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Cap products in string topology

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Chas and Sullivan showed that the homology of the free loop space LM of an oriented closed smooth manifold M admits the structure of a Batalin–Vilkovisky (BV) algebra equipped with an associative product (loop product) and a Lie bracket (loop bracket). We show that the cap product is compatible with the above two products in the loop homology. Namely, the cap product with cohomology classes coming from M via the circle action acts as derivations on the loop product as well as on the loop bracket. We show that Poisson identities and Jacobi identities hold for the cap product action, turning $H^*(M) \oplus \mathbb{H}_*(LM)$ into a BV algebra. Finally, we describe cap products in terms of the BV algebra structure in the loop homology.

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1 Introduction

Let M be a closed oriented smooth d–manifold. Let

$$D\colon H_*(M) \xrightarrow{\cong} H^{d-*}(M)$$

be the Poincaré duality map. Following a practice in string topology, we shift the homology grading downward by d and let $\mathbb{H}_{-*}(M) = H_{d-*}(M)$. The Poincaré duality now takes the form

$$D\colon \mathbb{H}_{-*}(M) \xrightarrow{\cong} H^*(M).$$

For a homology element a, let $|a|$ denote its $\mathbb{H}_*$–grading of a.

The intersection product $\cdot$ in homology is defined as the Poincaré dual of the cup product. Namely, for $a, b \in \mathbb{H}_*(M)$, $D(a \cdot b) = D(a) \cup D(b)$. If $\alpha \in H^*(M)$ is dual to a, then $\alpha \cap b = a \cdot b$ and its Poincaré dual is $\alpha \cup D(b)$. Thus, through Poincaré duality, the intersection product, the cap product and the cup product are all the same. In particular, the cap product and the intersection product commute:

$$\alpha \cap (b \cdot c) = (\alpha \cap b) \cdot c = (-1)^{|\alpha||b|} b \cdot (\alpha \cap c). \tag{1-1}$$

In fact, the direct sum $H^*(M) \oplus \mathbb{H}_*(M)$ can be made into a graded commutative associative algebra with unit, given by $1 \in H^0(M)$, using the cap and the cup product.

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For an infinite-dimensional manifold N, there is no longer Poincaré duality, and geometric intersections of finite dimensional cycles are all trivial. However, cap products can still be nontrivial and the homology $H_*(N)$ is a module over the cohomology ring $H^*(N)$.

When the infinite-dimensional manifold N is a free loop space LM of continuous maps from the circle $S^1 = \mathbb{R}/\mathbb{Z}$ to M, the homology $\mathbb{H}_*(LM) = H_{*+d}(LM)$ has a great deal more structure. As before, $|a|$ denotes the $\mathbb{H}_*$–grading of a homology element a of LM. Chas and Sullivan [1] showed that $\mathbb{H}_*(LM)$ has a degree-preserving associative graded commutative product $\cdot$ called the loop product, a Lie bracket $\{\ ,\ \}$ of degree 1 called the loop bracket compatible with the loop product and the BV operator Δ of degree 1 coming from the homology S^1 action. These structures turn $\mathbb{H}_*(LM)$ into a Batalin–Vilkovisky (BV) algebra. The purpose of this paper is to clarify the interplay between the cap product with cohomology elements and the BV structure in $\mathbb{H}_*(LM)$.

Let $p\colon LM \to M$ be the base point map $p(\gamma) = \gamma(0)$ for $\gamma \in LM$. For a cohomology class $\alpha \in H^*(M)$ in the base manifold, its pullback $p^*(\alpha) \in H^*(LM)$ is also denoted by α. Let $\Delta\colon S^1 \times LM \longrightarrow LM$ be the S^1–action map. This map induces a degree 1 map Δ in homology given by $\Delta a = \Delta_*([S^1] \times a)$ for $a \in \mathbb{H}_*(LM)$. For a cohomology class $\beta \in H^*(LM)$, the formula $\Delta^*(\beta) = 1 \times \beta + \{S^1\} \times \Delta\beta$ defines a degree -1 map Δ in cohomology, where $\{S^1\}$ is the fundamental cohomology class. Although we use the same notation Δ in three different but closely related situations, what is meant by Δ should be clear in the context.

Theorem A *Let $b, c \in \mathbb{H}_*(LM)$. The cap product with $\alpha \in H^*(M)$ graded commutes with the loop product. Namely*

$$\alpha \cap (b \cdot c) = (\alpha \cap b) \cdot c = (-1)^{|\alpha||b|} b \cdot (\alpha \cap c). \tag{1-2}$$

For $\alpha \in H^*(M)$, the cap product with $\Delta\alpha \in H^*(LM)$ acts as a derivation on the loop product and the loop bracket:

$$(\Delta\alpha) \cap (b \cdot c) = (\Delta\alpha \cap b) \cdot c + (-1)^{(|\alpha|-1)|b|} b \cdot (\Delta\alpha \cap c), \tag{1-3}$$

$$(\Delta\alpha) \cap \{b, c\} = \{\Delta\alpha \cap b, c\} + (-1)^{|\alpha|-1)(|b|+1)} \{b, \Delta\alpha \cap c\}. \tag{1-4}$$

The operator Δ acts as a derivation on the cap product. Namely, for $\alpha \in H^*(M)$ and $b \in \mathbb{H}_*(LM)$,

$$\Delta(\alpha \cap b) = \Delta\alpha \cap b + (-1)^{|\alpha|} \alpha \cap \Delta b. \tag{1-5}$$

We recall that in the BV algebra $\mathbb{H}_*(LM)$, the following identities are valid for $a, b, c \in \mathbb{H}_*(LM)$ [1]:

(BV identity) $\quad \Delta(a\cdot b) = (\Delta a)\cdot b + (-1)^{|a|} a\cdot \Delta b + (-1)^{|a|}\{a,b\}$

(Poisson identity) $\quad \{a, b\cdot c\} = \{a,b\}\cdot c + (-1)^{|b|(|a|+1)} b\cdot\{a,c\}$

(Commutativity) $\quad a\cdot b = (-1)^{|a||b|} b\cdot a, \qquad \{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\}$

(Jacobi identity) $\quad \{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}$

Here, $\deg a\cdot b = |a|+|b|$, $\deg \Delta a = |a|+1$ and $\deg\{a,b\} = |a|+|b|+1$.

We can extend the loop product and the loop bracket in $\mathbb{H}_*(LM)$ to include $H^*(M)$ in the following way. For $\alpha \in H^*(M)$ and $b \in \mathbb{H}_*(LM)$, we define the loop product and the loop bracket of α and b by

$$\alpha\cdot b = \alpha\cap b, \qquad \{\alpha, b\} = (-1)^{|\alpha|}(\Delta\alpha)\cap b. \tag{1-6}$$

Furthermore, the BV structure in $\mathbb{H}_*(LM)$ can be extended to the direct sum $A_* = H^*(M)\oplus\mathbb{H}_*(LM)$ by defining the BV operator $\boldsymbol{\Delta}$ on A_* to be trivial on $H^*(M)$ and to be the usual homological S^1 action Δ on $\mathbb{H}_*(LM)$. Here in A_*, elements in $H^k(M)$ are regarded as having homological degree $-k$.

Theorem B *The direct sum $H^*(M)\oplus\mathbb{H}_*(LM)$ has a structure of a BV algebra. In particular, for $\alpha\in H^*(M)$ and $b, c\in\mathbb{H}_*(LM)$, the following form of the Poisson identity and the Jacobi identity hold*:

$$\begin{aligned}\{\alpha\cdot b, c\} &= \alpha\cdot\{b,c\} + (-1)^{|b|(|c|+1)}\{\alpha,c\}\cdot b\\ &= \alpha\cdot\{b,c\} + (-1)^{|\alpha||b|} b\cdot\{\alpha,c\},\end{aligned} \tag{1-7}$$

$$\{\alpha,\{b,c\}\} = \{\{\alpha,b\},c\} + (-1)^{(|\alpha|+1)(|b|+1)}\{b,\{\alpha,c\}\}. \tag{1-8}$$

All the other possible forms of Poisson and Jacobi identities are also valid, and the above two identities are the most nontrivial ones. These identities are proved by using standard properties of the cap product and the BV identity above in $\mathbb{H}_*(LM)$ relating the BV operator Δ and the loop bracket $\{\ ,\ \}$, but without using Poisson identities nor Jacobi identities in the BV algebra $\mathbb{H}_*(LM)$.

The above identities may seem rather surprising, but they become transparent once we prove the following result.

Theorem C *For $\alpha \in H^*(M)$, let $a = \alpha \cap [M] \in \mathbb{H}_*(M)$ be its Poincaré dual. Then for $b \in \mathbb{H}_*(LM)$,*

$$\alpha \cap b = a \cdot b, \qquad (-1)^{|\alpha|}\Delta\alpha \cap b = \{a, b\}. \tag{1-9}$$

More generally, for cohomology elements $\alpha_0, \alpha_1, \dots, \alpha_r \in H^*(M)$, let $a_0, a_1, \dots, a_r \in \mathbb{H}_*(M)$ be their Poincaré duals. Then for $b \in \mathbb{H}_*(LM)$, we have

$$(\alpha_0 \cup \Delta\alpha_1 \cup \cdots \cup \Delta\alpha_r) \cap b = (-1)^{|a_1|+\cdots+|a_r|} a_0 \cdot \{a_1, \{a_2, \dots, \{a_r, b\}\cdots\}\}. \tag{1-10}$$

Since the cohomology $H^*(M)$ and the homology $\mathbb{H}_*(M)$ are isomorphic through Poincaré duality and $\mathbb{H}_*(M)$ is a subring of $\mathbb{H}_*(LM)$, the first formula in (1-9) is not surprising. However, the main difference between $H^*(M)$ and $\mathbb{H}_*(M)$ in our context is that the homology S^1 action Δ is trivial on $\mathbb{H}_*(M) \subset \mathbb{H}_*(LM)$, although cohomology S^1 action Δ is nontrivial on $H^*(M)$ and is related to loop bracket as in (1-9).

