Secondary characteristic classes of surface bundles

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The Miller–Morita–Mumford classes associate to an oriented surface bundle $E \to B$ a class $\kappa_i(E) \in H^{2i}(B;\mathbb{Z})$. It was proved in [1] that the mod p reduction $\kappa_i(E) \in H^{2i}(B;\mathbb{Z}/p)$ vanishes when $i+1$ is divisible by $(p-1)$. In this note we prove that the p^2 reduction $\kappa_i(E) \in H^{2i}(B;\mathbb{Z}/p^2)$ vanishes when $i+1$ is divisible by $p(p-1)$. We also define for each integer $i \geq 1$ a characteristic class $\lambda_i(E) \in H^{2i(p-1)-2}(B;\mathbb{Z}/p)$ which satisfies $p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B;\mathbb{Z}/p^2)$.

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1 Introduction and statement of results

This paper studies characteristic classes of surface bundles. By surface bundle we shall mean smooth fiber bundle $\pi\colon E \to B$ with closed oriented two-dimensional fibers. An important sequence of characteristic classes is the Miller–Morita–Mumford classes, or κ–classes. They associate to a smooth fiber bundle $\pi\colon E \to B$ a characteristic class $\kappa_i \in H^{2i}(B;\mathbb{Z})$ for all $i \geq 0$. They are natural with respect to pull back of surface bundles and also have other nice properties. The question under study in this paper and in the paper by the author, Madsen and Tillmann [1] is the question of *universal divisibility* of the classes κ_i. More precisely we have the following definition:

Definition 1.1 Let $D \geq 1$ be a natural number. Let us say that κ_i is divisible by D if there is a characteristic class μ with values in $H^{2i}(-;\mathbb{Z})$ such that $\kappa_i(E) = D\mu(E)$ for all surface bundles. Let us say that κ_i is divisible by D modulo torsion if there is a characteristic class μ with values in $H^{2i}(-;\mathbb{Z})$ such that $\kappa_i(E) - D\mu(E) \in H^{2i}(B;\mathbb{Z})$ is a torsion element for all surface bundles $\pi\colon E \to B$.

It is natural to ask for the maximal possible D for each i, both for integral divisibility and divisibility modulo torsion. This can be studied one prime at a time. We summarize the partial answer to this question given in [1].

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Theorem 1.2 [1] *Let p be a prime number and $v \geq 0$ a natural number.*

(i) *If κ_i is divisible by p^{v+1}, then $i+1$ is divisible by $p^v(p-1)$.*

(ii) *κ_i is divisible by p^{v+1} modulo torsion if and only if $i+1$ is divisible by $p^v(p-1)$.*

(iii) *κ_i is divisible by p if and only if $i+1$ is divisible by $(p-1)$.*

This completely determines the divisibility of κ_i modulo torsion. Part (i) is a consequence of part (ii) and gives an upper bound on the integral divisibility of κ_i, but the exact divisibility by p^v for $v \geq 2$ was left unanswered in [1]. The following theorem, which is our main theorem in this paper, settles the case $v = 2$.

Theorem 1.3 *κ_i is divisible by p^2 if and only if $i+1$ is divisible by $p(p-1)$.*

Let us rephrase the statement of the main theorem. The following theorem is obviously a consequence, but in fact turns out to be equivalent to Theorem 1.3. This is the form in which the main theorem will be proved.

Theorem 1.4 *Let p be a prime and $s \geq 1$. Then the reduction of $\kappa_{ps(p-1)-1}$ modulo p^2 vanishes,*

$$\kappa_{ps(p-1)-1}(E) = 0 \in H^*(B;\mathbb{Z}/p^2),$$

for all surface bundles $\pi\colon E \to B$.

