Chains on suspension spectra

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We define and study a homological version of Sullivan's rational de Rham complex for simplicial sets. This new functor can be generalised to simplicial symmetric spectra and in that context it has excellent categorical properties which promise to make a number of interesting applications much more straightforward.

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1 Introduction

In this paper we will define and study a functor Φ from simplicial sets to rational chain complexes, with the property that $H_*(\Phi_*(X))$ is just the ordinary rational homology of X.

Some background is needed to understand why this functor deserves attention. There is a much simpler functor called N_* (normalised simplicial chains) from simplicial sets to integral chain complexes that computes integral homology, and one can just tensor with $\mathbb{Q}$ to compute rational homology. There is a dual complex N^* that calculates integral cohomology. This is equipped with a natural product $N^*(X) \otimes N^*(X) \to N^*(X)$ which is commutative up to homotopy but not on the nose. The theory of Steenrod operations shows that if we work integrally then neither $N^*(X)$ nor any reasonable replacement can be given a strictly commutative product (even with the usual signs). Rationally, however, the situation is better: in [10] Sullivan developed a rational and simplicial version of de Rham theory giving a cochain complex $\Omega^*(X)$ with a strictly commutative product that computes the ordinary rational cohomology of X. This can be used as a starting point for the rich and powerful theory of rational homotopy (originally introduced by Quillen [8] using slightly different machinery). One can then stabilise and consider the category $\mathcal{S}_{\mathbb{Q}}$ of rational spectra, which makes things considerably simpler: it is well-known that the homotopy category of $\mathcal{S}_{\mathbb{Q}}$ is equivalent to the category of graded rational vector spaces. However, we can make things harder again by considering rational spectra with a ring structure or a group action. To handle these, we need to improve the homotopy classification of rational spectra to some kind of monoidal Quillen equivalence of $\mathcal{S}_{\mathbb{Q}}$ with a suitable model category $\mathrm{Ch}_{\mathbb{Q}}$ of rational chain complexes.

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Work of this type has been done especially by Greenlees [5], Greenlees and Shipley [6], Shipley [9] and Barnes [1], leading to very concrete and interesting descriptions of the homotopy theory of G–spectra for various compact Lie groups G, among other things. However, some of the arguments involved are more awkward than one might like, because they do not have a single symmetric monoidal Quillen functor $\Psi_*\colon \mathcal{S}_{\mathbb{Q}} \to \mathrm{Ch}_{\mathbb{Q}}$, but a zig-zag of Quillen functors whose monoidal properties fit together in an inconvenient way.

Recently, the author and Stefan Schwede independently discovered a functor Ψ_* as above, which promises to simplify many applications such as those of Greenlees et al. This will be explained in a separate paper by Schwede and the present author. It is then natural to ask for a calculation of $\Psi_*(T)$ for various popular spectra T, including suspension spectra. One of the most intriguing aspects of the story is that the complex $\Phi_*(X) = \Psi_*(\Sigma^\infty X_+)$ has a very natural description in terms of simplicial de Rham theory, although nothing of that kind is visible in the definition. In particular, we obtain a chain complex similar in spirit to $\Omega^*(X)$ that computes $H_*(X;\mathbb{Q})$ rather than $H^*(X;\mathbb{Q})$; this cannot reasonably be done by naive dualisation, as $\Omega^*(X)$ is infinite-dimensional (even when X is finite) and has no natural topology. This forms the main subject of the present paper.

It will be convenient for us to work in a slightly different order from that suggested by the above discussion. We will give a definition of $\Phi_*(X)$ that does use de Rham theory, and investigate the properties of Φ using that definition. Eventually, in Theorem 2.10 we will obtain a description of $\Phi_*(X)$ as a colimit of groups that do not involve differential forms. When we have defined Ψ (in a separate paper) it will be clear from that description that $\Psi_*(\Sigma^\infty X_+) = \Phi_*(X)$.

Appendix A contains some recollections and notational conventions about the simplicial category (especially the theory of shuffles) which will be in place throughout the paper. Appendix B contains formulae for integrals of polynomials over simplices. These are surely standard, but we do not know a convenient source.

2 de Rham chains

Let $\mathbb{K}$ be a field of characteristic zero. Some of our constructions will seem most natural for $\mathbb{K} = \mathbb{Q}$ and others for $\mathbb{K} = \mathbb{R}$, but in fact everything works for any $\mathbb{K}$.

Given a finite set I, we put

$$\widetilde{P}_I = \mathbb{K}[t_i \mid i \in I]$$
$$P_I = \widetilde{P}_I \Big/ \Big(1 - \sum_i t_i\Big),$$

so P_I is the ring of polynomial functions on an algebraic simplex $\Delta_I^{\mathrm{alg}} = \mathrm{spec}(P_I)$ of dimension $|I|-1$. We also put

$$W_I = \mathbb{K}\{dt_i \mid i \in I\} \Big/ \Big(\sum_i dt_i\Big)$$
$$\Omega_I^1 = P_I \otimes_{\mathbb{K}} W_I = P_I\{dt_i \mid i \in I\} \Big/ \big(\sum_i dt_i\Big)$$
$$\Omega_I^* = P_I \otimes_{\mathbb{K}} \Lambda^*(W_I) = \Lambda_{P_I}^*(\Omega_I^1).$$

Here Ω_I^* is graded with $|t_i| = 0$ and $|dt_i| = 1$, and we give Ω_I^* the standard de Rham differential, making it a differential graded algebra. All of these constructions are contravariantly functorial in I: a map $\alpha\colon I \to J$ of finite sets gives a ring map $\alpha^*\colon P_J \to P_I$ with $\alpha^*(t_j) = \sum_{\alpha(i)=j} t_i$, and this extends naturally to a map $\alpha^*\colon \Omega_J^* \to \Omega_I^*$. If α is just the inclusion of a subset, we write res_I^J for α^*.

In particular, the assignment $n \mapsto \Omega_{[n]}^*$ is a simplicial object in the category of DGA's, so for any simplicial set X we can define

$$\Omega^k(X) = \mathrm{sSet}(X, \Omega_\bullet^k)$$

and this gives us a differential graded algebra $\Omega^*(X)$. It is well-known that $H^*\Omega^*(X)$ is the usual cohomology $H^*(X;\mathbb{K})$.

We would like a version of this construction that is well-related to homology rather than cohomology. The most obvious approach is to dualise and put

$$\widehat{\Phi}_{I,k} = \mathrm{Hom}_{\mathbb{K}}(\Omega_I^k, \mathbb{K}),$$

giving a chain complex that is covariantly functorial in I. However, this is inconvenient because $\widehat{\Phi}_I^k$ is most naturally a product (rather than direct sum) of countably many copies of $\mathbb{Q}$, which introduces numerous technical complications. We will therefore use a smaller subcomplex $\Phi_{I,*} \leq \widehat{\Phi}_{I,*}$.

Definition 2.1 We define

$$\begin{aligned} W_I^\vee &= \mathrm{Hom}_{\mathbb{K}}(W_I, \mathbb{K}) \\ \Theta_{I,m} &= P_I \otimes \Lambda^m(W_I^\vee) = \Lambda^m_{P_I}(P_I \otimes W_I^\vee) \\ \Phi_{I,m} &= \bigoplus_{\emptyset \neq J \subseteq I} \Theta_{J,m}. \end{aligned}$$

We write i_J for the inclusion $\Theta_{J,m} \to \Phi_{I,m}$. We will occasionally use a bigrading on $\Phi_{I,*}$: we put

$$\Phi_{I,(p,q)} = \bigoplus_{|J|=p} \Theta_{J,p+q}$$

so that $\Phi_{I,m} = \bigoplus_{p+q=m} \Phi_{I,(p,q)}$.