Theorems A and C describe the cap product action of the cohomology $H^*(LM)$ on the BV algebra $\mathbb{H}_*(LM)$ for most elements in $H^*(LM)$. For example, for $\alpha \in H^*(M)$, the cap product with $\Delta\alpha$ is a derivation on the loop algebra $\mathbb{H}_*(LM)$ given by a loop bracket, and consequently the cap product with a cup product $\Delta\alpha_1 \cup \cdots \cup \Delta\alpha_r$ acts on the loop algebra as a composition of derivations, which is equal to a composition of loop brackets, according to (1-10). If $H^*(LM)$ is generated by elements α and $\Delta\alpha$ for $\alpha \in H^*(M)$ (for example, this is the case when $H^*(M)$ is an exterior algebra; see Remark 5.3), then Theorem C gives a complete description of the cap product with arbitrary elements in $H^*(LM)$ in terms of the BV algebra structure in $\mathbb{H}_*(LM)$. However, $H^*(LM)$ is in general bigger than the subalgebra generated by $H^*(M)$ and $\Delta H^*(M)$.

Since $\mathbb{H}_*(LM)$ is a BV algebra, in view of Theorem C, the validity of Theorem B may seem obvious. However, in the proof of Theorem B, we only used standard properties of the cap product and the BV identity. In fact, Theorem B gives an alternate elementary and purely homotopy theoretic proof of the Poisson and Jacobi identities in $\mathbb{H}_*(LM)$, when at least one of the elements a, b, c are in $\mathbb{H}_*(M)$. Similarly, Theorem C gives a purely homotopy theoretic interpretation of the loop product and the loop bracket if one of the elements is in $\mathbb{H}_*(M)$.

Our interest in cap products in string topology comes from an intuitive geometric picture that cohomology classes in LM are dual to finite codimension submanifolds of LM consisting of certain loop configurations. We can consider configurations of loops intersecting in particular ways (for example, two loops having their base points in common), or we can consider a family of loops intersecting transversally with submanifolds of M at certain points of loops. In a given family of loops, taking the cap product with a cohomology class selects a subfamily of a certain loop configuration, which are ready for certain loop interactions. In this context, roughly speaking, composition of two interactions of loops correspond to the cup product of corresponding cohomology classes.

The organization of this paper is as follows. In Section 2, we describe a geometric problem of describing a certain family of intersection configuration of loops in terms of cap products. This gives a geometric motivation for the remainder of the paper. In Section 3, we review the loop product in $\mathbb{H}_*(LM)$ in detail from the point of view of the intersection product in $\mathbb{H}_*(M)$. Here we pay careful attention to signs. In particular, we give a homotopy theoretic proof of graded commutativity in the BV algebra $\mathbb{H}_*(LM)$, which turned out to be not so trivial. In Section 4, we prove compatibility relations between the cap product and the BV algebra structure, and prove Theorems A and B. In the last section, we prove Theorem C.

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2 Cap products and intersections of loops

Let $A_1, A_2, \dots, A_r$ and $B_1, B_2, \dots, B_s$ be oriented closed submanifolds of M^d. Let $F \subset LM$ be a compact family of loops. We consider the following question.

Question Fix r points $0 \le t_1^*, t_2^*, \dots, t_r^* \le 1$ in $S^1 = \mathbb{R}/\mathbb{Z}$. Describe the homology class of the subset I of the compact family F consisting of loops γ in F such that γ intersects submanifolds $A_1, \dots, A_r$ at time $t_1^*, \dots, t_r^*$ and intersects $B_1, \dots, B_s$ at some unspecified time.

This subset $I \subset F$ can be described as follows. We consider the following diagram of an evaluation map and a projection map:

(2-1)
$$\begin{array}{ccc}
\overbrace{(S^1 \times \cdots \times S^1)}^{s} \times LM & \xrightarrow{\ e\ } & \overbrace{M \times \cdots \times M}^{r} \times \overbrace{M \times \cdots \times M}^{s} \\
\Big\downarrow{\scriptstyle \pi_2} & & \\
LM & &
\end{array}$$

given by $e\big((t_1,\dots,t_s),\gamma\big)=\big(\gamma(t_1^*),\dots,\gamma(t_r^*),\gamma(t_1),\dots,\gamma(t_s)\big)$. Then the pullback set $e^{-1}(\prod_i A_i\times\prod_j B_j)$ is a closed subset of $S^1\times\cdots\times S^1\times LM$. Let

$$\widetilde{I}=e^{-1}\bigg(\prod_i A_i\times\prod_j B_j\bigg)\cap(S^1\times\cdots\times S^1\times F).$$

The set I in question is given by $I=\pi_2(\widetilde{I})$. We want to understand this set I homologically, including multiplicity. Although $e^{-1}(\prod_i A_i\times\prod_j B_j)$ is infinite dimensional, it has finite codimension in $(S^1)^r\times LM$. So we work cohomologically.

Let $\alpha_i,\beta_j\in H^*(M)$ be cohomology classes dual to $[A_i],[B_j]$ for $1\le i\le r$ and $1\le j\le s$. Then the subset $e^{-1}(\prod_i A_i\times\prod_j B_j)$ is dual to the cohomology class $e^*(\prod_i\alpha_i\times\prod_j\beta_j)\in H^*((S^1)^s\times LM)$. Suppose the family F is parametrized by a closed oriented manifold K by an onto map $\lambda\colon K\longrightarrow F$ and let $b=\lambda_*([K])\in\mathbb{H}_*(LM)$ be the homology class of F in LM. Then the homology class of $\widetilde{I}$ in $(S^1)^s\times LM$ is given by

$$[\widetilde{I}]=e^*\bigg(\prod_i\alpha_i\times\prod_j\beta_j\bigg)\cap([S^1\times\cdots\times S^1]\times b). \tag{2-2}$$

Note that the homology class $(\pi_2)_*([\widetilde{I}])$ represents the homology class of I with multiplicity.

Proposition 2.1 *With the above notation, $(\pi_2)_*([\widetilde{I}])$ is given by the following formula in terms of the cap product or in terms of the BV structure:*

$$\begin{aligned}(\pi_2)_*([\widetilde{I}])&=(-1)^{\sum_j j|\beta_j|-s}\big(\alpha_1\cdots\alpha_r(\Delta\beta_1)\cdots(\Delta\beta_s)\big)\cap b\\&=(-1)^{\sum_j j|\beta_j|-s}[A_1]\cdots[A_s]\cdot\{[B_1],\{\cdots\{[B_s],b\}\cdots\}\in\mathbb{H}_*(LM).\end{aligned}\tag{2-3}$$

Proof The evaluation map e in (2-1) is given by the following composition.

$$\begin{aligned}\overbrace{S^1\times\cdots\times S^1}^{s}\times LM&\xrightarrow{1\times\phi}(S^1\times\cdots\times S^1)\times\overbrace{LM\times\cdots\times LM}^{r+s}\\&\xrightarrow{T}\overbrace{LM\times\cdots\times LM}^{r}\times\overbrace{(S^1\times LM)\times\cdots\times(S^1\times LM)}^{s}\\&\xrightarrow{1^r\times\Delta^s}(LM\times\cdots\times LM)\times(LM\times\cdots\times LM)\xrightarrow{p^r\times p^s}(M\times\cdots\times M)\times(M\times\cdots\times M),\end{aligned}$$

where ϕ is a diagonal map, T moves S^1 factors. Since we apply $(\pi_2)_*$ later, we only need terms in $e^*(\prod A_i\times\prod B_j)$ containing the factor $\{S^1\}\times\cdots\times\{S^1\}$. Since

$\Delta^* p^*(\beta_j) = 1 \times p^*(\beta_j) + \{S^1\} \times \Delta\beta_j$ for $1 \le j \le s$, following the above decomposition of e, we have

$$e^*(\alpha_1 \times \cdots \times \alpha_r \times \beta_1 \times \cdots \times \beta_s) = \varepsilon \{S^1\}^s \times \big(\alpha_1 \cdots \alpha_r (\Delta\beta_1) \cdots (\Delta\beta_s)\big) + \text{ other terms,}$$

where the sign ε is given by

$$\varepsilon = (-1)^{\sum_{\ell=1}^{s} (s-\ell)(|\beta_\ell|-1) + s \sum_{\ell=1}^{r} |\alpha_\ell|}.$$

Thus, taking the cap product with $[S^1]^s \times b$ and applying $(\pi_2)_*$, we get

$$\pi_{2*}\big(e^*(\alpha_1 \times \cdots \times \alpha_r \times \beta_1 \times \cdots \times \beta_s) \cap ([S^1] \times \cdots \times [S^1] \times b)\big)$$
$$= (-1)^{\sum_{\ell=1}^{s} \ell|\beta_\ell| - s} \alpha_1 \cdots \alpha_r (\Delta\beta_1) \cdots (\Delta\beta_s) \cap b.$$

The second equality follows from the formula (1-10). □

Remark 2.2 In the diagram (2-1), in terms of cohomology transfer ${\pi_2}^!$ we have

$$\text{(2-4)} \qquad {\pi_2}^! e^*(\alpha_1 \times \cdots \times \alpha_r \times \beta_1 \times \cdots \times \beta_s) = \pm \alpha_1 \cdots \alpha_r (\Delta\beta_1) \cdots (\Delta\beta_s),$$

where ${\pi_2}^!(\alpha) \cap b = (-1)^{s|\alpha|} \pi_{2*}\big(\alpha \cap \pi_{2!}(b)\big)$ for any $\alpha \in H^*((S^1)^s \times LM)$ and $b \in \mathbb{H}_*(LM)$. Here $\pi_{2!}(b) = [S^1]^s \times b$.