We explain how to deduce Theorem 1.3 from Theorem 1.4. The "only if" part is already contained in Theorem 1.2(i). For the "if" part we consider the long exact sequence in homology associated to the short exact sequence of coefficients $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p^2$. It follows that for each surface bundle $E \to B$ there is a class $\mu(E)$ in integral cohomology such that $p^2\mu(E) = \kappa_{ps(p-1)-1}(E)$. To see that we can choose $\mu(E)$ natural, we apply this argument in a universal situation. The classifying space $B\mathrm{Diff}(F)$ of the topological group of orientation preserving diffeomorphisms classifies surface bundles with fiber F in the sense that there is a natural bijection between the set of isomorphism classes of surface bundles $E \to B$ with fiber F and the set $[B, B\mathrm{Diff}(F)]$ of homotopy classes of maps $B \to B\mathrm{Diff}(F)$. There are universal classes $\kappa_i \in H^{2i}(B\mathrm{Diff}(F);\mathbb{Z})$ which, assuming Theorem 1.4, vanish after reduction modulo p^2 (if the reduction were nonzero there would be some map $B \to B\mathrm{Diff}(F)$ from a smooth manifold B, such that the pullback to B was also nonzero). Hence we can choose a universal $\mu \in H^*(B\mathrm{Diff}(F);\mathbb{Z})$. Thus Theorem 1.3 and Theorem 1.4 are equivalent. In the proof, we prove the statement of Theorem 1.4.

Thus for $v = 0, 1$ we have proved that κ_i is divisible by p^{v+1} if and only if it is divisible modulo torsion. It seems reasonable to conjecture that this is the case for all v. Hence we formulate the following conjecture, also mentioned in [1], which would completely settle the question of divisibility of κ–classes.

Conjecture 1.5 *Let $s \geq 1$ and $v \geq 0$. Then*

$$\kappa_{p^v s(p-1)-1}(E) = 0 \in H^*(B; \mathbb{Z}/p^{v+1}).$$

In the course of the proof of Theorem 1.4 we introduce certain new characteristic classes λ_i which might have some independent interest. Their main properties are given by the following theorem.

Theorem 1.6 *For each $i \geq 1$ there is a characteristic class λ_i which associates to a surface bundle $E \to B$ a class $\lambda_i(E) \in H^{2i(p-1)-2}(B; \mathbb{Z}/p)$ with the property that*

$$p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B; \mathbb{Z}/p^2). \tag{1-1}$$

Furthermore, the class λ_i satisfies the following properties:

(i) *If $\pi\colon E \to B$ and $\pi'\colon E' \to B$ are two surface bundles, then*

$$\lambda_i(E \amalg E') = \lambda_i(E) + \lambda_i(E').$$

(ii) *Let E_1, E_2 and E_2' be bundles of compact surfaces with boundary and assume that the oriented boundaries satisfy $\partial E_1 = \partial E_2 = \partial E_2'$. Then we can form the surface bundles*

$$\begin{aligned} E &= E_1 \cup_\partial \bar{E}_2, \\ E' &= E_1 \cup_\partial \bar{E}_2', \\ \text{and} \qquad D &= E_2' \cup_\partial \bar{E}_2, \end{aligned}$$

where the bars denote orientation reversal. In this case we have

$$\lambda_i(E') = \lambda_i(E) + \lambda_i(D).$$

The classes λ_i are defined using Toda brackets. In Section 2 we review general properties of Toda brackets and in Section 3 we give the definition of $\lambda_i(E)$ for a surface bundle E. It is a secondary class, and we prove that $\lambda_i(E) \subseteq H^*(B; \mathbb{Z})$ is defined with indeterminacy $\mathbb{Z}\kappa_{i(p-1)-1}$. Then we prove that the reduction modulo p has zero indeterminacy and satisfies the properties in Theorem 1.6. Finally, our main theorem (Theorem 1.3) is proved by showing that the reduction of the class

$\lambda_{ps}(E)$ modulo p vanishes for all surface bundles (after modifying the definition from Section 3 a little).

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2 Secondary composition

We recall the definition of secondary compositions (Toda brackets). For further details see Toda [3].

All spaces and all maps in this section are pointed. The reduced suspension SX is regarded as the pushout of the diagram

$$X \wedge [-1,0] \longleftarrow X \longrightarrow X \wedge [0,1],$$

where $-1 \in [-1,0]$ and $1 \in [0,1]$ are the basepoints. Thus, two nullhomotopies $F\colon X \wedge [-1,0] \to Y$ and $G\colon X \wedge [0,1] \to Y$ induce a map $G - F\colon SX \to Y$.