We want to interpret $\Phi_{I,*}$ as a subcomplex of $\widehat{\Phi}_{I,*}$, and for this we need to define various bilinear pairings. First, we define a pairing of $\Lambda^m(W_I^\vee)$ with $\Lambda^m(W_I)$ by the formula

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_m, \omega_1 \wedge \cdots \wedge \omega_m \rangle_I = (-1)^{m(m-1)/2} \det(\langle \alpha_i, \omega_j \rangle)_{i,j=1}^m.$$

This is a perfect pairing, and we will silently use it to identify $\Lambda^m(W_I^\vee)$ with $\Lambda^m(W_I)^\vee$. Next, we can extend this linearly over P_I to get a pairing

$$\langle \cdot, \cdot \rangle_I \colon \Theta_{I,m} \otimes \Omega_I^m \to P_I$$

given by essentially the same formula. Occasionally we will use the convention $\langle \alpha, \omega \rangle = 0$ if $\alpha \in \Theta_{I,m}$ and $\omega \in \Omega_I^p$ with $p \neq m$.

Remark 2.2 The factor $(-1)^{m(m-1)/2}$ is inserted to ensure that the term $\prod_i \langle \alpha_i, \omega_i \rangle$ in the determinant comes with the standard sign for converting the term

$$\alpha_1 \otimes \cdots \otimes \alpha_m \otimes \omega_1 \otimes \cdots \otimes \omega_m$$

to the term

$$\alpha_1 \otimes \omega_1 \otimes \alpha_2 \otimes \omega_2 \otimes \cdots \otimes \alpha_m \otimes \omega_m.$$

In other words, if we defined the pairing by a diagram in the usual notation of symmetric monoidal categories, then the sign would come from the twist maps and so would not need to be inserted explicitly.

We really want a pairing with values in $\mathbb{K}$ rather than P_I, and for this we need to integrate.

Definition 2.3 Given a monomial $t^\nu = \prod_{i\in I} t_i^{\nu_i}$, we put $n = |I| - 1$ and define

$$\int_I t^\nu = \Big(\prod_i \nu_i!\Big)/(n + \sum_i \nu_i)! \in \mathbb{K}.$$

This extends to a linear map $\int_I\colon \widetilde{P}_I \to \mathbb{K}$, and one can check (see Lemma B.1) that it factors through the quotient $P_I = \widetilde{P}_I/(1 - \sum_i t_i)$. It is often convenient to use the notation $\nu! = \prod_k (\nu_k!)$ and $t^{[\nu]} = t^\nu/\nu!$ and $|\nu| = \sum_i \nu_i$, so that $\int_I t^{[\nu]} = 1/(n+|\nu|)!$.

Remark 2.4 One can also check (see Lemma B.2) that in the case $\mathbb{K} = \mathbb{R}$, the map $\int_I\colon P_I \to \mathbb{R}$ is just integration over the simplex Δ_I with respect to a natural measure.

Remark 2.5 There is a theory of integration for functions on a space with a measure, and also a theory of integration for differential forms on a manifold with orientation. In discussing de Rham cohomology it is more usual to use integration of forms, but in our application it is painful to keep track of the orientations, so we have chosen to reformulate everything in terms of integration of functions.

Definition 2.6 We define a pairing

$$(\cdot\,,\,\cdot)\colon \Phi_{I,m} \otimes \Omega_I^m \to \mathbb{K}$$

by

$$(i_J(\alpha), \omega) = \int_J \langle \alpha, \mathrm{res}_J^I(\omega)\rangle.$$

In particular, for $\alpha \in \Theta_{I,m} \leq \Phi_{I,m}$ we just have $(\alpha, \omega) = \int_I \langle \alpha, \omega\rangle$. We let $\xi\colon \Phi_{I,m} \to \widehat{\Phi}_{I,m}$ be adjoint to $(\cdot\,,\,\cdot)$.

Our main results about Φ are summarised below; proofs will be given in the subsequent sections of the paper.

Theorem 2.7 (a) *The map ξ_I is injective, and the image (which we will identify with $\Phi_{I,*}$) is a subcomplex of $\widehat{\Phi}_{I,*}$.*

(b) *$\Phi_{I,*}$ is a covariant functor of I, and the maps $\alpha_*\colon \Phi_{I,*} \to \Phi_{J,*}$ are quasiisomorphisms.*

(c) *For the singleton $1 = \{0\}$ we have $\Phi_{1,*} = \mathbb{Q}$ (concentrated in degree zero).* □

Definition 2.8 If X is a simplicial set, we let $\Phi_*(X)$ be the coend of the functor $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathrm{Ch}_{\mathbb{K}}$ given by $(n, m) \mapsto \mathbb{Z}[X_n] \otimes \Phi_{[m],*}$.

Theorem 2.9 *Φ is a lax symmetric monoidal functor from spaces to chain complexes, with a natural isomorphism $H_*\Phi_*(X) = H_*(X;\mathbb{K})$. There is a natural $\mathbb{K}$–linear isomorphism*

$$\Phi_d(X) = \bigoplus_k N_k(X) \otimes \Theta_{[k],d},$$

where $N_*(X)$ is the group of normalised chains on X. □

Theorem 2.10 *There is a natural isomorphism*

$$\Phi_*(X) = \varinjlim_A \operatorname{Hom}(\widetilde{H}_*(S^A), \widetilde{N}_*(S^A \wedge X_+)),$$

where A runs over the category of finite sets and injective maps. □

3 The differential

We next introduce a differential $\delta\colon \Phi_{I,m+1} \to \Phi_{I,m}$. This involves interior multiplication, which we now recall.

Definition 3.1 Let U be a finitely generated free module over a ring R, with dual $U^\vee = \operatorname{Hom}_R(U,R)$. Given $u \in U$ and $a \in \Lambda^{k+1}(U^\vee)$, we let $u \vdash a \in \Lambda^k(U^\vee)$ denote the unique element such that

$$\langle u \vdash a, v\rangle = (-1)^{k+1}\langle a, u \wedge v\rangle \qquad \text{for all } v \in \Lambda^k(U)$$

(using the standard pairings described in Section 2).

Lemma 3.2 (a) *If $a \in U^\vee = \Lambda^1(U^\vee)$ we have $u \vdash a = -\langle u, a\rangle$.*

(b) *If $a \in \Lambda^p(U^\vee)$ and $b \in \Lambda^q(U^\vee)$ then $u \vdash (a \wedge b) = (u \vdash a) \wedge b + (-1)^p a \wedge (u \vdash b)$.*

(c) *If $u, v \in U$ and $a \in \Lambda^k(U^\vee)$ then $u \vdash (v \vdash a) + v \vdash (u \vdash a) = 0$.*

(d) *If $a \in \Lambda^{k+1}(U)$ then $u \vdash a \in \Lambda^k((U/u)^\vee) \leq \Lambda^k(U^\vee)$. Moreover, there is a well-defined multiplication $u \wedge (\cdot)\colon \Lambda^k(U/u) \to \Lambda^{k+1}(U)$ and in this context we again have $\langle u \vdash a, v\rangle = (-1)^{k+1}\langle a, u \wedge v\rangle$.*

Proof This is fairly standard multilinear algebra and is left to the reader. □

Definition 3.3 Suppose we have $\varnothing \neq J \subseteq I$ and $f \in P_J$ and $\alpha_0 \in \Lambda^d(W_J^\vee)$, so $i_J(f\,\alpha_0) \in \Phi_{I,d}$. Note that we have an interior product $\Omega^1_J \otimes_{P_J} \Theta_{J,d} \to \Theta_{J,d-1}$, so we can interpret $df \vdash \alpha_0$ as an element of $\Theta_{J,d-1}$. Also, if $j \in J$ we can interpret $dt_j \vdash \alpha_0$ as an element of $\Lambda^{d-1}((W_J/dt_j)^\vee) = \Lambda^{d-1}(W^\vee_{J\setminus\{j\}})$. We can thus put

$$\begin{aligned}
\delta'(i_J(f\,\alpha_0)) &= -i_J(df \vdash \alpha_0) \\
&= -\sum_{j\in J} i_J((\partial f/\partial t_j)\,dt_j \vdash \alpha_0) \\
\delta''(i_J(f\,\alpha_0)) &= -\sum_{j\in J} i_{J\setminus\{j\}}(\mathrm{res}^J_{J\setminus\{j\}}(f)\,dt_j \vdash \alpha_0) \\
\delta(\alpha) &= \delta'(\alpha) + \delta''(\alpha).
\end{aligned}$$

(Here the second description of $\delta'(i_J(f\alpha_0))$ relies on the choice of a lift of $f \in P_J$ to $\widetilde{P}_J$, but the first description shows that the result is independent of the lift.) This gives maps

$$\begin{tikzcd}
\Phi_{I,(p,q)} \arrow[r, "\delta'"] \arrow[d, "\delta''"'] & \Phi_{I,(p,q-1)} \arrow[d, "\delta''"] \\
\Phi_{I,(p-1,q)} \arrow[r, "\delta'"'] & \Phi_{I,(p-1,q-1)}
\end{tikzcd}$$

and thus $\delta\colon \Phi_{I,m} \to \Phi_{I,m-1}$. We will show that the square above anticommutes.

