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Homotopy theory of modules over operads in symmetric spectra

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We establish model category structures on algebras and modules over operads in symmetric spectra and study when a morphism of operads induces a Quillen equivalence between corresponding categories of algebras (resp. modules) over operads.

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1 Introduction

Operads parametrize simple and complicated algebraic structures and naturally arise in several areas of algebraic topology, homotopy theory and homological algebra; see Basterra and Mandell [1], Goerss and Hopkins [13], Hinich and Schechtman [18], Kriz and May [24], May [30] and McClure and Smith [31]. The symmetric monoidal category of symmetric spectra (see Hovey, Shipley and Smith [21]) provides a simple and convenient model for the classical stable homotopy category, and is an interesting setting where such algebraic structures naturally arise. Given an operad $\mathcal{O}$ in symmetric spectra, we are interested in the possibility of doing homotopy theory in the categories of $\mathcal{O}$–algebras and $\mathcal{O}$–modules in symmetric spectra, which in practice means putting a Quillen model structure on these categories of algebras and modules. In this setting, $\mathcal{O}$–algebras are the same as left $\mathcal{O}$–modules concentrated at 0 (Section 3.18). This paper establishes a homotopy theory for algebras and modules over operads in symmetric spectra.

This is the main theorem.

Theorem 1.1 *Let $\mathcal{O}$ be an operad in symmetric spectra. Then the category of $\mathcal{O}$–algebras and the category of left $\mathcal{O}$–modules both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the positive flat stable fibrations in symmetric spectra.*

Remark 1.2 For ease of notational purposes, we have followed Schwede [37] in using the term *flat* (eg, flat stable model structure) for what is called S (eg, stable S–model structure) in Hovey, Shipley and Smith [21], Schwede [36] and Shipley [39].

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The theorem remains true when the positive flat stable model structure on symmetric spectra is replaced by the positive stable model structure. This follows immediately from the proof of Theorem 1.1 since every (positive) stable cofibration is a (positive) flat stable cofibration.

Theorem 1.3 *Let $\mathcal{O}$ be an operad in symmetric spectra. Then the category of $\mathcal{O}$–algebras and the category of left $\mathcal{O}$–modules both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the positive stable fibrations in symmetric spectra.*

In Section 5 we prove that a morphism of operads which is an objectwise stable equivalence induces an equivalence between the corresponding homotopy categories of algebras (resp. modules).

Theorem 1.4 *Let $\mathcal{O}$ be an operad in symmetric spectra and let $\mathsf{Alg}_{\mathcal{O}}$ (resp. $\mathsf{Lt}_{\mathcal{O}}$) be the category of $\mathcal{O}$–algebras (resp. left $\mathcal{O}$–modules) with the model structure of Theorem 1.1 or Theorem 1.3. If $f\colon \mathcal{O} \longrightarrow \mathcal{O}'$ is a map of operads, then the adjunctions*

$$\mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'}, \qquad \mathsf{Lt}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'}, \tag{1.5}$$

are Quillen adjunctions with left adjoints on top and f^* the forgetful functor. If furthermore, f is an objectwise stable equivalence, then the adjunctions (1.5) are Quillen equivalences, and hence induce equivalences on the homotopy categories.

The properties of the flat stable model structure on symmetric spectra are fundamental to the results of this paper. For some of the good properties, see Hovey, Shipley and Smith [21, Theorem 5.3.7 and Corollary 5.3.10]. The *positive* flat stable model structure, compared to the flat stable model structure, arises very clearly in our arguments. See, for example, Proposition 4.28 and its proof, the following of which is a special case of particular interest.

Proposition 1.6 *If $i\colon X \longrightarrow Y$ is a cofibration between cofibrant objects in symmetric spectra with the positive flat stable model structure and $t \geq 1$, then $X^{\wedge t} \longrightarrow Y^{\wedge t}$ is a cofibration of Σ_t–diagrams in symmetric spectra with the positive flat stable model structure, and hence with the flat stable model structure.*

In Section 7 we summarize several constructions and results of particular interest for the special case of algebras over operads.

1.7 Relationship to previous work

One of the theorems of Schwede and Shipley [38] is that the category of monoids in symmetric spectra has a natural model structure inherited from the (flat) stable model structure on symmetric spectra. This result was improved by the author [16] to algebras and left modules over any non–Σ operad $\mathcal{O}$ in symmetric spectra.

One of the theorems of Shipley [39] (resp. Mandell, May, Schwede and Shipley [28]) is that the category of commutative monoids in symmetric spectra has a natural model structure inherited from the positive flat stable model structure (resp. positive stable model structure) on symmetric spectra. Theorem 1.1 and Theorem 1.3 improve these results to algebras and left modules over any operad $\mathcal{O}$ in symmetric spectra.

One of the theorems of Elmendorf and Mandell [6] is that for symmetric spectra the category of algebras over any operad $\mathcal{O}$ in simplicial sets has a natural model structure inherited from the positive stable model structure on symmetric spectra. Theorem 1.3 improves this result to algebras and left modules over any operad $\mathcal{O}$ in symmetric spectra. Their proof involves a filtration in the underlying category of certain pushouts of algebras. We have benefitted from their paper and our proofs of Theorem 1.1 and Theorem 1.3 exploit similar filtrations.

Another of the theorems of Elmendorf and Mandell [6] is that a morphism of operads in simplicial sets which is an objectwise weak equivalence induces a Quillen equivalence between categories of algebras over operads. Theorem 1.4 improves this result to algebras and left modules over operads in symmetric spectra.

Our approach to studying algebras and modules over operads is largely influenced by Rezk [35].

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2 Symmetric spectra

The purpose of this section is to recall some basic definitions and properties of symmetric spectra. A useful introduction to symmetric spectra is given in the original paper of Hovey, Shipley and Smith [21]; see also the development given by Schwede [37]. Define the sets $\mathbf{n} := \{1, \ldots, n\}$ for each $n \geq 0$, where $\mathbf{0} := \varnothing$ denotes the empty set.

Let S^1 denote the simplicial circle $\Delta[1]/\partial\Delta[1]$ and for each $n \geq 0$ define $S^n := (S^1)^{\wedge n}$ the n–fold smash power of S^1, where $S^0 := \Delta[0]_+ = \Delta[0] \amalg \Delta[0]$.

Definition 2.1 Let $n \geq 0$.

- Σ_n is the category with exactly one object $\mathbf{n}$ and morphisms the bijections of sets.
- S_* is the category of pointed simplicial sets and their maps.
- $\mathsf{S}_*^{\Sigma_n}$ is the category of functors $X\colon \Sigma_n \longrightarrow \mathsf{S}_*$ and their natural transformations.

In other words, an object in $\mathsf{S}_*^{\Sigma_n}$ is a pointed simplicial set X equipped with a basepoint preserving left action of the symmetric group Σ_n and a morphism in $\mathsf{S}_*^{\Sigma_n}$ is a map $f\colon X \longrightarrow Y$ in S_* such that f is Σ_n–equivariant.

2.2 Symmetric spectra

Recall the following definition from [21, Section 1.2].

Definition 2.3 A *symmetric spectrum* X consists of the following:

(1) a sequence of objects $X_n \in \mathsf{S}_*^{\Sigma_n}$ $(n \geq 0)$, and

(2) a sequence of maps $\sigma\colon S^1 \wedge X_n \longrightarrow X_{n+1}$ in S_* $(n \geq 0)$,

such that the iterated maps $\sigma^p\colon S^p \wedge X_n \longrightarrow X_{n+p}$ are $\Sigma_p \times \Sigma_n$–equivariant for $p \geq 1$ and $n \geq 0$. Here, $\sigma_p := \sigma(S^1 \wedge \sigma) \cdots (S^{p-1} \wedge \sigma)$ is the composition of the maps

$$S^i \wedge S^1 \wedge X_{n+p-1-i} \xrightarrow{S^i \wedge \sigma} S^i \wedge X_{n+p-i} \; .$$

The maps σ are the *structure maps* of the symmetric spectrum.

A *map of symmetric spectra* $f\colon X \longrightarrow Y$ is a sequence of maps $f_n\colon X_n \longrightarrow Y_n$ in $\mathsf{S}_*^{\Sigma_n}$ $(n \geq 0)$, such that the diagram

$$\begin{tikzcd}
S^1 \wedge X_n \arrow[r, "\sigma"] \arrow[d, "S^1 \wedge f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\
S^1 \wedge Y_n \arrow[r, "\sigma"] & Y_{n+1}
\end{tikzcd}$$

commutes for each $n \geq 0$.

Denote by Sp^Σ the category of symmetric spectra and their maps; the null object is denoted by $*$.

The *sphere spectrum* S is the symmetric spectrum defined by $S_n := S^n$, with left Σ_n–action given by permutation and structure maps $\sigma\colon S^1 \wedge S^n \longrightarrow S^{n+1}$ the natural isomorphisms.

2.4 Symmetric spectra as modules over the sphere spectrum

The purpose of this subsection is to recall the description of symmetric spectra as modules over the sphere spectrum. A similar tensor product construction will appear when working with algebras and left modules over operads in Section 3.

Definition 2.5 Let $n \geq 0$.

- Σ is the category of finite sets and their bijections.
- S_*^Σ is the category of functors $X\colon \Sigma \longrightarrow \mathsf{S}_*$ and their natural transformations.
- If $X \in \mathsf{S}_*^\Sigma$, define $X_n := X[\mathbf{n}]$ the functor X evaluated on the set $\mathbf{n}$.
- An object $X \in \mathsf{S}_*^\Sigma$ is *concentrated at n* if $X_r = *$ for all $r \neq n$.

If X is a finite set, define $|X|$ to be the number of elements in X.

Definition 2.6 Let X be a finite set and A in S_*. The *copowers* $A \cdot X$ and $X \cdot A$ in S_* are defined as follows:

$$A \cdot X := \coprod_X A \cong A \wedge X_+ , \qquad X \cdot A := \coprod_X A \cong X_+ \wedge A,$$

the coproduct in S_* of $|X|$ copies of A.

Definition 2.7 Let X, Y be objects in S_*^Σ. The *tensor product* $X \otimes Y \in \mathsf{S}_*^\Sigma$ is the left Kan extension of objectwise smash along coproduct of sets:

$$\begin{tikzcd} \Sigma \times \Sigma \arrow[r, "X \times Y"] \arrow[d, "\amalg"] & \mathsf{S}_* \times \mathsf{S}_* \arrow[r, "\wedge"] & \mathsf{S}_* \\ \Sigma \arrow[rr, "X \otimes Y", "\text{left Kan extension}"'] & & \mathsf{S}_* \end{tikzcd}$$

Useful details on Kan extensions and their calculation are given by Mac Lane [26, X], in particular see X.4. The following is a calculation of tensor product, whose proof is left to the reader.

Proposition 2.8 *Let X, Y be objects in S_*^Σ and $N \in \Sigma$, with $n := |N|$. There are natural isomorphisms*

$$(X \otimes Y)_n \cong (X \otimes Y)[N] \cong \coprod_{\substack{\pi\colon N \longrightarrow \mathbf{2} \\ \text{in Set}}} X[\pi^{-1}(1)] \wedge Y[\pi^{-1}(2)],$$

$$\cong \coprod_{n_1+n_2=n} \Sigma_n \underset{\Sigma_{n_1} \times \Sigma_{n_2}}{\cdot} X_{n_1} \wedge Y_{n_2}. \tag{2.9}$$

Remark 2.10 The coproduct is in the category S_*. Set is the category of sets and their maps.

The following is proved in [21, Section 2.1] and verifies that tensor product in the category S_*^Σ inherits many of the good properties of smash product in the category S_*.

Proposition 2.11 *$(\mathsf{S}_*^\Sigma, \otimes, S^0)$ has the structure of a closed symmetric monoidal category. All small limits and colimits exist and are calculated objectwise. The unit $S^0 \in \mathsf{S}_*^\Sigma$ is given by $S^0[\mathbf{n}] = *$ for each $n \geq 1$ and $S^0[\mathbf{0}] = S^0$.*

The sphere spectrum S has two naturally occurring maps $S \otimes S \longrightarrow S$ and $S^0 \longrightarrow S$ in S_*^Σ which give S the structure of a commutative monoid in $(\mathsf{S}_*^\Sigma, \otimes, S^0)$. Furthermore, any symmetric spectrum X has a naturally occurring map $m\colon S \otimes X \longrightarrow X$ which gives X a left action of S in $(\mathsf{S}_*^\Sigma, \otimes, S^0)$. The following is proved in [21, Section 2.2] and provides a useful interpretation of symmetric spectra.

Proposition 2.12 *Define the category $\Sigma' := \amalg_{n \geq 0} \Sigma_n$, a skeleton of Σ.*

(a) *The sphere spectrum S is a commutative monoid in $(\mathsf{S}_*^\Sigma, \otimes, S^0)$.*

(b) *The category of symmetric spectra is equivalent to the category of left S–modules in $(\mathsf{S}_*^\Sigma, \otimes, S^0)$.*

(c) *The category of symmetric spectra is isomorphic to the category of left S–modules in $(\mathsf{S}_*^{\Sigma'}, \otimes, S^0)$.*

In this paper we will not distinguish between these equivalent descriptions of symmetric spectra.

2.13 Smash product of symmetric spectra

The *smash product* $X \wedge Y \in \mathsf{Sp}^\Sigma$ of symmetric spectra X and Y is defined as the colimit

$$X \wedge Y := X \otimes_S Y := \operatorname{colim}\Big(X \otimes Y \underset{\mathrm{id} \otimes m}{\overset{m \otimes \mathrm{id}}{\leftleftarrows}} X \otimes S \otimes Y \Big). \tag{2.14}$$

Note that since S is a commutative monoid, a left action of S on X determines a right action $m\colon X \otimes S \longrightarrow X$ which gives X the structure of an (S, S)–bimodule. Hence the tensor product $X \otimes_S Y$ has the structure of a left S–module.

The following is proved in [21, Section 2.2] and verifies that smash products of symmetric spectra inherit many of the good properties of smash products of pointed simplicial sets.

Proposition 2.15 $(\mathsf{Sp}^\Sigma, \wedge, S)$ *has the structure of a closed symmetric monoidal category. All small limits and colimits exist and are calculated objectwise.*

Recall that by *closed* we mean there exists a functor which we call *mapping object* (or function spectrum),

$$(\mathsf{Sp}^\Sigma)^{\mathrm{op}} \times \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma, \qquad (Y, Z) \longmapsto \mathrm{Map}(Y, Z),$$

which fits into isomorphisms

$$\hom(X \wedge Y, Z) \cong \hom(X, \mathrm{Map}(Y, Z)), \tag{2.16}$$

natural in symmetric spectra X, Y, Z. These mapping objects will arise when we introduce mapping sequences associated to circle products in Section 3.

3 Algebras and modules over operads

In this section we recall certain definitions and constructions involving symmetric sequences, algebras, and modules over operads. A useful introduction to operads and their algebras is given by Kriz and May [24]. See also the original article of May [30]; other accounts include Berger and Moerdijk [2], Fresse [8], Ginzburg and Kapranov [11], Hinich [17], Markl, Shnider and Stasheff [29], McClure and Smith [32] and Spitzweck [41]. The circle product introduced in Section 3.3 goes back to Getzler and Jones [10] and Smirnov [40] and more recently appears in Fresse [7; 9], Goerss and Hopkins [12], Kapranov and Manin [22], Kelly [23] and Rezk [35]. A fuller account of the material in this section is given in [16] for the general context of a monoidal model category, which was largely influenced by the development in [35].