For a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $g \circ f \simeq 0$ and $h \circ g \simeq 0$, a choice of nullhomotopies $F\colon g \circ f \simeq 0$ and $G\colon h \circ g \simeq 0$ determines a map

$$h \circ F - G \circ (f \wedge [-1,0])\colon SX \to W.$$

We define the *secondary composition* to be the subset $\{h, g, f\} \subseteq [SX, W]$ of homotopy classes of maps obtained in this fashion, as F, G range over all nullhomotopies.

Recall that $[SX, W] = [X, \Omega W]$ is a group.

Lemma 2.1 *$\{h, g, f\}$ depends only on the homotopy classes of h, g, and f. If $\{h, g, f\}$ is defined, then it gives a unique element in the double coset,*

$$\{h, g, f\} \in h \circ [SX, Z] \setminus [SX, W] / [SY, W] \circ Sf.$$

If $[SX, W]$ is abelian, then

$$\{h, g, f\} \in [SX, W] / \big(h \circ [SX, Z] + [SY, W] \circ Sf\big).$$

Proof See Toda [3, Lemma 1.1]. □

Proposition 2.2 *For a sequence of maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

we have

(i) $\{k, h, g\} \circ f \subseteq \{k, h, g \circ f\}$
(ii) $\{k, h, g \circ f\} \subseteq \{k, h \circ g, f\}$
(iii) $\{k \circ h, g, f\} \subseteq \{k, h \circ g, f\}$
(iv) $k \circ \{h, g, f\} \subseteq \{k \circ h, g, f\}$.

Proof See Toda [3, Proposition 1.2]. $\square$

Proposition 2.3 *Let*

$$K(\mathbb{Z}, n) \xrightarrow{p} K(\mathbb{Z}, n) \xrightarrow{\rho} K(\mathbb{Z}/p, n) \xrightarrow{\beta} K(\mathbb{Z}, n+1)$$

represent multiplication by p, reduction mod p, and the mod p Bockstein, respectively. Then

$$\mathrm{id} \in \{\beta, \rho, p\} \subseteq [SK(\mathbb{Z}, n), K(\mathbb{Z}, n+1)] = [K(\mathbb{Z}, n), K(\mathbb{Z}, n)].$$

Proof Consider the diagram

$$\begin{tikzcd}
K(\mathbb{Z}, n) \arrow[r, "p"] \arrow[d, equal] & K(\mathbb{Z}, n) \arrow[r, "g"] \arrow[d, equal] & C(p) \arrow[r, "h"] \arrow[d] & SK(\mathbb{Z}, n) \arrow[d, "k"] \\
K(\mathbb{Z}, n) \arrow[r, "p"] & K(\mathbb{Z}, n) \arrow[r, "\rho"] & K(\mathbb{Z}/p, n) \arrow[r, "\beta"] & K(\mathbb{Z}, n+1)
\end{tikzcd}$$

where the top row is the Puppe sequence. It is immediate from the definition that $\mathrm{id} \in \{h, g, p\}$. Now apply Proposition 2.2 (iii)–(iv) to get $\{h, g, p\} \circ k \subseteq \{p, \rho, \beta\}$. $\square$

Corollary 2.4 *Let $c\colon X \to K(\mathbb{Z}, n)$ represent a cohomology class. Let ρ and β be as in Proposition 2.3. Then*

$$\{\beta, \rho, c\} = \tfrac{1}{p} c + \mathbb{Z} c \subseteq H^n(X) = [SX, K(\mathbb{Z}, n+1)],$$

where

$$\tfrac{1}{p} c = \{c' \mid pc' = c\}.$$

Proof The two sides have the same indeterminacy $\mathbb{Z}c + \beta H^{n-1}(X; \mathbb{Z}/p)$, so all we need to check is that if $pc' = c$, then $c' \in \{\beta, \rho, c\}$. But this follows from Proposition 2.2 and Proposition 2.3:

$$\{\beta, \rho, p \circ c'\} \supseteq \{\beta, \rho, p\} \circ c' \ni c'. \qquad \square$$

3 Secondary characteristic classes

In this section we first review the definition of the κ–classes in a convenient language. Then we define the new characteristic classes λ_i and prove that they satisfy Theorem 1.6.