Proposition 3.4 *We have $\delta'\delta'' + \delta''\delta' = 0$ and $(\delta')^2 = 0$ and $(\delta'')^2 = 0$ and $\delta^2 = 0$, so that $\Phi_{I,(*,*)}$ is a double complex.*

Proof The first three equations follow directly from the definitions, using the second description of δ', the commutation of partial derivatives and the rule $u \vdash (v \vdash a) + v \vdash (u \vdash a) = 0$. We can then expand out $(\delta' + \delta'')^2$ to see that $\delta^2 = 0$. $\square$

Proposition 3.5 *The map $\xi\colon \Phi_{I,*} \to \widehat{\Phi}_{I,*}$ is a chain map. Equivalently, for $\alpha \in \Phi_{I,d+1}$ and $\omega \in \Omega^d_I$ we have*

$$(\delta(\alpha), \omega) = (-1)^{d+1}(\alpha, d\omega).$$

In order to prove this, we need a definition and a lemma.

Definition 3.6 For any vector $x \in \mathbb{K}^I$ we write ∇_x for the operator $\sum_i x_i(\partial/\partial t_i)$ on $\widetilde{P}_I$. We note that this induces an operation on $P_I = \widetilde{P}_I/(1 - \sum_i t_i)$ if and only if $\sum_i x_i = 0$.

Lemma 3.7 *For $f \in P_I$ and $\sum_i x_i = 0$ we have*

$$\int_I \nabla_x f + \sum_i x_i \int_{I\setminus\{i\}} \mathrm{res}^I_{I\setminus\{i\}} f = 0.$$

(This is a version of Stokes' Theorem, but it is easier to prove it directly than to do the translation necessary to quote it from elsewhere.)

Proof It will suffice to prove this for a monomial $f = t^{[\nu]}$. Put $\epsilon = 1/(|\nu| + n - 1)$ and $J = \{i \in I \mid \nu_i > 0\}$, and suppose that $i \in J$. Let $\delta_i \colon I \to \{0,1\}$ be the Kronecker delta, so $\partial f/\partial t_i = t^{[\nu-\delta_i]}$ and $|\nu - \delta_i| = |\nu| - 1$. We then have $\int_I x_i \partial f/\partial t_i = x_i\epsilon$, but $\mathrm{res}^I_{I\setminus\{i\}} f = 0$. Suppose instead that $i \not\in J$. Then $\partial f/\partial t_i = 0$ but $\int_{I\setminus\{i\}} \mathrm{res}^I_{I\setminus\{i\}} f = \int_{I\setminus\{i\}} t^{[\nu]} = \epsilon$. Thus the first term in the claimed equation is $\sum_{i\in J} x_i\epsilon$, and the second term is $\sum_{i\not\in J} x_i\epsilon$, so altogether we have $\epsilon . \sum_I x_i = 0$. ◻

Lemma 3.8 *Proposition 3.5 holds when $\alpha \in \Theta_{I,d+1} \leq \Phi_{I,d+1}$.*

Proof We reduce by linearity to the case where $\alpha = f\,\alpha_0$ and $\omega = g\,\omega_0$ for some $f, g \in \widetilde{P}_I$ and $\alpha_0 \in \Lambda^{d+1}(W_I^\vee)$ and $\omega_0 \in \Lambda^d(W_I)$. Put

$$x_i = \langle dt_i \vdash \alpha_0, \omega_0\rangle = (-1)^{d+1}\langle \alpha_0, dt_i \wedge \omega_0\rangle \in \mathbb{K},$$

and observe that $\sum_i x_i = 0$ (because $\sum_i dt_i = 0$). We can thus apply Lemma 3.7 to the function fg giving

$$\int_I f\,.\,nabla_x(g) + \int_I \nabla_x(f)\,.\,g + \sum_i x_i \int_{I\setminus\{i\}} \mathrm{res}^I_{I\setminus\{i\}}(fg) = 0.$$

From the definitions we find that

$$\begin{aligned} f\,.\,\nabla_x(g) &= (-1)^{d+1} \sum_i f \frac{\partial g}{\partial t_i} \langle \alpha_0, dt_i \wedge \omega_0\rangle \\ &= (-1)^{d+1}\langle f\alpha_0, dg \wedge \omega_0\rangle = (-1)^{d+1}\langle \alpha, d\omega\rangle. \end{aligned}$$

By a similar argument, we have $\nabla_x(f)g = \langle df \vdash \alpha_0, \omega\rangle$. Next, recall that we can interpret $dt_i \vdash \alpha_0$ as an element of $\Lambda^d(W^\vee_{I\setminus\{i\}})$, and then we have

$$x_i = \langle dt_i \vdash \alpha_0, \mathrm{res}^I_{I\setminus\{i\}}(\omega_0)\rangle.$$

It follows that

$$x_i\,\mathrm{res}^I_{I\setminus\{i\}}(fg) = \langle \mathrm{res}^I_{I\setminus\{i\}}(f) dt_i \vdash \alpha, \mathrm{res}^I_{I\setminus\{i\}}(\omega)\rangle,$$

and thus that

$$\int_{I\setminus\{i\}} x_i \operatorname{res}^I_{I\setminus\{i\}}(fg) = (i_{I\setminus\{i\}}(\operatorname{res}^I_{I\setminus\{i\}}(f)dt_i \vdash \alpha_0), \omega).$$

The lemma now follows by combining these facts with the definition of $\delta(\alpha)$. □

Proof of Proposition 3.5 The element $\alpha \in \Phi_{I,m+1}$ can be written as $\sum_{\varnothing \neq J \subseteq I} i_J(\alpha_J)$, with $\alpha_J \in \Theta_J$. By applying Lemma 3.8 to the pairs $(\alpha_J, \operatorname{res}^I_J(\omega))$ we recover the statement of Proposition 3.5. □

Lemma 3.9 *The map*

$$\xi_I \colon \Phi_{I,k} \to \widehat{\Phi}_{I,k}.$$

is injective.

Proof If we can prove this for $\mathbb{K} = \mathbb{R}$ then it will follow for $\mathbb{K} = \mathbb{Q}$ by restriction, and then for arbitrary $\mathbb{K}$ by tensoring up again. We therefore take $\mathbb{K} = \mathbb{R}$ for the rest of the proof.