3 The intersection product and the loop product

Let M be a closed oriented smooth d–manifold. The loop product in $\mathbb{H}_*(LM)$ was discovered by Chas and Sullivan [1], in terms of transversal chains. Later, Cohen and Jones [2] gave a homotopy theoretic description of the loop product. The loop product is a hybrid of the intersection product in $\mathbb{H}_*(M)$ and the Pontrjagin product in the homology of the based loop spaces $H_*(\Omega M)$. In this section, we review and prove some properties of the loop product in preparation for the next section. Our treatment of the loop product follows [2]. However, we will be precise with signs and give a homotopy theoretic proof of the graded commutativity of the loop product, which [2] did not include. For the Frobenius compatibility formula with careful discussion of signs, see Tamanoi [9]. For homotopy theoretic deduction of the BV identity, see Tamanoi [8].

For our purpose, the free loop space LM is the space of *continuous* maps from $S^1 = \mathbb{R}/\mathbb{Z}$ to M. Our discussion is homotopy theoretic and does not require smoothness of loops, although we do need smoothness of M which is enough to allows us to have tubular neighborhoods for certain submanifolds in the space of continuous loops.

Recall that the space LM of continuous loops can be given a structure of a smooth manifold. See the discussion before Definition 3.2.

Let $p\colon LM \longrightarrow M$ be the base point map given by $p(\gamma)=\gamma(0)$. Let $s\colon M \longrightarrow LM$ be the constant loop map given by $s(x)=c_x$, where c_x is the constant loop at $x\in M$. Since $p_*\circ s_*=1$, $\mathbb{H}_*(M)$ is contained in $\mathbb{H}_*(LM)$ through s_* and we often regard $\mathbb{H}_*(M)$ as a subset of $\mathbb{H}_*(LM)$.

We start with a discussion on the intersection ring $\mathbb{H}_*(M)$ and later we compare it with the loop homology algebra $\mathbb{H}_*(LM)$. An exposition on intersection products in homology of manifolds can be found on Dold's book [4, Chapter VIII, Section 13]. Our sign convention (which follows Milnor [6]) is slightly different from Dold's.

Those who are familiar with intersection product and loop products can skip this section after checking Definition 3.2.

Let $D\colon \mathbb{H}_*(M) \xrightarrow{\cong} H^{d-*}(M)$ be the Poincaré duality map such that $D(a)\cap[M]=a$ for $a\in\mathbb{H}_*(M)$. We discuss two ways to define intersection product in $\mathbb{H}_*(M)$. The first method defines the intersection product as the Poincaré dual of the cohomology cup product. Thus, $D(a\cdot b)=D(a)\cup D(b)$ for $a,b\in\mathbb{H}_*(M)$. For example, we have $a\cdot b=(-1)^{|a||b|}b\cdot a$.

The second method uses the transfer map induced from the diagonal map $\phi\colon M\longrightarrow M\times M$. Let ν be the normal bundle to $\phi(M)$ in $M\times M$, and we orient ν by $\nu\oplus\phi_*(TM)\cong T(M\times M)|_{\phi(M)}$. Let $u'\in H^d(\phi(M)^\nu)$ be the Thom class of ν. Let N be a closed tubular neighborhood of $\phi(M)$ in $M\times M$ so that $D(\nu)\cong N$, where $D(\nu)$ is the associated closed disc bundle of ν. Let $\pi\colon N\longrightarrow M$ be the projection map. Then the above Thom class can be thought of as $u'\in\widetilde{H}^d(N/\partial N)$, and we have the following commutative diagram, where $c\colon M\times M\longrightarrow N/\partial N$ is the Thom collapse map, and ι_N and j are obvious maps.

$$\begin{array}{ccc} H^d(N,N-\phi(M)) & \xrightarrow{\cong} & H^d(N,\partial N)\ni u' \\ \cong\Big\uparrow \iota_N^* & & \Big\downarrow c^* \\ u''\in H^d\big(M\times M, M\times M-\phi(M)\big) & \xrightarrow{j^*} & H^d(M\times M)\ni u \end{array} \tag{3-1}$$

Let $u''\in H^d\big(M\times M, M\times M-\phi(M)\big)$ and $u\in H^d(M\times M)$ be the classes corresponding to the Thom class. We have $u=c^*(u')=j^*(u'')$. This class u is characterized by the property $u\cap[M\times M]=\phi_*([M])$, and $\phi^*(u)=e_M\in H^d(M)$ is the Euler class of M. See for example Section 11 of [6]. The transfer map $\phi_!$ is defined as the following composition:

$$\phi_!\colon H_*(M\times M)\xrightarrow{c_*}\widetilde{H}_*(N/\partial N)\xrightarrow[\cong]{u'\cap(\ )} H_{*-d}(N)\xrightarrow[\cong]{\pi_*} H_{*-d}(M). \tag{3-2}$$

For a homology element a, let $|a|'$ denote its regular homology degree of a, so that we have $a \in H_{|a|}(M)$ and $|a|' = |a| + d$.

Proposition 3.1 *Suppose M is a connected oriented closed d–manifold with a base point x_0. The transfer map $\phi_!\colon H_*(M \times M) \longrightarrow H_{*-d}(M)$ satisfies the following properties. For $a, b \in H_*(M)$,*

$$\phi_*\phi_!(a \times b) = u \cap (a \times b), \tag{3-3}$$

$$\phi_!\phi_*(a \times b) = \chi(M)[x_0]. \tag{3-4}$$

For $\alpha \in H^*(M \times M)$ and $b, c \in H_*(M)$, we have

$$\phi_!\big(\alpha \cap (b \times c)\big) = (-1)^{d|\alpha|}\phi^*(\alpha) \cap \phi_!(b \times c). \tag{3-5}$$

The intersection product and the transfer map coincide up to a sign:

$$a \cdot b = (-1)^{d(|a|'-d)}\phi_!(a \times b). \tag{3-6}$$

Proof For the first identity, we consider the following commutative diagram, where M^2 denotes $M \times M$.

$$\begin{tikzcd}
H_*(M^2) \arrow[r, "c_*"] \arrow[d, equal] & H_*(N, \partial N) \arrow[r, "u'\cap(\,)", "\cong"'] \arrow[d, "\iota_{N*}", "\cong"'] & H_{*-d}(N) \arrow[r, "\pi_*", "\cong"'] \arrow[d, "\iota_{N*}"] & H_{*-d}(M) \arrow[d, "\phi_*"] \\
H_*(M^2) \arrow[r, "j_*"] & H_*\big(M^2, M^2 - \phi(M)\big) \arrow[r, "u''\cap(\,)", "\cong"'] & H_{*-d}(M^2) \arrow[r, equal] & H_{*-d}(M^2)
\end{tikzcd}$$

The commutativity implies that for $a, b \in H_*(M)$, we have $\phi_*\phi_! = u'' \cap j_*(a \times b) = j^*(u'') \cap (a \times b) = u \cap (a \times b)$.

To check the second formula, we first compute $\phi_*\phi_!\phi_*([M])$. By the first formula, $\phi_*\phi_!\phi_*([M]) = u \cap \phi_*([M]) = \phi_*(\phi^*(u) \cap [M])$. Since $\phi^*(u)$ is the Euler class e_M, this is equal to $\phi_*(e_M \cap [M]) = \chi(M)[(x_0, x_0)]$. Since M is assumed to be connected, ϕ_* is an isomorphism in H_0. Hence $\phi_!\phi_*([M]) = \chi(M)[x_0] \in H_0(M)$.

For the next formula, we examine the commutative diagram

$$\begin{tikzcd}
H_*(M^2) \arrow[r, "c_*"] \arrow[d, "\alpha\cap(\,)"'] & H_*(N, \partial N) \arrow[r, "u'\cap(\,)", "\cong"'] \arrow[d, "\iota_N^*(\alpha)\cap(\,)"'] & H_{*-d}(N) \arrow[d, "\iota_N^*(\alpha)\cap(\,)"'] & H_{*-d}(M) \arrow[l, "\iota'_*"', "\cong"] \arrow[d, "\iota^*(\alpha)"'] \\
H_{*-|\alpha|}(M^2) \arrow[r, "c_*"] & H_{*-|\alpha|}(N, \partial N) \arrow[r, "u'\cap(\,)", "\cong"'] & H_{*-d-|\alpha|}(N) & H_{*-d-|\alpha|}(M) \arrow[l, "\iota'_*"', "\cong"]
\end{tikzcd}$$

where ι': $M \to N$ is an inclusion map and $\iota'_* = (\pi_*)^{-1}$. The middle square commutes up to the factor $(-1)^{|\alpha|d}$. The commutativity of this diagram immediately implies that $\iota^*(\alpha) \cap \phi_!(a \times b) = (-1)^{|\alpha|d}\phi_!\big(\alpha \cap (a \times b)\big)$.