3.1 Symmetric sequences

Definition 3.2 Let $n \geq 0$ and G be a finite group.

- A *symmetric sequence* in Sp^Σ is a functor $A\colon \Sigma^{\mathrm{op}} \longrightarrow \mathsf{Sp}^\Sigma$. SymSeq is the category of symmetric sequences in Sp^Σ and their natural transformations; the null object is denoted by $*$.
- SymSeq^G is the category of functors $X\colon G \longrightarrow \mathsf{SymSeq}$ and their natural transformations.
- A symmetric sequence A is *concentrated at n* if $A[\mathbf{r}] = *$ for all $r \neq n$.

3.3 Tensor product and circle product of symmetric sequences

Definition 3.4 Let X be a finite set and A in Sp^Σ. The *copowers* $A\cdot X$ and $X\cdot A$ in Sp^Σ are defined as follows:

$$A\cdot X := \coprod_X A \cong A\wedge X_+ , \qquad X\cdot A := \coprod_X A \cong X_+\wedge A,$$

the coproduct in Sp^Σ of $|X|$ copies of A.

Definition 3.5 Let $A_1,\dots,A_t$ be symmetric sequences. The *tensor products* $A_1\check{\otimes}\cdots\check{\otimes}A_t\in\mathsf{SymSeq}$ are the left Kan extensions of objectwise smash along coproduct of sets:

$$\begin{tikzcd}
(\Sigma^{\mathrm{op}})^{\times t} \arrow[r, "A_1\times\cdots\times A_t"] \arrow[d, "\coprod"] & (\mathsf{Sp}^\Sigma)^{\times t} \arrow[r, "\wedge"] & \mathsf{Sp}^\Sigma \\
\Sigma^{\mathrm{op}} \arrow[rr, "A_1\check{\otimes}\cdots\check{\otimes}A_t", "\text{left Kan extension}"'] & & \mathsf{Sp}^\Sigma.
\end{tikzcd}$$

This definition of tensor product in SymSeq is conceptually the same as the definition of tensor product in S_*^Σ given in Definition 2.7. The following is a calculation of tensor product, whose proof is left to the reader.

Proposition 3.6 *Let $A_1,\dots,A_t$ be symmetric sequences and $R\in\Sigma$, with $r:=|R|$. There are natural isomorphisms*

$$\begin{aligned}
(A_1\check{\otimes}\cdots\check{\otimes}A_t)[R] &\cong \coprod_{\substack{\pi\colon R\longrightarrow \mathbf{t}\\ \text{in } \mathsf{Set}}} A_1[\pi^{-1}(1)]\wedge\cdots\wedge A_t[\pi^{-1}(t)],\\
&\cong \coprod_{r_1+\cdots+r_t=r} A_1[\mathbf{r_1}]\wedge\cdots\wedge A_t[\mathbf{r_t}]\underset{\Sigma_{r_1}\times\cdots\times\Sigma_{r_t}}{\cdot}\Sigma_r. \qquad (3.7)
\end{aligned}$$

It will be useful to extend the definition of tensor powers $A^{\check{\otimes}t}$ to situations in which the integers t are replaced by a finite set T.

Definition 3.8 Let A be a symmetric sequence and $R,T\in\Sigma$. The *tensor powers* $A^{\check{\otimes}T}\in\mathsf{SymSeq}$ are defined objectwise by

$$\begin{aligned}
(A^{\check{\otimes}T})[R] &:= \coprod_{\substack{\pi\colon R\longrightarrow T\\ \text{in } \mathsf{Set}}} \wedge_{t\in T} A[\pi^{-1}(t)], \quad T\neq\varnothing, \qquad (3.9)\\
(A^{\check{\otimes}\varnothing})[R] &:= \coprod_{\substack{\pi\colon R\longrightarrow \varnothing\\ \text{in } \mathsf{Set}}} S.
\end{aligned}$$

Note that there are no functions $\pi\colon R \longrightarrow \varnothing$ in Set unless $R = \varnothing$. We will use the abbreviation $A^{\check{\otimes} 0} := A^{\check{\otimes}\varnothing}$.

Definition 3.10 Let A, B be symmetric sequences, $R \in \Sigma$, and define $r := |R|$. The *circle product* (or composition product) $A \circ B \in \mathsf{SymSeq}$ is defined objectwise by the coend

$$(A \circ B)[R] \ := \ A \wedge_\Sigma (B^{\check{\otimes} -})[R] \cong \coprod_{t \geq 0} A[\mathbf{t}] \wedge_{\Sigma_t} (B^{\check{\otimes} t})[\mathbf{r}]. \tag{3.11}$$

Definition 3.12 Let B, C be symmetric sequences, $T \in \Sigma$, and define $t := |T|$. The *mapping sequence* $\mathrm{Map}^\circ(B, C) \in \mathsf{SymSeq}$ and the *mapping object* $\mathrm{Map}^{\check{\otimes}}(B, C) \in \mathsf{SymSeq}$ are defined objectwise by the ends

$$\mathrm{Map}^\circ(B, C)[T] := \mathrm{Map}((B^{\check{\otimes} T})[-], C)^\Sigma \cong \prod_{r \geq 0} \mathrm{Map}((B^{\check{\otimes} \mathbf{t}})[\mathbf{r}], C[\mathbf{r}])^{\Sigma_r},$$

$$\mathrm{Map}^{\check{\otimes}}(B, C)[T] := \mathrm{Map}(B, C[T \amalg -])^\Sigma \cong \prod_{r \geq 0} \mathrm{Map}(B[\mathbf{r}], C[\mathbf{t} + \mathbf{r}])^{\Sigma_r}. \tag{3.13}$$

These mapping sequences and mapping objects are part of closed monoidal category structures on symmetric sequences and fit into isomorphisms

$$\hom(A \circ B, C) \cong \hom(A, \mathrm{Map}^\circ(B, C)),$$

$$\hom(A \check{\otimes} B, C) \cong \hom(A, \mathrm{Map}^{\check{\otimes}}(B, C)),$$

natural in symmetric sequences A, B, C. The mapping sequences also arise in describing algebras and modules over operads (3.22).

Proposition 3.14

(a) $(\mathsf{SymSeq}, \check{\otimes}, 1)$ *has the structure of a closed symmetric monoidal category. All small limits and colimits exist and are calculated objectwise. The unit* $1 \in \mathsf{SymSeq}$ *is given by* $1[\mathbf{n}] = *$ *for each* $n \geq 1$ *and* $1[\mathbf{0}] = S$.

(b) $(\mathsf{SymSeq}, \circ, I)$ *has the structure of a closed monoidal category with all small limits and colimits. Circle product is not symmetric. The (two-sided) unit* $I \in \mathsf{SymSeq}$ *is given by* $I[\mathbf{n}] = *$ *for each* $n \neq 1$ *and* $I[\mathbf{1}] = S$.

3.15 Symmetric sequences build functors

The category Sp^Σ embeds in SymSeq as the full subcategory of symmetric sequences concentrated at 0, via the functor $\hat{-}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{SymSeq}$ defined objectwise by

$$\hat{Z}[R] := \begin{cases} Z, & \text{for } |R| = 0, \\ *, & \text{otherwise.} \end{cases} \tag{3.16}$$

Definition 3.17 Let $\mathcal{O}$ be a symmetric sequence and $Z \in \mathsf{Sp}^\Sigma$. The corresponding functor $\mathcal{O}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma$ is defined objectwise by

$$\mathcal{O}(Z) := \mathcal{O} \circ (Z) := \coprod_{t \geq 0} \mathcal{O}[\mathbf{t}] \wedge_{\Sigma_t} Z^{\wedge t} \cong (\mathcal{O} \circ \hat{Z})[\mathbf{0}].$$

3.18 Algebras and modules and over operads

Definition 3.19 An *operad* is a monoid object in $(\mathsf{SymSeq}, \circ, I)$ and a *morphism of operads* is a morphism of monoid objects in $(\mathsf{SymSeq}, \circ, I)$.

Similar to the case of any monoid object, we study operads because we are interested in the objects they act on. A useful introduction to monoid objects and monoidal categories is given in [26, VII].

Definition 3.20 Let $\mathcal{O}$ be an operad. A *left $\mathcal{O}$–module* is an object in $(\mathsf{SymSeq}, \circ, I)$ with a left action of $\mathcal{O}$ and a *morphism of left $\mathcal{O}$–modules* is a map in SymSeq which respects the left $\mathcal{O}$–module structure.

Each operad $\mathcal{O}$ determines a functor $\mathcal{O}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma$ (Definition 3.17) together with natural transformations $m\colon \mathcal{O}\mathcal{O} \longrightarrow \mathcal{O}$ and $\eta\colon \mathrm{id} \longrightarrow \mathcal{O}$ which give the functor $\mathcal{O}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma$ the structure of a monad (or triple) in Sp^Σ. One perspective offered in [24, I.3] is that operads determine particularly manageable monads. A useful introduction to monads and their algebras is given in [26, VI]. Recall the following definition from [24, I.2 and I.3].

Definition 3.21 Let $\mathcal{O}$ be an operad. An *$\mathcal{O}$–algebra* is an object in Sp^Σ with a left action of the monad $\mathcal{O}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma$ and a *morphism of $\mathcal{O}$–algebras* is a map in Sp^Σ which respects the left action of the monad $\mathcal{O}\colon \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma$.

It is easy to verify that an $\mathcal{O}$–algebra is the same as an object $X \in \mathsf{Sp}^\Sigma$ with a left $\mathcal{O}$–module structure on $\hat{X}$, and if X and X' are $\mathcal{O}$–algebras, then a morphism of $\mathcal{O}$–algebras is the same as a map $f\colon X \longrightarrow X'$ in Sp^Σ such that $\hat{f}\colon \hat{X} \longrightarrow \hat{X}'$ is a morphism of left $\mathcal{O}$–modules. In other words, an algebra over an operad $\mathcal{O}$ is the same as a left $\mathcal{O}$–module which is concentrated at 0.

Giving a symmetric sequence Y a left $\mathcal{O}$–module structure is the same as giving a morphism of operads

$$m\colon \mathcal{O} \longrightarrow \mathrm{Map}^{\circ}(Y,Y). \tag{3.22}$$

Similarly, giving an object $X \in \mathsf{Sp}^{\Sigma}$ an $\mathcal{O}$–algebra structure is the same as giving a morphism of operads

$$m\colon \mathcal{O} \longrightarrow \mathrm{Map}^{\circ}(\hat{X},\hat{X}).$$

This is the original definition given in [30] of an $\mathcal{O}$–algebra structure on X, where $\mathrm{Map}^{\circ}(\hat{X},\hat{X})$ is called the *endomorphism operad* of X, and motivates the suggestion in [24; 30] that $\mathcal{O}[\mathbf{t}]$ should be thought of as parameter objects for t–ary operations.

Definition 3.23 Let $\mathcal{O}$ be an operad.

- $\mathsf{Alg}_{\mathcal{O}}$ is the category of $\mathcal{O}$–algebras and their morphisms.
- $\mathsf{Lt}_{\mathcal{O}}$ is the category of left $\mathcal{O}$–modules and their morphisms.

The category $\mathsf{Alg}_{\mathcal{O}}$ embeds in $\mathsf{Lt}_{\mathcal{O}}$ as the full subcategory of left $\mathcal{O}$–modules concentrated at 0, via the functor $\hat{-}\colon \mathsf{Alg}_{\mathcal{O}} \longrightarrow \mathsf{Lt}_{\mathcal{O}}$ defined objectwise by (3.16).

Proposition 3.24 *Let $\mathcal{O}$ be an operad in symmetric spectra. There are adjunctions*

$$\mathsf{Sp}^{\Sigma} \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}}, \qquad \mathsf{SymSeq} \underset{U}{\overset{\mathcal{O}\circ -}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}}, \tag{3.25}$$

with left adjoints on top and U the forgetful functor.

Proof The unit I for circle product is the initial operad, hence there is a unique map of operads $f\colon I \longrightarrow \mathcal{O}$. The desired adjunctions are the following special cases

$$\mathsf{Sp}^{\Sigma} = \mathsf{Alg}_{I} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}}, \qquad \mathsf{SymSeq} = \mathsf{Lt}_{I} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}},$$

of change of operads adjunctions. □

Definition 3.26 Let C be a category. A pair of maps of the form

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1$$

in C is called a *reflexive pair* if there exists $s_0\colon X_0 \longrightarrow X_1$ in C such that $d_0 s_0 = \mathrm{id}$ and $d_1 s_0 = \mathrm{id}$. A *reflexive coequalizer* is the coequalizer of a reflexive pair.

The following proposition is proved in [35, Proposition 2.3.5], and allows us to calculate certain colimits in algebras and modules over operads by working in the underlying category. It is also proved in [16] and is closely related to [5, Proposition II.7.2]. Since it plays a fundamental role in several of the main arguments in this paper, we have included a proof below.

Proposition 3.27 *Let $\mathcal{O}$ be an operad in symmetric spectra. Reflexive coequalizers and filtered colimits exist in $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Lt}_{\mathcal{O}}$, and are preserved by the forgetful functors.*

First we consider the following proposition which is proved in [35, Lemma 2.3.4]. It is also proved in [16] and follows from the proof of [5, Proposition II.7.2] or the arguments in [12, Section 1] as we indicate below.

Proposition 3.28

(a) *If $A_{-1} \longleftarrow A_0 \leftleftarrows A_1$ and $B_{-1} \longleftarrow B_0 \leftleftarrows B_1$ are reflexive coequalizer diagrams in SymSeq, then their objectwise circle product*

$$A_{-1} \circ B_{-1} \longleftarrow A_0 \circ B_0 \leftleftarrows A_1 \circ B_1$$

is a reflexive coequalizer diagram in SymSeq.

(b) *If $A, B\colon \mathsf{D} \longrightarrow \mathsf{SymSeq}$ are filtered diagrams, then objectwise circle product of their colimiting cones is a colimiting cone. In particular, there are natural isomorphisms*

$$\operatorname*{colim}_{d\in\mathsf{D}}(A_d \circ B_d) \cong (\operatorname*{colim}_{d\in\mathsf{D}} A_d) \circ (\operatorname*{colim}_{d\in\mathsf{D}} B_d)$$

in SymSeq.