Consider a nonzero element $\alpha = \sum_J i_J(\alpha_J)$ of the domain. Choose a set J of largest possible size with $\alpha_J \neq 0$ in $\Theta_{J,k}$. As α_J is nonzero, and $\Theta_{J,k}$ is dual over P_J to Ω^k_J, and the restriction map $\Omega^k_I \to \Omega^k_J$ is surjective, we can choose $\omega \in \Omega^k_I$ such that the element $f_0 = \langle \alpha_J, \operatorname{res}^I_J(\omega)\rangle \in P_J$ is nonzero. We can then choose $f \in P_I$ with $\operatorname{res}^I_J(f) = f_0$. We also put $g = \prod_{j\in J} t_j \in P_I$ and $\theta = fg\omega \in \Omega^k_I$. We claim that $\xi_I(\alpha)(\theta) = (\alpha, \theta) \neq 0$. Indeed, we have

$$(i_J(\alpha_J), \theta) = \int_J \langle \alpha_J, \operatorname{res}^I_J(fg\omega)\rangle = \int_J f_0^2 \operatorname{res}^I_J(g).$$

Now $g > 0$ on the interior of the simplex Δ_J, and f_0^2 is nonnegative everywhere and strictly positive on a nonempty open set, so the integral is strictly positive. However, we also need to consider the other terms $(i_K(\alpha_K), \theta)$ for $K \neq J$. If K is a strict superset of J then $\alpha_K = 0$ by our choice of J. If $K \not\supseteq J$ then we can choose $j \in J \setminus K$ and then $\operatorname{res}^I_K(t_j) = 0$ so $\operatorname{res}^I_K(g) = 0$. Either way we find that $(i_K(\alpha_K), \omega) = 0$. It follows that $(\alpha, \omega) = (i_J(\alpha_J), \omega) > 0$, as required. □

Definition 3.10 Let $\widetilde{W}_I$ be the vector space freely generated by $\{dt_i \mid i \in I\}$, so $W_I = \widetilde{W}_I / \sum_i dt_i$. Let $\{e_i \mid i \in I\}$ be the obvious basis for $\widetilde{W}_I^\vee$, so that $W_I^\vee$ is spanned by the elements $e_i - e_j$. Next, in the case $I = [n] = \{0, 1, \dots, n\}$ put

$$\begin{aligned}
\widetilde{\theta}_{[n]} &= e_0 \wedge e_1 \wedge \cdots \wedge e_n \in \Lambda^{n+1}(\widetilde{W}^\vee_{[n]}) \\
\theta_{[n]} &= (e_1 - e_0) \wedge (e_2 - e_0) \wedge \cdots \wedge (e_n - e_0) \\
&= (e_1 - e_0) \wedge (e_2 - e_1) \wedge \cdots \wedge (e_n - e_{n-1}) \in \Lambda^n(W^\vee_{[n]}) \leq \Theta_{[n],n}.
\end{aligned}$$

It is an exercise to check that the two expressions for $\theta_{[n]}$ are the same, and that $e_i \wedge \theta_{[n]} = \widetilde{\theta}_{[n]}$ for all i, and that $\theta_{[n]}$ is the unique element of $\Lambda^n(W_{[n]}^\vee)$ with this property.

If I is any finite ordered set with $|I| = n+1$ then there is a unique ordered bijection $[n] \to I$, and we use this to define $\widetilde{\theta}_I \in \Lambda^{n+1}(\widetilde{W}_I^\vee)$ and $\theta_I \in \Lambda^n(W_I^\vee)$. It is easy to see that $\Lambda^{n+1}(\widetilde{W}_I) = \mathbb{K}\,.\,\widetilde{\theta}_I$ and $\Lambda^n(W_I) = \mathbb{K}\,.\,\theta_I$.

Lemma 3.11 *We have $\delta'(\theta_{[n]}) = 0$ and*

$$\delta''(\theta_{[n]}) = \delta(\theta_{[n]}) = -\sum_{j \in [n]} (-1)^j i_{[n]\setminus\{j\}}(\theta_{[n]\setminus\{j\}}).$$

Proof By inspection of the definitions, this reduces to the claim that

$$dt_j \vdash \theta_{[n]} = (-1)^j \theta_{[n]\setminus\{j\}}.$$

For $j = 0$ it is most convenient to use the expression

$$\theta_{[n]} = (e_1 - e_0) \wedge (e_2 - e_1) \wedge \cdots \wedge (e_n - e_{n-1})$$

and the derivation property

$$dt_0 \vdash (a \wedge b) = (dt_0 \vdash a) \wedge b + (-1)^{|a|} a \wedge (dt_0 \vdash b).$$

We have $dt_0 \vdash (e_1 - e_0) = -\langle dt_0, e_1 - e_0 \rangle = 1$ and $dt_0 \vdash (e_{k+1} - e_k) = 0$ for $k > 0$. It follows that

$$dt_0 \vdash \theta_{[n]} = (e_2 - e_1) \wedge (e_3 - e_2) \wedge \cdots \wedge (e_n - e_{n-1}) = \theta_{[n]\setminus\{0\}}$$

as claimed.

For $j > 0$ we instead use the expression

$$\theta_{[n]} = (e_1 - e_0) \wedge (e_2 - e_0) \wedge \cdots \wedge (e_n - e_0).$$

We have $dt_j \vdash (e_k - e_0) = 0$ for $k \neq j$, so only the term $dt_j \vdash (e_j - e_0)$ contributes, and this has a factor $(-1)^{j-1}$ because of its position in the list. We also have $dt_j \vdash (e_j - e_0) = -\langle dt_j, e_j - e_0 \rangle = -1$ which gives one more sign change, so $dt_j \vdash \theta_{[n]} = (-1)^j \theta_{[n]\setminus\{j\}}$ as claimed. □

Lemma 3.12 *For any totally ordered set J we have $H_*(\Theta_{J,*}; \delta') = \mathbb{K}\,.\,\theta_J$.*

(The ordering is only used here to fix the sign of the generator.)

Proof We may assume that $J=[m]$ for some m, so $P_J=\mathbb{K}[t_1,\dots,t_m]$ and $W_J=\mathbb{K}\{dt_1,\dots,dt_m\}$. Let $\{w_1,\dots,w_m\}$ be the dual basis for $W_J^\vee$ and put $C(i)_*=\mathbb{K}[t_i]\{1,w_i\}$, so that $\Theta_{J,*}=\bigotimes_i C(i)_*$. It is not hard to see that this decomposition is compatible with the differentials, and that in $C(i)_*$ we have $\delta'(f(t_i)w_i)=f'(t_i)$ and $\delta'(g(t_i))=0$. It follows that $H_*(C(i)_*;\delta')=\mathbb{K}\,.\,w_i$, and thus, by the Künneth theorem, that $H_*(\Theta_{J,*};\delta')=K\,.\,\bigwedge_i w_i=\mathbb{K}\,.\,\theta_J$. □

We can now calculate the homology of $\Phi_{I,*}$. Note that for $j\in I$ we have $\Theta_{\{j\},*}=\mathbb{K}$ (concentrated in degree zero), so we have an element $i_{\{j\}}(1)\in\Phi_{I,0}$, which is a cycle for degree reasons.

Proposition 3.13 *The elements $i_{\{j\}}(1)$ are all homologous to each other, and the corresponding homology class generates $H_0(\Phi_{I,*};\delta)$ freely over $\mathbb{K}$. Moreover, we have $H_d(\Phi_{I,*};\delta)=0$ for all $d\neq 0$.*

Proof We may assume that I is totally ordered, which gives an ordering on each subset $J\subseteq I$ and thus defines elements θ_J as before.