For the last identity, we apply ϕ_* on both sides and compare. Since the formula $a \cdot b = \phi^*(D(a) \times D(b)) \cap [M]$ holds, we have

$$\begin{aligned}\phi_*(a \cdot b) &= \big(D(a) \times D(b)\big) \cap \phi_*([M]) \\ &= \big(D(a) \times D(b)\big) \cap \big(u \cap [M]\big) \\ &= (-1)^{d(|a|'-d)} u \cap (a \times b) = (-1)^{d(|a|'-d)} \phi_* \phi_!(a \times b).\end{aligned}$$

Since ϕ_* is injective, we have $a \cdot b = (-1)^{d(|a|'-d)}\phi_!(a \times b)$. □

These two intersection products differ only in signs. However, the formulas for graded commutativity take different forms:

$$a \cdot b = (-1)^{(d-|a|')(d-|b|')} b \cdot a \tag{3-7}$$

$$\phi_!(a \times b) = (-1)^{|a|'|b|'+d}\phi_!(b \times a) \tag{3-8}$$

The sign $(-1)^d$ in the second formula above comes from the fact that the Thom class $u \in H^d(M \times M)$ satisfies $T^*(u) = (-1)^d u$, where T is the switching map of factors.

Next we turn to the loop product in $H_*(LM)$. We consider the diagram

$$\begin{tikzcd} LM \times LM \arrow[d, "p \times p"'] & LM \times_M LM \arrow[l, "j"'] \arrow[r, "\iota"] \arrow[d, "q"] & LM \\ M \times M & M \arrow[l, "\phi"'] & \end{tikzcd} \tag{3-9}$$

where $LM \times_M LM = (p \times p)^{-1}(\phi(M))$ consists of pairs of loops (γ, η) with the same base point, and $\iota(\gamma, \eta) = \gamma \cdot \eta$ is the product of composable loops. Let $\widetilde{N} = (p \times p)^{-1}(N)$ and let $\widetilde{c}$: $LM \times LM \longrightarrow \widetilde{N}/\partial\widetilde{N}$ be the Thom collapse map. Let $\widetilde{\pi}$: $\widetilde{N} \longrightarrow LM \times_M LM$ be a projection map defined as follows. For $(\gamma, \eta) \in \widetilde{N}$, let their base points be $(x, y) \in N$. Let $\pi(x, y) = (z, z) \in \phi(M)$. Since $N \cong D(\nu)$ has a bundle structure, let $\ell(t) = (\ell_1(t), \ell_2(t))$ be the straight ray in the fiber over (z, z) from (z, z) to (x, y). Then let $\widetilde{\pi}\big((\gamma, \eta)\big) = (\ell_1 \cdot \gamma \cdot \ell_1^{-1}, \ell_2 \cdot \eta \cdot \ell_2^{-1})$. By considering $\ell_{[t,1]}$, we see that $\widetilde{\pi}$ is a deformation retraction.

In fact, more is true. Stacey [7, Proposition 5.3] showed that when $L_{\text{smooth}}M$ is the space of *smooth* loops, $\widetilde{N}$ has an actual structure of a tubular neighborhood of $LM \times_M LM$ inside of $LM \times LM$ equipped with a diffeomorphism $p^*\big(D(\nu)\big) \cong \widetilde{N}$. His proof only uses the smoothness of M and exactly the same proof applies to the space LM of *continuous* loops and $\widetilde{N}$ still has the structure of a tubular neighborhood and we again have a diffeomorphism $p^*\big(D(\nu)\big) \cong \widetilde{N}$ between spaces of continuous loops.

Let $\widetilde{u}' = (p\times p)^*(u') \in \widetilde{H}^d(\widetilde{N}/\partial\widetilde{N})$, and $\widetilde{u} = (p\times p)^*(u) \in H^d(LM\times LM)$ be pullbacks of Thom classes. Define the transfer map $j_!$ by the following composition of maps:

$$j_!\colon H_*(LM\times LM) \xrightarrow{\widetilde{c}_*} \widetilde{H}_*(\widetilde{N}/\partial\widetilde{N}) \xrightarrow[\cong]{\widetilde{u}'\cap(\ )} H_{*-d}(\widetilde{N}) \xrightarrow[\cong]{\widetilde{\pi}_*} H_{*-d}(LM\times_M LM).$$

The tubular neighborhood structure of $\widetilde{N}$ implies that the middle map is a genuine Thom isomorphism.

Definition 3.2 Let M be a closed oriented d–manifold. For $a,b\in\mathbb{H}_*(LM)$, their loop product, denoted by $a\cdot b$, is defined by

$$a\cdot b = (-1)^{d(|a|'-d)}\iota_* j_!(a\times b) = (-1)^{d|a|}\iota_* j_!(a\times b). \tag{3-10}$$

The sign $(-1)^{d(|a|'-d)}$ appears in [3] in the commutative diagram (1-7). We include this sign explicitly in the definition of the loop product for at least three reasons. The most trivial reason is that on the left hand side, the dot representing the loop product is between a and b. On the right hand side, $j_!$ of degree $-d$ representing the loop product is in front of a. Switching a and $j_!$ gives the sign $(-1)^{d|a|'}$. The other part of the sign $(-1)^d$ comes from our choice of orientation of ν and ensures that $s_*([M])\in\mathbb{H}_0(LM)$, with the $+$ sign, is the unit of the loop product.

The second reason is that this choice of sign for the loop product is the same sign appearing in the formula for the intersection product defined in terms of the transfer map (3-6). This makes the loop product compatible with the intersection product. See Proposition 3.3 below.

The third reason of the sign for the loop product is that it gives the correct graded commutativity, as given in [1] proved in terms of chains. We discuss a homotopy theoretic proof of graded commutativity (Proposition 3.4 below) because [2] did not include it, and because the homotopy theoretic proof itself is not so trivial with careful treatment of transfers and signs. Contrast the present homotopy theoretic proof with the simple geometric proof given in [1].

We verify the second and third reasons above.

Proposition 3.3 *Both of the following maps are algebra maps preserving units between the loop algebra $\mathbb{H}_*(LM)$ and the intersection ring $\mathbb{H}_*(M)$:*

$$p_*\colon \mathbb{H}_*(LM)\longrightarrow\mathbb{H}_*(M),\qquad s_*\colon\mathbb{H}_*(M)\longrightarrow\mathbb{H}_*(LM). \tag{3-11}$$

Proof The proof is more or less straightforward, but we discuss it briefly. We consider the following diagram.

$$\begin{array}{ccccc}
LM \times LM & \xleftarrow{j} & LM \times_M LM & \xrightarrow{\iota} & LM \\
{\scriptstyle p\times p}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
M \times M & \xleftarrow{\phi} & M & = & M \\
{\scriptstyle s\times s}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle s}\downarrow \\
LM \times LM & \xleftarrow{j} & LM \times_M LM & \xrightarrow{\iota} & LM
\end{array}$$

Since the Thom classes for embeddings j and ϕ are compatible via $(p \times p)^*$, the induced homology diagram with transfers $j_!$ and $\phi_!$ is commutative. Then by diagram chasing, we can easily check that p_* and s_* preserve products because of the same signs appearing in (3-6) and (3-10). □

Details of the homotopy proof of graded commutativity are given next.

Proposition 3.4 *For $a, b \in \mathbb{H}_*(LM)$, the following graded commutativity relation holds*:

$$a \cdot b = (-1)^{(|a|'-d)(|b|'-d)} b \cdot a = (-1)^{|a||b|} b \cdot a. \tag{3-12}$$

Proof We consider the following commutative diagram, where $R_{1/2}$ is the rotation of loops by $1/2$, that is, $R_{1/2}(\gamma)(t) = \gamma(t + 1/2)$.

$$\begin{array}{ccccc}
LM \times LM & \xleftarrow{j} & LM \times_M LM & \xrightarrow{\iota} & LM \\
{\scriptstyle T}\downarrow & & {\scriptstyle T}\downarrow & & {\scriptstyle R_{1/2}}\downarrow \\
LM \times LM & \xleftarrow{j} & LM \times_M LM & \xrightarrow{\iota} & LM
\end{array}$$

Since $R_{1/2}$ is homotopic to the identity map, we have ${R_{1/2}}_* = 1$. Hence

$$a \cdot b = (-1)^{d(|a|'-d)} \iota_* j_!(a \times b) = (-1)^{d(|a|'-d)} \iota_* T_* j_!(a \times b).$$

Next we show that the induced homology square with transfer $j_!$, we have $T_* j_! = (-1)^d j_! T_*$. Since the left square in the above diagram commutes on space level, we have that $T_* j_!$ and $j_! T_*$ coincides up to a sign. To determine this sign, we compose j_* on the left of these maps and compare. Since the homology square with induced homology maps commute,

$$j_* T_* j_!(a \times b) = T_* j_* j_!(a \times b) = T_*\big(\widetilde{u} \cap (a \times b)\big).$$

On the other hand,

$$j_* j_! T_*(a \times b) = \widetilde{u} \cap T_*(a \times b) = T_*\big(T^*(\widetilde{u}) \cap (a \times b)\big).$$

We compare $T^*(\widetilde{u})$ and $\widetilde{u}$. Since $\widetilde{u} = (p \times p)^*(u)$, we have $T^*(\widetilde{u}) = (p \times p)^* T^*(u)$. Since u is characterized by the property $u \cap [M \times M] = \phi_*([M])$ and $T \circ \phi = \phi$, we have

$$\phi_*([M]) = T_*\phi_*([M]) = T^*(u) \cap T_*([M \times M]) = T^*(u) \cap (-1)^d [M \times M].$$

Thus $T^*(u) = (-1)^d u$. Hence $T^*(\widetilde{u}) = (-1)^d \widetilde{u}$. In view of the above two identities, this implies that $j_* T_* j_! = (-1)^d j_* j_! T_*$, or $T_* j_! = (-1)^d j_! T_*$.

