+–realizable by a map $\bar{f}\colon Y \to X$ in PD^n_+ with $\tau_+\bar{f} = f$.*

For the proof of Lemma 3.5, we use:

Lemma 3.6 *Let $X = X' \cup e^n$ be a weakly standard PD^n–complex. Then $\widehat{C}_n(X)$ has a generator $[e]$, corresponding to the cell e^n, such that $\widehat{C}_nX = \widehat{C}_nX' \oplus \Lambda[e]$ and that the cycle $1 \otimes [e] \in \mathbb{Z}^\omega \otimes_\Lambda \widehat{C}_nX$ represents the fundamental class $[X]$. Let $\{e_m\}_{m\in M}$ be a basis of $\widehat{C}_{n-1}X = \widehat{C}_{n-1}X'$. Then the coefficients $\{a_m\}_{m\in M}, a_m \in \Lambda$ for $m \in M$, of the linear combination $d_n[e] = \sum a_m[e_m]$, generate $\overline{I(\pi_1X)}$ as a right Λ–module, where $I(\pi)$ denotes the augmentation ideal* $\ker(\mathrm{aug}\colon \Lambda \to \mathbb{Z})$.

Proof Poincaré duality implies $\mathrm{H}_n(X, \mathbb{Z}^\omega) \cong H^0(X, \mathbb{Z}) \cong \mathbb{Z}$. Hence $1 \otimes d$ maps a multiple of the generator $1 \otimes [e]$ of $\mathbb{Z}^\omega \otimes_\Lambda \widehat{C}_n(X) = \mathbb{Z}^\omega \otimes_\Lambda \Lambda[e] \cong \mathbb{Z}$ to zero, that is, there is an $\ell \in \mathbb{N}$ such that

$$\begin{aligned} 0 = 1 \otimes d(\ell(1 \otimes [e])) = \ell(1 \otimes d[e]) &= \ell(1 \otimes \sum_{m\in M} a_m[e_m]) \\ = \ell \sum 1.a_m \otimes [e_m] = \ell \sum_{m\in M} \mathrm{aug}(\overline{a_m}) \otimes [e_m]. \end{aligned}$$

Since $\mathbb{Z}^\omega \otimes_\Lambda \widehat{C}_{n-1}(X) = \mathbb{Z}^\omega \otimes_\Lambda \bigoplus_{m\in M} \Lambda[e_m] \cong \bigoplus_{m\in M} \mathbb{Z}^\omega \otimes_\Lambda \Lambda[e_m] = \bigoplus_{m\in M} \mathbb{Z}$ is free as abelian group, $\mathrm{aug}(\overline{a_m}) = 0$ and hence $\overline{a_m} \in I(\pi_1X)$ for every $m \in M$. Therefore $1 \otimes d(1 \otimes [e]) = 0$ and $1 \otimes [e] \in \mathbb{Z}^\omega \otimes_\Lambda \widehat{C}_n(X)$ is a cycle representing a generator of the group $\mathrm{H}_n(X, \mathbb{Z}^\omega)$. We may assume, without loss of generality, that the orientation of e is such that $1 \otimes e$ represents the fundamental class $[X]$. Further, Poincaré duality implies that $\mathrm{H}^n(X, {}^{\omega}\Lambda) \cong \mathbb{Z}$ and hence $I(\pi_1 X) \cong \mathrm{im}(d^*)[e]^*$, where $[e]^*\colon \Lambda[e] \to \Lambda, [e] \mapsto 1$. But, for every $\varphi \in {}^{\omega}\mathrm{Hom}_{\Lambda}(\widehat{C}_{n-1}(X), {}^{\omega}\Lambda)$,

$$\big(d^*\varphi\big)[e] = \varphi(d[e]) = \varphi\big(\sum a_m[e_m]\big) = \sum a_m \varphi[e_m] = \big(\sum \overline{\varphi[e_m]}\,\overline{a_m}[e]^*\big)[e],$$

and hence $I(\pi_1 X)$ is generated by $\{\overline{a_m}\}_{m \in M}$ as a left Λ–module. Thus $\overline{I(\pi_1 X)}$ is generated by $\{a_m\}_{m \in M}$ as a right Λ–module. □

Lemma 3.7 *Let $\bar{X} = X' \cup_f e^3$ be a weakly standard* PD^3*–complex. Then we can choose a homotopy $f \simeq g$ so that $X = X' \cup_g e^3$ admits a splitting, $\widehat{C}_2 X = S \oplus d_3(\widehat{C}_3 X')$, as a direct sum of Λ–modules satisfying $d_3[e] \in S$.*

Proof As X' is homologically 2–dimensional, $\widehat{C}(\bar{X})$ admits a splitting,

$$\widehat{C}_2(\bar{X}) = \mathrm{im}\, d_3' \oplus S,$$

as direct sum of Λ–modules, where $d_3'\colon \widehat{C}_3(X') \to \widehat{C}_2(X')$. Thus $d_3[e] \in \widehat{C}_2(\bar{X}) = \mathrm{im}\, d_3' \oplus S$ decomposes as a sum $d_3[e] = \alpha + \beta$, with $\alpha \in \mathrm{im}\, d_3'$ and $\beta \in S$. Since α, viewed as a map $S^2 \to X'$, is homotopically trivial in X', there is a homotopy $f \simeq g$, where g represents β, such that $X = X' \cup_g e^3$ has the stated properties. □

We turn to proving Lemma 3.5. Certain aspects of the proof for the case $n = 3$ differ from that for the case $n \geq 4$. Those parts of the proof pertaining to the case $n = 3$ appear in square brackets [...]. [For $n = 3$ we assume that $X = X' \cup_g e^3$ is chosen as in Lemma 3.7.]

Proof of Lemma 3.5 Given $X = X' \cup_g e^n$ and $Y = Y' \cup_{g'} e'^n$ and a morphism $\varphi = \{f\}\colon \tau(Y) = (Q, \omega_Y, t_Y) \to \tau(X) = (P, \omega_X, t_X)$ in $\mathbf{Trp}^n_+$, the diagram

$$\begin{array}{ccc} X^{n-1} \subseteq X' \subset X & \xrightarrow{p} & P = P_{n-2}X \\ \uparrow{\scriptstyle \bar{\eta}} & & \uparrow{\scriptstyle f} \\ Y^{n-1} \subseteq Y' \subset Y & \xrightarrow{p'} & Q = P_{n-2}Y, \end{array}$$

commutes in $\mathbf{CW}_0$, where p and p' coincide with the identity morphisms on the $(n-1)$–skeleta, and where $\bar{\eta}$ is the restriction of f. For $n \geq 4$, we have $X' = X^{n-1}$ and $Y' = Y^{n-1}$. We obtain the following commutative diagram of chain complexes in $\mathbf{H}_0$:

$$\begin{array}{ccc} \widehat{C} X^{n-1} \subset \widehat{C} X & \xrightarrow{p_*} & \widehat{C} P \\ \uparrow{\scriptstyle \bar{\eta}_*} & & \uparrow{\scriptstyle f_*} \\ \widehat{C} Y^{n-1} \subset \widehat{C} Y & \xrightarrow{p'_*} & \widehat{C} Q. \end{array}$$

For $n \geq 4$, we construct a morphism $(\xi, \eta)\colon r(Y) \to r(X)$ in the category $\mathbf{H}^c_{n-1}$ of homotopy systems of order $(n-1)$ (see Section 8), rendering the diagram

(3–4)
$$\begin{tikzcd} r(X) \arrow[r, "r(p)"] & r(P) \\ r(Y) \arrow[u, "(\xi,\eta)"] \arrow[r, "r(p')"] & r(Q) \arrow[u, "r(f)"'] \end{tikzcd}$$

homotopy commutative in $\mathbf{H}^c_{n-1}$. Here $\xi\colon \widehat{C}Y \to \widehat{C}X$ and $\eta\colon Y^{n-2} \to X^{n-2}$ is the restriction of $\overline{\eta}$ above.

[For $n = 3$, the map $\overline{\eta}$ itself need not extend to a map $Y' \to X'$. But, since Y' is homologically 2–dimensional, there is a map $\eta'\colon Y' \to X'$ inducing $\pi_1\eta' = \pi_1\varphi$. Since we may assume that Q is obtained from Y by attaching cells of dimension ≥ 3, we can choose f representing φ with $p\eta' = fp'$.]

We write $\pi = \pi_1 X, \pi' = \pi_1 Y, \Lambda = \mathbb{Z}[\pi]$ and $\Lambda' = \mathbb{Z}[\pi']$ and let $[e'] \in \widehat{C}_n Y$ and $[e] \in \widehat{C}_n X$ be the elements corresponding to the n–cells e_n and e'_n, respectively, $n \geq 3$. Since $\{f\}$ is a morphism in $\mathbf{Trp}^n_+$, we obtain $f_* p'_*[Y] = p_*[X]$ in $\mathrm{H}_n(P, \mathbb{Z}^\omega)$ and hence
$$f_* p'_*[e'] - p_*[e] \in \operatorname{im}(d\colon \widehat{C}_{n+1} P \to \widehat{C}_n P) + \overline{I(\pi)}\widehat{C}_n P.$$
Thus there are elements $x \in \widehat{C}_{n+1} P$ and $y \in \overline{I(\pi)}\widehat{C}_n P$ with

$$f_* p'_*[e'] - p_*[e] = dx + y. \tag{3–5}$$

Let $\{e'_m\}_{m\in M}$ be a basis of $\widehat{C}_{n-1} Y$. By Lemma 3.6,

$$d[e'] = \sum a_m[e'_m], \tag{3–6}$$

for some $a_m \in \Lambda', m \in M$, where $\{a_m\}_{m\in M}$ generate $\overline{I(\pi')}$ as right Λ'–module. Since $\varphi = \pi_1(f)$ is surjective, $\overline{I(\pi)}$ is generated by $\{\varphi(a_m)\}_{m\in M}$ as right Λ–module, and we may write

$$y = \sum_{m\in M} \varphi(a_m) z_m, \tag{3–7}$$

for some $z_m \in \widehat{C}_n P, m \in M$, since there is a surjection $\bigoplus_{m\in M} \Lambda[m] \twoheadrightarrow \overline{I(\pi)}$ of right Λ–modules which maps the generator $[m]$ to $\varphi(a_m)$. Then (3–5) implies that $d(f_* p'_*[e'] - p_*[e]) = dy = \sum_{m\in M} \varphi(a_m) dz_m$, whence

$$p_* d[e] = \sum_{m\in M} \varphi(a_m) f_* p'_*[e'_m] - \sum_{m\in M} \varphi(a_m) dz_m. \tag{3–8}$$