We now regard Φ_I as a double complex under δ' and δ'', and use the resulting spectral sequence. We write C_* for the E_1 page, which is just

$$C_*=H_*(\Phi_{I,*};\delta')=\mathbb{K}\{\theta_J\mid\varnothing\neq J\subseteq I\}.$$

The differential is given by Lemma 3.11. Note also that

$$\Lambda^*(\widetilde{W}_I^\vee)=\Lambda^*(e_i\mid i\in I)=\mathbb{K}\{\widetilde{\theta}_J\mid J\subseteq I\}$$

(and here we do have a term for $J=\varnothing$). We can make this a differential graded ring with $d(e_i)=1$ for all i, and the resulting homology is zero. We can then define $\phi\colon\Lambda^*(\widetilde{W}_I^\vee)\to\Sigma C_*$ by $\phi(\widetilde{\theta}_J)=\Sigma\theta_J$ when $J\neq\varnothing$, and $\phi(1)=\phi(\widetilde{\theta}_\varnothing)=0$. It follows from Lemma 3.11 that ϕ is a chain map. The short exact sequence

$$\mathbb{K}\to\Lambda^*(\widetilde{W}_I^\vee)\xrightarrow{\phi}\Sigma C_*$$

gives a long exact sequence in homology. This in turn shows that $H_i(C_*)=0$ for $i\neq 0$, and gives an isomorphism $H_0C_*=H_1(\Sigma C_*)=\mathbb{K}$. Our spectral sequence must therefore collapse at the E_2 page, so $H_i(\Phi_{I,*})=0$ for all $i\neq 0$, and the construction gives an isomorphism $H_0(\Phi_{I,*})\to\mathbb{K}$. We leave it to the reader to check that this sends $i_{\{j\}}(1)$ to 1 for all j. □

4 Functorality of Φ_I

Definition 4.1 Let $\sigma\colon I \to J$ be a surjective map. As in Section 2 this gives maps $\sigma^*\colon P_J \to P_I$ and $\sigma^*\colon W_J \to W_I$ and $\sigma^*\colon \Omega^*_J \to \Omega^*_I$. Next, for any map $\nu\colon I \to \mathbb{Z}$ we define $\sigma_*\nu\colon J \to \mathbb{Z}$ by $(\sigma_*\nu)(j) = \sum_{\sigma(i)=j} \nu(i)$. We then define a map $\sigma_*\colon \widetilde{P}_I \to \widetilde{P}_J$ (of abelian groups, not of rings) by $\sigma_*(t^{[\nu]}) = t^{[\sigma_*(\nu+1)-1]}$. We also let $\sigma_*\colon \Lambda^*(W_I^\vee) \to \Lambda^*(W_J^\vee)$ be dual to the map $\sigma^*\colon \Lambda^*(W_J) \to \Lambda^*(W_I)$, and we again write σ_* for the map

$$\sigma_* \otimes \sigma_*\colon \widetilde{P}_I \otimes \Lambda^*(W_I)^\vee \to \widetilde{P}_J \otimes \Lambda^*(W_J)^\vee.$$

Remark 4.2 It is easy to check that in all the contexts mentioned we have $(\tau\sigma)_* = \tau_*\sigma_*$ for any pair of surjective maps

$$I \xrightarrow{\sigma} J \xrightarrow{\tau} K.$$

Lemma 4.3 *The map $\sigma_*\colon \widetilde{P}_I \to \widetilde{P}_J$ induces a map $\sigma_*\colon P_I \to P_J$ which satisfies $\int_J \sigma_*(f) = \int_I f$.*

Proof Put $r_I = \sum_{i\in I} t_i$, so that $P_I = \widetilde{P}_I/(1-r_I)\widetilde{P}_I$ and $r_I t^{[\nu]} = \sum_i (\nu_i+1)t^{[\nu+e_i]}$. A straightforward calculation shows that $\sigma_*(r_I t^{[\nu]}) = r_J \sigma_*(t^{[\nu]})$, and it follows that σ_* induces a map $P_I \to P_J$.

For the integral formula, put $n = |I|-1$ and $m = |J|-1$, so $\int_I t^{[\nu]} = 1/(n+|\nu|)!$ and $\int_J t^{[\mu]} = (m+|\mu|)!$. It will suffice to show that $n+|\nu| = m+|\mu|$, which is again straightforward. □

Remark 4.4 If we let $\gamma\colon I \to 1$ be the unique map to a singleton, we find that $P_1 = \mathbb{K}$ and $\gamma_*(f) = \int_I f$. This gives another way to see that $\int_J \sigma_*(f) = \int_I f$.

Lemma 4.5 *More generally, for $f \in P_I$ and $g \in P_J$ we have $\int_I f.\sigma^*(g) = \int_J \sigma_*(f).g$.*

Remark 4.6 One can deduce that in the case $\mathbb{K} = \mathbb{R}$, the map σ_* is given by integrating over fibres of the map $\sigma_*\colon \Delta_I \to \Delta_J$ of simplices.

Proof We may assume that $f = t^{[\nu]}$ and $g = t^{[\mu]}$ for some $\nu\colon I \to \mathbb{N}$ and $\mu\colon J \to \mathbb{N}$. Put $n = |I|-1$ and $m = |J|-1$ and $\epsilon = 1/(n+|\nu|+|\mu|)!$. Put $\overline{\nu} = \sigma_*(\nu+1)-1$, so that $\sigma_*(t^{[\nu]}) = t^{[\overline{\nu}]}$ and $|\overline{\nu}| = |\nu|+n-m$ and $|\overline{\nu}|+|\mu|+m = |\nu|+|\mu|+n$. Put $u_j = (\overline{\nu}_j, \mu_j)$ so $\sigma_*(t^{[\nu]})t^{[\mu]} = (\prod_j u_j)t^{[\overline{\nu}+\mu]}$. and so $\int_J \sigma_*(t^{[\nu]})t^{[\mu]} = (\prod_j u_j)\epsilon$.

Next, put $I_j = \sigma^{-1}\{j\}$, and let Λ_j be the set of maps $\lambda\colon I_j \to \mathbb{N}$ with $|\lambda| = \mu_j$. The binomial expansion tells us that

$$\sigma^* t_j^{[\mu_j]} = \left(\sum_{i \in I_j} t_i\right)^{[\mu_j]} = \sum_{\lambda \in \Lambda_j} t^{[\lambda]}.$$

Next, for $\lambda \in \Lambda_j$ put $c_\lambda = \prod_{i \in I_j} (v_i, \lambda_i)$, and then put $v_j = \sum_{\lambda \in \Lambda_j} c_\lambda$. Put

$$\Lambda = \prod_j \Lambda_j \simeq \{\lambda\colon I \to \mathbb{N} \mid \sigma_*(\lambda) = \mu\},$$

and for $\lambda = (\lambda_j)_{j \in J}$ put $c_\lambda = \prod_j c_{\lambda_j}$, and then put $v = \sum_{\lambda \in \Lambda} c_\lambda = \prod_j v_j$. We find that $t^{[\nu]}\sigma^* t^{[\mu]} = \sum_{\lambda \in \Lambda} c_\lambda t^{[\nu+\lambda]}$. For these terms we have $|\lambda| = |\sigma_*(\lambda)| = |\mu|$ and so $\int_I t^{[\nu+\lambda]} = \epsilon$. It follows that

$$\int_I t^{[\nu]}\sigma^*(t^{[\mu]}) = \left(\sum_\lambda c_\lambda\right)\epsilon = \left(\prod_j v_j\right)\epsilon,$$

so it will suffice to show that $u_j = v_j$.

For this, we choose an identification of I_j with the set $[d] = \{0, 1, \dots, d\}$ for some $d \geq 0$. Let N_i be a totally ordered set of size ν_i, and put $N = \coprod_{i \in [d]} N_i$, ordered so that N_i comes before N_{i+1}. Now put $D = \{i + \frac{1}{2} \mid 0 \leq i < d\}$ and call this the set of "dividers"; we order $N \amalg D$ so that $i + \frac{1}{2}$ comes between N_i and N_{i+1}. Let M be a totally ordered set of size μ_j, and let U be the set of total orderings of $N \amalg D \amalg M$ that are compatible with the given orderings of $N \amalg D$ and M. Now $|N \amalg D| = \overline{v}_j$ and $|M| = \mu_j$ so $|U| = (\overline{v}_j, \mu_j) = u_j$. Given an ordering in U we can split M along the dividers to get a decomposition $M = M_0 \amalg \cdots \amalg M_d$. Here the sets M_i are consecutive intervals, so the decomposition is completely determined by the numbers $\lambda_i = |M_i|$, which satisfy $\sum_i \lambda_i = \mu_i$. Given the decomposition $M = \coprod_i M_i$, the order on $N \amalg D \amalg M$ is determined by the relative order of M_i and N_i within $N_i \amalg M_i$, for which the number of choices is $\prod_i (\nu_i, \lambda_i) = c_\lambda$. Using this, one can check that $|U| = v_j$, so $u_j = v_j$ as required. □

Definition 4.7 Let $\sigma\colon I \to I'$ be an arbitrary map of finite sets. Given a subset $J \subseteq I$ and an element $\alpha \in \Theta_{J,*}$ we can interpret σ as a surjection $J \to \sigma(J)$ and thus get an element $i_{\sigma(J)}(\sigma_*(\alpha)) \in \Phi_{I',*}$. We define a map $\sigma_*\colon \Phi_{I,*} \to \Phi_{I',*}$ by $\sigma_*(i_J(\alpha)) = i_{\sigma(J)}(\sigma_*(\alpha))$.