Continuing our computation,

$$a \cdot b = (-1)^{d|a|'} \iota_* j_! T_*(a \times b) = (-1)^{|a|'|b|' + d|\alpha|} \iota_* j_!(b \times a) = (-1)^{(|a|'-d)(|b|'-d)} b \cdot a.$$

This completes the homotopy theoretic proof of commutativity formula. □

The transfer map $j_!$ enjoys the following properties similar to those satisfies by $\phi_!$ as given in Proposition 3.1. The proof is similar, and we omit it.

Proposition 3.5 *For $a, b \in \mathbb{H}_*(LM)$ and $\alpha \in H^*(LM \times LM)$, the following formulas are valid:*

$$j_* j_!(a \times b) = \widetilde{u} \cap (a \times b) \tag{3-13}$$

$$j_!\big(\alpha \cap (a \times b)\big) = (-1)^{d|\alpha|} j^*(\alpha) \cap j_!(b \times c) \tag{3-14}$$

The second formula says that $j_!$ is a $H^*(LM \times LM)$–module map.

4 Cap products and extended BV algebra structure

We examine compatibility of the cap product with the various structures in the BV algebra $\mathbb{H}_*(LM) = H_{*+d}(LM)$.

We recall that a BV algebra A_* is an associative graded commutative algebra equipped with a degree 1 Lie bracket $\{\ ,\ \}$ and a degree 1 operator Δ satisfying the following relations for $a, b, c \in A_*$:

(BV identity) $\quad \Delta(a \cdot b) = (\Delta a) \cdot b + (-1)^{|a|} a \cdot \Delta b + (-1)^{|a|}\{a, b\}$

(Poisson identity) $\quad \{a, b \cdot c\} = \{a, b\} \cdot c + (-1)^{|b|(|a|+1)} b \cdot \{a, c\}$

(Commutativity) $\quad a \cdot b = (-1)^{|a||b|} b \cdot a, \qquad \{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$

(Jacobi identity) $\quad \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}$

Here, degrees of elements are given by $\Delta a \in A_{|a|+1}, a \cdot b \in A_{|a|+|b|}$, and $\{a, b\} \in A_{|a|+|b|+1}$. One way to view these relations is to consider operators D_a and M_a acting on A_* for each $a \in A_*$ given by $D_a(b) = \{a, b\}$ and $M_a(b) = a \cdot b$. Let $[x, y] = xy - (-1)^{|x||y|}yx$ be the graded commutator of operators. Then the Poisson identity and the Jacobi identity take the following forms:

$$[D_a, M_b] = M_{\{a,b\}}, \qquad [D_a, D_b] = D_{\{a,b\}}, \tag{4-1}$$

where degrees of operators are $|D_a| = |a| + 1$ and $|M_b| = |b|$.

One nice context to understand BV identity is in the context of odd symplectic geometry [5, Section 2], where BV operator Δ appears as a mixed second order odd differential operator, and BV identity can be simply understood as Leibnitz rule in differential calculus. This context actually arises in loop homology. In [10], we explicitly computed the BV structure of $\mathbb{H}_*(LM)$ for the Lie group $\mathrm{SU}(n+1)$ and complex Stiefel manifolds. There, the BV operator Δ is given by second order mixed odd differential operator as above, and $\mathbb{H}_*(LM)$ is interpreted as the space of polynomial functions on the odd symplectic vector space.

The fact that the loop algebra $\mathbb{H}_*(LM)$ is a BV algebra was proved in [1]. Note that the above BV relations are satisfied with respect to $\mathbb{H}_*$–grading, rather than the usual homology grading. The same is true for compatibility relations with cap products.

First we discuss the cohomological S^1 action operator Δ on $H^*(LM)$. Consider the S^1 action map $\Delta\colon S^1 \times LM \longrightarrow LM$ given by $\Delta(t, \gamma) = \gamma_t$, where $\gamma_t(s) = \gamma(s+t)$ for $s, t \in S^1 = \mathbb{R}/\mathbb{Z}$. The degree -1 operator $\Delta\colon H^*(LM) \longrightarrow H^{*-1}(LM)$ is defined by the following formula for $\alpha \in H^*(LM)$:

$$\Delta^*(\alpha) = 1 \times \alpha + \{S^1\} \times \Delta\alpha \tag{4-2}$$

where $\{S^1\}$ is the fundamental cohomology class of S^1. The homological S^1 action Δ is not a derivation with respect to the loop product and the deviation from being a derivation is given by the loop bracket. However, the cohomology S^1–operator Δ is a derivation with respect to the cup product.

Proposition 4.1 *The cohomology S^1–operator Δ satisfies $\Delta^2 = 0$, and it acts as a derivation on the cohomology ring $H^*(LM)$. That is, for $\alpha, \beta \in H^*(LM)$,*

$$\Delta(\alpha \cup \beta) = (\Delta\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \cup \Delta\beta. \tag{4-3}$$

Proof The property $\Delta^2 = 0$ is straightforward using the following diagram:

$$\begin{array}{ccc} S^1 \times S^1 \times LM & \xrightarrow{1\times\Delta} & S^1 \times LM \\ {\scriptstyle \mu\times 1}\big\downarrow & & {\scriptstyle \Delta}\big\downarrow \\ S^1 \times LM & \xrightarrow{\Delta} & LM \end{array}$$

Comparing both sides of $(1\times\Delta)^*\Delta^*(\alpha) = (\mu\times 1)^*\Delta^*(\alpha)$, we obtain $\Delta^2(\alpha) = 0$.

For the derivation property, we consider the following diagram.

$$\begin{array}{ccccc} S^1 \times LM & \xrightarrow{\phi\times\phi} & (S^1\times S^1)\times(LM\times LM) & \xrightarrow{1\times T\times 1} & (S^1\times LM)\times(S^1\times LM) \\ {\scriptstyle \Delta}\big\downarrow & & & & {\scriptstyle \Delta\times\Delta}\big\downarrow \\ LM & \xrightarrow{\phi} & LM\times LM & = & LM\times LM \end{array}$$

On the one hand, $\Delta^*\phi^*(\alpha\times\beta) = \Delta^*(\alpha\cup\beta) = 1\times(\alpha\cup\beta) + \{S^1\}\times\Delta(\alpha\cup\beta)$. On the other hand,

$$(\phi\times\phi)^*(1\times T\times 1)^*(\Delta\times\Delta)^*(\alpha\times\beta) = 1\times(\alpha\cup\beta) + (-1)^{|\alpha|}\{S^1\}\times\big(\alpha\cup\Delta\beta + \Delta\alpha\cup\beta\big).$$

Comparing the above two identities, we obtain the derivation formula. □

We can regard the cohomology ring $H^*(LM)$ together with cohomological S^1 action Δ as a BV algebra with trivial bracket product.

Now we show that the cap product is compatible with the loop product in the BV algebra $\mathbb{H}_*(LM)$. The following theorem describes the behavior of the cap product with those elements in the subalgebra of $H^*(LM)$ generated by $H^*(M)$ and $\Delta\big(H^*(M)\big)$.

Theorem 4.2 *Let $\alpha \in H^*(M)$ and $b, c \in \mathbb{H}_*(LM)$. The cap product with $p^*(\alpha)$ behaves associatively and graded commutatively with respect to the loop product. Namely*

$$p^*(\alpha)\cap(b\cdot c) = (p^*(\alpha)\cap b)\cdot c = (-1)^{|\alpha||b|} b\cdot(p^*(\alpha)\cap c). \tag{4-4}$$

The cap product with $\Delta\big(p^*(\alpha)\big)$ is a derivation on the loop product. Namely,

$$\Delta\big(p^*(\alpha)\big)\cap(b\cdot c) = \big(\Delta\big(p^*(\alpha)\big)\cap b\big)\cdot c + (-1)^{(|\alpha|-1)|b|} b\cdot\big(\Delta\big(p^*(\alpha)\big)\cap c\big). \tag{4-5}$$

Proof For (4-4), we consider the following diagram, where π_i is the projection onto the i–th factor for $i = 1, 2$.

$$\begin{tikzcd}
LM \arrow[d, "p"'] & LM \times LM \arrow[l, "\pi_i"'] \arrow[d, "p\times p"'] & LM \times_M LM \arrow[l, "j"'] \arrow[r, "\iota"] \arrow[d, "q"'] & LM \arrow[d, "p"'] \\
M & M \times M \arrow[l, "\pi_i"'] & M \arrow[l, "\phi"'] \arrow[r, equal] & M
\end{tikzcd}$$

Since $p^*(\alpha)\cap(b\cdot c)=(-1)^{d|b|}\iota_*\big(\iota^* p^*(\alpha)\cap j_!(b\times c)\big)$, we need to understand $\iota^* p^*(\alpha)$. From the above commutative diagram, we have $\iota^* p^*(\alpha) = j^*\pi_i^* p^*(\alpha)$, which is equal to either $j^*(p^*(\alpha)\times 1)$ or $j^*(1\times p^*(\alpha))$. In the first case, continuing our computation using (3-14), we have

$$\begin{aligned}
p^*(\alpha)\cap(b\cdot c) &= (-1)^{d|b|}\iota_*\big(j^*(p^*(\alpha)\times 1)\cap j_!(b\times c)\big)\\
&= (-1)^{d|b|+d|\alpha|}\iota_* j_!\big((p^*(\alpha)\times 1)\cap(b\times c)\big)\\
&= (-1)^{d|b|+d|\alpha|}\iota_* j_!\big((p^*(\alpha)\cap b)\times c\big)\\
&= (p^*(\alpha)\cap b)\cdot c.
\end{aligned}$$

Similarly, using $\iota^* p^*(\alpha) = j^*\big(1\times p^*(\alpha)\big)$, we get the other identity. This proves (4-4).