We define the φ–equivariant homomorphism

$$\bar{\alpha}_n\colon \hat{C}_{n-1}Y \to \hat{C}_n P \quad \text{by} \quad \bar{\alpha}_n([e'_m]) = -z_m. \tag{3–9}$$

For $n \geq 4$, we define $\xi\colon \hat{C}Y \to \hat{C}X$ by $\xi[e'] = [e]$ and

$$\xi_i = \begin{cases} \hat{C}_{n-1}(\bar{\eta}) + d\bar{\alpha}_n & \text{for } i = n-1, \\ \hat{C}_i(\bar{\eta}) & \text{for } i < n-1. \end{cases} \tag{3–10}$$

[For $n = 3$ we use the splitting $\hat{C}_2 Y = S \oplus d_3\hat{C}_3 Y'$ in Lemma 3.7 and define $\xi_i\colon \hat{C}_i Y \to \hat{C}_i X$ by $\xi_3[e'] = [e], \xi_3|\hat{C}_3 Y' = \hat{C}_3\eta'$, and

$$\begin{aligned} &\xi_2|S = (\hat{C}_2\eta' + d\bar{\alpha}_3)|S, \\ &\xi_2|d_3\hat{C}_3Y' = \hat{C}_2\eta'|d_3\hat{C}_3Y', \\ &\xi_i = \hat{C}_i\eta \quad \text{for } i < 2.] \end{aligned}$$

To ensure that ξ is a chain map, it is now enough to show that $d\xi[e'] = \xi d[e']$. But, for the injection $\hat{C}(p)$, we obtain

$$\begin{aligned} \hat{C}_{n-1}(p)\xi d[e'] &= \hat{C}_{n-1}(p)(\hat{C}_{n-1}(\bar{\eta}) + d\bar{\alpha}_n)d[e'] \\ &= \hat{C}_{n-1}(p\circ\bar{\eta})d[e'] + \hat{C}_{n-1}(p)\big(d\bar{\alpha}_n\big(\sum_{m\in M} a_m[e'_m]\big)\big) \\ &= \hat{C}_{n-1}(f\circ p')d[e'] + \hat{C}_{n-1}(p)\sum_{m\in M}\varphi(a_m)d\bar{\alpha}_n[e'_m] \\ &= \sum_{m\in M}\varphi(a_m)\hat{C}_{n-1}(f\circ p')[e'_m] - \hat{C}_{n-1}(p)\sum_{m\in M}\varphi(a_m)dz_m \\ &= \hat{C}_{n-1}(p)d[e] = \hat{C}_{n-1}(p)d\xi[e'], \quad \text{by (3–8)}. \end{aligned}$$

[For $n = 3$, Theorem 4.3 now implies that there is a map $\bar{f}\colon Y \to X$ such that $\hat{C}(\bar{f}) = \xi$. Then $\tau(\bar{f}) = f$, $\bar{f}$ is a degree 1 map and the proof is complete for $n = 3$.]

Now let $n \geq 4$. To check that (ξ, η) is a morphism in $\mathbf{H}^c_{n-1}$, note that the attaching map satisfies the cocycle condition and hence, by its definition, the map ξ_{n-1} commutes with attaching maps in $r(X)$ and $r(Y)$, since $\hat{C}_{n-1}\bar{\eta}$ has this property. We must show that Diagram (3–4) is homotopy commutative. But $r(f) = (f_*, \eta)$ and $r(p) = (p_*, j), r(p') = (p'_*, j')$, where j and j' are the identity morphisms on $X^{n-2} = P^{n-2}$ and $Y^{n-2} = Q^{n-2}$, respectively. Hence we must find a homotopy $\alpha\colon (p_*\xi, \eta) \simeq (f_* p'_*, \eta)$ in $\mathbf{H}^c_{n-1}$, that is, φ–equivariant maps

$$\alpha_{i+1}\colon \hat{C}_i Y \to \hat{C}_{i+1} P, \ i \geq n-1,$$

such that

$$\{\eta\} + g_{n-1}\alpha_{n-1} = \{\eta\}, \tag{3–11}$$

$$(p_*\xi)_i - (f \circ p')_i = \alpha_i d + d\alpha_{i+1} \quad \text{for} \quad i \geq n-1, \tag{3–12}$$

where g_{n-1} is the attaching map of $(n-1)$–cells in P. Define α by $\alpha_{n+1}[e'] = -x$ (see (3–5)) and

$$\alpha_i = \begin{cases} \overline{\alpha}_n & \text{for } i = n, \\ 0 & \text{for } i < n. \end{cases} \tag{3–13}$$

Then α satisfies (3–11) trivially. For $i = n-1$, we obtain

$$\begin{aligned} (p_*\xi)_{n-1} - (f \circ p')_{n-1} &= \xi_{n-1} - \widehat{C}_{n-1}(f) \\ &= \xi_{n-1} - \widehat{C}_{n-1}(\overline{\eta}) \\ &= d\alpha_n, \quad \text{by (3–10) and (3–13)}. \end{aligned}$$

For $i = n$, we evaluate (3–12) on $[e']$. By (3–5),

$$(p_*\xi - f_* p'_*)[e'] = p_*[e] - f_* p'_*[e'] = -dx - y.$$

On the other hand,

$$\begin{aligned} (d\alpha_{n+1} + \alpha_n d)[e'] &= d\alpha_{n+1}[e'] + \alpha_n \sum_{m\in M} a_m[e'_m], && \text{by (3–6)}, \\ &= -dx - \sum_{m\in M} \varphi(a_m) z_m, && \text{by (3–13) and (3–9)}, \\ &= -dx - y && \text{by (3–7)}. \end{aligned}$$

Hence α satisfies (3–12) and Diagram (3–4) is homotopy commutative.

To construct a morphism $\overline{f}\colon Y \to X$ in $\mathbf{PD}^n_+$ with $\tau(\overline{f}) = f$, consider the obstruction $\mathcal{O}(\xi,\eta) \in \mathrm{H}^n(Y, \Gamma_{n-1}X)$ (see Section 8) and note that p induces an isomorphism $p_*\colon \Gamma_{n-1}X \to \Gamma_{n-1}P$ (see Baues [1, II.4.8]). Hence the obstruction for the composite $r(p)(\xi,\eta)$ coincides with $p_*\mathcal{O}(\xi,\eta)$, where p_* is an isomorphism. On the other hand, the obstruction for $r(f)r(p')$ vanishes, since this map is λ–realizable. Thus, by the homotopy commutativity of (3–4), $p_*\mathcal{O}(\xi,\eta) = \mathcal{O}(r(f)r(p')) = 0$, so that $\mathcal{O}(\xi,\eta) = 0$ and there is a λ–realization $(\xi, \widetilde{\eta}')$ of (ξ,η) in $\mathbf{H}^c_n$. Since $\mathrm{H}^{n+1}(Y, \Gamma_n X) = 0$, there is a λ–realization $(\xi, \overline{f})$ of $(\xi, \widetilde{\eta}')$ in $\mathbf{H}^c_{n+1}$. As $Y = Y^n$, $X = X^n$ and ξ is, by construction, compatible with fundamental classes, $\overline{f}\colon Y \to X$ is a degree 1 map in $\mathbf{PD}^n_+$ realizing the map f in $\mathbf{Trp}^n_+$. □

4 PD^3–complexes

The fundamental triple of a PD^3–complex consists of a group π, an orientation ω and an element $t \in \mathrm{H}_3(\pi, \mathbb{Z}^\omega)$. Here we use the fact that the homology of a group π coincides with the homology of the corresponding Eilenberg–Mac Lane space $K(\pi, 1)$. In general, it is a difficult problem to actually compute $\mathrm{H}_3(\pi, \mathbb{Z}^\omega)$. The homotopy type of a PD^3–complex is characterized by its fundamental triple, but not every fundamental triple occurs as the fundamental triple of a PD^3–complex. Turaev [18] uses the invariant $\nu_C(t)$ to characterize those fundamental triples which are realizable by a PD^3–complex. Let $\mathbf{Trp}^3_{+,\nu}$ be the full subcategory of $\mathbf{Trp}^3_+$ consisting of fundamental triples satisfying Turaev's realization condition. Then Theorem 3.1 implies:

Theorem 4.1 *The functor*

$$\tau_+\colon \mathbf{PD}^3_+/\simeq \to \mathbf{Trp}^3_{+,\nu}$$

reflects isomorphisms and is representative and full.

Remark Turaev does not mention that the functor τ_+ is actually full and thus only proves the first part of the following corollary, which is one of the consequences to Theorem 4.1.

Corollary 4.2 *The functor τ_+ yields a 1–1 correspondence between oriented homotopy types of PD^3–complexes and isomorphism types of fundamental triples satisfying Turaev's realization condition. Moreover, for every PD^3–complex X, there is a surjection of groups*

$$\tau_+\colon \mathrm{Aut}_+(X) \to \mathrm{Aut}(\tau(X)),$$

where $\mathrm{Aut}_+(X)$ is the group of oriented homotopy equivalences of X in $\mathbf{PD}^3_+/\simeq$ and $\mathrm{Aut}(\tau(X))$ is the group of automorphisms of the triple $\tau(X)$ in $\mathbf{Trp}^3_+$ which is a subgroup of $\mathrm{Aut}(\pi_1 X)$.

As every 3–manifold has the homotopy type of a finite standard PD^3–complex, the question arises which fundamental triples in $\mathbf{Trp}^3_+$ correspond to finite standard PD^3–complexes. While Turaev does not discuss this question, we use the concept of PD^3–chain complexes (see Section 2) in the category $\mathbf{PD}^3_*$ to do so.