Proof Consider part (a). The corresponding statement for smash products of symmetric spectra follows from the proof of [5, Proposition II.7.2] or the argument appearing between Definition 1.8 and Lemma 1.9 in [12, Section 1]. Using this together with (3.9) and (3.11), the statement for circle products easily follows by verifying the universal property of a colimit. Consider part (b). It is easy to verify the corresponding statement for smash products of symmetric spectra, and the statement for circle products easily follows as in part (a). □

Proof of Proposition 3.27 Suppose $A_0 \leftleftarrows A_1$ is a reflexive pair in $\mathsf{Lt}_{\mathcal{O}}$ and consider the solid commutative diagram

$$\begin{tikzcd}
\mathcal{O}\circ\mathcal{O}\circ A_{-1} \arrow[d, dotted, shift right, "d_0"'] \arrow[d, dotted, shift left, "d_1"] & \mathcal{O}\circ\mathcal{O}\circ A_0 \arrow[l] \arrow[d, shift right, "m\circ\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}\circ m"] & \mathcal{O}\circ\mathcal{O}\circ A_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift right, "m\circ\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}\circ m"] \\
\mathcal{O}\circ A_{-1} \arrow[d, dotted, "m"] & \mathcal{O}\circ A_0 \arrow[l] \arrow[d, "m"] & \mathcal{O}\circ A_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "m"] \\
A_{-1} \arrow[u, dotted, bend left, "s_0"] & A_0 \arrow[l] \arrow[u, bend left, "\eta\circ\mathrm{id}"] & A_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[u, bend left, "\eta\circ\mathrm{id}"]
\end{tikzcd}$$

in SymSeq, with bottom row the reflexive coequalizer diagram of the underlying reflexive pair in SymSeq. By Proposition 3.28, the rows are reflexive coequalizer diagrams and hence there exist unique dotted arrows m, s_0, d_0, d_1 in SymSeq which make the diagram commute. By uniqueness, $s_0 = \eta\circ\mathrm{id}$, $d_0 = m\circ\mathrm{id}$, and $d_1 = \mathrm{id}\circ m$. It is easy to verify that m gives A_{-1} the structure of a left $\mathcal{O}$–module and that the bottom row is a reflexive coequalizer diagram in $\mathsf{Lt}_{\mathcal{O}}$; it is easy to check the diagram lives in $\mathsf{Lt}_{\mathcal{O}}$ and that the colimiting cone is initial with respect to all cones in $\mathsf{Lt}_{\mathcal{O}}$. The case for filtered colimits is similar. □

The next proposition is proved in [35, Proposition 2.3.5]. It verifies the existence of all small colimits in algebras and left modules over an operad, and provides one approach to their calculation. The proposition also follows from the argument in [5, Proposition II.7.4]. To keep the paper relatively self-contained, we have included a proof at the end of Section 6.

Proposition 3.29 *Let $\mathcal{O}$ be an operad in symmetric spectra. All small colimits exist in $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Lt}_{\mathcal{O}}$. If $A\colon \mathsf{D}\longrightarrow\mathsf{Lt}_{\mathcal{O}}$ is a small diagram, then* $\operatorname{colim} A$ *in $\mathsf{Lt}_{\mathcal{O}}$ may be calculated by a reflexive coequalizer of the form*

$$\operatorname{colim} A \cong \operatorname{colim}\Big(\mathcal{O}\circ\big(\operatorname*{colim}_{d\in D} A_d\big) \leftleftarrows \mathcal{O}\circ\big(\operatorname*{colim}_{d\in D}(\mathcal{O}\circ A_d)\big) \Big)$$

in the underlying category SymSeq; the colimits appearing inside the parenthesis are in the underlying category SymSeq.

The proof of the following is left to the reader.

Proposition 3.30 *Let $\mathcal{O}$ be an operad in symmetric spectra. All small limits exist in $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Lt}_{\mathcal{O}}$, and are preserved by the forgetful functors.*

4 Model structures

The purpose of this section is to prove Theorem 1.1 and Theorem 1.3, which establish certain model category structures on algebras and left modules over an operad. Model categories provide a setting in which one can do homotopy theory, and in particular, provide a framework for constructing and calculating derived functors. A useful introduction to model categories is given in Dwyer and Spalinski [4]; see also the original articles of Quillen [33; 34] and the more recent by Goerss and Jardine [15], Hirschhorn [19] and Hovey [20]. When we refer to the extra structure of a monoidal model category, we are using Schwede and Shipley [38, Definition 3.1]; an additional condition involving the unit is assumed in Lewis and Mandell [25, Definition 2.3] which we will not require in this paper.

In this paper, our primary method of establishing model structures is to use a small object argument together with the extra structure enjoyed by a cofibrantly generated model category [19, Chapter 11; 20, Section 2.1; 38, Section 2]. The reader unfamiliar with the small object argument may consult [4, Section 7.12] for a useful introduction, followed by the (possibly transfinite) versions described in [19, Chapter 10; 20, Section 2.1; 38, Section 2].

In [38, Section 2] an account of these techniques is provided which will be sufficient for our purposes; our proofs of Theorem 1.1 and Theorem 1.3 will reduce to verifying the conditions of [38, Lemma 2.3(1)]. This verification amounts to a homotopical analysis of certain pushouts (Section 4.3) which lies at the heart of this paper. The reader may contrast this with a path object approach explored in [2], which amounts to verifying the conditions of [38, Lemma 2.3(2)]; compare also [17; 41].

A first step is to recall just enough notation so that we can describe and work with the (positive) flat stable model structure on symmetric spectra, and the corresponding projective model structures on the diagram categories SymSeq and SymSeq^G, for G a finite group. The functors involved in such a description are easy to understand when defined as the left adjoints of appropriate functors, which is how they naturally arise in this context.

For each $m \geq 0$ and subgroup $H \subset \Sigma_m$ denote by $l\colon H \longrightarrow \Sigma_m$ the inclusion of groups and define the *evaluation* functor $\mathrm{ev}_m\colon \mathsf{S}_*^{\Sigma} \longrightarrow \mathsf{S}_*^{\Sigma_m}$ objectwise by $\mathrm{ev}_m(X) := X_m$. There are adjunctions

$$\mathsf{S}_* \underset{\lim_H}{\overset{}{\rightleftarrows}} \mathsf{S}_*^H \underset{l^*}{\overset{\Sigma_m \cdot_H -}{\rightleftarrows}} \mathsf{S}_*^{\Sigma_m} \underset{\mathrm{ev}_m}{\overset{}{\rightleftarrows}} \mathsf{S}_*^{\Sigma}$$

with left adjoints on top. Define $G_m^H\colon \mathsf{S}_* \longrightarrow \mathsf{S}_*^\Sigma$ to be the composition of the three top functors, and define $\lim_H \mathrm{ev}_m\colon \mathsf{S}_*^\Sigma \longrightarrow \mathsf{S}_*$ to be the composition of the three bottom functors; we have dropped the restriction functor l^* from the notation. It is easy to check that if $K \in \mathsf{S}_*$, then $G_m^H(K)$ is the object in S_*^Σ which is concentrated at m with value $\Sigma_m \cdot_H K$. Consider the forgetful functor $\mathsf{Sp}^\Sigma \longrightarrow \mathsf{S}_*^\Sigma$. It follows from Proposition 2.12 that there is an adjunction

$$\mathsf{S}_*^\Sigma \underset{}{\overset{S\otimes -}{\rightleftarrows}} \mathsf{Sp}^\Sigma$$

with left adjoint on top.

For each $p \geq 0$, define the *evaluation* functor $\mathrm{Ev}_p\colon \mathsf{SymSeq} \longrightarrow \mathsf{Sp}^\Sigma$ objectwise by $\mathrm{Ev}_p(A) := A[\mathbf{p}]$, and for each finite group G, consider the forgetful functor $\mathsf{SymSeq}^G \longrightarrow \mathsf{SymSeq}$. There are adjunctions

$$\mathsf{Sp}^\Sigma \underset{\mathrm{Ev}_p}{\overset{G_p}{\rightleftarrows}} \mathsf{SymSeq} \underset{}{\overset{G\cdot -}{\rightleftarrows}} \mathsf{SymSeq}^G$$

with left adjoints on top. It is easy to check that if $X \in \mathsf{Sp}^\Sigma$, then $G_p(X)$ is the symmetric sequence concentrated at p with value $X \cdot \Sigma_p$.

Putting it all together, there are adjunctions

$$\mathsf{S}_* \underset{\lim_H \mathrm{ev}_m}{\overset{G_m^H}{\rightleftarrows}} \mathsf{S}_*^\Sigma \underset{}{\overset{S\otimes -}{\rightleftarrows}} \mathsf{Sp}^\Sigma \underset{\mathrm{Ev}_p}{\overset{G_p}{\rightleftarrows}} \mathsf{SymSeq} \underset{}{\overset{G\cdot -}{\rightleftarrows}} \mathsf{SymSeq}^G \tag{4.1}$$

with left adjoints on top. We are now in a good position to describe several useful model structures. It is proved in [39] that the following two model category structures exist on symmetric spectra.

Definition 4.2

(a) The *flat stable model structure* on Sp^Σ has weak equivalences the stable equivalences, cofibrations the retracts of (possibly transfinite) compositions of pushouts of maps

$$S \otimes G_m^H \partial\Delta[k]_+ \longrightarrow S \otimes G_m^H \Delta[k]_+ \quad (m \geq 0,\ k \geq 0,\ H \subset \Sigma_m \text{ subgroup}),$$

and fibrations the maps with the right lifting property with respect to the acyclic cofibrations.

(b) The *positive flat stable model structure* on Sp^Σ has weak equivalences the stable equivalences, cofibrations the retracts of (possibly transfinite) compositions of pushouts of maps

$$S\otimes G^H_m\partial\Delta[k]_+ \longrightarrow S\otimes G^H_m\Delta[k]_+ \quad (m\geq 1,\ k\geq 0,\ H\subset\Sigma_m \text{ subgroup}),$$

and fibrations the maps with the right lifting property with respect to the acyclic cofibrations.

It follows immediately from the above description that every positive flat stable cofibration is a flat stable cofibration. Several useful properties of the flat stable model structure are proved in [21, Section 5.3]; here, we remind the reader of Remark 1.2.

The *stable model structure* on Sp^Σ is defined by fixing H in Definition 4.2(a) to be the trivial subgroup. This is one of several model category structures that is proved in [21] to exist on symmetric spectra.

The *positive stable model structure* on Sp^Σ is defined by fixing H in Definition 4.2(b) to be the trivial subgroup. This model category structure is proved in [28] to exist on symmetric spectra. It follows immediately that every (positive) stable cofibration is a (positive) flat stable cofibration.

These model structures on symmetric spectra enjoy several good properties, including that smash products of symmetric spectra mesh nicely with each of the model structures defined above. More precisely, each model structure above is cofibrantly generated in which the generating cofibrations and acyclic cofibrations have small domains, and that with respect to each model structure $(\mathsf{Sp}^\Sigma,\wedge,S)$ is a monoidal model category. There is also a model structure on Sp^Σ which has weak equivalences the stable equivalences and cofibrations the monomorphisms [21, Section 5.3]; this model structure is not a monoidal model structure on $(\mathsf{Sp}^\Sigma,\wedge,S)$.

If G is a finite group, it is easy to check that the diagram categories SymSeq and SymSeq^G inherit corresponding projective model category structures, where the weak equivalences (resp. fibrations) are the objectwise weak equivalences (resp. objectwise fibrations). We refer to these model structures by the names above (eg, the *positive flat stable* model structure on SymSeq^G). Each of these model structures is cofibrantly generated in which the generating cofibrations and acyclic cofibrations have small domains. Furthermore, with respect to each model structure $(\mathsf{SymSeq},\otimes,1)$ is a monoidal model category; this is proved in [16], but can easily be verified directly using (3.13).

Proof of Theorem 1.1 Consider SymSeq and Sp^Σ both with the positive flat stable model structure. We will prove that the model structure on $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) is created by the adjunction

$$\mathsf{SymSeq} \underset{U}{\overset{\mathcal{O}\circ -}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}} \qquad \Big(\text{resp.} \quad \mathsf{Sp}^{\Sigma} \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}}\Big)$$

with left adjoint on top and U the forgetful functor.

Define a map f in $\mathsf{Lt}_{\mathcal{O}}$ to be a weak equivalence (resp. fibration) if $U(f)$ is a weak equivalence (resp. fibration) in SymSeq. Similarly, define a map f in $\mathsf{Alg}_{\mathcal{O}}$ to be a weak equivalence (resp. fibration) if $U(f)$ is a weak equivalence (resp. fibration) in Sp^{Σ}. Define a map f in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) to be a cofibration if it has the left lifting property with respect to all acyclic fibrations in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$).

Consider the case of $\mathsf{Lt}_{\mathcal{O}}$. We want to verify the model category axioms (MC1)–(MC5) in [4]. By Proposition 3.29 and Proposition 3.30, we know that (MC1) is satisfied, and verifying (MC2) and (MC3) is clear. The (possibly transfinite) small object arguments described in the proof of [38, Lemma 2.3] reduce the verification of (MC5) to the verification of Proposition 4.4 below. The first part of (MC4) is satisfied by definition, and the second part of (MC4) follows from the usual lifting and retract argument, as described in the proof of [38, Lemma 2.3]. This verifies the model category axioms. By construction, the model category is cofibrantly generated. Argue similarly for the case of $\mathsf{Alg}_{\mathcal{O}}$ by considering left $\mathcal{O}$–modules concentrated at 0. □

Proof of Theorem 1.3 Consider SymSeq and Sp^{Σ} both with the positive stable model structure. We will prove that the model structure on $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) is created by the adjunction

$$\mathsf{SymSeq} \underset{U}{\overset{\mathcal{O}\circ -}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}} \qquad \Big(\text{resp.} \quad \mathsf{Sp}^{\Sigma} \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}}\Big)$$

with left adjoint on top and U the forgetful functor. Define a map f in $\mathsf{Lt}_{\mathcal{O}}$ to be a weak equivalence (resp. fibration) if $U(f)$ is a weak equivalence (resp. fibration) in SymSeq. Similarly, define a map f in $\mathsf{Alg}_{\mathcal{O}}$ to be a weak equivalence (resp. fibration) if $U(f)$ is a weak equivalence (resp. fibration) in Sp^{Σ}. Define a map f in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) to be a cofibration if it has the left lifting property with respect to all acyclic fibrations in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$).

The model category axioms are verified exactly as in the proof of Theorem 1.1; (MC5) is verified by Proposition 4.4 below since every cofibration in SymSeq (resp. Sp^{Σ}) with the positive stable model structure is a cofibration in SymSeq (resp. Sp^{Σ}) with the positive flat stable model structure. □

4.3 Homotopical analysis of certain pushouts

The purpose of this section is to prove the following proposition which we used in the proofs of Theorem 1.1 and Theorem 1.3. The constructions developed here will also be important for homotopical analyses in other sections of this paper.

Proposition 4.4 *Let $\mathcal{O}$ be an operad in symmetric spectra, $A \in \mathsf{Lt}_{\mathcal{O}}$, and $i\colon X \longrightarrow Y$ a generating acyclic cofibration in SymSeq with the positive flat stable model structure. Consider any pushout diagram in $\mathsf{Lt}_{\mathcal{O}}$ of the form*

$$\begin{tikzcd} \mathcal{O}\circ X \arrow[r, "f"] \arrow[d, "\mathrm{id}\circ i"'] & A \arrow[d, "j"] \\ \mathcal{O}\circ Y \arrow[r] & A \amalg_{(\mathcal{O}\circ X)} (\mathcal{O}\circ Y). \end{tikzcd} \tag{4.5}$$

Then j is a monomorphism and a weak equivalence.

Symmetric arrays arise naturally when calculating certain coproducts and pushouts of left modules and algebras over operads (Proposition 4.7 and Proposition 4.20).

Definition 4.6

- A *symmetric array* in Sp^Σ is a symmetric sequence in SymSeq; ie a functor $A\colon \Sigma^{\mathrm{op}} \longrightarrow \mathsf{SymSeq}$.
- $\mathsf{SymArray} := \mathsf{SymSeq}^{\Sigma^{\mathrm{op}}} \cong (\mathsf{Sp}^\Sigma)^{\Sigma^{\mathrm{op}}\times\Sigma^{\mathrm{op}}}$ is the category of symmetric arrays in Sp^Σ and their natural transformations.