For (4-5), we first note that the element $\Delta\big(p^*(\alpha)\big)\cap(b\cdot c)$ is equal to

$$\Delta\big(p^*(\alpha)\big)\cap(-1)^{d|b|}\iota_* j_!(b\times c) = (-1)^{d|b|}\iota_*\big(\iota^*\big(\Delta(p^*(\alpha))\big)\cap j_!(b\times c)\big).$$

Thus, we need to understand the element $\iota^*\big(\Delta(p^*(\alpha))\big)$. We need some notation. Let $I = I_1\cup I_2$, where $I_1 = [0,\frac12]$ and $I_2 = [\frac12, 1]$, and set $S^1_i = I_i/\partial I_i$ for $i = 1, 2$. Let $r\colon S^1 = I/\partial I \longrightarrow I/\{0,\frac12,1\} = S^1_1\vee S^1_2$ be an identification map, and let $\iota_i\colon S^1_i \longrightarrow S^1_1\vee S^1_2$ be the inclusion map into the i–th wedge summand. We examine the diagram

$$\begin{tikzcd}
S^1\times(LM \underset{M}{\times} LM) \arrow[r, "r\times 1"] \arrow[d, "1\times\iota"'] & (S^1_1\vee S^1_2)\times(LM \underset{M}{\times} LM) \arrow[d, "e'"'] & \{0\}\times(LM \underset{M}{\times} LM) \arrow[l] \arrow[d, "\iota"'] \\
S^1\times LM \arrow[r, "e"] & M & LM \arrow[l, "p"']
\end{tikzcd}$$

where $e = p\circ\Delta$ is the evaluation map for $S^1\times LM$, and the other evaluation map e' is given by

$$e'(t,\gamma,\eta) = \begin{cases}\gamma(2t) & 0\le t\le\frac12,\\ \eta(2t-1) & \frac12\le t\le 1.\end{cases}$$

For $\alpha \in H^*(M)$, we let

$$e'^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{s_1^1\} \times \Delta_1(\alpha) + \{S_2^1\} \times \Delta_2(\alpha)$$

for some $\Delta_i(\alpha) \in H^*(LM \times_M LM)$ for $i = 1, 2$. The first term in the right hand side is identified using the right square of the above commutative diagram. Since $r^*(\{S_i^1\}) = \{S^1\}$ for $i = 1, 2$,

$$(r \times 1)^* e'^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{S^1\} \times \big(\Delta_1(\alpha) + \Delta_2(\alpha)\big).$$

The commutativity of the left square implies that this must be equal to

$$(1 \times \iota)^* \Delta^* p^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{S^1\} \times \iota^* \Delta\big(p^*(\alpha)\big).$$

Hence $$\iota^*\big(\Delta(p^*(\alpha))\big) = \Delta_1(\alpha) + \Delta_2(\alpha) \in H_*(LM \times_M LM).$$

To understand elements $\Delta_i(\alpha)$, we consider the following commutative diagram, where $\ell_1(t) = 2t$ for $0 \le t \le \frac{1}{2}$ and $\ell_2(t) = 2t - 1$ for $\frac{1}{2} \le t \le 1$.

$$\begin{array}{ccccc}
S_i^1 \times (LM \times_M LM) & \xrightarrow{\ell_i \times j} & S^1 \times (LM \times LM) & \xrightarrow{1 \times \pi_i} & S^1 \times LM \\
\Big\downarrow{\scriptstyle \iota_i \times 1} & & & & \Big\downarrow{\scriptstyle \Delta} \\
(S_1^1 \vee S_2^1) \times (LM \times_M LM) & \xrightarrow{e'} & M & \xleftarrow{p} & LM
\end{array}$$

On the one hand, $(\iota_1 \times 1)^* e'^*(\alpha) = 1 \times \iota^* p^*(\alpha) + \{S_1^1\} \times \Delta_1(\alpha)$. On the other hand,

$$(\ell_1 \times j)^*(1 \times \pi_1)^* \Delta^* p^*(\alpha) = 1 \times j^*\big(p^*(\alpha) \times 1\big) + \{S_1^1\} \times j^*\big(\Delta(p^*(\alpha)) \times 1\big).$$

By the commutativity of the diagram, we get $\Delta_1(\alpha) = j^*\big(\Delta(p^*(\alpha)) \times 1\big)$. Similarly, $i = 2$ case implies $\Delta_2(\alpha) = j^*\big(1 \times \Delta(p^*(\alpha))\big)$. Hence we finally obtain

$$\iota^*\big(\Delta(p^*(\alpha))\big) = j^*\big(\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta(p^*(\alpha))\big).$$

With this identification of $\iota^*\big(\Delta(p^*(\alpha))\big)$ as j^* of some other element, we can continue our initial computation.

$$\begin{aligned}
\Delta\big(p^*(\alpha)\big) \cap (b \cdot c) &= (-1)^{d|b|} \iota_*\big(j^*\big(\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta\big(p^*(\alpha)\big)\big) \cap j_!(b \times c)\big) \\
&= (-1)^{d|b| + (|\alpha|-1)d} \iota_* j_!\Big(\big(\Delta(p^*(\alpha)) \times 1 + 1 \times \Delta\big(p^*(\alpha)\big)\big) \cap (b \times c)\Big) \\
&= (-1)^{d(|\alpha|+|b|-1)} \iota_* j_!\Big(\big(\Delta(p^*(\alpha)) \cap b\big) \times c + (-1)^{(|b|+d)(|\alpha|-1)} b \times \big(\Delta(p^*(\alpha)) \cap c\big)\Big) \\
&= \big(\Delta(p^*(\alpha)) \cap b\big) \cdot c + (-1)^{(|\alpha|-1)|b|} b \cdot \big(\Delta(p^*(\alpha)) \cap c\big).
\end{aligned}$$

This completes the proof of the derivation property of the cap product with respect to the loop product. □

Next we describe the relation between the cap product and the BV operator in homology and cohomology.

Proposition 4.3 *For $\alpha \in H^*(LM)$ and $b \in \mathbb{H}_*(LM)$, the BV-operator Δ satisfies*

$$\Delta(\alpha \cap b) = (\Delta\alpha) \cap b + (-1)^{|\alpha|}\alpha \cap \Delta b. \tag{4-6}$$

Proof On the one hand, the S^1–action map $\Delta\colon S^1 \times LM \longrightarrow LM$ satisfies

$$\Delta_*\big(\Delta^*(\alpha) \cap ([S^1] \times b)\big) = \alpha \cap \Delta_*([S^1] \times b) = \alpha \cap \Delta b.$$

On the other hand, since $\Delta^*(\alpha) = 1 \times \alpha + \{S^1\} \times \Delta\alpha$, we have

$$\begin{aligned}\Delta_*\big(\Delta^*(\alpha) \cap ([S^1] \times b)\big) &= \Delta_*\big((-1)^{|\alpha|}[S^1] \times (\alpha \cap b) + (-1)^{|\alpha|-1}[pt] \times (\Delta\alpha \cap b)\big)\\ &= (-1)^{|\alpha|}\Delta(\alpha \cap b) + (-1)^{|\alpha|-1}\Delta\alpha \cap b.\end{aligned}$$

Comparing the above two formulas, we obtain $\Delta(\alpha \cap b) = \Delta\alpha \cap b + (-1)^{|\alpha|}\alpha \cap \Delta b$. □

Since homology BV operator Δ on $\mathbb{H}_*(LM)$ acts trivially on $\mathbb{H}_*(M)$, the following corollary is immediate.

Corollary 4.4 *For $\alpha \in H^*(M)$, the cap product of $\Delta\alpha$ with $\mathbb{H}_*(M) \subset \mathbb{H}_*(LM)$ is trivial.*

Proof For $b \in \mathbb{H}_*(M)$, the operator Δ acts trivially on both $\alpha \cap b$ and b. Hence formula (4-6) implies $(\Delta\alpha) \cap b = 0$. □

Next, we discuss a behavior of the cap product with respect to the loop bracket.