To define κ–classes we use the Pontrjagin–Thom construction, which we first review. Any surface bundle $\pi\colon E \to B$ admits an embedding $j\colon E \to B \times \mathbb{R}^{N+2}$ over B, for some N. For N large, j is unique up to isotopy. A choice of embedding j induces a transfer ("collapse") map

$$B_+ \wedge S^{N+2} \xrightarrow{\pi_!} \mathrm{Th}(\nu j),$$

where νj is the normal bundle of j and $\mathrm{Th}(\nu j)$ is its Thom space. The embedding $j\colon E \to B \times \mathbb{R}^{N+2}$ also induces classifying maps

$$\begin{tikzcd} T^\pi E \arrow[r, "\mathrm{cl}(T^\pi E)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times \mathrm{SO}(2)} \mathbb{R}^2 \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/(\mathrm{SO}(N)\times\mathrm{SO}(2)) \end{tikzcd}$$

and

$$\begin{tikzcd} \nu j \arrow[r, "\mathrm{cl}(\nu j)"] \arrow[d] & \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times \mathrm{SO}(2)} \mathbb{R}^N \arrow[d] \\ E \arrow[r] & \mathrm{SO}(N+2)/(\mathrm{SO}(N)\times\mathrm{SO}(2)). \end{tikzcd}$$

For brevity, write

$$U = U_N = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times \mathrm{SO}(2)} \mathbb{R}^2,$$
$$U^\perp = U_N^\perp = \mathrm{SO}(N+2) \times_{\mathrm{SO}(N)\times \mathrm{SO}(2)} \mathbb{R}^N.$$

We get the composition

$$\alpha = \alpha_E = \mathrm{Th}(\mathrm{cl}(\nu j)) \circ \pi_!\colon B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp).$$

By Thom isomorphism, there is a Thom class $u_{U^\perp} \in H^N(\mathrm{Th}(U^\perp), \star; \mathbb{Z})$ and we have $H^{N+*}(\mathrm{Th}(U^\perp), \star; \mathbb{Z}) = \mathbb{Z}[e(U)].u_{U^\perp}$ for $* < N$. Here $e(U)$ is the Euler class of U. The definition of the κ–classes is

$$\kappa_i(E) = \alpha^*(e(U)^{i+1}.u_{U^\perp}) = \pi_!^*(e(T^\pi E)^{i+1}.u_{\nu j}) \in H^{2i}(B; \mathbb{Z}).$$

The following lemma is the key to defining the classes λ_i. For an odd prime p, we write $\mathcal{P}^i$ for the Steenrod power operation. For $p=2$ we write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$.

Lemma 3.1 *Let p be a prime and let $\pi\colon E \to B$ be a surface bundle. Let $\alpha = \alpha_E\colon B_+ \wedge S^{N+2} \to \mathrm{Th}(U_N^\perp)$ be as above and let $u\colon \mathrm{Th}(U_N^\perp) \to K(\mathbb{Z}, N)$ be the Thom class. Then the Toda bracket*

$$\{\beta\mathcal{P}^i, u, \alpha\} \subseteq H^{2i(p-1)-2+N}(B_+ \wedge S^{N+2}; \mathbb{Z}) = H^{2i(p-1)-2}(B; \mathbb{Z})$$

is defined with indeterminacy $\mathbb{Z}\kappa_{i(p-1)-1}(E)$.

Definition 3.2 With notation as in Lemma 3.1 define

$$\lambda_i(E) = (-1)^i\{\beta\mathcal{P}^i, u, \alpha\} \in H^{2i(p-1)-2}(B; \mathbb{Z})/\mathbb{Z}\kappa_{i(p-1)-1}(E).$$

Since $\kappa_{i(p-1)-1}(E)$ is divisible by p, the indeterminacy vanishes if we reduce $\lambda_i(E)$ modulo p. We will use the same notation for the reduced class $\lambda_i(E) \in H^*(B; \mathbb{Z}/p)$. Before proving Lemma 3.1, we need the following lemma from [1]. As before $e = e(U)$ denotes the Euler class in $H^2(\mathrm{SO}(N+2)/(\mathrm{SO}(N) \times \mathrm{SO}(2)))$.