Theorem 4.3 *The functor $\widehat{C}\colon \mathbf{PD}^3/\simeq \longrightarrow \mathbf{PD}^3_*/\simeq$ reflects isomorphisms and is representative and full.*

Proof This follows from Theorem 10.1 and Theorem 10.2 in Section 10. □

Corollary 4.4 *The functor $\widehat{C}$ yields a 1–1 correspondence between homotopy types of PD^3–complexes and homotopy types of PD^3–chain complexes. Moreover, for every PD^3–complex X there is a surjection of groups*

$$\widehat{C}\colon \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\widehat{C}(X)).$$

Remark 4.5 Corollary 4.4 implies that the diagonal of every PD^3–chain complex is, in fact, homotopy associative and homotopy commutative.

Connecting the functor $\widehat{C}$ and the functor τ_+, we obtain the diagram

$$\begin{tikzcd}
\mathbf{PD}^3_+/\simeq \arrow[rr, "\widehat{C}"] \arrow[dr, "\tau_+"'] & & \mathbf{PD}^3_{*+}/\simeq \arrow[dl, "\tau_*"] \\
& \mathbf{Trp}^3_{+,\nu} &
\end{tikzcd}$$

where τ_+ determines τ_* together with a natural isomorphism $\tau_*\widehat{C} \cong \tau_+$.

Corollary 4.6 *Each of the functors $\widehat{C}, \tau_+$ and τ_* reflects isomorphisms and is full and representative.*

By Remark 2.4, the functor $\widehat{C}$ yields a 1–1 correspondence between homotopy types of finite standard PD^3–complexes and finite standard PD^3–chain complexes, respectively.

5 Realizability of PD^4–chain complexes

Given a PD^4–chain complex C, we define an invariant $\mathcal{O}(C)$ which vanishes if and only if C is realizable by a PD^4–complex. To this end we recall the *quadratic functor* Γ (see also (4.1) on page 13 in [1]). A function $f\colon A \to B$ between abelian groups is called a *quadratic map* if $f(-a) = f(a)$, for $a \in A$, and if the function $A \times A \to B$, $(a,b) \mapsto f(a+b) - f(a) - f(b)$ is bilinear. There is a *universal quadratic map*

$$\gamma\colon A \to \Gamma(A),$$

such that for all quadratic maps $f\colon A \to B$ there exists a unique homomorphism $f^{\square}\colon \Gamma(A) \to B$ satisfying $f^{\square}\gamma = f$. Using γ, we obtain the *Whitehead product map*

$$\begin{aligned} P\colon A \otimes A &\longrightarrow \Gamma(A), \\ a \otimes b &\longmapsto [a,b] = \gamma(a+b) - \gamma(a) - \gamma(b). \end{aligned}$$

With the *exterior product* $\wedge^2 A$ of the abelian group A we obtain the natural exact sequence

$$\Gamma(A) \xrightarrow{H} A \otimes A \longrightarrow \wedge^2 A \longrightarrow 0, \tag{5–1}$$

where H maps $\gamma(a)$ to $a \otimes a$ for $a \in A$ (see also page 14 in [1]). The composite $PH\colon \Gamma(A) \to \Gamma(A)$ coincides with $2\mathrm{id}_{\Gamma(A)}$. In fact, PH maps $\gamma(a)$ to $[a,a] = 2\gamma(a)$. J H C Whitehead [23] introduced the functor $\Gamma_k, k \geq 3$, assigning to each CW–complex the image of the inclusion homomorphism for homotopy groups of skeleta, $\pi_k(X^{k-1}) \to \pi_k(X^k)$, and showed that there is a natural isomorphism $\Gamma_3(X) \cong \Gamma(\pi_2 X)$.

Theorem 5.1 *Let $C = ((\pi, C), \omega, [C], \Delta)$ be a* PD^4*–chain complex with homology module* $\mathrm{H}_2(C,\Lambda) = H_2$*. Then there is an invariant*

$$\mathcal{O}(C) \in \mathrm{H}_0(\pi, \wedge^2 H_2^{\omega})$$

with $\mathcal{O}(C) = 0$ if and only if there is a PD^4*–complex X such that $\widehat{C}(X)$ is isomorphic to C in* $\mathbf{PD}^4_*/\simeq$*. Moreover, if $\mathcal{O}(C) = 0$, the group*

$$\ker\big(H_*\colon \mathrm{H}_0(\pi, \Gamma(H_2^{\omega})) \longrightarrow \mathrm{H}_0(\pi, H_2^{\omega} \otimes H_2^{\omega})\big)$$

acts transitively and effectively on the set $\mathrm{Real}_{\widehat{C}}(C)$ *of realizations of C in* $\mathbf{PD}^4/\simeq$*. Here* $\ker H_*$ *is 2–torsion.*

Proof First note that

$$\mathrm{H}^4(C, \wedge^2 H_2) \cong \mathrm{H}_0(C, \wedge^2 H_2^{\omega}) \cong \mathrm{H}_0(\pi, \wedge^2 H_2^{\omega}). \tag{5–2}$$

By Lemma 2.1, we may assume that C is 2–realizable. By Remark 1.1 and Proposition 8.3, there is a 4–dimensional CW–complex X together with an isomorphism $\widehat{C}X \cong (\pi, C)$. The CW–complex X yields the homotopy systems $\bar{\bar{X}}$ in $\mathbf{H}^c_3$ and $\bar{X}$ in $\mathbf{H}^c_4$ with $\bar{X} = r(X)$ and $\bar{\bar{X}} = \lambda X$. By Theorem 10.1, we may choose a diagonal $\bar{\bar{\Delta}}\colon \bar{\bar{X}} \to \bar{\bar{X}} \otimes \bar{\bar{X}}$ inducing $\Delta\colon C \to C \otimes C$, whose homotopy class is determined by Δ. However, $\bar{\bar{\Delta}}$ need not be λ–realizable. Lemma 9.1 shows that there is an obstruction

$$\mathcal{O}' = \mathcal{O}_{\bar{X}, \bar{X}\otimes\bar{X}}(\bar{\bar{\Delta}}) \in \mathrm{H}^4(C, \Gamma_3(\bar{X} \otimes \bar{X})) \tag{5–3}$$

which vanishes if and only if there is a diagonal $\bar{\Delta}\colon \bar{X} \to \bar{X} \otimes \bar{X}$ realizing $\bar{\bar{\Delta}}$. Note that $\mathcal{O}'$ is determined by the diagonal Δ on C, since the obstruction only depends on the homotopy class of $\bar{\bar{\Delta}}$. By Theorem 10.2, the existence of $\bar{\Delta}$ realizing $\bar{\bar{\Delta}}$ also implies the existence of $\Delta_X\colon X \to X \times X$ realizing $\bar{\Delta}$. But

$$\begin{aligned}\Gamma_3(\bar{X} \otimes \bar{X}) &\cong \Gamma(\pi_2(\bar{X} \otimes \bar{X}))\\ &\cong \Gamma(\pi_2(X \times X))\\ &\cong \Gamma(\pi_2 \oplus \pi_2) \quad \text{where } \pi_2 = \pi_2 X.\end{aligned}$$

Applying Lemma 9.2 (1), we see that

$$\mathcal{O}' \in \ker p_{i*} \quad (i = 1, 2),$$

where $p_i\colon \pi_2 \oplus \pi_2 \to \pi_2$ is the i–th projection. Now

$$\Gamma(\pi_2 \oplus \pi_2) = \Gamma(\pi_2) \oplus \pi_2 \otimes \pi_2 \oplus \Gamma(\pi_2)$$

and hence $\mathcal{O}'$ yields $\mathcal{O}'' \in \mathrm{H}^4(C, \pi_2 \otimes \pi_2)$. While the homotopy type of $\bar{\bar{\bar{X}}}$ is determined by C, the homotopy type of $\bar{X}$ is an element of $\mathrm{Real}_\lambda(\bar{\bar{\bar{X}}})$ and the group $\mathrm{H}^4(C, \Gamma(\pi_2))$ acts transitively and effectively on this set of realizations. To describe the behaviour of the obstruction under this action using Lemma 9.3, we first consider the homomorphism

$$\nabla = \Delta_* - \iota_{1*} - \iota_{2*}\colon \Gamma(\pi_2) \longrightarrow \Gamma(\pi_2 \oplus \pi_2),$$

where $\Delta\colon \pi_2 \to \pi_2 \oplus \pi_2$ maps $x \in \pi_2$ to $\iota_1(x) + \iota_2(x)$, and $\iota_i\colon \pi_2 \to \pi_2 \oplus \pi_2$ denotes the i–th inclusion. For $x \in \pi_2$, we obtain

$$\begin{aligned}\nabla(\gamma(x)) &= \gamma(\iota_1(x) + \iota_2(x)) - \gamma(\iota_1(x)) - \gamma(\iota_2(x))\\ &= [\iota_1(x), \iota_2(x)]\\ &= x \otimes x \in \pi_2 \otimes \pi_2 \subset \Gamma(\pi_2 \oplus \pi_2),\end{aligned}$$

showing that ∇ coincides with $H\colon \Gamma(\pi_2) \to \pi_2 \otimes \pi_2$. Given $\alpha \in \mathrm{H}^4(C, \Gamma(\pi_2))$, the obstruction $\mathcal{O}''_\alpha = \mathcal{O}_{\bar{Y}, \bar{Y} \otimes \bar{Y}}(\bar{\bar{\Delta}})$ with $\bar{Y} = \bar{X} + \alpha$ satisfies

$$\mathcal{O}''_\alpha = \mathcal{O}'' + H_*\alpha,$$

by Lemma 9.3. The exact sequence

$$\mathrm{H}^4(C, \Gamma(\pi_2)) \longrightarrow \mathrm{H}^4(C, \pi_2 \otimes \pi_2) \longrightarrow \mathrm{H}^4(C, \wedge^2 \pi_2) \longrightarrow 0$$

allows us to identify the coset of im H_* represented by $\mathcal{O}''$ with an element

$$\mathcal{O} \in \mathrm{H}^4(C, \wedge^2 H_2),$$

where $H_2 = \mathrm{H}_2(C, \Lambda) \cong \pi_2$. By the isomorphisms (5–2), this element yields the invariant

$$\mathcal{O} \in \mathrm{H}_0(\pi, \wedge^2 H_2^\omega)$$

with the stated properties. Given that $\mathcal{O}''$ vanishes, the obstruction $\mathcal{O}''_\alpha$ vanishes if and only if $\alpha \in \ker H_*$, and Proposition 8.3 yields the result on $\mathrm{Real}_{\widehat{C}}(C)$. We observe that $\ker H_*$ is 2–torsion as $H_*(x) = 0$ implies $2x = P_* H_* x = 0$. □

Theorem 5.2 *Let $C = ((\pi, C), \omega, [C], \Delta)$ be a PD^4–chain complex for which Δ is homotopy commutative. Then the obstruction $\mathcal{O}(C)$ is 2–torsion, that is, $2\mathcal{O}(C) = 0$.*

Proof Lemma 9.2 (2) states

$$\mathcal{O}' \in \ker(\mathrm{id}_* - T_*)_*,$$

where id is the identity on $\pi_2 \oplus \pi_2$ and T is the interchange map on $\pi_2 \oplus \pi_2$ with $T\iota_1 = \iota_2$ and $T\iota_2 = \iota_1$. So T induces the map $-\mathrm{id}$ on $\wedge^2 \pi_2$ and the result follows. □

Remark Lemma 9.2 (3) concerning homotopy associativity of the diagonal does not yield a restriction of the invariant $\mathcal{O}(C)$.