Remark 4.8 Let $\delta_j\colon [n-1]\to[n]$ be the unique increasing map with image $[n]\setminus\{j\}$. We can now rewrite Lemma 3.11 as

$$\delta''(\theta_{[n]})=\delta(\theta_{[n]})=-\sum_{j\in[n]}(-1)^j(\delta_j)_*(\theta_{[n-1]}).$$

Proposition 4.9 *For $\alpha\in\Phi_{I,m}$ and $\omega\in\Omega^m_{I'}$ we have $(\sigma_*(\alpha),\omega)_{I'}=(\alpha,\sigma^*(\omega))_I$.*

Proof We may assume that $\alpha=i_J(f\alpha_0)$ for some $J\subseteq I$ and some $f\in P_J$ and $\alpha_0\in\Lambda^m(W_J^\vee)$. Similarly, we may assume that $\omega=g\omega_0$ for some $g\in P_{I'}$ and $\omega_0\in\Lambda^m(W_{I'})$. Put $J'=\sigma(J)$ and let σ' denote the surjective map $\sigma\colon I'\to J'$. Put $f'=\sigma'_*(f)\in P_{J'}$ and $\alpha'_0=\sigma'_*(\alpha_0)\in\Lambda^m(W_{J'})^\vee$. Let $i\colon J'\to I'$ be the inclusion, so that $i\sigma'=\sigma$. Put $g'=i^*g$ and $\omega'_0=i^*\omega$. From the definitions we then have

$$(\sigma_*(\alpha),\omega)_{I'}=\int_{J'}\langle f'\alpha'_0,g'\omega'_0\rangle=\langle\alpha'_0,\omega'_0\rangle\int_{J'}f'g'.$$

It is elementary that

$$\langle\alpha'_0,\omega'_0\rangle=\langle\sigma'_*(\alpha_0),i^*(\omega_0)\rangle=\langle\alpha_0,(\sigma')^*i^*\omega_0\rangle=\langle\alpha_0,\sigma^*\omega_0\rangle.$$

Similarly, we see from Lemma 4.5 that

$$\int_{J'}f'g'=\int_{J'}\sigma'_*(f)i^*(g)=\int_{I'}f\,.\,(\sigma')^*i^*(g)=\int_{I'}f\sigma^*(g).$$

The claim follows directly from this. □

Corollary 4.10 *The map $\sigma_*\colon\Phi_{I,*}\to\Phi_{I',*}$ is a chain map and a quasiisomorphism.*

Proof We can now identify the above map as a restriction of the map $\sigma_*\colon\widehat{\Phi}_{I,*}\to\widehat{\Phi}_{I',*}$, which is dual to the chain map $\sigma^*\colon\Omega^*_{I'}\to\Omega^*_I$ and so is itself a chain map. It follows from Proposition 3.13 that σ_* is also a quasiisomorphism. □

5 de Rham chains on a simplicial set

We are now in a position to implement Definition 2.8: a simplicial set X gives a functor $\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}\to\mathrm{Ch}$ by $(n,m)\mapsto\mathbb{Z}[X_n]\otimes\Phi_{[m],*}$, and we write $\Phi_*(X)$ for the coend. Thus Φ is a functor from simplicial sets to chain complexes that preserves all colimits, and $\Phi_*(\Delta_n)=\Phi_{[n],*}$, and these properties characterise $\Phi_*(X)$. Any generator of $\Phi_d(X)$ can be written as $x\otimes\alpha$ for some $x\in X_m$ and $\alpha\in\Phi_{[m],d}$, subject to the relations that $x\otimes\alpha$ is a $\mathbb{K}$–linear function of α and $\rho^*(x)\otimes\alpha=x\otimes\rho_*(\alpha)$ for all $\rho\colon[n]\to[m]$ and $\alpha\in\Phi_{[n],d}$. The differential is just $\delta(x\otimes\alpha)=x\otimes\delta(\alpha)$.

Recall that $\Omega^d(X)$ is the set of maps $X_n \to \Omega^d_{[n]}$ that are natural for $[n] \in \mathbf{\Delta}$. There is a natural pairing

$$(\cdot\,,\cdot)_X\colon \Phi_d(X) \otimes \Omega^d(X) \to \mathbb{K}$$

given by $(x \otimes \alpha, \omega)_X = (\alpha, \omega(x))_{[m]}$ (for $x \in X_m$ and $\alpha \in \Phi_{[m],d}$ and $\omega \in \Omega^d_X$).

Definition 5.1 We write $\widehat{\Phi}_*(X) = \mathrm{Hom}_{\mathbb{K}}(\Omega^*(X), \mathbb{K})$, so the above pairing gives a natural chain map $\xi\colon \Phi_*(X) \to \widehat{\Phi}_*(X)$.

Remark 5.2 In the rest of this paper, we will have a number of constructions related to $\Phi_{I,*}$ that depend on having a total order on I. If I is totally ordered and $|I| = n+1$ then there is a unique order-preserving bijection between I and $[n] = \{0, \dots, n\}$. Because of this, we can work with the sets $[n]$ where convenient, and we will transfer the results to all other finite ordered sets without explicit comment.

We next compare $\Phi_*(X)$ with the usual normalised chain complex $N_*(X)$. (We recall the definition: an n–simplex $x \in X_n$ is called *degenerate* if it can be written as $\alpha^* y$ for some $y \in X_m$ and some noninjective map $\alpha \in \mathbf{\Delta}([n],[m])$, and $N_n(X)$ is freely generated over $\mathbb{K}$ by the n–simplices modulo the degenerate ones.)

Proposition 5.3 *There is a natural chain map $\phi\colon N_*(X) \to \Phi_*(X)$ given by $\phi(x) = (-1)^n x \otimes \theta_{[n]} \in \Phi_n(X)$ for all $x \in X_n$. (Here $\theta_{[n]}$ is as in Definition 3.10.)*

Proof The formula $\phi(x) = (-1)^n x \otimes \theta_{[n]}$ certainly defines a natural map $X_n \to \Phi_n(X)$ of sets, which extends linearly to give a map $\phi\colon C_n(X) = \mathbb{K}\{X_n\} \to \Phi_n(X)$ of vector spaces. We make $C_*(X)$ into a chain complex using the alternating sum of face maps in the usual way. We claim that ϕ is then a chain map. Indeed, we have

$$\phi(d_i x) = \phi((\delta_i)^* x) = (-1)^{n-1}(\delta_i)^* x \otimes \theta_{[n-1]} = (-1)^{n-1} x \otimes (\delta_i)_* \theta_{[n-1]}.$$

By taking alternating sums and using Remark 4.8 we obtain

$$\phi(dx) = (-1)^n x \otimes \Big(-\sum_i (-1)^i (\delta_i)_* \theta_{[n-1]} \Big) = (-1)^n x \otimes \delta(\theta_{[n]}) = \delta(\phi(x)).$$

Now suppose that x is degenerate, say $x = \sigma^*(y)$ for some surjective map $\sigma\colon [n] \to [m]$ with $m < n$. Then $\phi(x) = \pm\sigma^*(x) \otimes \theta_{[n]} = \pm x \otimes \sigma_*(\theta_{[n]})$ and $\sigma_*(\theta_{[n]}) \in \Lambda^n(W^\vee_{[m]}) = 0$ so $\phi(x) = 0$. There is thus an induced chain map $\phi\colon N_*(X) \to \Phi_*(X)$ as claimed. □

Proposition 5.4 *There is a natural isomorphism of graded groups*

$$\bigoplus_m N_m(X) \otimes \Theta_{[m],d} \to \Phi_d(X).$$

(The interaction with differentials is complicated and will not be made explicit.)