First we analyze certain coproducts of modules over operads. The following proposition is proved in [16] in the more general context of monoidal model categories, and was motivated by a similar argument given in [14, Section 2.3] and [27, Section 13] in the context of algebras over an operad. Since the proposition is important to several results in this paper, and in an attempt to keep the paper relatively self-contained, we have included a proof below.

Proposition 4.7 *Let $\mathcal{O}$ be an operad in symmetric spectra, $A \in \mathsf{Lt}_{\mathcal{O}}$, and $Y \in \mathsf{SymSeq}$. Consider any coproduct in $\mathsf{Lt}_{\mathcal{O}}$ of the form*

$$A \amalg (\mathcal{O}\circ Y). \tag{4.8}$$

There exists a symmetric array $\mathcal{O}_A$ and natural isomorphisms

$$A \amalg (\mathcal{O}\circ Y) \cong \coprod_{q\geq 0} \mathcal{O}_A[\mathbf{q}] \check{\otimes}_{\Sigma_q} Y^{\check{\otimes} q}$$

in the underlying category SymSeq*. If* $q \geq 0$*, then* $\mathcal{O}_A[\mathbf{q}]$ *is naturally isomorphic to a colimit of the form*

$$\mathcal{O}_A[\mathbf{q}] \cong \operatorname{colim}\Big(\coprod_{p\geq 0} \mathcal{O}[\mathbf{p}+\mathbf{q}] \wedge_{\Sigma_p} A^{\check{\otimes} p} \overset{d_0}{\underset{d_1}{\leftleftarrows}} \coprod_{p\geq 0} \mathcal{O}[\mathbf{p}+\mathbf{q}] \wedge_{\Sigma_p} (\mathcal{O}\circ A)^{\check{\otimes} p} \Big),$$

in SymSeq *where* d_0 *is induced by operad multiplication and* d_1 *is induced by* $m\colon \mathcal{O}\circ A \longrightarrow A$*.*

Remark 4.9 Other possible notations for $\mathcal{O}_A$ include $\mathrm{U}_{\mathcal{O}}(A)$ or $\mathrm{U}(A)$; these are closer to the notation used in [6; 27] and are not to be confused with the forgetful functors.

First we make the following observation.

Proposition 4.10 *Let* $\mathcal{O}$ *be an operad in symmetric spectra and* $A \in \mathsf{Lt}_{\mathcal{O}}$*. Then*

$$A \xleftarrow{m} \mathcal{O}\circ A \overset{m\circ \mathrm{id}}{\underset{\mathrm{id}\circ m}{\leftleftarrows}} \mathcal{O}\circ\mathcal{O}\circ A \tag{4.11}$$

is a reflexive coequalizer diagram in $\mathsf{Lt}_{\mathcal{O}}$*.*

Proof We use a split fork argument. The unit map $\eta\colon I \longrightarrow \mathcal{O}$ induces a map $s_0 := \mathrm{id}\circ\eta\circ\mathrm{id}\colon \mathcal{O}\circ A \longrightarrow \mathcal{O}\circ\mathcal{O}\circ A$ in $\mathsf{Lt}_{\mathcal{O}}$. Relabeling the three maps in (4.11) as $d_0 := m$, $d_0 := m\circ\mathrm{id}$, $d_1 := \mathrm{id}\circ m$, it is easy to verify that $d_0 s_0 = \mathrm{id}$ and $d_1 s_0 = \mathrm{id}$. Hence the pair of maps is a reflexive pair in $\mathsf{Lt}_{\mathcal{O}}$, and by Proposition 3.27 it is enough to verify that (4.11) is a coequalizer diagram in the underlying category SymSeq. The unit map $\eta\colon I \longrightarrow \mathcal{O}$ also induces maps

$$s_{-1} := \eta\circ\mathrm{id}\colon A \longrightarrow \mathcal{O}\circ A$$
$$s_{-1} := \eta\circ\mathrm{id}\circ\mathrm{id}\colon \mathcal{O}\circ A \longrightarrow \mathcal{O}\circ\mathcal{O}\circ A$$

in the underlying category SymSeq which satisfy the relations

$$d_0 d_0 = d_0 d_1, \qquad d_0 s_{-1} = \mathrm{id}, \qquad d_1 s_{-1} = s_{-1} d_0.$$

Using these relations, it is easy to check that (4.11) is a coequalizer diagram in SymSeq by verifying the universal property of colimits. □

Proof of Proposition 4.7 The objectwise coproduct of two reflexive coequalizer diagrams is a reflexive coequalizer diagram, hence by Proposition 4.10 the coproduct

(4.8) may be calculated by a reflexive coequalizer in $\mathsf{Lt}_{\mathcal{O}}$ of the form

$$A \amalg (\mathcal{O}\circ Y) \cong \operatorname{colim}\Big((\mathcal{O}\circ A)\amalg(\mathcal{O}\circ Y) \underset{d_1}{\overset{d_0}{\leftleftarrows}} (\mathcal{O}\circ\mathcal{O}\circ A)\amalg(\mathcal{O}\circ Y) \Big).$$

The maps d_0 and d_1 are induced by maps $m\colon \mathcal{O}\circ\mathcal{O}\longrightarrow\mathcal{O}$ and $m\colon \mathcal{O}\circ A\longrightarrow A$, respectively. By Proposition 3.27, this reflexive coequalizer may be calculated in the underlying category SymSeq. There are natural isomorphisms

$$\begin{aligned}(\mathcal{O}\circ A)\amalg(\mathcal{O}\circ Y) &\cong \mathcal{O}\circ(A\amalg Y)\\ &\cong \coprod_{t\geq 0}\mathcal{O}[\mathbf{t}]\wedge_{\Sigma_t}(A\amalg Y)^{\check{\otimes} t}\\ &\cong \coprod_{q\geq 0}\Big(\coprod_{p\geq 0}\mathcal{O}[\mathbf{p}+\mathbf{q}]\wedge_{\Sigma_p}A^{\check{\otimes} p}\Big)\check{\otimes}_{\Sigma_q}Y^{\check{\otimes} q},\end{aligned}$$

and similarly,

$$(\mathcal{O}\circ\mathcal{O}\circ A)\amalg(\mathcal{O}\circ Y)\cong \coprod_{q\geq 0}\Big(\coprod_{p\geq 0}\mathcal{O}[\mathbf{p}+\mathbf{q}]\wedge_{\Sigma_p}(\mathcal{O}\circ A)^{\check{\otimes} p}\Big)\check{\otimes}_{\Sigma_q}Y^{\check{\otimes} q},$$

in the underlying category SymSeq. The maps d_0 and d_1 similarly factor in the underlying category SymSeq. □

Remark 4.12 We have used the natural isomorphisms

$$(A\amalg Y)^{\check{\otimes} t}\cong \coprod_{p+q=t}\Sigma_{p+q}\cdot_{\Sigma_p\times\Sigma_q}A^{\check{\otimes} p}\check{\otimes}Y^{\check{\otimes} q}$$

in the proof of Proposition 4.7.

Definition 4.13 Let $i\colon X\longrightarrow Y$ be a morphism in SymSeq and $t\geq 1$. Define $Q_0^t := X^{\check{\otimes} t}$ and $Q_t^t := Y^{\check{\otimes} t}$. For $0<q<t$ define Q_q^t inductively by the pushout diagrams

$$\begin{tikzcd} \Sigma_t\cdot_{\Sigma_{t-q}\times\Sigma_q}X^{\check{\otimes}(t-q)}\check{\otimes}Q_{q-1}^q \arrow[r, "\mathrm{pr}_*"] \arrow[d, "i_*"] & Q_{q-1}^t \arrow[d] \\ \Sigma_t\cdot_{\Sigma_{t-q}\times\Sigma_q}X^{\check{\otimes}(t-q)}\check{\otimes}Y^{\check{\otimes} q} \arrow[r] & Q_q^t \end{tikzcd} \tag{4.14}$$

in $\mathsf{SymSeq}^{\Sigma_t}$. We sometimes denote Q_q^t by $Q_q^t(i)$ to emphasize in the notation the map $i\colon X\longrightarrow Y$. The maps pr_* and i_* are the obvious maps induced by i and the appropriate projection maps.

Remark 4.15 For instance, to construct Q_2^3, first construct Q_1^2 via the pushout diagram

$$\begin{tikzcd}
\Sigma_2 \cdot_{\Sigma_1\times\Sigma_1} X\check{\otimes}X \arrow[r] \arrow[d, "\mathrm{id}\cdot_{\Sigma_1\times\Sigma_1}\mathrm{id}\check{\otimes} i"] & \Sigma_2\cdot_{\Sigma_2} X^{\check{\otimes}2} \arrow[r, "\cong"] & X^{\check{\otimes}2} \arrow[d] \\
\Sigma_2 \cdot_{\Sigma_1\times\Sigma_1} X\check{\otimes}Y \arrow[rr] & & Q_1^2
\end{tikzcd} \tag{4.16}$$

in $\mathsf{SymSeq}^{\Sigma_2}$, then construct Q_1^3 by the pushout diagram

$$\begin{tikzcd}
\Sigma_3 \cdot_{\Sigma_2\times\Sigma_1} X^{\check{\otimes}2}\check{\otimes}X \arrow[r] \arrow[d, "\mathrm{id}\cdot_{\Sigma_2\times\Sigma_1}\mathrm{id}\check{\otimes} i"] & \Sigma_3\cdot_{\Sigma_3} X^{\check{\otimes}3} \arrow[r, "\cong"] & X^{\check{\otimes}3} \arrow[d] \\
\Sigma_3 \cdot_{\Sigma_2\times\Sigma_1} X^{\check{\otimes}2}\check{\otimes}Y \arrow[rr] & & Q_1^3
\end{tikzcd}$$

in $\mathsf{SymSeq}^{\Sigma_3}$, and finally construct Q_2^3 by the pushout diagram

$$\begin{tikzcd}
\Sigma_3 \cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}Q_1^2 \arrow[r, "\mathrm{pr}_*"] \arrow[d, "\mathrm{id}\cdot_{\Sigma_1\times\Sigma_2}\mathrm{id}\check{\otimes} i_*"] & Q_1^3 \arrow[d] \\
\Sigma_3 \cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}Y^{\check{\otimes}2} \arrow[r] & Q_2^3
\end{tikzcd} \tag{4.17}$$

in $\mathsf{SymSeq}^{\Sigma_3}$. The map i_* in (4.17) is induced via (4.16) by the two maps

$$X^{\check{\otimes}2} \longrightarrow Y^{\check{\otimes}2},$$
$$\Sigma_2\cdot_{\Sigma_1\times\Sigma_1} X\check{\otimes}Y \longrightarrow \Sigma_2\cdot_{\Sigma_1\times\Sigma_1} Y\check{\otimes}Y \longrightarrow \Sigma_2\cdot_{\Sigma_2} Y^{\check{\otimes}2} \cong Y^{\check{\otimes}2}.$$

The pushout diagram

$$\begin{tikzcd}
\Sigma_3 \cdot_{\Sigma_1\times\Sigma_1\times\Sigma_1} X\check{\otimes}X\check{\otimes}X \arrow[r] \arrow[d] & \Sigma_3\cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}X^{\check{\otimes}2} \arrow[d] \\
\Sigma_3 \cdot_{\Sigma_1\times\Sigma_1\times\Sigma_1} X\check{\otimes}X\check{\otimes}Y \arrow[r] & \Sigma_3\cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}Q_1^2
\end{tikzcd} \tag{4.18}$$

in $\mathsf{SymSeq}^{\Sigma_3}$ is obtained by applying $\Sigma_3\cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}-$ to (4.16); the map pr_* in (4.17) is induced via (4.18) by the two maps

$$\Sigma_3\cdot_{\Sigma_1\times\Sigma_2} X\check{\otimes}X^{\check{\otimes}2} \longrightarrow \Sigma_3\cdot_{\Sigma_3} X^{\check{\otimes}3} \cong X^{\check{\otimes}3} \longrightarrow Q_1^3,$$
$$\Sigma_3\cdot_{\Sigma_1\times\Sigma_1\times\Sigma_1} X\check{\otimes}X\check{\otimes}Y \longrightarrow \Sigma_3\cdot_{\Sigma_2\times\Sigma_1} X^{\check{\otimes}2}\check{\otimes}Y \longrightarrow Q_1^3.$$

Remark 4.19 The construction Q_{t-1}^t can be thought of as a Σ_t–equivariant version of the colimit of a punctured t–cube [16]. There is a natural isomorphism $Y^{\check{\otimes} t}/Q_{t-1}^t \cong (Y/X)^{\check{\otimes} t}$.

The following proposition is proved in [16] in the more general context of monoidal model categories, and was motivated by a similar construction given in [6, Section 12] in the context of simplicial multifunctors of symmetric spectra. Since several results in this paper require both the proposition and its proof, and in an effort to keep the paper relatively self-contained, we have included a proof below.

Proposition 4.20 *Let $\mathcal{O}$ be an operad in symmetric spectra, $A \in \mathsf{Lt}_{\mathcal{O}}$, and $i\colon X \longrightarrow Y$ in SymSeq. Consider any pushout diagram in $\mathsf{Lt}_{\mathcal{O}}$ of the form*

$$\begin{tikzcd} \mathcal{O} \circ X \arrow[r, "f"] \arrow[d, "\mathrm{id} \circ i"] & A \arrow[d, "j"] \\ \mathcal{O} \circ Y \arrow[r] & A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y). \end{tikzcd} \tag{4.21}$$

The pushout in (4.21) is naturally isomorphic to a filtered colimit of the form

$$A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y) \cong \mathrm{colim}\Big(A_0 \xrightarrow{j_1} A_1 \xrightarrow{j_2} A_2 \xrightarrow{j_3} \cdots \Big) \tag{4.22}$$

in the underlying category SymSeq, with $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$ and A_t defined inductively by pushout diagrams in SymSeq of the form

$$\begin{tikzcd} \mathcal{O}_A[\mathbf{t}] \check{\otimes}_{\Sigma_t} Q_{t-1}^t \arrow[r, "f_*"] \arrow[d, "\mathrm{id} \check{\otimes}_{\Sigma_t} i_*"] & A_{t-1} \arrow[d, "j_t"] \\ \mathcal{O}_A[\mathbf{t}] \check{\otimes}_{\Sigma_t} Y^{\check{\otimes} t} \arrow[r, "\xi_t"] & A_t \end{tikzcd} \tag{4.23}$$

Proof It is easy to verify that the pushout in (4.21) may be calculated by a reflexive coequalizer in $\mathsf{Lt}_{\mathcal{O}}$ of the form

$$A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y) \cong \mathrm{colim}\Big(A \amalg (\mathcal{O} \circ Y) \underset{\bar{f}}{\overset{\bar{i}}{\leftleftarrows}} A \amalg (\mathcal{O} \circ X) \amalg (\mathcal{O} \circ Y) \Big).$$

By Proposition 3.27, this reflexive coequalizer may be calculated in the underlying category SymSeq. Hence it suffices to reconstruct this coequalizer in SymSeq via a suitable filtered colimit in SymSeq. A first step is to understand what it means to give a cone in SymSeq out of this diagram.