Theorem 5.3 *The functor $\widehat{C}$ induces a 1–1 correspondence between homotopy types of PD^4–complexes with finite fundamental group of odd order and homotopy types of PD^4–chain complexes with homotopy commutative diagonal and finite fundamental group of odd order.*

Proof Since π is of odd order, the cohomology $\mathrm{H}^0(\pi, M)$ is odd torsion and the result follows from Theorem 5.1. □

Remark By Theorem 5.3, every PD^4–chain complex with homotopy commutative diagonal and odd fundamental group has a homotopy associative diagonal.

Up to 2–torsion, Theorem 5.1 yields a correspondence between homotopy types of PD^4–complexes and homotopy types of PD^4–chain complexes. In Section 7 we provide a precise condition for a PD^4–chain complex to be realizable by a PD^4–complex.

6 The chains of a 2–type

The fundamental triple of a PD^4–complex X comprises its 2–type $T = P_2 X$ and an element of the homology $\mathrm{H}_4(T, \mathbb{Z}^\omega)$. To compute $\mathrm{H}_4(T, \mathbb{Z}^\omega)$, we construct a chain complex $P(T)$ which approximates the chain complex $\widehat{C}(T)$ up to dimension 4. Our construction uses a presentation of the fundamental group as well as the concepts of *pre-crossed module* and *Peiffer commutator*. To introduce these concepts, we work with right group actions as in [1], and define $P(T)$ as a chain complex of right Λ–modules.

With any left Λ–module M we associate a right Λ–module in the usual way by setting $x.\alpha = \alpha^{-1}.x$, for $\alpha \in \pi$ and $x \in M$, and vice versa.

A *pre-crossed module* is a group homomorphism $\partial\colon \rho_2 \to \rho_1$ together with a right action of ρ_1 on ρ_2, such that

$$\partial(x^\alpha) = -\alpha + \partial x + \alpha \quad \text{for } x \in \rho_2, \alpha \in \rho_1,$$

where we use additive notation for the group law in ρ_1 and ρ_2, as in [1]. For $x, y \in \rho_2$, the *Peiffer commutator* is given by

$$\langle x, y\rangle = -x - y + x + y^{\partial x}.$$

A pre-crossed module is a *crossed module*, if all Peiffer commutators vanish. A *map of pre-crossed modules*, $(m,n)\colon \partial \to \partial'$ is given by a commutative diagram

$$\begin{tikzcd} \rho_2 \arrow[r, "m"] \arrow[d, "\partial"'] & \rho_2' \arrow[d, "\partial'"] \\ \rho_1 \arrow[r, "n"] & \rho_1' \end{tikzcd}$$

in the category of groups, where m is n–equivariant. Let **cross** be the category of crossed modules and such morphisms. A *weak equivalence* in **cross** is a map $(m,n)\colon \partial \to \partial'$, which induces isomorphisms $\mathrm{coker}\partial \cong \mathrm{coker}\partial'$ and $\ker\partial \cong \ker\partial'$, and we denote the localization of **cross** with respect to weak equivalences by $\mathbf{Ho}(\mathbf{cross})$. By an old result of Whitehead–Mac Lane, there is an equivalence of categories

$$\overline{\rho}\colon 2\text{–}\mathbf{types} \longrightarrow \mathbf{Ho}(\mathbf{cross})$$

(compare Theorem III 8.2 in [1]). The functor $\overline{\rho}$ carries a 2–type T to the crossed module $\partial\colon \pi_2(T, T^1) \to \pi_1(T^1)$.

A pre-crossed module is *totally free*, if $\rho_1 = \langle E_1\rangle$ is a free group generated by a set E_1 and $\rho_2 = \langle E_2 \times \rho_1\rangle$ is a free group generated by a free ρ_1–set $E_2 \times \rho_1$ with the obvious right action of ρ_1. A function $f\colon E_2 \to \langle E_1\rangle$ yields the *associated totally free pre-crossed module* $\partial_f\colon \rho_2 \to \rho_1$ with $\partial_f(x) = f(x)$ for $x \in E_2$. Let $\mathrm{Pei}_n(\partial_f) \subset \rho_2$ be the subgroup generated by n–fold Peiffer commutators and put $\overline{\rho}_2 = \rho_2/\mathrm{Pei}_2(\partial_f)$. Let $\mathbf{cross}^=$ be the category whose objects are pairs (∂_f, B), where ∂_f is a totally free pre-crossed module $\partial_f\colon \rho_2 \to \rho_1$ and B is a submodule of $\ker(\partial\colon \overline{\rho}_2 \to \rho_1)$. Further, a morphism $m\colon (\partial_f, B) \to (\partial_{f'}, B')$ in $\mathbf{cross}^=$ is a map $\partial_f \to \partial_{f'}$ which maps B into B'. Then there is a functor

$$q\colon \mathbf{cross}^= \longrightarrow \mathbf{cross} \longrightarrow \mathbf{Ho}(\mathbf{cross}),$$

which assigns to (∂_f, B) the crossed module $\overline{\rho}_2/B \to \rho_1$, and one can check that q is full and representative. Given any map $g\colon T \to T'$ between 2–types, we may choose a map $\overline{\overline{g}}\colon (\partial_f, B) \to (\partial_{f'}, B')$ in $\mathbf{cross}^=$ representing the homotopy class of g via the functor q and the equivalence $\overline{\rho}$. We call $\overline{\overline{g}}$ a *map associated with* g.

Given an action of the group π on the group M and a group homomorphism $\varphi\colon N \to \pi$, a *φ–crossed homomorphism* $h\colon N \to M$ is a function satisfying

$$h(x+y) = (h(x))^{\varphi(y)} + h(y) \quad \text{for } x, y \in N.$$

By an old result of Whitehead [22], the totally free crossed module $\overline{\rho}_2 \to \rho_1$ enjoys the following properties.

Lemma 6.1 *Let X^2 be a 2–dimensional* CW*–complex in* $\mathbf{CW}_0$ *with attaching map of 2–cells $f\colon E_2 \to \langle E_1 \rangle = \pi_1(X^1)$. Then there is a commutative diagram*

$$\begin{tikzcd} \pi_2(X^2, X^1) \arrow[r, "\partial"] \arrow[d, equal] & \pi_1(X^1) \arrow[d, equal] \\ \overline{\rho}_2 \arrow[r, "\partial_f"] & \rho_1 \end{tikzcd}$$

identifying ∂ with the totally free crossed module ∂_f. Moreover, the abelianization of $\overline{\rho}_2$ coincides with $\widehat{C}_2(X^2)$, identifying the kernel of ∂_f with the kernel of $d_2\colon \widehat{C}_2(X^2) \to \widehat{C}_1(X^2)$, and ∂_f determines the boundary d_2 via the commutative diagram

$$\begin{tikzcd} \overline{\rho}_2 \arrow[r, "\partial_f"] \arrow[d, "h_2"'] & \rho_1 \arrow[d, "h_1"] \\ \widehat{C}_2(X^2) \arrow[r, "d_2"] & \widehat{C}_1(X^2). \end{tikzcd}$$

Here h_2 is the quotient map and h_1 is the $(q\colon \rho_1 \to \pi_1(X^2))$–crossed homomorphism which is the identity on the generating set E_1. Each map $\partial_f \to \partial_{f'}$ induces a chain map $\widehat{C}_2(X^2) \to \widehat{C}_2(X'^2)$ where X^2 and X'^2 are the 2–dimensional CW*–complexes with attaching maps f and f', respectively.*

In addition to Lemma 6.1, we need the following result on Peiffer commutators, which was originally proved in IV (1.8) of [1] and generalized in a paper with Conduché [2].

Lemma 6.2 *With the notation in Lemma 6.1, there is a short exact sequence*

$$0 \longrightarrow \Gamma(K) \longrightarrow \widehat{C}_2(X^2) \otimes \widehat{C}_2(X^2) \xrightarrow{w} \mathrm{Pei}_2(\partial_f)/\,\mathrm{Pei}_3(\partial_f) \longrightarrow 0,$$

where $K = \ker d_2 = \pi_2 X^2$ and w maps $x \otimes y$ to the Peiffer commutator $\langle \xi, \eta \rangle$ with $\xi, \eta \in \rho_2$ representing x and y, respectively.