Theorem 4.5 *The cap product with $\Delta\big(p^*(\alpha)\big)$ is a derivation on the loop bracket. Namely, for $\alpha \in H^*(M)$ and $b, c \in \mathbb{H}_*(LM)$,*

$$\Delta\big(p^*(\alpha)\big) \cap \{b, c\} = \{\Delta\big(p^*(\alpha)\big) \cap b, c\} + (-1)^{(|\alpha|-1)(|b|-1)}\{b, \Delta\big(p^*(\alpha)\big) \cap c\}. \tag{4-7}$$

Proof Our proof is computational using previous results. We use the BV identity as the definition of the loop bracket. Thus,

$$\{b, c\} = (-1)^{|b|}\Delta(b \cdot c) - (-1)^{|b|}(\Delta b) \cdot c - b \cdot \Delta c.$$

We compute the right hand side of (4-7). For simplicity, we write $\Delta\alpha$ for $\Delta\big(p^*(\alpha)\big)$. Each term in the right hand side of (4-7) gives

$$\begin{aligned}&\{\Delta\alpha \cap b, c\}\\ &\qquad = (-1)^{|b|-|\alpha|+1}\Delta\big((\Delta\alpha \cap b) \cdot c\big) - (-1)^{|b|}(\Delta\alpha \cap \Delta b) \cdot c - (\Delta\alpha \cap b) \cdot \Delta c,\\ &\{b, \Delta\alpha \cap c\}\\ &\qquad = (-1)^{|b|}\Delta\big(b \cdot (\Delta\alpha \cap c)\big) - (-1)^{|b|}\Delta b \cdot (\Delta\alpha \cap c) - (-1)^{|\alpha|-1}b \cdot (\Delta\alpha \cap \Delta c),\end{aligned}$$

Here we used (4-6) for the second term in the first identity and in the third term in the second identity. Combining these formulas, we get

$$\begin{aligned}
&\{\Delta\alpha \cap b, c\} + (-1)^{(|\alpha|-1)(|b|+1)}\{b, \Delta\alpha \cap c\} \\
&\quad = \big((-1)^{|b|-|\alpha|+1}\Delta\big((\Delta\alpha \cap b)\cdot c\big) + (-1)^{|b|+(|\alpha|-1)(|b|+1)}\Delta\big(b\cdot(\Delta\alpha \cap c)\big)\big) \\
&\qquad - \big((-1)^{|b|}(\Delta\alpha \cap \Delta b)\cdot c + (-1)^{(|\alpha|-1)(|b|+1)+|b|}\Delta b\cdot(\Delta\alpha \cap c)\big) \\
&\qquad\qquad - \big((\Delta\alpha \cap b)\cdot \Delta c + (-1)^{(|\alpha|-1)(|b|+1)+|\alpha|-1} b\cdot(\Delta\alpha \cap \Delta c)\big).
\end{aligned}$$

Using the derivation formula for $\Delta\alpha \cap (\;)$ with respect to the loop product (4-5), three pairs of terms above become

$$\begin{aligned}
&(-1)^{|b|-|\alpha|+1}\Delta\big(\Delta\alpha \cap (b\cdot c)\big) - (-1)^{|b|}\Delta\alpha \cap (\Delta b\cdot c) - \Delta\alpha \cap (b\cdot \Delta c) \\
&\qquad = \Delta\alpha \cap \big((-1)^{|b|}\Delta(b\cdot c) - (-1)^{|b|}\Delta b\cdot c - b\cdot\Delta c\big) = \Delta\alpha \cap \{b, c\}.
\end{aligned}$$

This completes the proof of the derivation formula for the loop bracket. □

Recall that in the BV algebra $\mathbb{H}_*(LM)$, for every $a \in \mathbb{H}_*(LM)$ the operation $\{a, \cdot\}$ of taking the loop bracket with a is a derivation with respect to both the loop product and the loop bracket, in view of the Poisson identity and the Jacobi identity. Since we have proved that the cap product with $\Delta p^*(\alpha)$ for $\alpha \in H^*(M)$ is a derivation with respect to both the loop product and the loop bracket, we wonder if we can extend the BV structure in $\mathbb{H}_*(LM)$ to a BV structure in $H^*(M) \oplus \mathbb{H}_*(LM)$. Indeed this is possible by extending the loop product and the loop bracket to elements in $H^*(M)$ as follows.

Definition 4.6 For $\alpha, \beta \in H^*(M)$ and $b \in \mathbb{H}_*(LM)$, we define their loop product and loop bracket by

$$\begin{aligned}
\alpha\cdot b &= \alpha \cap b, \quad \{\alpha, b\} = (-1)^{|\alpha|}(\Delta\alpha)\cap b, \\
\alpha\cdot\beta &= \alpha \cup \beta, \quad \{\alpha, \beta\} = 0.
\end{aligned} \tag{4-8}$$

This defines an associative graded commutative loop product by (4-4), and a bracket product on $H^*(M) \oplus \mathbb{H}_*(LM)$.

Note that this loop product on $H^*(M) \oplus \mathbb{H}_*(LM)$ reduces to the ring structure on $H^*(M) \oplus \mathbb{H}_*(M)$ mentioned in the introduction.

With this definition, the Poisson identities and the Jacobi identities are still valid in $H^*(M) \oplus \mathbb{H}_*(LM)$.

Theorem 4.7 *Let $\alpha,\beta\in H^*(M)$, and let $b,c\in\mathbb{H}_*(LM)$.*

(I) *The following Poisson identities are valid in $H^*(M)\oplus\mathbb{H}_*(LM)$:*

$$\{\alpha,\beta\cdot c\}=\{\alpha,\beta\}\cdot c+(-1)^{|\beta|(|\alpha|-1)}\beta\cdot\{\alpha,c\} \tag{4-9}$$

$$\{\alpha\beta,c\}=\alpha\cdot\{\beta,c\}+(-1)^{|\alpha||\beta|}\beta\cdot\{\alpha,c\} \tag{4-10}$$

$$\{\alpha,b\cdot c\}=\{\alpha,b\}\cdot c+(-1)^{(|b|-d)(|\alpha|-1)}b\cdot\{\alpha,c\} \tag{4-11}$$

$$\{\alpha\cdot b,c\}=\alpha\cdot\{b,c\}+(-1)^{|\alpha|(|b|-d)}b\cdot\{\alpha,c\}. \tag{4-12}$$

(II) *The following Jacobi identities are valid in $H^*(M)\oplus\mathbb{H}_*(LM)$:*

$$\{\alpha,\{\beta,c\}\}=\{\{\alpha,\beta\},c\}+(-1)^{(|\alpha|-1)(|\beta|-1)}\{\beta,\{\alpha,c\}\} \tag{4-13}$$

$$\{\alpha,\{b,c\}\}=\{\{\alpha,b\},c\}+(-1)^{(|\alpha|-1)(|b|-d+1)}\{b,\{\alpha,c\}\}. \tag{4-14}$$

Proof If we unravel definitions, we see that (4-9) and (4-13) are really the same as the graded commutativity of the cup product of the form

$$(\Delta\alpha)\cap(b\cap c)=(-1)^{|\beta|(|\alpha|-1)}\beta\cap(\Delta\alpha\cap c),$$
$$(\Delta\alpha)\cap(\Delta\beta\cap c)=(-1)^{(|\alpha|-1)(|\beta|-1)}(\Delta\beta)\cap\big((\Delta\alpha)\cap c\big).$$

The identity (4-10) is equivalent to the derivation formula (4-3) of the cohomology S^1 action operator with respect to the cup product.

$$\Delta(\alpha\cup\beta)=(\Delta\alpha)\cup\beta+(-1)^{|\alpha|}\alpha\cup(\Delta\beta).$$

The identity (4-11) says that $\Delta\alpha\cap(\ )$ is a derivation with respect to the loop product, and the identity (4-14) says that $\Delta\alpha\cap(\ )$ is a derivation with respect to the loop bracket. We have already verified both of these cases. Thus, what remains to be checked is formula (4-12), which says

$$\{\alpha\cap b,c\}=\alpha\cap\{b,c\}+(-1)^{|\alpha||b|+|\alpha|}b\cdot(\Delta\alpha\cap c).$$

Using the BV identity, the derivation formula (4-6) of the BV operator with respect to the cap product, and properties of $\alpha\cap(\ )$ and $\Delta\alpha\cap(\ )$, we can prove this identity:

$$\begin{aligned}(-1)^{|b|-|\alpha|}\{\alpha\cap b,c\}&=\Delta\big((\alpha\cap b)\cdot c\big)-\Delta(\alpha\cap b)\cdot c-(-1)^{|b|-|\alpha|}(\alpha\cap b)\cdot\Delta c\\&=\Delta\big(\alpha\cap(b\cdot c)\big)-(\Delta\alpha\cap b+(-1)^{|\alpha|}\alpha\cap\Delta b)\cdot c-(-1)^{|b|-|\alpha|}\alpha\cap(b\cdot\Delta c)\\&=(\Delta\alpha)\cap(b\cdot c)-(\Delta\alpha\cap b)\cdot c+(-1)^{|\alpha|}\alpha\cap\Delta(b\cdot c)\\&\qquad-(-1)^{|\alpha|}\alpha\cap(\Delta b\cdot c)-(-1)^{|b|-|\alpha|}\alpha\cap(b\cdot\Delta c)\\&=(-1)^{(|\alpha|-1)|b|}b\cdot(\Delta\alpha\cap c)+(-1)^{|\alpha|+|b|}\alpha\cap\{b,c\}.\end{aligned}$$

Canceling some signs, we get the desired formula. This completes the proof. □

Other Poisson and Jacobi identities with cohomology elements in the second argument formally follow from above identities by making following definitions for $\alpha \in H^*(M)$ and $b \in \mathbb{H}_*(LM)$:

$$b \cdot \alpha = (-1)^{|\alpha||b|}\alpha \cdot b, \qquad \{b, \alpha\} = -(-1)^{(|\alpha|+1)(|b|+1)}\{\alpha, b\}.$$

For $\alpha \in H^*(M)$ we showed that $\Delta\alpha \cap (\ )$ is a derivation for both the loop product and the loop bracket, and $\alpha \cap (\ )$ is graded commutative and associative with respect to the loop product. What is the behavior of $\alpha \cap (\ )$ is with respect to the loop bracket? Formula (4-12) says that $\alpha \cap (\cdot)$ on loop bracket is not a derivation or graded commutativity: it is a Poisson identity!