Lemma 3.3 *In $H^*(\mathrm{Th}(U^\perp), \star; \mathbb{Z}/p)$ we have that*

$$\mathcal{P}^i u_{U^\perp} = (-1)^i e^{i(p-1)} u_{U^\perp}.$$

Proof Let $\mathcal{P} = \sum_i \mathcal{P}^i$. We first calculate the action of $\mathcal{P}$ in the Thom space of the two-dimensional bundle U. We claim that

$$\mathcal{P}(u_U) = (1 + e(U)^{p-1})u_U.$$

To see this we identify $\mathbb{R}^2 = \mathbb{C}$ and $\mathrm{SO}(2) = \mathrm{U}(1)$. This gives a complex structure on U and hence a classifying map to the universal complex line bundle $L \to \mathbb{C}P^\infty$. This in turn gives a map $\mathrm{Th}(U) \to \mathrm{Th}(L)$ so it suffices to calculate the Steenrod action in $\mathrm{Th}(L)$. There is a well known homeomorphism $\mathrm{Th}(L) \cong \mathbb{C}P^\infty$ under which u_U corresponds to $c_1(L)$ and $e^{i-1}u_U$ corresponds to c_1^i. The formula for $\mathcal{P}(u_U)$ above now follows from the following obvious formula in $H^*(\mathbb{C}P^\infty; \mathbb{Z}/p)$:

$$\mathcal{P}^i(c_1) = \begin{cases} c_1 & \text{if } i = 0, \\ c_1^p & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, since $u_{U \oplus U^\perp} = u_U u_{U^\perp}$ we get

$$u_U u_{U^\perp} = u_{U\oplus U^\perp} = \mathcal{P}(u_{U\oplus U^\perp}) = \mathcal{P}(u_U)\mathcal{P}(u_{U^\perp}) = (1 + e(U)^{p-1})u_U\mathcal{P}(u_{U^\perp})$$

and hence

$$\mathcal{P}(u_{U^\perp}) = (1+e(U)^{p-1})^{-1} u_{U^\perp} = \Big(\sum_i (-1)^i e(U)^{i(p-1)}\Big) u_{U^\perp}. \qquad \square$$

Proof of Lemma 3.1 Clearly $u \circ \alpha \simeq 0$. It follows from Lemma 3.3 that $\mathcal{P}^i u$ is the reduction of an integral class, so $\beta\mathcal{P}^i \circ u \simeq 0$. Therefore $\{\beta\mathcal{P}^i, u, \alpha\}$ is defined.

The indeterminacy can be computed from Lemma 2.1. Indeed we have

$$\beta\mathcal{P}^i[B_+ \wedge S^{N+3}, K(\mathbb{Z}, N)] = 0$$

and

$$\begin{aligned}[S\mathrm{Th}(U^\perp), K(\mathbb{Z}, N+2i(p-1)+1)]\circ\alpha &= \alpha^* H^{N+2i(p-1)}(\mathrm{Th}(U^\perp);\mathbb{Z}) \\ &= \alpha^*(\mathbb{Z}e^{i(p-1)}u_{U^\perp}) = \mathbb{Z}\kappa_{i(p-1)-1}(E). \quad \square\end{aligned}$$

Proof of Theorem 1.6 The property (1-1) follows from Proposition 2.2 and Corollary 2.4 and the diagram

$$\begin{tikzcd}
B_+ \wedge S^{N+2} \arrow[r, "\alpha"] \arrow[dr, "\kappa_{i(p-1)-1}"'] & \mathrm{Th}(U_N^\perp) \arrow[r, "u"] \arrow[d, "e^{i(p-1)}u"] & K(\mathbb{Z}, N) \arrow[d, "\mathcal{P}^i"] \\
 & K(\mathbb{Z}, N+2i(p-1)) \arrow[r, "\rho"] & K(\mathbb{Z}/p, N+2i(p-1)) \arrow[d, "\beta"] \\
 & & K(\mathbb{Z}, N+2i(p-1)+1).
\end{tikzcd}$$

Indeed, Proposition 2.2 gives the inclusions

$$\begin{aligned}\{\beta, \rho, \kappa_{i(p-1)-1}(E)\} = \{\beta, \rho, (e^{i(p-1)}u)\circ\alpha\} &\subseteq \{\beta, \rho\circ(e^{i(p-1)}u), \alpha\} \\ &= (-1)^i\{\beta, \mathcal{P}^i u, \alpha\} \supseteq (-1)^i\{\beta\mathcal{P}^i, u, \alpha\} = \lambda_i(E).\end{aligned}$$