The maps $\overline{i}$ and $\overline{f}$ are induced by maps $\mathrm{id}\circ i_*$ and $\mathrm{id}\circ f_*$ which fit into the commutative diagram

$$\textbf{(4.24)}\qquad \begin{tikzcd} A \amalg \big(\mathcal{O}\circ(X\amalg Y)\big) \arrow[d, shift right, "\overline{i}"'] \arrow[d, shift left, "\overline{f}"] & \mathcal{O}\circ(A\amalg X\amalg Y) \arrow[l] \arrow[d, shift right, "\mathrm{id}\circ i_*"'] \arrow[d, shift left, "\mathrm{id}\circ f_*"] & \mathcal{O}\circ\big((\mathcal{O}\circ A)\amalg X\amalg Y\big) \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \arrow[d, shift right, "\mathrm{id}\circ i_*"'] \arrow[d, shift left, "\mathrm{id}\circ f_*"] \\ A\amalg(\mathcal{O}\circ Y) & \mathcal{O}\circ(A\amalg Y) \arrow[l] & \mathcal{O}\circ\big((\mathcal{O}\circ A)\amalg Y\big) \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \end{tikzcd}$$

in $\mathsf{Lt}_{\mathcal{O}}$, with rows reflexive coequalizer diagrams, and maps i_* and f_* in SymSeq induced by $i\colon X\longrightarrow Y$ and $f\colon X\longrightarrow A$ in SymSeq. Here we have used the same notation for both f and its adjoint (3.25). By Proposition 3.27, the pushout in (4.21) may be calculated by the colimit of the left-hand column of (4.24) in the underlying category SymSeq. By (4.24) and Proposition 4.7, f induces maps $\overline{f}_{q,p}$ which make the diagrams

$$\begin{tikzcd} A \amalg \big(\mathcal{O}\circ(X\amalg Y)\big) \cong \coprod\limits_{q\geq 0}\coprod\limits_{p\geq 0}\big(\ \big) \arrow[d, "\overline{f}"] & \Big(\mathcal{O}_A[\mathbf{p}+\mathbf{q}]\check{\otimes}_{\Sigma_p\times\Sigma_q} X^{\check{\otimes}p}\check{\otimes} Y^{\check{\otimes}q}\Big) \arrow[l, "\mathrm{in}_{q,p}"'] \arrow[d, dotted, "\overline{f}_{q,p}"] \\ A\amalg(\mathcal{O}\circ Y)\cong \coprod\limits_{t\geq 0}\big(\ \big) & \Big(\mathcal{O}_A[\mathbf{q}]\check{\otimes}_{\Sigma_q} Y^{\check{\otimes}q}\Big) \arrow[l, "\mathrm{in}_q"'] \end{tikzcd}$$

in SymSeq commute. Similarly, i induces maps $\overline{i}_{q,p}$ which make the diagrams

$$\begin{tikzcd} A \amalg \big(\mathcal{O}\circ(X\amalg Y)\big) \cong \coprod\limits_{q\geq 0}\coprod\limits_{p\geq 0}\big(\ \big) \arrow[d, "\overline{i}"] & \Big(\mathcal{O}_A[\mathbf{p}+\mathbf{q}]\check{\otimes}_{\Sigma_p\times\Sigma_q} X^{\check{\otimes}p}\check{\otimes} Y^{\check{\otimes}q}\Big) \arrow[l, "\mathrm{in}_{q,p}"'] \arrow[d, dotted, "\overline{i}_{q,p}"] \\ A\amalg(\mathcal{O}\circ Y)\cong \coprod\limits_{t\geq 0}\big(\ \big) & \Big(\mathcal{O}_A[\mathbf{p}+\mathbf{q}]\check{\otimes}_{\Sigma_{p+q}} Y^{\check{\otimes}(p+q)}\Big) \arrow[l, "\mathrm{in}_{p+q}"'] \end{tikzcd}$$

in SymSeq commute. We can now describe more explicitly what it means to give a cone in SymSeq out of the left-hand column of (4.24). Let $\varphi\colon A\amalg(\mathcal{O}\circ Y)\longrightarrow\cdot$ be a morphism in SymSeq and define $\varphi_q := \varphi\mathrm{in}_q$. Then $\varphi\overline{i}=\varphi\overline{f}$ if and only if the diagrams

$$\textbf{(4.25)}\qquad \begin{tikzcd} \mathcal{O}_A[\mathbf{p}+\mathbf{q}]\check{\otimes}_{\Sigma_p\times\Sigma_q} X^{\check{\otimes}p}\check{\otimes} Y^{\check{\otimes}q} \arrow[r, "\overline{f}_{q,p}"] \arrow[d, "\overline{i}_{q,p}"] & \mathcal{O}_A[\mathbf{q}]\check{\otimes}_{\Sigma_q} Y^{\check{\otimes}q} \arrow[d, "\varphi_q"] \\ \mathcal{O}_A[\mathbf{p}+\mathbf{q}]\check{\otimes}_{\Sigma_{p+q}} Y^{\check{\otimes}(p+q)} \arrow[r, "\varphi_{p+q}"] & \cdot \end{tikzcd}$$

commute for every $p,q\geq 0$. Since $\overline{i}_{q,0}=\mathrm{id}$ and $\overline{f}_{q,0}=\mathrm{id}$, it is sufficient to consider $q\geq 0$ and $p>0$.

The next step is to reconstruct the colimit of the left-hand column of (4.24) in SymSeq via a suitable filtered colimit in SymSeq. The diagrams (4.25) suggest how to proceed. We will describe two filtration constructions that calculate the pushout (4.21) in the underlying category SymSeq. The purpose of presenting the filtration construction (4.27) is to provide motivation and intuition for the filtration construction (4.23) that we are interested in. Since (4.27) does not use the gluing construction in Definition 4.13 it is simpler to verify that (4.22) is satisfied and provides a useful warm-up for working with (4.23).

For each $t\geq 1$, there are natural isomorphisms

$$(X\amalg Y)^{\check{\otimes}t}-Y^{\check{\otimes}t}\cong\coprod_{\substack{p+q=t\\ q\geq 0,\ p>0}}\Sigma_{p+q}\cdot_{\Sigma_p\times\Sigma_q}X^{\check{\otimes}p}\check{\otimes}Y^{\check{\otimes}q}. \tag{4.26}$$

Here, $(X\amalg Y)^{\check{\otimes}t}-Y^{\check{\otimes}t}$ denotes the coproduct of all factors in $(X\amalg Y)^{\check{\otimes}t}$ except $Y^{\check{\otimes}t}$. Define $A_0:=\mathcal{O}_A[\mathbf{0}]\cong A$ and for each $t\geq 1$ define A_t by the pushout diagram

$$\begin{tikzcd}
\mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t}\big[(X\amalg Y)^{\check{\otimes}t}-Y^{\check{\otimes}t}\big] \arrow[r, "f_*"] \arrow[d, "i_*"] & A_{t-1} \arrow[d, "j_t"] \\
\mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t}Y^{\check{\otimes}t} \arrow[r, "\xi_t"] & A_t
\end{tikzcd} \tag{4.27}$$

in SymSeq. The maps f_* and i_* are induced by the appropriate maps $\overline{f}_{q,p}$ and $\overline{i}_{q,p}$. We want to use (4.26), (4.27) and (4.25) to verify that (4.22) is satisfied; it is sufficient to verify the universal property of colimits. By Proposition 4.7, the coproduct $A\amalg(\mathcal{O}\circ Y)$ is naturally isomorphic to a filtered colimit of the form

$$A\amalg(\mathcal{O}\circ Y)\cong\mathrm{colim}\big(\,B_0\longrightarrow B_1\longrightarrow B_2\longrightarrow\cdots\big)$$

in the underlying category SymSeq, with $B_0:=\mathcal{O}_A[\mathbf{0}]$ and B_t defined inductively by pushout diagrams in SymSeq of the form

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & B_{t-1} \arrow[d] \\
\mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t}Y^{\check{\otimes}t} \arrow[r] & B_t
\end{tikzcd}$$

For each $t \geq 1$, there are naturally occurring maps $B_t \longrightarrow A_t$, induced by the appropriate ξ_i and j_i maps in (4.27), which fit into the commutative diagram

$$\begin{tikzcd}
 & & & & A \amalg \big(\mathcal{O} \circ (X \amalg Y)\big) \arrow[d, shift right, "\overline{i}"'] \arrow[d, shift left, "\overline{f}"] \\
B_0 \arrow[r] \arrow[d, equal] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \operatorname{colim}_t B_t \arrow[r, "\cong"] \arrow[d] & A \amalg (\mathcal{O} \circ Y) \arrow[d, "\overline{\xi}"] \\
A_0 \arrow[r, "j_1"] & A_1 \arrow[r, "j_2"] & A_2 \arrow[r, "j_3"] & \cdots \arrow[r] & \operatorname{colim}_t A_t \arrow[r, equal] & \operatorname{colim}_t A_t
\end{tikzcd}$$

in SymSeq; the morphism of filtered diagrams induces a map $\overline{\xi}$. We claim that the right-hand column is a coequalizer diagram in SymSeq. To verify that $\overline{\xi}$ satisfies $\overline{\xi}\,\overline{i} = \overline{\xi}\,\overline{f}$, by (4.25) it is enough to check that the diagram

$$\begin{tikzcd}
\mathcal{O}_A[\mathbf{p}+\mathbf{q}] \check{\otimes}_{\Sigma_p \times \Sigma_q} X^{\check{\otimes} p} \check{\otimes} Y^{\check{\otimes} q} \arrow[r, "\overline{f}_{q,p}"] \arrow[d, "\overline{i}_{q,p}"] & \mathcal{O}_A[\mathbf{q}] \check{\otimes}_{\Sigma_q} Y^{\check{\otimes} q} \arrow[d, "\overline{\xi}\,\mathrm{in}_q"] \\
\mathcal{O}_A[\mathbf{p}+\mathbf{q}] \check{\otimes}_{\Sigma_{p+q}} Y^{\check{\otimes}(p+q)} \arrow[r, "\overline{\xi}\,\mathrm{in}_{p+q}"] & \operatorname{colim}_t A_t
\end{tikzcd}$$

commute for every $q \geq 0$ and $p > 0$; this is easily verified using (4.26) and (4.27), and is left to the reader. Let $\varphi\colon A \amalg (\mathcal{O} \circ Y) \longrightarrow \cdot$ be a morphism in SymSeq such that $\varphi \overline{i} = \varphi \overline{f}$. We want to verify that there exists a unique map $\overline{\varphi}\colon \operatorname{colim}_t A_t \longrightarrow \cdot$ in SymSeq such that $\varphi = \overline{\varphi}\,\overline{\xi}$. Consider the corresponding maps φ_i in (4.25) and define $\overline{\varphi}_0 := \varphi_0$. For each $t \geq 1$, the maps φ_i induce maps $\overline{\varphi}_t\colon A_t \longrightarrow \cdot$ such that $\overline{\varphi}_t\, j_t = \overline{\varphi}_{t-1}$ and $\overline{\varphi}_t\, \xi_t = \varphi_t$. In particular, the maps $\overline{\varphi}_t$ induce a map $\overline{\varphi}\colon \operatorname{colim}_t A_t \longrightarrow \cdot$ in SymSeq. Using (4.25) it is an easy exercise (which the reader should verify) that $\overline{\varphi}$ satisfies $\varphi = \overline{\varphi}\,\overline{\xi}$ and that $\overline{\varphi}$ is the unique such map. Hence the filtration construction (4.27) satisfies (4.22). One drawback of (4.27) is that it may be difficult to analyze homotopically. A hint at how to improve the construction is given by the observation that the collection of maps $\overline{f}_{q,p}$ and $\overline{i}_{q,p}$ satisfy many compatibility relations. To obtain a filtration construction we can homotopically analyze, the idea is to replace $(X \amalg Y)^{\check{\otimes} t} - Y^{\check{\otimes} t}$ in (4.27) with the gluing construction Q^t_{t-1} in Definition 4.13 as follows.

Define $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$ and for each $t \geq 1$ define A_t by the pushout diagram (4.23) in SymSeq. The maps f_* and i_* are induced by the appropriate maps $\overline{f}_{q,p}$ and $\overline{i}_{q,p}$. Arguing exactly as above for the case of (4.27), it is easy to use the diagrams (4.25) to verify that (4.22) is satisfied. The only difference is that the naturally occurring maps $B_t \longrightarrow A_t$ are induced by the appropriate ξ_i and j_i maps in (4.23) instead of in (4.27). $\square$

The following proposition illustrates some of the good properties of the positive flat stable model structure on SymSeq. The statement in part (b) is motivated by [6, Lemma 12.7] in the context of symmetric spectra with the positive stable model structure. We defer the proof to Section 6.

Proposition 4.28 *Let $B \in \mathsf{SymSeq}^{\Sigma_t^{\mathrm{op}}}$ and $t \geq 1$. If $i\colon X \longrightarrow Y$ is a cofibration between cofibrant objects in SymSeq with the positive flat stable model structure, then*

(a) *$X^{\check{\otimes} t} \longrightarrow Y^{\check{\otimes} t}$ is a cofibration in $\mathsf{SymSeq}^{\Sigma_t}$ with the positive flat stable model structure, and hence with the flat stable model structure,*

(b) *the map $B \check{\otimes}_{\Sigma_t} Q_{t-1}^t \longrightarrow B \check{\otimes}_{\Sigma_t} Y^{\check{\otimes} t}$ is a monomorphism.*

We will prove the following proposition in Section 6.

Proposition 4.29 *Let G be a finite group and consider SymSeq, SymSeq^G and $\mathsf{SymSeq}^{G^{\mathrm{op}}}$ each with the flat stable model structure.*

(a) *If $B \in \mathsf{SymSeq}^{G^{\mathrm{op}}}$, then the functor*

$$B \check{\otimes}_G -\colon \mathsf{SymSeq}^G \longrightarrow \mathsf{SymSeq}$$

preserves weak equivalences between cofibrant objects, and hence its total left derived functor exists.

(b) *If $Z \in \mathsf{SymSeq}^G$ is cofibrant, then the functor*

$$- \check{\otimes}_G Z\colon \mathsf{SymSeq}^{G^{\mathrm{op}}} \longrightarrow \mathsf{SymSeq}$$

preserves weak equivalences.

We are now in a good position to give a homotopical analysis of the pushout in Proposition 4.4.

Proposition 4.30 *If the map $i\colon X \longrightarrow Y$ in Proposition 4.20 is a generating acyclic cofibration in SymSeq with the positive flat stable model structure, then each map j_t is a monomorphism and a weak equivalence. In particular, the map j is a monomorphism and a weak equivalence.*

Proof The generating acyclic cofibrations in SymSeq have cofibrant domains. Proposition 4.28 implies that each j_t is a monomorphism. We know that $A_t/A_{t-1} \cong \mathcal{O}_A[\mathbf{t}] \check{\otimes}_{\Sigma_t} (Y/X)^{\check{\otimes} t}$ and that $* \longrightarrow Y/X$ is an acyclic cofibration in SymSeq with the positive flat stable model structure. It follows from Proposition 4.28 and Proposition 4.29 that j_t is a weak equivalence. □

Proof of Proposition 4.4 By assumption, the map $i\colon X \longrightarrow Y$ is a generating acyclic cofibration in SymSeq with the positive flat stable model structure, hence Proposition 4.30 finishes the proof. □

5 Relations between homotopy categories

The purpose of this section is to prove Theorem 1.4, which establishes an equivalence between certain homotopy categories of algebras (resp. modules) over operads. Our argument is a verification of the conditions in [4, Theorem 9.7] for an adjunction to induce an equivalence between the corresponding homotopy categories, and amounts to a homotopical analysis (Section 5.1) of the unit of the adjunction.