Definition 6.3 Given a 2–type T in 2–**types**, we define the chain complex $P(T) = P(\partial_f, B)$ as follows. Let $f\colon E_2 \to \langle E_1 \rangle$ be the attaching map of 2–cells in T and put $C_i = \widehat{C}_i(T)$. Then the 2–skeleton of $P(T)$ coincides with $\widehat{C}(T^2)$, that is, $P_i(T) = C_i$ for $i \le 2$, and $P_i(T) = 0$ for $i > 4$. To define $P_4(T)$, let H be the map in (5–1) and put $B = \mathrm{im}(d\colon C_3 \to C_2)$ and $\nabla_B = B \otimes B + H[B, C_2]$ as a submodule of $C_2 \otimes C_2$. Then $P_4(T)$ is given by the quotient

$$P_4(T) = C_2 \otimes C_2/\nabla_B.$$

To define $P_3(T)$, we use Lemma 6.1, Lemma 6.2 and the identification $\pi_2 T^2 = \ker(d\colon C_2 \to C_1)$ and put $\sigma_2 = \rho_2/\,\mathrm{Pei}_3(\partial_f)$. Then $P_3(T)$ is given by the pullback diagram

$$\begin{tikzcd}
P_3(T) \arrow[rr, tail] \arrow[d, "\bar{d}"'] & & \sigma_2/w\nabla_B \arrow[d, two heads] \\
B \arrow[r, tail] & \pi_2 T^2 \arrow[r, tail] & \bar{\rho}_2 .
\end{tikzcd}$$

The chain complex $P(T)$ is determined by the commutative diagram

$$\begin{tikzcd}
P_4(T) \arrow[r, "d"] \arrow[d, equal] & P_3(T) \arrow[rr] \arrow[d] \arrow[dr, "\bar{d}"] & & P_2(T) \arrow[r] \arrow[d, equal] & P_1(T) \arrow[r] \arrow[d, equal] & P_0(T) \arrow[d, equal] \\
C_2 \otimes C_2/\nabla_B \arrow[r, "-w"] & \sigma_2/w\nabla_B & B \arrow[r, tail] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 .
\end{tikzcd}$$

Clearly, $P(T) = P(\partial_f, B)$ depends only on the pair (∂_f, B) and yields a functor

$$P\colon \mathbf{cross}^{=} \longrightarrow \mathbf{H}_0.$$

The homology of $P(T)$ is given by

$$\mathrm{H}_i(P(T)) = \begin{cases} 0 & \text{for } i = 1 \text{ and } i = 3, \\ \mathrm{H}_2 C = \pi_2 T & \text{for } i = 2, \\ \Gamma(\pi_2(T)) & \text{for } i = 4. \end{cases}$$

Lemma 6.4 *Given a 2–type T, there is a chain map*

$$\bar{\beta}\colon \hat{C}(T) \longrightarrow P(T)$$

inducing isomorphisms in homology in degree ≤ 4. The map $\bar{\beta}$ is natural in T up to homotopy, that is, a map $g\colon T \to T'$ between 2–types yields a homotopy commutative diagram

$$\begin{tikzcd} \hat{C}(T) \arrow[r, "g_*"] \arrow[d, "\bar{\beta}"'] & \hat{C}(T') \arrow[d, "\bar{\beta}"] \\ P(T) \arrow[r, "\bar{\bar{g}}_*"] & P(T'), \end{tikzcd}$$

where $\bar{\bar{g}}_*$ is induced by a map $\bar{\bar{g}}\colon \partial_f \to \partial_{f'}$ associated with g.

For a proof of Lemma 6.4, we refer the reader to diagram (1.2) in Chapter V of [1]. In order to compute the fourth homology or cohomology of a 2–type T with coefficients, choose a pair (∂_f, B) representing T and a free chain complex C together with a weak equivalence of chain complexes

$$C \xrightarrow{\sim} P(\partial_f, B).$$

Then, for right Λ–modules M and left Λ–modules N,

$$\mathrm{H}_4(T, M) = \mathrm{H}_4(C \otimes M),$$
$$\mathrm{H}^4(T, N) = \mathrm{H}^4(\mathrm{Hom}_\Lambda(C, N)).$$

This allows the computation of H_4 in terms of chain complexes only, as is the case for the computation of group homology in Section 4. Of course, it is also possible to compute the homology of T in terms of a spectral sequence associated with the fibration

$$K(\pi_2(T), 2) \longrightarrow T \longrightarrow K(\pi_1(T), 1).$$

However, in general, this yields nontrivial differentials, which may be related to the properties of the chain complex $P(\partial_f, B)$.

7 Algebraic models of PD^4–complexes

Let X be a 4–dimensional CW–complex and let

$$p_2\colon X \longrightarrow P_2X = T$$

be the map to the 2–type of X, as in (3–1). Then p_2 yields the chain map

$$\beta\colon \hat{C}(X) \xrightarrow{p_{2*}} \hat{C}(T) \xrightarrow{\bar{\beta}_*} P(T) = P(\partial_f, B),$$

where ∂_f is given by the attaching map of 2–cells in X and $B = \mathrm{im}(d_3\colon \widehat{C}_3(X) \to \widehat{C}_2(X))$. We call the chain map β the *cellular boundary invariant of* X.

Lemma 7.1 *Suppose X and X' are 4–dimensional* CW*–complexes. A chain map $\varphi\colon \widehat{C}(X) \to \widehat{C}(X')$ is realizable by a map $g\colon X \to X'$ in* $\mathbf{CW_0}$*, that is, $\varphi = g_*$, if and only if the diagram*

$$\begin{tikzcd} \widehat{C}(X) \arrow[r, "\varphi"] \arrow[d, "\beta"'] & \widehat{C}(X') \arrow[d, "\beta'"] \\ P(\partial_f, B) \arrow[r, "\overline{\overline{\varphi}}"] & P(\partial_{f'}, B') \end{tikzcd}$$

commutes up to homotopy. Here $\overline{\overline{\varphi}}\colon \partial_f \to \partial_{f'}$ is a map in $\mathbf{cross}^=$ *inducing the map $\varphi_{\leq 2}\colon \widehat{C}(X^2) \to \widehat{C}(X'^2)$ as in Lemma 6.1.*

Proof By Lemma 6.4, the diagram

$$\begin{tikzcd} \widehat{C}(X) \arrow[r, "\varphi"] \arrow[d, "p_{2*}"'] & \widehat{C}(X') \arrow[d, "p_{2*}"] \\ \widehat{C}(T) \arrow[r, "g_*"] & \widehat{C}(T') \end{tikzcd}$$

is homotopy commutative, where g is given by $q(\overline{\overline{g}})$ in $\mathbf{Ho(cross)}$. Since p_{2*} and g_* are realizable, the obstruction $\mathcal{O}_{X,X'}(\varphi)$ vanishes. $\square$

The next definition relies on the theory of quadratic chain complexes from [1], in particular, we use the tensor product of quadratic chain complexes defined in [1]. We hope to discuss explicit examples of this definition elsewhere.

Definition 7.2 A β–PD^4–chain complex is a PD^4–chain complex $((\pi, C), \omega, [C], \Delta)$ together with a totally free pre-crossed module ∂_f inducing $d_2\colon C_2 \to C_1$ and a chain map

$$\beta\colon C \longrightarrow P(\partial_f, B)$$

which is the identity in degree ≤ 2. Here $B = \mathrm{im}(d_3\colon C_3 \to C_2)$, the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\beta"'] & C \otimes C \arrow[d, "\beta^{\otimes}"] \\ P(\partial_f, B) \arrow[r, "\overline{\overline{\Delta}}_*"] & P(\partial_{f \otimes f}, B^{\otimes}) \end{tikzcd}$$

commutes up to homotopy and β is the cellular boundary invariant β_σ of a totally free quadratic chain complex σ defined in V (1.8) of [1]. Further, $\beta^\otimes$ is the cellular boundary invariant of the quadratic chain complex $\sigma \otimes \sigma$ defined in Section IV 12 of [1], and there is an explicit formula expressing $\beta^\otimes$ in terms of β, which we do not recall here. The function $f \otimes f$ is the attaching map of 2–cells in the product $X^2 \times X^2$, where X^2 is given by f, and $B^\otimes$ is the image of d_3 in $C \otimes C$. The map $\overline{\overline{\Delta}}$ in **cross**$^=$ is chosen such that $\overline{\overline{\Delta}}$ induces Δ in degree ≤ 2 as in Lemma 7.1. Let $\mathbf{PD}^4_{*,\beta}$ be the category whose objects are β–PD^4–chain complexes and whose morphisms are maps φ in $\mathbf{PD}^4_*$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\beta"'] & C' \arrow[d, "\beta'"] \\ P(\partial_f, B) \arrow[r, "\overline{\overline{\varphi}}"] & P(\partial_{f'}, B') \end{tikzcd}$$

is homotopy commutative, where $\overline{\overline{\varphi}}$ induces $\varphi_{\leq 2}$ as in Lemma 7.1.

Theorem 7.3 *The functor $\widehat{C}$ yields a functor*

$$\widehat{C}\colon \mathbf{PD}^4/\simeq \longrightarrow \mathbf{PD}^4_{*,\beta}/\simeq$$

which reflects isomorphisms and is representative and full.

Proof Since C is 2–realizable, there is a 4–dimensional CW–complex X with $\widehat{C}(X) = C$ and cellular boundary invariant β. Compare Remark 1.1. By Lemma 7.1, the diagonal Δ is realizable by a diagonal $X \to X \times X$, showing that X is a PD^4–complex. By Lemma 7.1, a map φ is realizable by a map $X \to X'$. □

Corollary 7.4 *The functor $\widehat{C}$ induces a 1–1 correspondence between homotopy types of* PD^4*–complexes and homotopy types of β–*PD^4*–chain complexes.*

The functor τ in Section 3 yields the diagram of functors

(7–1)
$$\begin{tikzcd} \mathbf{PD}^4_+/\simeq \arrow[rr, "\widehat{C}"] \arrow[dr, "\tau_+"'] & & \mathbf{PD}^4_{*+,\beta}/\simeq \arrow[dl, "\tau_*"] \\ & \mathbf{Trp}^4_+ & \end{tikzcd}$$

where τ_+ determines τ_* together with a natural isomorphism $\tau_*\widehat{C} \cong \tau_+$.

Corollary 7.5 *The functor τ_* in (7–1) reflects isomorphisms and is full.*

8 Homotopy systems of order $(k+1)$

To investigate questions of realizability, we work in the category $\mathbf{H}^c_{k+1}$ of homotopy systems of order $(k+1)$. Let $\mathbf{CW}_0^k$ be the full subcategory of $\mathbf{CW}_0$ consisting of k–dimensional CW–complexes. A 0–homotopy H in $\mathbf{CW}_0$, denoted by $\simeq^0$, is a homotopy for which H_t is cellular for each $t, 0 \leq t \leq 1$.