Then Lemma 2.1 proves that the first inclusion is an equality since the two sides have the same indeterminacy $\mathrm{Im}(\beta) + \mathbb{Z}\kappa_{i(p-1)-1}$. Therefore Corollary 2.4 gives the inclusion

$$\lambda_i(E) \subseteq \{\beta, \rho, \kappa_{i(p-1)-1}(E)\} = \tfrac{1}{p}\kappa_{i(p-1)-1}(E) + \mathbb{Z}\kappa_{i(p-1)-1}(E),$$

and hence $p\lambda_i(E) \subseteq (1+p\mathbb{Z})\kappa_{i(p-1)-1}(E)$. Here the two sides of the inclusion have the same indeterminacy $p\mathbb{Z}\kappa_{i(p-1)-1}$ so they are equal and we get

$$p\lambda_i(E) = (1+p\mathbb{Z})\kappa_{i(p-1)-1}(E) \in H^*(B;\mathbb{Z})/p\mathbb{Z}\kappa_{i(p-1)-1}. \tag{3-1}$$

Firstly, (3-1) reproduces the fact that $\kappa_{i(p-1)-1}$ is divisible by p. Using this, we see that the indeterminacy vanishes after reducing (3-1) modulo p^2. This proves formula (1-1).

Now, (i) follows from the additivity of α under disjoint union, ie the property that

$$\alpha(E \amalg E') = \alpha(E) + \alpha(E') \in [B_+ \wedge S^{N+2}, \mathrm{Th}(U_N^\perp)].$$

Similarly (ii) follows from the "additivity" of α under gluing. Explicitly, a choice of embedding $j_\partial\colon \partial E_1 \to B \times \mathbb{R}^{N+1}$ over B will induce a map

$$\alpha_\partial\colon B_+ \wedge S^{N+1} \to \mathrm{Th}(U^\perp).$$

A choice of embedding $j_{E_1}\colon E_1 \to B \times [0,\infty) \times \mathbb{R}^{N+1}$ extending j_∂ will induce a nullhomotopy α_{E_1} of α_∂. Then we have

$$\begin{aligned} \alpha_E &= \alpha_{E_1} - \alpha_{E_2}, \\ \alpha_{E'} &= \alpha_{E_1} - \alpha_{E_2'}, \\ \alpha_D &= \alpha_{E_2'} - \alpha_{E_2}. \end{aligned}$$

Thus we get

$$\alpha_{E'} = \alpha_E + \alpha_D \in [B_+ \wedge S^{N+2}, \mathrm{Th}(U^\perp)]. \qquad \square$$

Remark 3.4 The proof in [1] of the "upper bound", Theorem 1.2(i), is based on maps $\varphi\colon B(\mathbb{Z}/p^n) \to B\mathrm{Diff}(\Sigma)$ for a suitable action of a cyclic group of order p^n on a Riemann surface Σ, first constructed in [2]. This gives a class

$$\varphi^*(\kappa_i) \in H^{2i}(B(\mathbb{Z}/p^n);\mathbb{Z}) = \mathbb{Z}/p^n,$$

and it follows from the calculations in [2] and [1] that for $n > v$, $\varphi^*(\kappa_i)$ is divisible by p^{v+1} if and only if $i+1$ is divisible by $p^v(p-1)$. Let us set $p=3$, $v=0$, $n=2$, and $i=1$. Then

$$\varphi^*(\kappa_1) \in H^2(B(\mathbb{Z}/9);\mathbb{Z}) = \mathbb{Z}/9$$

is divisible by 3 but not 9, so it is 3 times a generator of $\mathbb{Z}/9$. Property (1-1) says in this case that $3\lambda_1 = \kappa_1$, so $\varphi^*(\lambda_1)$ must be a generator of $\mathbb{Z}/9$ (with indeterminacy $3\mathbb{Z}/9$). Now let B be the lens space $B = S^3/(\mathbb{Z}/3)$ and let $B \to B(\mathbb{Z}/9)$ be the map which multiplies by 3 in π_1 and H^2. We get a map

$$\psi\colon B \to B\mathrm{Diff}(\Sigma)$$

such that $\psi^*(\kappa_1) = 0 \in H^2(B;\mathbb{Z}) = \mathbb{Z}/3$, but $\psi^*(\lambda_1)$ is a generator of $\mathbb{Z}/3$. The surface bundle $E \to B$ classified by ψ is an example of a bundle whose nontriviality is detected by λ_1 but not by any κ_i.