Proof Let $\mathbb{E}$ be the subcategory of $\mathbf{\Delta}$ which contains all the objects but only the surjective morphisms, and let $i\colon \mathbb{E} \to \mathbf{\Delta}$ be the inclusion. We find that Θ can be regarded as a functor from $\mathbb{E}$ to the category $\mathcal{V}_*$ of graded vector spaces over $\mathbb{K}$, and if we ignore the differential then Φ is just the left Kan extension $\varinjlim_i \Theta$. Now consider a simplicial set X and an object $V_* \in \mathcal{V}_*$. We can define a functor $T\colon \mathbf{\Delta} \to \mathcal{V}_*$ by $T_n = \mathrm{Map}(X_n, V_*)$ and from the universal properties of coends and Kan extensions we see that

$$\mathcal{V}_*(\Phi_*(X), V_*) = [\mathbf{\Delta}, \mathcal{V}_*](\Phi, T) = [\mathbf{\Delta}, \mathcal{V}_*]\left(\varinjlim_i \Theta, T\right) = [\mathbb{E}, \mathcal{V}_*](\Theta, i^*T).$$

Now let $\mathrm{ND}_n(X)$ BE the set of nondegenerate n–simplices in X. There is an evident map $\coprod_m \mathbb{E}(n,m) \times \mathrm{ND}_m(X) \to X_n$ sending (α, x') to $\alpha^* x'$, and it is a standard fact that this is bijective. (The original reference is Eilenberg and Zilber [2, 8.3], and we have given a proof as Lemma A.10 for convenience.) We therefore have $T_n = \prod_m \mathrm{Map}(\mathbb{E}(n,m), T'_m)$, where $T'_m = \mathrm{Map}(X'_m, V_*)$. It follows using the Yoneda Lemma that

$$\begin{aligned}
[\mathbb{E}, \mathcal{V}_*](\Theta, i^*T) &= \prod_m \mathcal{V}_*(\Theta_{[m],*}, T'_m) \\
&= \prod_m \mathcal{V}_*(\mathbb{Z}\{X'_m\} \otimes \Theta_{[m],*}, V_*) \\
&= \mathcal{V}_*\left(\bigoplus_m N_m(X) \otimes \Theta_{[m],*}, V_*\right).
\end{aligned}$$

We now see that $\Phi_d(X)$ and $\bigoplus_m N_m(X) \otimes \Theta_{[m],d}$ represent the same functor, so they are isomorphic in a canonical way. □

Proposition 5.5 *The map $\phi_X\colon N_*(X) \to \Phi_*(X)$ is a quasiisomorphism.*

Remark 5.6 The case where X is a point is easy. One way to prove the general case would be to show that the functor $H_*\Phi_*(X)$ is homotopy invariant, has Mayer–Vietoris sequences, and preserves filtered colimits; then the claim would reduce to the usual uniqueness argument for homology theories. Our proof will be slightly different; we will rearrange the uniqueness proof so as not to rely on homotopy invariance, which instead we deduce as a byproduct.

Proof Put $\mathcal{X} = \{X \mid \phi_X \text{ is a quasiisomorphism }\}$; we must show that this contains all simplicial sets. It is easy to see that $\mathcal{X}$ is closed under coproducts and filtered colimits. Proposition 3.13 tells us that $\Delta_n \in \mathcal{X}$ for all n. Now let Z be an n–dimensional simplicial set, and suppose inductively that all $(n-1)$–dimensional simplicial sets lie in $\mathcal{X}$. Let Y be the $(n-1)$–skeleton of Z, so we have a pushout square of the form

$$\begin{tikzcd}
A \times \partial\Delta_n \arrow[r, tail, "i"] \arrow[d, "f"'] & A \times \Delta_n \arrow[d, "g"] \\
Y \arrow[r, tail, "j"'] & Z
\end{tikzcd}$$

for some set A. This in turn gives a diagram

$$\begin{tikzcd}
N_*(A \times \partial\Delta_n) \arrow[r] \arrow[d, "\phi"'] & N_*(A \times \Delta_n) \oplus N_*(Y) \arrow[r] \arrow[d, "\phi\oplus\phi"'] & N_*(Z) \arrow[d, "\phi"] \\
\Phi_*(A \times \partial\Delta_n) \arrow[r] & \Phi_*(A \times \Delta_n) \oplus N_*(Y) \arrow[r] & \Phi_*(Z).
\end{tikzcd}$$

It is standard that the top row is short exact (giving a Mayer–Vietoris sequence in ordinary homology). Using Proposition 5.4 we see that $\Phi_n(X)$ can be split naturally as a direct sum of functors of the form $N_m(X)$ for various m, and it follows that the bottom row is also short exact. The first two vertical maps are quasiisomorphisms by the induction hypothesis and Proposition 3.13. It follows that ϕ_Z must also be a quasiisomorphism, so $Z \in \mathcal{X}$. By induction on dimension and passage to colimits we see that $\mathcal{X}$ contains all simplicial sets, as required. $\square$

5.1 Monoidal properties

We now define natural maps

$$\begin{aligned}
\mu_{X,Y} &: \Omega^*(X) \otimes \Omega^*(Y) \to \Omega^*(X \times Y), \\
\mu_{X,Y} &: \Phi_*(X) \otimes \Phi_*(Y) \to \Phi_*(X \times Y),
\end{aligned}$$

in several stages.

The cohomological version is straightforward.

Definition 5.7 Given $\omega \in \Omega^d(X)$ and $\upsilon \in \Omega^e(Y)$ we define $\omega \wedge \upsilon$ to be the composite

$$X_n \times Y_n \xrightarrow{\omega\wedge\upsilon} \Omega^d_{[n]} \times \Omega^e_{[n]} \xrightarrow{\text{mult}} \Omega^{d+e}_{[n]}.$$

This is natural for $n \in \mathbf{\Delta}$ and so gives $\omega \wedge \upsilon \in \Omega^{d+e}(X \times Y)$. This construction makes Ω into a symmetric monoidal functor from simplicial sets to cochain complexes.

For the homological version, we need to use the set $\Sigma(n,m)$ of (n,m)–shuffles; see Appendix A for details of our approach to this and various other preliminaries about the simplicial category.

Definition 5.8 In the ring $P_{[n]} = \mathbb{K}[t_0,\dots,t_n]/(1-\sum_i t_i)$ we put $s_i = \sum_{j<i} t_j$, so that $s_0 = 0$ and $s_{n+1} = 1$ and $P_{[n]} = \mathbb{K}[s_1,\dots,s_n]$. This gives a basis $\{ds_1,\dots,ds_n\}$ for $W_{[n]}$. Recall that $\widetilde{W}^\vee_{[n]}$ has basis $e_0,\dots,e_n$, and that $W^\vee_{[n]}$ is the subspace spanned by the differences $e_i - e_j$. We put $w_i = e_{i-1} - e_i$, and observe that $w_1,\dots,w_n$ is a basis for $W^\vee_{[n]}$, with $\langle w_i, s_j\rangle = \delta_{ij}$.

The following observation is immediate from the definitions.