Let $k \geq 2$. A *homotopy system of order* $(k+1)$ is a triple $X = (C, f_{k+1}, X^k)$, where X^k is an object in $\mathbf{CW}_0^k$, C is a chain complex of free $\pi_1(X^k)$–modules, which coincides with $\widehat{C}(X^k)$ in degree $\leq k$, and where f_{k+1} is a homomorphism of left $\pi_1(X^k)$–modules such that

$$\begin{tikzcd} C_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "d"'] & \pi_k(X^k) \arrow[d, "j"] \\ C_k & \pi_k(X^k, X^{k-1}) \arrow[l, "h_k"'] \end{tikzcd}$$

commutes. Here d is the boundary in C,

$$h_k \colon \pi_k(X^k, X^{k-1}) \xrightarrow[\cong]{p_*^{-1}} \pi_k(\widehat{X}^k, \widehat{X}^{k-1}) \xrightarrow[\cong]{h} \mathrm{H}_k(\widehat{X}^k, \widehat{X}^{k-1}),$$

given by the Hurewicz isomorphism h and the inverse of the isomorphism on the relative homotopy groups induced by the universal covering $p \colon \widehat{X} \to X$. Moreover, f_{k+1} satisfies the *cocycle condition*

$$f_{k+1} d(C_{k+2}) = 0.$$

Given an object X in $\mathbf{CW}_0$, the triple $r(X) = (\widehat{C}(X), f_{k+1}, X^k)$ is a homotopy system of order $(k+1)$, where X^k is the k–skeleton of X, and

$$f_{k+1} \colon \widehat{C}_{k+1}(X) \cong \pi_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial} \pi_k(X^k)$$

is the attaching map of $(k{+}1)$–cells in X. A *morphism* or *map* between homotopy systems of order $(k+1)$ is a pair

$$(\xi, \eta) \colon (C, f_{k+1}, X^k) \to (C', g_{k+1}, Y^k),$$

where $\eta\colon X^k \to Y^k$ is a morphism in $\mathbf{CW}_0/\simeq^0$ and the $\pi_1(\eta)$–equivariant chain map $\xi\colon C \to C'$ coincides with $\widehat{C}_*(\eta)$ in degree $\leq k$ such that

$$\begin{tikzcd} C_{k+1} \arrow[r, "\xi_{k+1}"] \arrow[d, "f_{k+1}"] & C'_{k+1} \arrow[d, "g_{k+1}"] \\ \pi_k(X^k) \arrow[r, "\eta_*"'] & \pi_k(Y^k) \end{tikzcd}$$

commutes. We also write $\pi_1 X = \pi_1(X^k)$ for an object $X = (C, f_{k+1}, X^k)$ in $\mathbf{H}^c_{k+1}$.

To define the homotopy relation in $\mathbf{H}^c_{k+1}$, we use the action

$$[X^k, Y]_\varphi \times \widehat{\mathrm{H}}^k(X^k, \varphi^*\pi_k Y) \to [X^k, Y]_\varphi, \quad (F, \{\alpha\}) \mapsto F + \{\alpha\}, \tag{8–1}$$

where $[X^n, Y]_\varphi$ is the set of elements in $[X^n, Y]$ which induce φ on the fundamental groups (see (2.4)(3) on page 45 in [1]). Two morphisms

$$(\xi, \eta), (\xi', \eta')\colon (C, f_{k+1}, X^k) \to (C', g_{k+1}, Y^k)$$

are *homotopy equivalent* in $\mathbf{H}^c_{k+1}$ if $\pi_1(\eta) = \pi_1(\eta') = \varphi$ and if there are φ–equivariant homomorphisms $\alpha_{j+1}\colon C_j \to C'_{j+1}$ for $j \geq k$ such that

$$\{\eta\} + g_{k+1}\alpha_{k+1} = \{\eta'\},$$
$$\xi'_i - \xi_i = \alpha_i d + d\alpha_{i+1}, \quad i \geq k+1,$$

where $\{\eta\}$ denotes the homotopy class of η in $[X^k, Y^k]$ and $+$ is the action (8–1).

Given homotopy systems $X = (C, f_{k+1}, X^k)$ and $Y = (C', g_{k+1}, Y^k)$, consider

$$X \otimes Y = (C \otimes_{\mathbb{Z}} C', h_{k+1}, (X^k \times Y^k)^k),$$

where we choose CW–complexes X^{k+1} and Y^{k+1} with attaching maps f_{k+1} and g_{k+1}, respectively, and h_{k+1} is given by the attaching maps of $(k+1)$–cells in $X^{k+1} \times Y^{k+1}$. Then $X \otimes Y$ is a homotopy system of order $(k+1)$, and

$$\otimes\colon \mathbf{H}^c_{k+1} \times \mathbf{H}^c_{k+1} \to \mathbf{H}^c_{k+1}$$

is a bifunctor, called the *tensor product of homotopy systems*. The two projections $p_1\colon X \otimes Y \to X$ and $p_2\colon X \otimes Y \to Y$ in $\mathbf{H}^c_{k+1}$ are given by the projections of the tensor product and the product of CW–complexes. Similarly, we obtain the inclusions $\iota_1\colon X \to X \otimes Y$ and $\iota_2\colon Y \to X \otimes Y$. Then $p_1\iota_1 = \mathrm{id}_X$ and $p_2\iota_2 = \mathrm{id}_Y$, while $p_1\iota_2$ and $p_2\iota_1$ yield the trivial maps.

There are functors

$$\mathbf{CW}_0 \xrightarrow{r} \mathbf{H}^c_{k+1} \xrightarrow{\lambda} \mathbf{H}^c_k \xrightarrow{C} \mathbf{H}_0 \tag{8–2}$$

for $k \geq 3$, with $r(X) = (\widehat{C}(X), f_{k+1}, X^k)$ such that $r = \lambda r$. We write $\lambda X = \overline{X}$ for objects X in $\mathbf{H}^c_{k+1}$. As $\overline{X \otimes Y} = \lambda(X \otimes Y) = \overline{X} \otimes \overline{Y}$, the functor λ, like r and C, is a monoidal functor between monoidal categories. There is a homotopy relation defined on the category $\mathbf{H}^c_{k+1}$ such that these functors induce functors between homotopy categories

$$\mathbf{CW}_0/\simeq \xrightarrow{r} \mathbf{H}^c_{k+1}/\simeq \xrightarrow{\lambda} \mathbf{H}^c_k/\simeq \xrightarrow{C} \mathbf{H}_0/\simeq .$$

For $k \geq 3$, Whitehead's functor Γ_k factors through the functor $r\colon \mathbf{CW} \to \mathbf{H}^c_k$, so that the cohomology $\widehat{\mathrm{H}}_m(\overline{X}, \varphi^*\Gamma_k(\overline{Y})) = \mathrm{H}^m(C, \varphi^*\Gamma_k(\overline{Y}))$ is defined, where $\varphi\colon \pi_1\overline{X} \to \pi_1\overline{Y}$ and $\overline{X}$ and $\overline{Y}$ are objects in $\mathbf{H}^c_k$.

Consider $f = (\xi, \eta)\colon \overline{X} \to \overline{Y}$ in $\mathbf{H}^c_k$, where $\overline{X} = \lambda X$ and $\overline{Y} = \lambda Y$. To describe the obstruction to realizing f by a map $X \to Y$ in $\mathbf{H}^c_{k+1}$ for objects $X = (C, f_{k+1}, X^k)$ and $Y = (C', g_{k+1}, Y^k)$, choose $F\colon X^k \to Y^k$ in $\mathbf{CW}/\simeq^0$ extending $\eta\colon X^{k-1} \to Y^{k-1}$ and for which $\widehat{C}_* F$ coincides with ξ in degree $\leq k$. Then

$$\begin{array}{ccc} C_{k+1} & \xrightarrow{\xi_{k+1}} & C'_{k+1} \\ \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle g_{k+1}} \\ \pi_k(X^k) & \xrightarrow{F_*} & \pi_k(Y^k) \end{array}$$

need not commute and the difference $\mathcal{O}(F) = -g_{k+1}\xi_{k+1} + F_* f_{k+1}$ is a cocycle in $\mathrm{Hom}_\varphi(C_{k+1}, \Gamma_k(\overline{Y}))$. Theorem II 3.3 in [1] implies:

Proposition 8.1 *The map $f = (\xi, \eta)\colon \overline{X} \to \overline{Y}$ in $\mathbf{H}^c_k$ can be realized by a map $f_0 = (\xi, \eta_0)\colon X \to Y$ in $\mathbf{H}^c_{k+1}$ if and only if $\mathcal{O}_{X,Y}(f) = \{\mathcal{O}(F)\} \in \widehat{\mathrm{H}}^{k+1}(\overline{X}, \varphi^*\Gamma_k\overline{Y})$ vanishes. The obstruction $\mathcal{O}$ is a derivation, that is, for $f\colon \overline{X} \to \overline{Y}$ and $g\colon \overline{Y} \to \overline{Z}$,*

$$\mathcal{O}_{X,Z}(gf) = g_*\mathcal{O}_{X,Y}(f) + f^*\mathcal{O}_{Y,Z}(g), \tag{8–3}$$

and $\mathcal{O}_{X,Y}(f)$ depends on the homotopy class of f only.