Proof of Theorem 1.4 Let $f\colon \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism of operads and consider the case of left modules. We know (1.5) is a Quillen adjunction since the forgetful functor f^* preserves fibrations and acyclic fibrations. Assume furthermore that f is a weak equivalence in the underlying category SymSeq with the positive flat stable model structure; let's verify the Quillen adjunction (1.5) is a Quillen equivalence. By [4, Theorem 9.7], it is enough to verify: for cofibrant $Z \in \mathsf{Lt}_{\mathcal{O}}$ and fibrant $B \in \mathsf{Lt}_{\mathcal{O}'}$, a map $\xi\colon Z \longrightarrow f^*B$ is a weak equivalence in $\mathsf{Lt}_{\mathcal{O}}$ if and only if its adjoint map $\eta\colon f_*Z \longrightarrow B$ is a weak equivalence in $\mathsf{Lt}_{\mathcal{O}'}$. Noting that ξ factors as

$$Z \longrightarrow f^*f_*Z \xrightarrow{f^*\eta} f^*B$$

together with Proposition 5.2 below finishes the proof. Argue similarly for the case of algebras by considering left modules concentrated at 0. ☐

5.1 Homotopical analysis of the unit of the adjunction

The purpose of this subsection is to prove the following proposition which we used in the proof of Theorem 1.4. Our argument is motivated by the proof of [6, Theorem 12.5].

Proposition 5.2 *Let $f\colon \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism of operads and consider $\mathsf{Lt}_{\mathcal{O}}$ with the positive flat stable model structure. If $Z \in \mathsf{Lt}_{\mathcal{O}}$ is cofibrant and f is a weak equivalence in the underlying category SymSeq with the positive flat stable model structure, then the natural map $Z \longrightarrow f^*f_*Z$ is a weak equivalence in $\mathsf{Lt}_{\mathcal{O}}$.*

First we make the following observation.

Proposition 5.3 *Consider SymSeq with the positive flat stable model structure. If $W \in \mathsf{SymSeq}$ is cofibrant, then the functor*

$$-\circ W\colon \mathsf{SymSeq} \longrightarrow \mathsf{SymSeq}$$

preserves weak equivalences.

Proof Let $A \longrightarrow B$ be a weak equivalence in SymSeq; we want to verify

$$A[\mathbf{t}] \wedge_{\Sigma_t} (W^{\check{\otimes} t})[\mathbf{r}] \longrightarrow B[\mathbf{t}] \wedge_{\Sigma_t} (W^{\check{\otimes} t})[\mathbf{r}]$$

is a weak equivalence in Sp^Σ with the flat stable model structure for each $r, t \geq 0$. By Proposition 4.28 we know $W^{\check{\otimes} t}$ is cofibrant in $\mathsf{SymSeq}^{\Sigma_t}$ with the flat stable model structure for each $t \geq 1$. By considering symmetric sequences concentrated at 0, Proposition 4.29 finishes the proof. □

Proof of Proposition 5.2 Let $X \longrightarrow Y$ be a generating cofibration in SymSeq with the positive flat stable model structure, and consider the pushout diagram

(5.4)
$$\begin{tikzcd} \mathcal{O} \circ X \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ \mathcal{O} \circ Y \arrow[r] & Z_1 \end{tikzcd}$$

in $\mathsf{Lt}_{\mathcal{O}}$. For each $W \in \mathsf{SymSeq}$ consider the natural maps

(5.5) $$Z_0 \amalg (\mathcal{O} \circ W) \longrightarrow f^* f_* \big(Z_0 \amalg (\mathcal{O} \circ W)\big),$$

(5.6) $$Z_1 \amalg (\mathcal{O} \circ W) \longrightarrow f^* f_* \big(Z_1 \amalg (\mathcal{O} \circ W)\big),$$

and note that the left-hand (resp. right-hand) diagram

$$\begin{tikzcd} \mathcal{O} \circ X \arrow[r] \arrow[d] & Z_0 \amalg (\mathcal{O} \circ W) =: A \arrow[d] \\ \mathcal{O} \circ Y \arrow[r] & Z_1 \amalg (\mathcal{O} \circ W) \cong A_\infty \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}' \circ X \arrow[r] \arrow[d] & f_* Z_0 \amalg (\mathcal{O}' \circ W) =: A' \arrow[d] \\ \mathcal{O}' \circ Y \arrow[r] & f_* Z_1 \amalg (\mathcal{O}' \circ W) \cong f_* A_\infty \end{tikzcd}$$

is pushout diagram in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Lt}_{\mathcal{O}'}$). Assume (5.5) is a weak equivalence for every cofibrant $W \in \mathsf{SymSeq}$; let's verify (5.6) is a weak equivalence for every cofibrant $W \in \mathsf{SymSeq}$. Suppose $W \in \mathsf{SymSeq}$ is cofibrant. By Proposition 4.20 there are corresponding filtrations

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "\xi_0"] & A_1 \arrow[r] \arrow[d, dotted, "\xi_1"] & A_2 \arrow[r] \arrow[d, dotted, "\xi_2"] & \cdots \arrow[r] & \mathrm{colim}_t\, A_t \arrow[r, equal] \arrow[d] & A_\infty \arrow[d] \\ A'_0 \arrow[r] & A'_1 \arrow[r] & A'_2 \arrow[r] & \cdots \arrow[r] & \mathrm{colim}_t\, A'_t \arrow[r, "\cong"] & f^* f_* A_\infty, \end{tikzcd}$$

together with induced maps ξ_t $(t \geq 1)$ which make the diagram in SymSeq commute. By assumption we know ξ_0 is a weak equivalence, and to verify (5.6) is a weak equivalence, it is enough to check that ξ_t is a weak equivalence for each $t \geq 1$. Since the horizontal maps are monomorphisms and we know $A_t/A_{t-1} \cong \mathcal{O}_A[\mathbf{t}] \check{\otimes}_{\Sigma_t} (Y/X)^{\check{\otimes} t}$, it is enough to verify that

$$A \amalg (\mathcal{O} \circ (Y/X)) \longrightarrow A' \amalg (\mathcal{O}' \circ (Y/X))$$

is a weak equivalence, which is the same as verifying that

$$Z_0 \amalg (\mathcal{O} \circ W) \amalg (\mathcal{O} \circ (Y/X)) \longrightarrow f^* f_* \big(Z_0 \amalg (\mathcal{O} \circ W) \amalg (\mathcal{O} \circ (Y/X))\big)$$

is a weak equivalence. Noting that $W \amalg (Y/X)$ is cofibrant finishes the argument that (5.6) is a weak equivalence. Consider a sequence

$$Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \cdots$$

of pushouts of maps as in (5.4). Assume Z_0 makes (5.5) a weak equivalence for every cofibrant $W \in \mathsf{SymSeq}$; we want to show that for $Z_\infty := \operatorname{colim}_k Z_k$ the natural map

$$Z_\infty \amalg (\mathcal{O} \circ W) \longrightarrow f^* f_* \big(Z_\infty \amalg (\mathcal{O} \circ W)\big) \tag{5.7}$$

is a weak equivalence for every cofibrant $W \in \mathsf{SymSeq}$. Consider the diagram

$$\begin{tikzcd}
Z_0 \amalg (\mathcal{O} \circ W) \arrow[r] \arrow[d] & Z_1 \amalg (\mathcal{O} \circ W) \arrow[r] \arrow[d] & Z_2 \amalg (\mathcal{O} \circ W) \arrow[r] \arrow[d] & \cdots \\
f^* f_* \big(Z_0 \amalg (\mathcal{O} \circ W)\big) \arrow[r] & f^* f_* \big(Z_1 \amalg (\mathcal{O} \circ W)\big) \arrow[r] & f^* f_* \big(Z_2 \amalg (\mathcal{O} \circ W)\big) \arrow[r] & \cdots
\end{tikzcd}$$

in $\mathsf{Lt}_{\mathcal{O}}$. The horizontal maps are monomorphisms and the vertical maps are weak equivalences, hence the induced map (5.7) is a weak equivalence. Noting that every cofibration $\mathcal{O} \circ * \longrightarrow Z$ in $\mathsf{Lt}_{\mathcal{O}}$ is a retract of a (possibly transfinite) composition of pushouts of maps as in (5.4), starting with $Z_0 = \mathcal{O} \circ *$, together with Proposition 5.3, finishes the proof. □

6 Proofs

The purpose of this section is to prove Proposition 4.28 and Proposition 4.29; we have also included a proof of Proposition 3.29 at the end of this section. First we establish a characterization of flat stable cofibrations.

6.1 Flat stable cofibrations

The purpose of this subsection is to prove Proposition 6.6, which identifies flat stable cofibrations in SymSeq^G, for G a finite group.

It is proved in [39] that the following model category structure exists on left Σ_n–objects in pointed simplicial sets.

Definition 6.2 Let $n \geq 0$.

- The *mixed Σ_n–equivariant model structure* on $\mathsf{S}_*^{\Sigma_n}$ has weak equivalences the underlying weak equivalences of simplicial sets, cofibrations the retracts of (possibly transfinite) compositions of pushouts of maps

$$\Sigma_n/H \cdot \partial\Delta[k]_+ \longrightarrow \Sigma_n/H \cdot \Delta[k]_+ \quad (k \geq 0,\ H \subset \Sigma_n \text{ subgroup}),$$

  and fibrations the maps with the right lifting property with respect to the acyclic cofibrations.

Furthermore, it is proved in [39] that this model structure is cofibrantly generated in which the generating cofibrations and acyclic cofibrations have small domains, and that the cofibrations are the monomorphisms. It is easy to prove that the diagram category of $(\Sigma_r^{\mathrm{op}} \times G)$–shaped diagrams in $\mathsf{S}_*^{\Sigma_n}$ appearing in the following proposition inherits a corresponding projective model structure. This proposition, whose proof is left to the reader, will be needed for identifying flat stable cofibrations in SymSeq^G.

Proposition 6.3 *Let G be a finite group and consider any $n, r \geq 0$. The diagram category $(\mathsf{S}_*^{\Sigma_n})^{\Sigma_r^{\mathrm{op}} \times G}$ inherits a corresponding projective model structure from the mixed Σ_n–equivariant model structure on $\mathsf{S}_*^{\Sigma_n}$. The weak equivalences (resp. fibrations) are the underlying weak equivalences (resp. fibrations) in $\mathsf{S}_*^{\Sigma_n}$ and the cofibrations are the monomorphisms such that $\Sigma_r^{\mathrm{op}} \times G$ acts freely on the simplices of the codomain not in the image.*

Definition 6.4 Define $\overline{S} \in \mathsf{Sp}^\Sigma$ such that $\overline{S}_n := S_n$ for $n \geq 1$ and $\overline{S}_0 := *$. The structure maps are the naturally occurring ones such that there exists a map of symmetric spectra $i \colon \overline{S} \longrightarrow S$ satisfying $i_n = \mathrm{id}$ for each $n \geq 1$.

The following calculation, which follows easily from (2.9) and (2.14), will be needed for characterizing flat stable cofibrations in SymSeq^G below.

Calculation 6.5 Let $m, p \geq 0$, $H \subset \Sigma_m$ a subgroup, and K a pointed simplicial set. Define $X := G \cdot G_p(S \otimes G_m^H K) \in \mathsf{SymSeq}^G$. Here, X is obtained by applying the indicated functors in (4.1) to K. Then for $r = p$ we have

$$(\bar{S} \wedge X[\mathbf{r}])_n \cong \begin{cases} G \cdot \big(\Sigma_n \cdot_{\Sigma_{n-m} \times \Sigma_m} \bar{S}_{n-m} \wedge (\Sigma_m / H \cdot K)\big) \cdot \Sigma_p & \text{for } n > m, \\ * & \text{for } n \le m, \end{cases}$$

$$X[\mathbf{r}]_n \cong \begin{cases} G \cdot \big(\Sigma_n \cdot_{\Sigma_{n-m} \times \Sigma_m} S_{n-m} \wedge (\Sigma_m / H \cdot K)\big) \cdot \Sigma_p & \text{for } n > m, \\ G \cdot (\Sigma_m / H \cdot K) \cdot \Sigma_p & \text{for } n = m, \\ * & \text{for } n < m, \end{cases}$$

and for $r \neq p$ we have $X[\mathbf{r}] = * = \bar{S} \wedge X[\mathbf{r}]$.

The following characterization of flat stable cofibrations in SymSeq^G is motivated by [21, Proposition 5.2.2]; we have benefitted from the discussion and corresponding characterization in [37] of cofibrations in Sp^Σ with the flat stable model structure.

Proposition 6.6 *Let G be a finite group.*

(a) *A map $f\colon X \longrightarrow Y$ in SymSeq^G with the flat stable model structure is a cofibration if and only if the induced maps*

$$X[\mathbf{r}]_0 \longrightarrow Y[\mathbf{r}]_0, \quad r \ge 0,\ n = 0,$$
$$(\bar{S} \wedge Y[\mathbf{r}])_n \amalg_{(\bar{S} \wedge X[\mathbf{r}])_n} X[\mathbf{r}]_n \longrightarrow Y[\mathbf{r}]_n, \quad r \ge 0,\ n \ge 1,$$

are cofibrations in $(\mathsf{S}_*^{\Sigma_n})^{\Sigma_r^{\mathrm{op}} \times G}$ with the model structure of Proposition 6.3.

(b) *A map $f\colon X \longrightarrow Y$ in SymSeq^G with the positive flat stable model structure is a cofibration if and only if the maps $X[\mathbf{r}]_0 \longrightarrow Y[\mathbf{r}]_0$, $r \ge 0$, are isomorphisms, and the induced maps*

$$(\bar{S} \wedge Y[\mathbf{r}])_n \amalg_{(\bar{S} \wedge X[\mathbf{r}])_n} X[\mathbf{r}]_n \longrightarrow Y[\mathbf{r}]_n, \quad r \ge 0,\ n \ge 1,$$

are cofibrations in $(\mathsf{S}_*^{\Sigma_n})^{\Sigma_r^{\mathrm{op}} \times G}$ with the model structure of Proposition 6.3.