The Poisson and Jacobi identities we have just proved in $A_* = H^*(M) \oplus \mathbb{H}_*(LM)$ show that A_* is a Gerstenhaber algebra. In fact, A_* can be formally turned into a BV algebra by defining a BV operator $\mathbf{\Delta}$ on A_* to be trivial on $H^*(M)$ and to be the usual one on $\mathbb{H}_*(LM)$ coming from the homological S^1 action.

Corollary 4.8 *The direct sum $A_* = H^*(M) \oplus \mathbb{H}_*(LM)$ has the structure of a BV algebra.*

Proof Since $\mathbb{H}_*(LM)$ is a BV algebra and since we have already verified the Poisson identities and the Jacobi identities in A_*, we only have to verify BV identities in A_*. For $\alpha, \beta \in H^*(M)$, an identity

$$\mathbf{\Delta}(\alpha \cup \beta) = (\mathbf{\Delta}\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \cup (\mathbf{\Delta}\beta) + (-1)^{|\alpha|}\{\alpha, \beta\}$$

is trivially satisfied since all terms are zero by definition of BV operator $\mathbf{\Delta}$ and the loop bracket on $H^*(M) \subset A_*$.

Next, let $\alpha \in H^*(M)$ and $b \in \mathbb{H}_*(LM)$. Since the BV operator $\mathbf{\Delta}$ on A_* acts trivially on $H^*(M)$, an identity

$$\mathbf{\Delta}(\alpha \cap b) = (\mathbf{\Delta}\alpha) \cap b + (-1)^{|\alpha|}\alpha \cap (\mathbf{\Delta}b) + (-1)^{|\alpha|}\{\alpha, b\}$$

is really a restatement of the derivative formula of the homology S^1 action operator Δ on cap product: $\Delta(\alpha \cap b) = (-1)^{|\alpha|}\alpha \cap (\Delta b) + (\Delta\alpha) \cap b$ in formula (4-6). □

In connection with the above Corollary, we can ask whether $H^*(LM) \oplus \mathbb{H}_*(LM)$ has a structure of a BV algebra. Of course, $H^*(LM)$ together with the cohomological S^1 action operator Δ, which is a derivation, is a BV algebra with trivial bracket product. Thus, as a direct sum of BV algebras, $H^*(LM) \oplus \mathbb{H}_*(LM)$ is a BV algebra, although products between $H^*(LM)$ and $\mathbb{H}_*(LM)$ are trivial. More meaningful question would be to ask whether the direct sum $H^*(LM) \oplus \mathbb{H}_*(LM)$ has a BV algebra structure extending the one on A_* described in Corollary 4.8. If we want to use the cap product as an extension of the loop product, the answer is no. This is because the cap product with an arbitrary element $\alpha \in H^*(LM)$ does not behave associatively with respect to the loop product in $\mathbb{H}_*(LM)$: if α is of the form $\alpha = \Delta\beta$ for some $\beta \in H^*(M)$, then $\alpha \cap (\cdot)$ acts as a derivation on loop product in $\mathbb{H}_*(LM)$ due to (4-5) and does not satisfy associativity.

Remark 4.9 In the course of our investigation, we noticed the following curious identity, which is in some sense symmetric in three variables, for $\alpha \in H^*(M)$ and $b, c \in \mathbb{H}_*(LM)$.

$$\begin{aligned}\{\alpha, b\cdot c\} + (-1)^{|b|}\alpha\cdot\{b,c\} &= \{\alpha, b\}\cdot c + (-1)^{|b|}\{\alpha\cdot b, c\} \\ &= (-1)^{(|\alpha|+1)|b|}\big(b\cdot\{\alpha, c\} + (-1)^{|\alpha|}\{b, \alpha\cdot c\}\big).\end{aligned} \tag{4-15}$$

This identity is easily proved using the Poisson identities. But we wonder the meaning of this symmetry.

5 Cap products in terms of BV algebra structure

In the previous section, we showed that the BV algebra structure in $\mathbb{H}_*(LM)$ can be extended to the BV algebra structure in $H^*(M) \oplus \mathbb{H}_*(LM)$ by proving the Poisson identities and the Jacobi identities. This may be a bit surprising. But this turns out to be very natural through Poincaré duality in the following way. For $a \in \mathbb{H}_*(M)$, we denote the element $s_*(a) \in \mathbb{H}_*(LM)$ by a, where $s\colon M \to LM$ is the inclusion map.

Theorem 5.1 *For $a \in \mathbb{H}_*(M)$, let $\alpha = D(a) \in H^*(M)$ be its Poincaré dual. Then for any $b \in \mathbb{H}_*(LM)$, the following identities hold.*

$$p^*(\alpha)\cap b = a\cdot b, \qquad (-1)^{|\alpha|}\Delta\big(p^*(\alpha)\big)\cap b = \{a, b\}.$$

Proof Let $1 = s_*([M]) \in \mathbb{H}_0(LM)$ be the unit of the loop product. Since $p^*(\alpha)\cap b = p^*(\alpha)\cap(1\cdot b) = \big(p^*(\alpha)\cap 1\big)\cdot b$ by (4-4), and since

$$p^*(\alpha)\cap 1 = p^*(\alpha)\cap s_*([M]) = s_*\big(s^*p^*(\alpha)\cap[M]\big) = s_*(\alpha\cap[M]) = a,$$

we have $p^*(\alpha)\cap b = a\cdot b$. This proves the first identity.

For the second identity, in the BV identity

$$(-1)^{|a|}\{a, b\} = \Delta(a\cdot b) - (\Delta a)\cdot b - (-1)^{|a|}a\cdot\Delta b,$$

the first term in the right hand side gives

$$\Delta(a\cdot b)=\Delta(p^*(\alpha)\cap b)=\Delta\big(p^*(\alpha)\big)\cap b+(-1)^{|\alpha|}p^*(\alpha)\cap\Delta b$$

in view of the first identity we just proved and the derivation property of the homological A^1 action operator on cap products. Here $p^*(\alpha)\cap\Delta b=a\cdot\Delta b$. Since $a\in\mathbb{H}_*(M)$ is a homology class of constant loops, we have $\Delta a=0$. Thus,

$$(-1)^{|a|}\{a,b\}=\Delta\big(p^*(\alpha)\big)\cap b+(-1)^{|\alpha|}a\cdot\Delta b-(-1)^{|a|}a\cdot\Delta b=\Delta\big(p^*(\alpha)\big)\cap b,$$

since $|\alpha|=-|a|$. Thus, $\{a,b\}=(-1)^{|\alpha|}\Delta\big(p^*(\alpha)\big)\cap b$. This completes the proof. □

In view of this theorem, since $\mathbb{H}_*(LM)$ is already a BV algebra, the Poisson identities and the Jacobi identities we proved in Section 4 may seem obvious. However, what we did in Section 4 is that we gave a *new and elementary homotopy theoretic proof* of the Poisson identities and the Jacobi identities using only basic properties of the cap product and the BV identity, when at least one of the elements is from $\mathbb{H}_*(M)$.

The above theorem shows that loop products and loop brackets with elements in $\mathbb{H}_*(M)$ can be written as cap products with cohomology elements in LM. Thus, compositions of loop products and loop brackets with elements in $\mathbb{H}_*(M)$ corresponds to a cap product with the product of corresponding cohomology classes in $H^*(LM)$. Namely:

Corollary 5.2 *Let $a_0,a_1,\dots,a_r\in\mathbb{H}_*(M)$, and let $\alpha_0,\alpha_1,\dots,\alpha_r\in H^*(M)$ be their Poincaré duals. Then for $b\in\mathbb{H}_*(LM)$,*

$$a_0\cdot\{a_1,\{a_2,\dots,\{a_r,b\}\cdots\}\}=(-1)^{|a_1|+\cdots+|a_r|}\big(\alpha_0(\Delta\alpha_1)(\Delta\alpha_2)\cdots(\Delta\alpha_r)\big)\cap b.$$

In Section 2, we considered a problem of intersections of loops with submanifolds in certain configurations, and we saw that the homology class of the intersections of interest can be given by a cap product with cohomology cup products of the above form (Proposition 2.1). The above corollary computes this homology class in terms of BV structure in $\mathbb{H}_*(LM)$ using the homology classes of these submanifolds.

Remark 5.3 In general, elements $\alpha,\Delta\alpha$ for $\alpha\in H^*(M)$ do not generate the entire cohomology ring $H^*(LM)$. However, if $H^*(M;\mathbb{Q})=\Lambda_{\mathbb{Q}}(\alpha_1,\alpha_2,\dots,\alpha_r)$ is an exterior algebra, over $\mathbb{Q}$, then using minimal models or spectral sequences, we have

$$H^*(LM;\mathbb{Q})=\Lambda_{\mathbb{Q}}(\alpha_1,\alpha_2,\dots,\alpha_r)\otimes\mathbb{Q}[\Delta\alpha_1,\Delta\alpha_2,\dots,\Delta\alpha_r],$$

and thus we have the complete description of the cap products with any elements in $H^*(LM;\mathbb{Q})$ in terms of the BV structure in $\mathbb{H}_*(LM;\mathbb{Q})$.

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