4 A variant of λ_{ps}

The goal of this section is to prove Theorem 1.4. We have already seen that $\kappa_{i(p-1)-1}$ is divisible by p. When $i = ps$ for some $s > 0$, a variant of λ_{ps} can be used to prove that $\kappa_{ps(p-1)-1}$ is divisible by p^2. Again, let $\mathfrak{A}_p$ be the Steenrod algebra. When $p = 2$ we write $\mathcal{P}^i = \mathrm{Sq}^{2i}$ and $\beta\mathcal{P}^i = \mathrm{Sq}^{2i+1}$ as before.

Definition 4.1 Let $s \geq 0$ and define $\theta_s \in \mathfrak{A}_p$ by

$$\theta_s = \sum_{j=0}^{s} (-1)^j \binom{(p-1)(s-j)}{j} \mathcal{P}^{ps-j}\mathcal{P}^j = \mathcal{P}^{ps} + \text{terms of length 2.}$$

Define vectors $v_s, w_s \in \mathfrak{A}_p^{s+1}$ by

$$w_s = (\mathcal{P}^0, \dots \mathcal{P}^s), \ \ v_s = (\mathcal{P}^{ps}, \dots, (-1)^j \binom{(p-1)(s-j)-1}{j} \mathcal{P}^{ps-j}, \dots, \mathcal{P}^{(p-1)s}).$$

Lemma 4.2

(i) *In $H^*(\mathrm{Th}(U^\perp), \star; \mathbb{Z}/p)$ we have that $\theta_s u_{U^\perp} = e^{ps(p-1)} u_{U^\perp}$.*

(ii) $v_s^T \beta w_s = \beta\theta_s$.

Proof (i) This is similar to Lemma 3.3, using the fact that the admissible terms of length 2 act trivially on $u_{U^\perp}$. Formula (ii) is the Adem relation for $\mathcal{P}^{(p-1)s}\beta\mathcal{P}^s$. □

Definition 4.3 Let α, u, θ_s be as above. Define the secondary characteristic class

$$\widetilde{\lambda}_{ps}(E) = (-1)^s\{\beta\theta_s, u, \alpha\} \in H^{2ps(p-1)-2}(B, \mathbb{Z})/\mathbb{Z}\kappa_{ps(p-1)-1}(E).$$

Notice that $\widetilde{\lambda}_{ps}$ satisfies the same formal properties as λ_{ps}. In particular $p\widetilde{\lambda}_{ps} = (1 + p\mathbb{Z})\kappa_{ps(p-1)-1}$.

Proof of Theorem 1.4 We have

$$\begin{aligned}(-1)^s \rho \circ \{\beta\theta_s, u, \alpha\} \subseteq (-1)^s\{\rho \circ \beta\theta_s, u, \alpha\} &= (-1)^s\{v_s^T \beta w_s, u, \alpha\} \\ &\supseteq (-1)^s v_s^T\{\beta w_s, u, \alpha\}\end{aligned}$$

and we see that all the inclusions are equalities since the indeterminacies vanish. Since

$$(-1)^s\{\beta w_s, u, \alpha\} \in \prod_{i=0}^{s} H^{N+2i(p-1)}(B_+ \wedge S^{N+2}; \mathbb{Z}/p) = \prod_{i=0}^{s} H^{2i(p-1)-2}(B; \mathbb{Z}/p),$$

v^T will vanish because $H^*(B;\mathbb{Z}/p)$ is an unstable $\mathfrak{A}_p$–module.

Hence the mod p reduction of $\widetilde{\lambda}_{ps}(E)$ vanishes, so $\kappa_{ps(p-1)-1}(E) = p\widetilde{\lambda}_{ps}(E) = 0 \in H^*(B;\mathbb{Z}/p^2)$. □

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