Proof It suffices to prove part (a). Consider any $f\colon X \longrightarrow Y$ in SymSeq^G with the flat stable model structure. We want a sufficient condition for f to be a cofibration. The first step is to rewrite a lifting problem as a sequential lifting problem.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] \arrow[ur, dotted] & B \end{tikzcd} \qquad \begin{tikzcd} X[\mathbf{r}]_n \arrow[r] \arrow[d] & E[\mathbf{r}]_n \arrow[d] \\ Y[\mathbf{r}]_n \arrow[r] \arrow[ur, dotted] & B[\mathbf{r}]_n \end{tikzcd} \qquad \begin{tikzcd} (\bar{S} \otimes Y[\mathbf{r}])_n \arrow[r] \arrow[d] & Y[\mathbf{r}]_n \arrow[d] \\ (\bar{S} \otimes E[\mathbf{r}])_n \arrow[r] & E[\mathbf{r}]_n \end{tikzcd}$$

The left-hand solid commutative diagram in SymSeq^G has a lift if and only if the right-hand sequence of lifting problems in $(\mathsf{S}_*^{\Sigma_n})^{\Sigma_r^{\mathrm{op}}\times G}$ has a solution, if and only if the sequence of lifting problems

$$\begin{tikzcd} X[\mathbf{r}]_n \arrow[r] \arrow[d] & E[\mathbf{r}]_n \arrow[d] \\ Y[\mathbf{r}]_n \arrow[r] \arrow[ur, dotted] & B[\mathbf{r}]_n \end{tikzcd} \qquad \begin{tikzcd} (\bar{S}\wedge Y[\mathbf{r}])_n \arrow[r] \arrow[d] & Y[\mathbf{r}]_n \arrow[d] \\ (\bar{S}\wedge E[\mathbf{r}])_n \arrow[r] & E[\mathbf{r}]_n \end{tikzcd}$$

in $(\mathsf{S}_*^{\Sigma_n})^{\Sigma_r^{\mathrm{op}}\times G}$ has a solution, if and only if the sequence of lifting problems

$$\begin{tikzcd} X[\mathbf{r}]_0 \arrow[r] \arrow[d, "(*)_0"'] & E[\mathbf{r}]_0 \arrow[d] \\ Y[\mathbf{r}]_0 \arrow[r] \arrow[ur, dotted] & B[\mathbf{r}]_0 \end{tikzcd} \qquad \begin{tikzcd} (\bar{S}\wedge Y[\mathbf{r}])_n \amalg_{(\bar{S}\wedge X[\mathbf{r}])_n} X[\mathbf{r}]_n \arrow[r] \arrow[d, "(*)_n"'] & E[\mathbf{r}]_n \arrow[d] \\ Y[\mathbf{r}]_n \arrow[r] \arrow[ur, dotted] & B[\mathbf{r}]_n \end{tikzcd} \quad (n\geq 1)$$

has a solution. If each $(*)_n$ is a cofibration then f has the left lifting property with respect to all acyclic fibrations, and hence f is a cofibration. Conversely, suppose f is a cofibration. We want to verify that each $(*)_n$ is a cofibration. Every cofibration is a retract of a (possibly transfinite) composition of pushouts of generating cofibrations, and hence by a reduction argument that we leave to the reader, it is sufficient to verify for f a generating cofibration. Let $g\colon K\longrightarrow L$ be a monomorphism in S_*, $m,p\geq 0$, $H\subset\Sigma_m$ a subgroup, and define $f\colon X\longrightarrow Y$ in SymSeq^G to be the induced map

$$G\cdot G_p(S\otimes G_m^H K)\xrightarrow{g_*} G\cdot G_p(S\otimes G_m^H L)\ .$$

Here, the map g_* is obtained by applying the indicated functors in (4.1) to the map g. We know $(*)_0$ is a cofibration. Consider $n\geq 1$. By Calculation 6.5: $(*)_n$ is an isomorphism for the case $r\neq p$ and for the case ($r=p$ and $n\neq m$). For the case ($r=p$ and $n=m$), $(*)_n$ is the map

$$G\cdot(\Sigma_m/H\cdot K)\cdot\Sigma_p\xrightarrow{G\cdot(\Sigma_m/H\cdot g)\cdot\Sigma_p} G\cdot(\Sigma_m/H\cdot L)\cdot\Sigma_p$$

Hence in all cases $(*)_n$ is a cofibration. $\square$

6.7 Proofs

Proof of Proposition 4.29 Consider part (b). Let $g\colon K \longrightarrow L$ be a monomorphism in S_*, $m, p \geq 0$, $H \subset \Sigma_m$ a subgroup, and consider the pushout diagram

$$\begin{tikzcd} G \cdot G_p(S \otimes G_m^H K) \arrow[r] \arrow[d, "g_*"] & Z_0 \arrow[d] \\ G \cdot G_p(S \otimes G_m^H L) \arrow[r] & Z_1 \end{tikzcd} \tag{6.8}$$

in SymSeq^G. Here, the map g_* is obtained by applying the indicated functors in (4.1) to the map g. Consider the functors

$$- \check{\otimes}_G Z_0\colon \mathsf{SymSeq}^{G^{\mathrm{op}}} \longrightarrow \mathsf{SymSeq}, \tag{6.9}$$

$$- \check{\otimes}_G Z_1\colon \mathsf{SymSeq}^{G^{\mathrm{op}}} \longrightarrow \mathsf{SymSeq}, \tag{6.10}$$

and assume (6.9) preserves weak equivalences; let's verify (6.10) preserves weak equivalences. Suppose $A \longrightarrow B$ in $\mathsf{SymSeq}^{G^{\mathrm{op}}}$ is a weak equivalence. Applying $A \check{\otimes}_G -$ to (6.8) gives the pushout diagram

$$\begin{tikzcd} A\check{\otimes} G_p(S \otimes G_m^H K) \arrow[r] \arrow[d, "(*)"] & A \check{\otimes}_G Z_0 \arrow[d, "(**)"] \\ A\check{\otimes} G_p(S \otimes G_m^H L) \arrow[r] & A \check{\otimes}_G Z_1 \end{tikzcd}$$

in SymSeq. Let's check $(*)$ is a monomorphism. This amounts to a calculation:

$$\big(A\check{\otimes} G_p(S \otimes G_m^H K)\big)[\mathbf{r}] \cong \begin{cases} A[\mathbf{r}-\mathbf{p}] \wedge (S \otimes G_m^H K) \cdot_{\Sigma_{r-p}\times 1} \Sigma_r & \text{for } r \geq p, \\ * & \text{for } r < p. \end{cases}$$

Since the map $S \otimes G_m^H K \longrightarrow S \otimes G_m^H L$ is a cofibration in Sp^Σ with the flat stable model structure, smashing with any symmetric spectrum gives a monomorphism. It follows that $(*)$ is a monomorphism, and hence $(**)$ is a monomorphism. Consider the commutative diagram

$$\begin{tikzcd} A \check{\otimes}_G Z_0 \arrow[r] \arrow[d] & A \check{\otimes}_G Z_1 \arrow[r] \arrow[d] & A\check{\otimes} G_p(S \otimes G_m^H (L/K)) \arrow[d] \\ B \check{\otimes}_G Z_0 \arrow[r] & B \check{\otimes}_G Z_1 \arrow[r] & B\check{\otimes} G_p(S \otimes G_m^H (L/K)). \end{tikzcd}$$

Since $S \otimes G_m^H (L/K)$ is cofibrant in Sp^Σ with the flat stable model structure, smashing with it preserves weak equivalences. It follows that the right-hand vertical map is a weak equivalence. By assumption, the left-hand vertical map is a weak equivalence, hence the middle vertical map is a weak equivalence and we get that (6.10) preserves weak equivalences. Consider a sequence

$$Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \cdots$$

of pushouts of maps as in (6.8). Assume (6.9) preserves weak equivalences; we want to show that for $Z_\infty := \mathrm{colim}_k Z_k$ the functor

$$- \check{\otimes}_G Z_\infty \colon \mathsf{SymSeq}^{G^{\mathrm{op}}} \longrightarrow \mathsf{SymSeq}$$

preserves weak equivalences. Suppose $A \longrightarrow B$ in $\mathsf{SymSeq}^{G^{\mathrm{op}}}$ is a weak equivalence and consider the diagram

$$\begin{tikzcd}
A \check{\otimes}_G Z_0 \arrow[r] \arrow[d] & A \check{\otimes}_G Z_1 \arrow[r] \arrow[d] & A \check{\otimes}_G Z_2 \arrow[r] \arrow[d] & \cdots \\
B \check{\otimes}_G Z_0 \arrow[r] & B \check{\otimes}_G Z_1 \arrow[r] & B \check{\otimes}_G Z_2 \arrow[r] & \cdots
\end{tikzcd}$$

in SymSeq. The horizontal maps are monomorphisms and the vertical maps are weak equivalences, hence the induced map $A \check{\otimes}_G Z_\infty \longrightarrow B \check{\otimes}_G Z_\infty$ is a weak equivalence. Noting that every cofibration $* \longrightarrow Z$ in SymSeq^G is a retract of a (possibly transfinite) composition of pushouts of maps as in (6.8), starting with $Z_0 = *$, finishes the proof of part (b). Consider part (a). Suppose $X \longrightarrow Y$ in SymSeq^G is a weak equivalence between cofibrant objects; we want to show that $B \check{\otimes}_G X \longrightarrow B \check{\otimes}_G Y$ is a weak equivalence. The map $* \longrightarrow B$ factors in $\mathsf{SymSeq}^{G^{\mathrm{op}}}$ as

$$* \longrightarrow B^c \longrightarrow B$$

a cofibration followed by an acyclic fibration, the diagram

$$\begin{tikzcd}
B^c \check{\otimes}_G X \arrow[r] \arrow[d] & B^c \check{\otimes}_G Y \arrow[d] \\
B \check{\otimes}_G X \arrow[r] & B \check{\otimes}_G Y
\end{tikzcd}$$

commutes, and since three of the maps are weak equivalences, so is the fourth. □

Proposition 6.11 *Let G be a finite group. If $B \in \mathsf{SymSeq}^{G^{\mathrm{op}}}$, then the functor*

$$B \check{\otimes}_G - \colon \mathsf{SymSeq}^G \longrightarrow \mathsf{SymSeq}$$

sends cofibrations in SymSeq^G with the flat stable model structure to monomorphisms.

Proof Let $g\colon K \longrightarrow L$ be a monomorphism in S_*, $m, p \geq 0$, $H \subset \Sigma_m$ a subgroup, and consider the pushout diagram

(6.12)
$$\begin{tikzcd} G \cdot G_p(S \otimes G_m^H K) \arrow[r] \arrow[d, "g_*"] & Z_0 \arrow[d] \\ G \cdot G_p(S \otimes G_m^H L) \arrow[r] & Z_1 \end{tikzcd}$$

in SymSeq^G. Here, the map g_* is obtained by applying the indicated functors in (4.1) to the map g. Applying $B \check{\otimes}_G -$ gives the pushout diagram

$$\begin{tikzcd} B \check{\otimes} G_p(S \otimes G_m^H K) \arrow[r] \arrow[d, "(*)"] & B \check{\otimes}_G Z_0 \arrow[d, "(**)"] \\ B \check{\otimes} G_p(S \otimes G_m^H L) \arrow[r] & B \check{\otimes}_G Z_1 \end{tikzcd}$$

in SymSeq. The map $(*)$ is a monomorphism by the same arguments used in the proof of Proposition 4.29, hence $(**)$ is a monomorphism. Noting that every cofibration in SymSeq^G is a retract of a (possibly transfinite) composition of pushouts of maps as in (6.12) completes the proof. □

The following two propositions are exercises left to the reader.

Proposition 6.13 *Let $t \geq 1$. Suppose the left-hand diagram is a pushout diagram in* SymSeq*:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"] & A \arrow[d, "j"] \\ Y \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} Q_{t-1}^t(i) \arrow[r] \arrow[d] & Q_{t-1}^t(j) \arrow[d] \\ Y^{\check{\otimes} t} \arrow[r] & B^{\check{\otimes} t} \end{tikzcd}$$

Then the corresponding right-hand diagram is a pushout diagram in $\mathsf{SymSeq}^{\Sigma_t}$.

Proposition 6.14 *Let $t \geq 1$ and consider a commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "i"] & B \arrow[r, "r"] \arrow[d, "j"] & C \arrow[d, "k"] \\ X \arrow[r, "s"] & Y \arrow[r, "r"] & Z \end{tikzcd}$$

in SymSeq*. Then the corresponding diagram*

$$\begin{tikzcd}
Q^t_{t-1}(i) \arrow[r, "\overline{s}"] \arrow[d] & Q^t_{t-1}(j) \arrow[r, "\overline{r}"] \arrow[d] & Q^t_{t-1}(k) \arrow[d] \\
X^{\check{\otimes} t} \arrow[r] & Y^{\check{\otimes} t} \arrow[r] & Z^{\check{\otimes} t}
\end{tikzcd}$$

in $\mathsf{SymSeq}^{\Sigma_t}$ *commutes. Furthermore,* $\overline{r}\,\overline{s} = \overline{rs}$ *and* $\overline{\mathrm{id}} = \mathrm{id}$*.*

The following calculation, which follows easily from (2.9), (2.14) and (3.7), will be needed in the proof of Proposition 4.28 below.

Calculation 6.15 Let $k, m, p \geq 0$, $H \subset \Sigma_m$ a subgroup, and $t \geq 1$. Let the map $g\colon \partial\Delta[k]_+ \longrightarrow \Delta[k]_+$ be a generating cofibration for S_* and define $X \longrightarrow Y$ in SymSeq to be the induced map

$$G_p(S \otimes G^H_m \partial\Delta[k]_+) \xrightarrow{g_*} G_p(S \otimes G^H_m \Delta[k]_+) \;.$$

Here, the map g_* is obtained by applying the indicated functors in (4.1) to the map g. For $r = tp$ we have the calculation

$$\big((Y^{\check{\otimes} t})[\mathbf{r}]\big)_n \cong \begin{cases} \Sigma_n \cdot_{\Sigma_{n-tm} \times H^{\times t}} S_{n-tm} \wedge (\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n > tm, \\ \Sigma_{tm} \cdot_{H^{\times t}} (\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n = tm, \\ * & \text{for } n < tm, \end{cases}$$

$$\big(\overline{S} \wedge (Y^{\check{\otimes} t})[\mathbf{r}]\big)_n \cong \begin{cases} \Sigma_n \cdot_{\Sigma_{n-tm} \times H^{\times t}} \overline{S}_{n-tm} \wedge (\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n > tm, \\ * & \text{for } n \leq tm, \end{cases}$$

$$\big(Q^t_{t-1}[\mathbf{r}]\big)_n \cong \begin{cases} \Sigma_n \cdot_{\Sigma_{n-tm} \times H^{\times t}} S_{n-tm} \wedge \partial(\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n > tm, \\ \Sigma_{tm} \cdot_{H^{\times t}} \partial(\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n = tm, \\ * & \text{for } n < tm, \end{cases}$$

$$\big(\overline{S} \wedge Q^t_{t-1}[\mathbf{r}]\big)_n \cong \begin{cases} \Sigma_n \cdot_{\Sigma_{n-tm} \times H^{\times t}} \overline{S}_{n-tm} \wedge \partial(\Delta[k]^{\times t})_+ \cdot \Sigma_{tp} & \text{for } n > tm, \\ * & \text{for } n \leq tm, \end{cases}$$

and for $r \neq tp$ we have $(Y^{\check{\otimes} t})[\mathbf{r}] = * = \overline{S} \wedge (Y^{\check{\otimes} t})[\mathbf{r}]$ and $Q^t_{t-1}[\mathbf{r}] = * = \overline{S} \wedge Q^t_{t-1}[\mathbf{r}]$.

The following proposition is proved in [3, I.2] and will be useful below for verifying that certain induced maps are cofibrations.

Proposition 6.16 *Let M be a model category and consider a commutative diagram of the form*

$$\begin{tikzcd} A_0 \arrow[d] & A_1 \arrow[l] \arrow[r] \arrow[d] & A_2 \arrow[d] \\ B_0 & B_1 \arrow[l] \arrow[r] & B_2 \end{tikzcd}$$

in M. If the maps $A_0 \longrightarrow B_0$ and $B_1 \amalg_{A_1} A_2 \longrightarrow B_2$ are cofibrations, then the induced map

$$A_0 \amalg_{A_1} A_2 \longrightarrow B_0 \amalg_{B_1} B_2$$

is a cofibration.