Denoting the set of morphisms $X \to Y$ in $\mathbf{H}^c_{k+1}/\simeq$ by $[X,Y]$, and the subset of morphisms inducing φ on the fundamental groups by $[X,Y]_\varphi \subseteq [X,Y]$, there is a group action

$$[X,Y]_\varphi \times \widehat{\mathrm{H}}^k(\overline{X}, \varphi^*\Gamma_k\overline{Y}) \xrightarrow{+} [X,Y]_\varphi,$$

where $\overline{X} = \lambda X$ and $\overline{Y} = \lambda Y$. Theorem II 3.3 in [1] implies:

Proposition 8.2 *Given morphisms $f_0, f'_0 \in [X,Y]_\varphi$, then $\lambda f_0 = \lambda f'_0 = f$ if and only if there is an $\alpha \in \widehat{\mathrm{H}}^k(\bar{X}, \varphi^*\Gamma_k\bar{Y})$ with $f'_0 = f_0 + \alpha$. In other words, $\widehat{\mathrm{H}}^k(\bar{X}, \varphi^*\Gamma_k\bar{Y})$ acts transitively on the set of realizations of f. Further, the action satisfies the linear distributivity law*

(8–4) $$(f_0+\alpha)(g_0+\beta) = f_0g_0 + f_*\beta + g^*\alpha.$$

For the functor λ in (8–2), Theorem II 3.3 and Proposition II 3.13 in [1] imply:

Proposition 8.3 *For all objects X in $\mathbf{H}^c_{k+1}$ and for all $\alpha \in \widehat{\mathrm{H}}^{k+1}(\bar{X}, \Gamma_k\bar{X})$, there is an object X' in $\mathbf{H}^c_{k+1}$ with $\lambda(X') = \lambda(X) = \bar{X}$ and $\mathcal{O}_{X,X'}(\mathrm{id}_{\bar{X}}) = \alpha$. We then write $X' = X + \alpha$.*

Now let Y be an object in $\mathbf{H}^c_k$. Then the group $\widehat{\mathrm{H}}^{k+1}(Y, \Gamma_k Y)$ acts transitively and effectively on $\mathrm{Real}_\lambda(Y)$ via $+$, provided $\mathrm{Real}_\lambda(Y)$ is nonempty. Moreover, $\mathrm{Real}_\lambda(Y)$ is nonempty if and only if an obstruction $\mathcal{O}(Y) \in \widehat{\mathrm{H}}^{k+2}(Y, \Gamma_k Y)$ vanishes.

For objects X and Y in $\mathbf{H}^c_{k+1}$ and a morphism $f\colon \bar{X} \to \bar{Y}$ in $\mathbf{H}^c_k$, Proposition 8.1 and Proposition 8.3 yield

(8–5) $$\mathcal{O}_{X+\alpha,Y+\beta}(f) = \mathcal{O}_{X,Y}(f) - f_*\alpha + f^*\beta$$

for all $\alpha \in \widehat{\mathrm{H}}^{k+1}(\bar{X}, \Gamma_k\bar{X})$ and $\beta \in \widehat{\mathrm{H}}^{k+1}(\bar{Y}, \Gamma_k\bar{Y})$. Given another object Z in $\mathbf{H}^c_{k+1}$ with $\lambda Z = \bar{Z}$,

(8–6) $$\mathcal{O}_{X\otimes Z, Y\otimes Z}(f \otimes \mathrm{id}_{\bar{Z}}) = \bar{\iota}_{1*}\bar{p}_1^*\mathcal{O}_{X,Y}(f),$$

(8–7) $$\mathcal{O}_{Z\otimes X, Z\otimes Y}(\mathrm{id}_{\bar{Z}} \otimes f) = \bar{\iota}_{2*}\bar{p}_2^*\mathcal{O}_{X,Y}(f),$$

where $\bar{\iota}_1\colon \bar{X} \to \bar{X}\otimes\bar{Z}$ and $\bar{p}_1\colon \bar{X}\otimes\bar{Z} \to \bar{X}$ are, respectively, the inclusion of and projection onto the first factor and $\bar{\iota}_2$ and $\bar{p}_2$ are defined analogously. We obtain

(8–8) $$(X+\alpha)\otimes(Y+\beta) = (X\otimes Y) + \bar{\iota}_{1*}\bar{p}_1^*\alpha + \bar{\iota}_{2*}\bar{p}_2^*\beta.$$

9 Obstructions to the diagonal

Let $k \geq 2$. A *diagonal* on $X = (C, f_{k+1}, X^k)$ in $\mathbf{H}^c_{k+1}$ is a morphism, $\Delta\colon X \to X \otimes X$, such that, for $i = 1, 2$, the diagram

(9–1) $$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & X\otimes X \arrow[d, "p_i"] \\ & X \end{tikzcd}$$

commutes up to homotopy in $\mathbf{H}^c_{k+1}$. Applying the functor $r\colon \mathbf{CW}_0 \to \mathbf{H}^c_k$ to a diagonal $\Delta\colon X \to X \times X$ in $\mathbf{CW}_0$, we obtain the diagonal $r(\Delta)\colon r(X) \to r(X) \otimes r(X)$ in $\mathbf{H}^c_k$.

Lemma 9.1 *Suppose X is an object in $\mathbf{H}^c_{k+1}$. Then every λ–realizable diagonal $\bar{\Delta}\colon \bar{X} = \lambda X \to \bar{X} \otimes \bar{X}$ in $\mathbf{H}^c_k/\simeq$ has a λ–realization $\Delta\colon X \to X \otimes X$ in $\mathbf{H}^c_{k+1}/\simeq$ which is a diagonal in $\mathbf{H}^c_{k+1}$.*

Proof Suppose $\Delta'\colon X \to X \otimes X$ is a λ–realization of $\bar{\Delta}$ in $\mathbf{H}^c_{k+1}$. The projection $p_\ell\colon X \to X \otimes X$ realizes the projection $\bar{p}_\ell\colon \bar{X} \to \bar{X} \otimes \bar{X}$ and hence $p_\ell \Delta'$ realizes $\bar{p}_\ell \bar{\Delta}$ for $\ell = 1, 2$. Now the identity on X realizes the identity on $\bar{X}$ and $\bar{p}_\ell \Delta$ is homotopic to the identity on $\bar{X}$ by assumption. Hence $p_\ell \Delta'$ and the identity on X realize the same homotopy class of maps for $\ell = 1, 2$. The group $\widehat{\mathrm{H}}^k(\bar{X}, \Gamma_k \bar{X})$ acts transitively on the set of realizations of this homotopy class by Proposition 8.2, whence there are elements $\alpha_\ell \in \widehat{\mathrm{H}}^k(\bar{X}, \Gamma_k \bar{X})$ such that

$$\{p_\ell \Delta'\} + \alpha_\ell = \{\mathrm{id}_X\} \quad \text{for } \ell = 1, 2,$$

where $\{f\}$ denotes the homotopy class of the morphism f in $\mathbf{H}^c_{k+1}$. We put

$$\{\Delta\} = \{\Delta'\} + \iota_1 \alpha_1 + \iota_2 \alpha_2.$$

By Proposition 8.2,

$$\begin{aligned}
\{p_\ell \Delta\} &= \{p_\ell\}(\{\Delta'\} + \iota_1 \alpha_1 + \iota_2 \alpha_2) \\
&= \{p_\ell \Delta'\} + \bar{p}_{\ell *} \iota_1 \alpha_1 + \bar{p}_{\ell *} \iota_2 \alpha_2 \\
&= \{p_\ell \Delta'\} + \alpha_\ell = \{\mathrm{id}_X\}.
\end{aligned}$$

□

Lemma 9.2 *For X in $\mathbf{H}^c_{k+1}$, let $\Delta_{\bar{X}}\colon \bar{X} \to \bar{X} \otimes \bar{X}$ be a diagonal on $\bar{X} = \lambda X$ in $\mathbf{H}^c_k$. Then we obtain, in $\mathrm{H}^{k+1}(\bar{X}, \Gamma_k(\bar{X} \otimes \bar{X}))$,*

(1) $\mathcal{O}_{X, X\otimes X}(\Delta_{\bar{X}}) \in \ker \bar{p}_{i*}$ *for $i = 1, 2$,*

(2) $\mathcal{O}_{X, X\otimes X}(\Delta_{\bar{X}}) \in \ker(\mathrm{id}_{\bar{X}*} - T_*)_*$ *if $\Delta_{\bar{X}}$ is homotopy commutative and*

(3) $\mathcal{O}_{X, X\otimes X}(\Delta_{\bar{X}}) \in \ker\big(\bar{\iota}_{1,2*} - \bar{\iota}_{2,3*} + (\Delta_{\bar{X}} \otimes \mathrm{id}_{\bar{X}})_* - (\mathrm{id}_{\bar{X}} \otimes \Delta_{\bar{X}})_*\big)_*$ *if $\Delta_{\bar{X}}$ is homotopy associative.*

Proof By definition, $\bar{p}_i \Delta_{\bar{X}} \simeq \mathrm{id}_{\bar{X}}$ for $i = 1, 2$. As the identity on $\bar{X}$ is realized by the identity on X and $\bar{p}_i\colon \bar{X} \otimes \bar{X} \to \bar{X}$ is realized by $p_i\colon X \otimes X \to X$, Proposition 8.1 implies $\mathcal{O}_{X, X\otimes X}(\mathrm{id}_{\bar{X}}) = 0$ and $\mathcal{O}_{X\otimes X, X}(\bar{p}_i) = 0$. Since $\mathcal{O}$ is a derivation, we obtain

$$0 = \mathcal{O}_{X,X}(\bar{p}_i \Delta_{\bar{X}}) = \bar{p}_{i*} \mathcal{O}_{X, X\otimes X}(\Delta_{\bar{X}}) + \Delta^*_{\bar{X}} \mathcal{O}_{X\otimes X, X}(\bar{p}_i) = \bar{p}_{i*} \mathcal{O}_{X, X\otimes X}(\Delta_{\bar{X}}),$$

and hence $O_{X,X\otimes X}(\Delta_{\bar{X}}) \in \ker \bar{p}_{i*}$ for $i=1,2$. If $\Delta_{\bar{X}}$ is homotopy commutative, then

$$\mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}) = \mathcal{O}_{X,X\otimes X}(T\Delta_{\bar{X}}) = T_*\mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}),$$

since $\mathcal{O}_{X\otimes X,X\otimes X}(T)=0$, as T is λ–realizable. So $\mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}) \in \ker(\mathrm{id}_{\bar{X}*} - T_*)_*$. For $1 \le k < \ell, \le 3$, let $\iota_{k,\ell}\colon X\otimes X \to X\otimes X\otimes X$ denote the inclusion of the k–th and ℓ–th factors and suppose $\Delta_{\bar{X}}$ is a homotopy commutative diagonal in $\mathbf{H}^c_k$. Then $\mathcal{O}_{X,X\otimes X\otimes X}((\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})\Delta_{\bar{X}}) = \mathcal{O}_{X,X\otimes X\otimes X}((\mathrm{id}_{\bar{X}}\otimes\Delta_{\bar{X}})\Delta_{\bar{X}})$, as the obstruction depends on the homotopy class of a morphism only, and