Proof of Proposition 4.28 Consider part (a). The argument is by induction on t. Let $m \geq 1$, $H \subset \Sigma_m$ a subgroup, and $k, p \geq 0$. Let $g\colon \partial\Delta[k]_+ \longrightarrow \Delta[k]_+$ be a generating cofibration for S_* and consider the pushout diagram

$$\text{(6.17)}\qquad \begin{tikzcd} G_p(S \otimes G_m^H \partial\Delta[k]_+) \arrow[r] \arrow[d, "g_*"] & Z_0 \arrow[d, "i_0"] \\ D := G_p(S \otimes G_m^H \Delta[k]_+) \arrow[r] & Z_1 \end{tikzcd}$$

in SymSeq with Z_0 cofibrant. Here, the map g_* is obtained by applying the indicated functors in (4.1) to the map g. By Proposition 6.13, the corresponding diagram

$$\begin{tikzcd} Q^t_{t-1}(g_*) \arrow[r] \arrow[d, "(*)"] & Q^t_{t-1}(i_0) \arrow[d, "(**)"] \\ D^{\check{\otimes} t} \arrow[r] & Z_1^{\check{\otimes} t} \end{tikzcd}$$

is a pushout diagram in $\mathsf{SymSeq}^{\Sigma_t}$. Since $m \geq 1$, it follows from Proposition 6.6 and Calculation 6.15 that $(*)$ is a cofibration in $\mathsf{SymSeq}^{\Sigma_t}$, and hence $(**)$ is a cofibration. Consider a sequence

$$\text{(6.18)}\qquad Z_0 \xrightarrow{i_0} Z_1 \xrightarrow{i_1} Z_2 \xrightarrow{i_2} \cdots$$

of pushouts of maps as in (6.17), define $Z_\infty := \operatorname{colim}_q Z_q$, and consider the naturally occurring map $i_\infty\colon Z_0 \longrightarrow Z_\infty$. Using Proposition 6.16 and (4.14), it is easy to verify that each $Z_q^{\check{\otimes} t} \longrightarrow Q^t_{t-1}(i_q)$ is a cofibration in $\mathsf{SymSeq}^{\Sigma_t}$. By above we know that each $Q^t_{t-1}(i_q) \longrightarrow Z_{q+1}^{\check{\otimes} t}$ is a cofibration. It follows immediately that each $Z_q^{\check{\otimes} t} \longrightarrow Z_{q+1}^{\check{\otimes} t}$ is a cofibration in $\mathsf{SymSeq}^{\Sigma_t}$, and hence the map $Z_0^{\check{\otimes} t} \longrightarrow Z_\infty^{\check{\otimes} t}$ is a cofibration. Noting that every cofibration between cofibrant objects in SymSeq with the positive flat stable model structure is a retract of a (possibly transfinite) composition of pushouts of maps as in (6.17) finishes the proof for part (a). Consider part (b). Proceed as above for part (a) and consider the commutative diagram

$$\text{(6.19)}\qquad \begin{tikzcd} Z_0^{\check{\otimes} t} \arrow[r] \arrow[d, equal] & Q_{t-1}^t(i_0) \arrow[r] \arrow[d] & Q_{t-1}^t(i_1 i_0) \arrow[r] \arrow[d] & Q_{t-1}^t(i_2 i_1 i_0) \arrow[r] \arrow[d] & \cdots \\ Z_0^{\check{\otimes} t} \arrow[r] & Z_1^{\check{\otimes} t} \arrow[r] & Z_2^{\check{\otimes} t} \arrow[r] & Z_3^{\check{\otimes} t} \arrow[r] & \cdots \end{tikzcd}$$

in $\mathsf{SymSeq}^{\Sigma_t}$. We claim that (6.19) is a diagram of cofibrations. By part (a), the bottom row is a diagram of cofibrations. Using Proposition 6.16 and (4.14), it is easy to verify that if i and j are composable cofibrations between cofibrant objects in SymSeq, then the induced maps

$$Q_{t-1}^t(i) \longrightarrow Q_{t-1}^t(ji) \longrightarrow Q_{t-1}^t(j)$$

are cofibrations in $\mathsf{SymSeq}^{\Sigma_t}$; it follows easily that the vertical maps and the top row maps are cofibrations. Applying $B \check{\otimes}_{\Sigma_t} -$ to (6.19) gives the commutative diagram

$$\text{(6.20)}\qquad \begin{tikzcd} B \check{\otimes}_{\Sigma_t} Z_0^{\check{\otimes} t} \arrow[r] \arrow[d, equal] & B \check{\otimes}_{\Sigma_t} Q_{t-1}^t(i_0) \arrow[r] \arrow[d] & B \check{\otimes}_{\Sigma_t} Q_{t-1}^t(i_1 i_0) \arrow[r] \arrow[d] & \cdots \\ B \check{\otimes}_{\Sigma_t} Z_0^{\check{\otimes} t} \arrow[r] & B \check{\otimes}_{\Sigma_t} Z_1^{\check{\otimes} t} \arrow[r] & B \check{\otimes}_{\Sigma_t} Z_2^{\check{\otimes} t} \arrow[r] & \cdots \end{tikzcd}$$

in SymSeq. By Proposition 6.11, (6.20) is a diagram of monomorphisms, hence the induced map $B \check{\otimes}_{\Sigma_t} Q_{t-1}^t(i_\infty) \longrightarrow B \check{\otimes}_{\Sigma_t} Z_\infty^{\check{\otimes} t}$ is a monomorphism. Noting that every cofibration between cofibrant objects in SymSeq is a retract of a (possibly transfinite) composition of pushouts of maps as in (6.17), together with Proposition 6.14, finishes the proof for part (b). $\square$

Proof of Proposition 3.29 Suppose $A\colon \mathsf{D} \longrightarrow \mathsf{Lt}_{\mathcal{O}}$ is a small diagram. We want to show that $\operatorname{colim} A$ exists. It is easy to verify, using Proposition 4.10, that this colimit may be calculated by a reflexive coequalizer in $\mathsf{Lt}_{\mathcal{O}}$ of the form

$$\operatorname{colim} A \cong \operatorname{colim}\Bigl(\operatorname*{colim}_{d \in \mathsf{D}} (\mathcal{O} \circ A_d) \underset{(\mathrm{id} \circ m)_*}{\overset{(m \circ \mathrm{id})_*}{\longleftleftarrows}} \operatorname*{colim}_{d \in \mathsf{D}} (\mathcal{O} \circ \mathcal{O} \circ A_d) \Bigr),$$

provided that the indicated colimits appearing in this reflexive pair exist in $\mathsf{Lt}_{\mathcal{O}}$. The underlying category SymSeq has all small colimits, and left adjoints preserve colimiting cones, hence there is a commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{d\in \mathsf{D}}(\mathcal{O}\circ A_d) \arrow[d, "\cong"] & \operatorname*{colim}_{d\in \mathsf{D}}(\mathcal{O}\circ\mathcal{O}\circ A_d) \arrow[l, shift right, "(m\circ \mathrm{id})_*"'] \arrow[l, shift left, "(\mathrm{id}\circ m)_*"] \arrow[d, "\cong"] \\
\mathcal{O}\circ\big(\operatorname*{colim}_{d\in D} A_d\big) & \mathcal{O}\circ\big(\operatorname*{colim}_{d\in D}(\mathcal{O}\circ A_d)\big) \arrow[l, shift right] \arrow[l, shift left]
\end{tikzcd}$$

in $\mathsf{Lt}_{\mathcal{O}}$; the colimits in the bottom row exist since they are in the underlying category SymSeq (we have dropped the notation for the forgetful functor U), hence the colimits in the top row exist in $\mathsf{Lt}_{\mathcal{O}}$. Therefore $\operatorname{colim} A$ exists and Proposition 3.27 completes the proof. □

7 Constructions in the special case of algebras over an operad

Some readers may only be interested in the special case of algebras over an operad and may wish to completely avoid working with the circle product and the left $\mathcal{O}$–module constructions. It is easy to translate the constructions and proofs in this paper into the special case of algebras while avoiding the circle product notation. Usually, this amounts to replacing $(\mathsf{SymSeq},\check{\otimes})$ with $(\mathsf{Sp}^{\Sigma},\wedge)$, replacing the left adjoint $\mathcal{O}\circ -\colon \mathsf{SymSeq}\longrightarrow \mathsf{Lt}_{\mathcal{O}}$ with the left adjoint $\mathcal{O}(-)\colon \mathsf{Sp}^{\Sigma}\longrightarrow \mathsf{Alg}_{\mathcal{O}}$ (Definition 3.17), and then replacing the symmetric array $\mathcal{O}_A$ in Proposition 4.7 with the symmetric sequence $\mathcal{O}_A$ in Proposition 7.2. We illustrate below with several special cases of particular interest.

7.1 Special cases

Proposition 4.7 has the following special case.

Proposition 7.2 *Let $\mathcal{O}$ be an operad in symmetric spectra, $A\in\mathsf{Alg}_{\mathcal{O}}$, and $Y\in\mathsf{Sp}^{\Sigma}$. Consider any coproduct in $\mathsf{Alg}_{\mathcal{O}}$ of the form*

(7.3)
$$A\amalg\mathcal{O}(Y).$$

There exists a symmetric sequence $\mathcal{O}_A$ and natural isomorphisms

$$A\amalg\mathcal{O}(Y)\cong\coprod_{q\geq 0}\mathcal{O}_A[\mathbf{q}]\wedge_{\Sigma_q}Y^{\wedge q}$$

in the underlying category Sp^Σ*. If* $q \geq 0$*, then* $\mathcal{O}_A[\mathbf{q}]$ *is naturally isomorphic to a colimit of the form*

$$\mathcal{O}_A[\mathbf{q}] \cong \operatorname{colim}\Big(\coprod_{p\geq 0} \mathcal{O}[\mathbf{p}+\mathbf{q}] \wedge_{\Sigma_p} A^{\wedge p} \underset{d_1}{\overset{d_0}{\leftleftarrows}} \coprod_{p\geq 0} \mathcal{O}[\mathbf{p}+\mathbf{q}] \wedge_{\Sigma_p} (\mathcal{O}(A))^{\wedge p} \Big),$$

in Sp^Σ*, with* d_0 *induced by operad multiplication and* d_1 *induced by* $m\colon \mathcal{O}(A) \longrightarrow A$*.*

Definition 4.13 has the following special case.

Definition 7.4 Let $i\colon X \longrightarrow Y$ be a morphism in Sp^Σ and $t \geq 1$. Define $Q_0^t := X^{\wedge t}$ and $Q_t^t := Y^{\wedge t}$. For $0 < q < t$ define Q_q^t inductively by the pushout diagrams

$$\begin{tikzcd} \Sigma_t \cdot_{\Sigma_{t-q}\times\Sigma_q} X^{\wedge(t-q)} \wedge Q_{q-1}^q \arrow[r, "\mathrm{pr}_*"] \arrow[d, "i_*"] & Q_{q-1}^t \arrow[d] \\ \Sigma_t \cdot_{\Sigma_{t-q}\times\Sigma_q} X^{\wedge(t-q)} \wedge Y^{\wedge q} \arrow[r] & Q_q^t \end{tikzcd}$$

in $(\mathsf{Sp}^\Sigma)^{\Sigma_t}$. We sometimes denote Q_q^t by $Q_q^t(i)$ to emphasize in the notation the map $i\colon X \longrightarrow Y$. The maps pr_* and i_* are the obvious maps induced by i and the appropriate projection maps.

Proposition 4.20 has the following special case.

Proposition 7.5 *Let* $\mathcal{O}$ *be an operad in symmetric spectra,* $A \in \mathsf{Alg}_\mathcal{O}$*, and* $i\colon X \longrightarrow Y$ *in* Sp^Σ*. Consider any pushout diagram in* $\mathsf{Alg}_\mathcal{O}$ *of the form*

(7.6)
$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "f"] \arrow[d, "\mathrm{id}(i)"] & A \arrow[d, "j"] \\ \mathcal{O}(Y) \arrow[r] & A \amalg_{\mathcal{O}(X)} \mathcal{O}(Y). \end{tikzcd}$$

The pushout in (7.6) is naturally isomorphic to a filtered colimit of the form

$$A \amalg_{\mathcal{O}(X)} \mathcal{O}(Y) \cong \operatorname{colim}\Big(A_0 \xrightarrow{j_1} A_1 \xrightarrow{j_2} A_2 \xrightarrow{j_3} \cdots \Big)$$

in the underlying category Sp^Σ, *with* $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$ *and* A_t *defined inductively by pushout diagrams in* Sp^Σ *of the form:*

$$\begin{tikzcd}
\mathcal{O}_A[\mathbf{t}] \wedge_{\Sigma_t} Q^t_{t-1} \arrow[r, "f_*"] \arrow[d, "\mathrm{id}\wedge_{\Sigma_t} i_*"] & A_{t-1} \arrow[d, "j_t"] \\
\mathcal{O}_A[\mathbf{t}] \wedge_{\Sigma_t} Y^{\wedge t} \arrow[r, "\xi_t"] & A_t
\end{tikzcd}$$

Propositions 4.28, 4.29 and 4.30 have the following special cases, respectively.

Proposition 7.7 *Let* $B \in (\mathsf{Sp}^\Sigma)^{\Sigma_t^{\mathrm{op}}}$ *and* $t \geq 1$. *If* $i\colon X \longrightarrow Y$ *is a cofibration between cofibrant objects in* Sp^Σ *with the positive flat stable model structure, then*

(a) $X^{\wedge t} \longrightarrow Y^{\wedge t}$ *is a cofibration in* $(\mathsf{Sp}^\Sigma)^{\Sigma_t}$ *with the positive flat stable model structure, and hence with the flat stable model structure,*

(b) *the map* $B \wedge_{\Sigma_t} Q^t_{t-1} \longrightarrow B \wedge_{\Sigma_t} Y^{\wedge t}$ *is a monomorphism.*

Proposition 7.8 *Let* G *be a finite group and consider* Sp^Σ, $(\mathsf{Sp}^\Sigma)^G$, *and* $(\mathsf{Sp}^\Sigma)^{G^{\mathrm{op}}}$ *each with the flat stable model structure.*

(a) *If* $B \in (\mathsf{Sp}^\Sigma)^{G^{\mathrm{op}}}$, *then the functor*

$$B \wedge_G -\colon (\mathsf{Sp}^\Sigma)^G \longrightarrow \mathsf{Sp}^\Sigma$$

preserves weak equivalences between cofibrant objects, and hence its total left derived functor exists.

(b) *If* $Z \in (\mathsf{Sp}^\Sigma)^G$ *is cofibrant, then the functor*

$$- \wedge_G Z\colon (\mathsf{Sp}^\Sigma)^{G^{\mathrm{op}}} \longrightarrow \mathsf{Sp}^\Sigma$$

preserves weak equivalences.

Proposition 7.9 *If the map* $i\colon X \longrightarrow Y$ *in Proposition 7.5 is a generating acyclic cofibration in* Sp^Σ *with the positive flat stable model structure, then each map* j_t *is a monomorphism and a weak equivalence. In particular, the map* j *is a monomorphism and a weak equivalence.*

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