$$\begin{aligned}
\mathcal{O}_{X,X\otimes X\otimes X}(\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}}) &= \bar{\iota}_{1,2*}\bar{p}_1^*\mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}})\\
\mathcal{O}_{X,X\otimes X\otimes X}(\mathrm{id}_{\bar{X}}\otimes \Delta_{\bar{X}}) &= \bar{\iota}_{2,3*}\bar{p}_2^*\mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}),
\end{aligned}$$

by (8–6) and (8–7). Omitting the objects in the notation for the obstruction, we obtain

$$\begin{aligned}
\mathcal{O}((\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})\Delta_{\bar{X}}) &= \Delta_{\bar{X}}^*\mathcal{O}(\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}}) + (\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})_*\mathcal{O}(\Delta_{\bar{X}})\\
&= \Delta_{\bar{X}}^*\bar{\iota}_{1,2*}\bar{p}_1^*\mathcal{O}(\Delta_{\bar{X}}) + (\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})_*\mathcal{O}(\Delta_{\bar{X}})\\
&= \bar{\iota}_{1,2*}(\bar{p}_1\Delta_{\bar{X}})^*\mathcal{O}(\Delta_{\bar{X}}) + (\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})_*\mathcal{O}(\Delta_{\bar{X}})\\
&= \bar{\iota}_{1,2*}\mathcal{O}(\Delta_{\bar{X}}) + (\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})_*\mathcal{O}(\Delta_{\bar{X}}).
\end{aligned}$$

Similarly, we obtain

$$\mathcal{O}((\mathrm{id}_{\bar{X}}\otimes\Delta_{\bar{X}})\Delta_{\bar{X}}) = \bar{\iota}_{2,3*}\mathcal{O}(\Delta_{\bar{X}}) + (\Delta_{\bar{X}}\otimes \mathrm{id}_{\bar{X}})_*\mathcal{O}(\Delta_{\bar{X}}),$$

which proves (3). □

Question Given a λ–realizable object $\bar{X}$ with a diagonal $\Delta_{\bar{X}}\colon \bar{X}\to\bar{X}\otimes\bar{X}$ in $\mathbf{H}^c_k$, is there an object X with $\lambda X = \bar{X}$ and a diagonal $\Delta_X\colon X\to X\otimes X$ in $\mathbf{H}^c_{k+1}$ such that $\lambda\Delta_X = \Delta_{\bar{X}}$?

Let X in $\mathbf{H}^c_{k+1}$ be a λ–realization of $\bar{X}$. By Proposition 8.3, any λ–realization X' of $\bar{X}$ is of the form $X' = X+\alpha$ for some $\alpha\in\widehat{\mathrm{H}}^{k+1}(\bar{X},\Gamma_k\bar{X})$. By (8–8), $X'\otimes X' = (X\otimes X) + \bar{\iota}_{1*}\bar{p}_1^*\alpha + \bar{\iota}_{2*}\bar{p}_2^*\alpha$ and as the obstruction $\mathcal{O}$ is a derivation, we obtain

$$\begin{aligned}
\mathcal{O}_{X',X'\otimes X'}(\Delta_{\bar{X}}) &= \mathcal{O}_{X+\alpha,(X+\alpha)\otimes(X+\alpha)}(\Delta_{\bar{X}})\\
&= \mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}) - \Delta_{\bar{X}*}\alpha + \Delta_{\bar{X}}^*(\bar{\iota}_{1*}\bar{p}_1^*\alpha + \iota_{2*}\bar{p}_2^*\alpha)\\
&= \mathcal{O}_{X,X\otimes X}(\Delta_{\bar{X}}) - (\Delta_{\bar{X}*} - \bar{\iota}_{1*} - \bar{\iota}_{2*})\alpha,
\end{aligned}$$

since $\Delta^*\bar{\iota}_{i*}\bar{p}_i^* = \bar{\iota}_{i*}(\bar{p}_i\Delta)^* = \bar{\iota}_{i*}$, for $i=1,2$.

Lemma 9.3 *For X in $\mathbf{H}^c_{k+1}$, let $\Delta_{\bar X}\colon \bar X \to \bar X \otimes \bar X$ be a diagonal on $\bar X = \lambda X$ in $\mathbf{H}^c_k$ and let $X' = X + \alpha$ for some $\alpha \in \widehat{\mathrm{H}}^{k+1}(\bar X, \Gamma_k \bar X)$. Then we obtain, in $\mathrm{H}^{k+1}(\bar X, \Gamma_k(\bar X \otimes \bar X)$,*

$$\mathcal{O}_{X',X'\otimes X'}(\Delta_{\bar X}) = \mathcal{O}_{X,X\otimes X}(\Delta_{\bar X}) - (\Delta_{\bar X *} - \bar\iota_{1*} - \bar\iota_{2*})\alpha.$$

10 PD^{n}–homotopy systems

A PD^{n}*–homotopy system* $X = (X, \omega_X, [X], \Delta_X)$ of order $(k+1)$ consists of an object $X = (C, f_{k+1}, X^k)$ in $\mathbf{H}^c_{k+1}$, a group homomorphism $\omega_X\colon \pi_1 X \to \mathbb{Z}/2\mathbb{Z}$, a fundamental class $[X] \in \mathrm{H}_n(C, \mathbb{Z}^\omega)$ and a diagonal $\Delta\colon X \to X \otimes X$ in $\mathbf{H}^c_{k+1}$ such that $(C, \omega_X, [X], \Delta_X)$ is a PD^n–chain complex. A map $f\colon (X, \omega_X, [X], \Delta_X) \to (Y, \omega_Y, [Y], \Delta_Y)$ of PD^{n}–homotopy systems of order $(k+1)$ is a morphism in $\mathbf{H}^c_{k+1}$ such that $\omega_X = \omega_Y \pi_1(f)$ and $(f \otimes f)\Delta_X \simeq \Delta_Y f$, and we thus obtain the category $\mathbf{PD}^n_{[k+1]}$ of PD^{n}–homotopy systems of order $(k+1)$. Homotopies in $\mathbf{PD}^n_{[k+1]}$ are homotopies in $\mathbf{H}^c_{k+1}$, and restricting the functors in (8–2), we obtain, for $k \geq 3$, the functors

$$\mathbf{PD}^n \xrightarrow{r} \mathbf{PD}^n_{[k+1]} \xrightarrow{\lambda} \mathbf{PD}^n_{[k]} \xrightarrow{C} \mathbf{PD}^n_*. \tag{10–1}$$

These functors induce functors between homotopy categories:

$$\mathbf{PD}^n/\simeq \xrightarrow{r} \mathbf{PD}^n_{[k+1]}/\simeq \xrightarrow{\lambda} \mathbf{PD}^n_{[k]}/\simeq \xrightarrow{C} \mathbf{PD}^n_*/\simeq.$$

Theorem 10.1 *The functor $C\colon \mathbf{PD}^n_{[3]}/\simeq \longrightarrow \mathbf{PD}^n_*/\simeq$ is an equivalence of categories for $n \geq 3$.*

Proof The functor C is full and faithful by Theorem III 2.9 and Theorem III 2.12 in [1]. By Lemma 2.1, every PD^n–chain complex, $\bar X = (D, \omega, [D], \bar\Delta)$, in $\mathbf{PD}^n_*$ is 2–realizable, that is, there is an object X^2 in CW^2_0 such that $\hat C(X^2) = D_{\leq 2}$, and we obtain the object $X = (D, f_3, X^2)$ in $\mathbf{H}^c_3$. As C is monoidal, full and faithful, the diagonal $\bar\Delta$ on $\bar X$ is realized by a diagonal Δ on X and hence $(X, \omega, [D], \Delta)$ is an object in $\mathbf{PD}^n_{[3]}$ with $C(X) = \bar X$. □

Theorem 10.2 *For $n \geq 3$, the functor $r\colon \mathbf{PD}^n/\simeq \longrightarrow \mathbf{PD}^n_{[n]}/\simeq$ reflects isomorphisms, is representative and full.*

Proof That r reflects isomorphisms follows from Whitehead's Theorem.

Poincaré duality implies $\widehat{\mathrm{H}}^{n+1}(Y, \Gamma_n Y) = \widehat{\mathrm{H}}^{n+2}(Y, \Gamma_n Y) = 0$, for every object $Y = (Y, \omega_Y, [Y], \Delta_Y)$ in $\mathbf{PD}^n_{[n]}$. Hence, by Proposition 8.3, $Y = \lambda(X)$ for some object X in $\mathbf{H}^c_{n+1}$, and, by Proposition 8.1, the diagonal Δ_Y is λ–realizable. Thus Lemma 9.1 guarantees the existence of a diagonal $\Delta_X \colon X \to X \otimes X$ in $\mathbf{H}^c_{n+1}$ with $\lambda\Delta_X = \Delta_Y$. The homomorphism ω_Y and the fundamental class $[Y]$ determine a homomorphism $\omega_X \colon \pi_1 X \to \mathbb{Z}/2\mathbb{Z}$ and a fundamental class $[X] \in \mathrm{H}_n(C, \mathbb{Z}^\omega)$, such that $X = (X, \omega_X, [X], \Delta_X)$ is an object in $\mathbf{PD}^n_{[n+1]}$. Inductively, we obtain an object $(X_k, \omega_{X_k}, [X_k], \Delta_{X_k})$ realizing $(Y, \omega_Y, [Y], \Delta_Y)$ in $\mathbf{PD}^n_{[k]}$ for $k > n$, and in the limit an object $X = (X, \omega_X, [X], \Delta_X)$ in $\mathbf{PD}^n$ with $r(x) = Y$.

Proposition 8.1 together with the fact that, by Poincaré duality, $\widehat{\mathrm{H}}^k(X, B) = 0$ for $k > n$ and every Λ–module B, implies that r is full. □

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