Lemma 5.9 *If $\alpha\colon [n]\to[m]$ is surjective then $\alpha^*(s_i) = s_{\alpha^\dagger(i)}$ and so $\alpha^*(ds_i) = ds_{\alpha^\dagger(i)}$.* □

Lemma 5.10 *If $(\zeta,\xi)\in\Sigma(n,m)$, then the resulting maps*

$$W_{[n]}\oplus W_{[m]} \xrightarrow{(\zeta^*,\xi^*)} W_{[n+m]}$$
$$\Lambda^*(W_{[n]})\otimes\Lambda^*(W_{[m]}) \xrightarrow{\zeta^*\otimes\xi^*} \Lambda^*(W_{[n+m]})\otimes\Lambda^*(W_{[n+m]}) \xrightarrow{\text{mult}} \Lambda^*(W_{[n+m]})$$
$$P_{[n]}\otimes P_{[m]} \xrightarrow{\zeta^*\otimes\xi^*} P_{[n+m]}\otimes P_{[n+m]} \xrightarrow{\text{mult}} P_{[n+m]}$$
$$\Omega^*_{[n]}\otimes\Omega^*_{[m]} \xrightarrow{\zeta^*\otimes\xi^*} \Omega^*_{[n+m]}\otimes\Omega^*_{[n+m]} \xrightarrow{\text{mult}} \Omega^*_{[n+m]}$$

are isomorphisms. (We will write $\mu_{\zeta\xi}$ for any of these maps.)

Proof The maps

$$[n]' \xrightarrow{\zeta^\dagger} [n+m]' \xleftarrow{\xi^\dagger} [m]'$$

give a coproduct decomposition by Lemma A.13. The claim follows by Lemma 5.9. □

Definition 5.11 Given a nondecreasing surjective map $\sigma\colon[n]\to[m]$, we define $\sigma^\bullet\colon W^\vee_{[m]}\to W^\vee_{[n]}$ by $\sigma^\bullet(w_j) = w_{\sigma^\dagger(i)}$. We also write $\sigma^\bullet$ for $\Lambda^k(\sigma^\bullet)\colon \Lambda^*(W^\vee_{[m]})\to\Lambda^*(W^\vee_{[n]})$ or for

$$\sigma^*\otimes\sigma^\bullet\colon \Theta_{[m],*} = P_{[m]}\otimes\Lambda^*(W^\vee_{[m]}) \to P_{[n]}\otimes\Lambda^*(W^\vee_{[n]}) = \Theta_{[n],*}.$$

Remark 5.12 One can check directly from the definitions that $\langle\sigma^\bullet(\alpha),\sigma^*(\omega)\rangle_{[n]} = \langle\alpha,\omega\rangle_{[m]}$ and $\sigma^*(u)\vdash\sigma^\bullet(\alpha) = \sigma^\bullet(u\vdash\alpha)$.

Definition 5.13 Given a shuffle $(\zeta,\xi)\colon [n+m]\to[n]\times[m]$ we define an isomorphism

$$\mu_{\zeta\xi}\colon \Lambda^*(W_{[n]}^\vee)\otimes\Lambda^*(W_{[m]}^\vee)\to\Lambda^*(W_{[n+m]}^\vee)$$

by $\mu_{\zeta\xi}(\alpha\otimes\beta)=\zeta^\bullet(\alpha)\wedge\xi^\bullet(\beta)$. This extends to an isomorphism $\Theta_{[n],*}\otimes\Theta_{[m],*}\to\Theta_{[n+m],*}$ by putting

$$\mu_{\zeta\xi}(f\alpha_0\otimes g\beta_0)=\zeta^*(f)\xi^*(g)\zeta^\bullet(\alpha_0)\wedge\xi^\bullet(\beta_0).$$

Lemma 5.14 *For all $\alpha\in\Theta_{[n],d}$ and $\beta\in\Theta_{[m],e}$ and $\omega\in\Omega_{[n]}^d$ and $\upsilon\in\Omega_{[m]}^e$ we have*

$$\begin{aligned}\langle\mu_{\zeta\xi}(\alpha\otimes\beta),\mu_{\zeta\xi}(\omega\otimes\upsilon)\rangle_{[n+m]}&=\langle\mu_{\zeta\xi}(\alpha\otimes\beta),\zeta^*(\omega)\wedge\xi^*(\upsilon)\rangle_{[n+m]}\\&=(-1)^{|\beta||\omega|}\zeta^*(\langle\alpha,\omega\rangle_{[n]})\xi^*(\langle\beta,\upsilon\rangle_{[m]}).\end{aligned}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\Theta_{[n],*}\otimes\Theta_{[m],*} \arrow[rr, "\tau"] \arrow[dr, "\mu_{\zeta\xi}"'] & & \Theta_{[m],*}\otimes\Theta_{[n],*} \arrow[dl, "\mu_{\xi\zeta}"] \\
& \Theta_{[n+m],*} &
\end{tikzcd}$$

(Here τ is the usual twist map $\tau(a\otimes b)=(-1)^{|a||b|}b\otimes a$.)

Proof Left to the reader. □

Definition 5.15 We let $\mathrm{sgn}(\zeta,\xi)\in\{\pm1\}$ be the number such that

$$\mu_{\zeta\xi}(\theta_{[n]}\otimes\theta_{[m]})=\mathrm{sgn}(\zeta,\xi)\theta_{[n+m]}.$$

We now recall the standard way to make N_* into a symmetric monoidal functor (see for example May [7, Section 29]).

Definition 5.16 We define a map $\mu\colon N_n(X)\otimes N_m(Y)\to N_{n+m}(X\times Y)$ (called the *shuffle product*) by

$$\mu(x\otimes y)=\sum_{(\zeta,\xi)\in\Sigma(n,m)}\mathrm{sgn}(\zeta,\xi)(\zeta^*(x),\xi^*(y)).$$

There are a number of known generalisations of this construction; for example, the same formula gives a well-behaved map $R_n\otimes R_m\to R_{n+m}$ for any simplicial ring $R_\bullet$. As far as we understand it, none of these generalisations can be applied directly to our situation, but nonetheless we can give a definition along the same lines.

Definition 5.17 We define $\mu\colon \Phi_*(X)\otimes\Phi_*(Y)\to\Phi_*(X\times Y)$ by

$$\mu((x\otimes\alpha_1)\otimes(y\otimes\beta_1)) = \sum_{(\zeta,\xi)\in\Sigma(n,m)} (\zeta^* x, \xi^* y)\otimes\mu_{\zeta,\xi}(\alpha_1\otimes\beta_1)$$

for $x\in X_n$, $y\in Y_m$, $\alpha_1\in\Theta_{[n],*}$ and $\beta_1\in\Theta_{[m],*}$. To see that this is well-defined and has good properties, we repeat the definition in a more long-winded form as follows. We note that a shuffle (ζ,ξ) gives a nondegenerate $(n+m)$–simplex $x_{\zeta\xi}\in(\Delta_n\times\Delta_m)_{n+m}$, and thus a basis element in $N_{n+m}(\Delta_n\times\Delta_m)$. We then define

$$\mu\colon \Theta_{[n],*}\otimes\Theta_{[m],*}\to\Phi_*(\Delta_n\otimes\Delta_m)=\bigoplus_d N_d(\Delta_n\times\Delta_m)\otimes\Theta_{[d],*}$$

by $$\mu(\alpha_1\otimes\beta_1)=\sum_{\zeta,\xi} x_{\zeta\xi}\otimes\mu_{\zeta\xi}(\alpha_1\otimes\beta_1).$$

By a slight change of notation, if J and K are any finite, nonempty, totally ordered sets we get natural maps $\mu\colon \Theta_{J,*}\otimes\Theta_{K,*}\to\Phi_*(\Delta_J\times\Delta_K)$. If $J\subseteq[n]$ and $K\subseteq[m]$ then $\Delta_J\times\Delta_K\subseteq\Delta_n\times\Delta_m$, so we get a map $\mu\colon\Theta_{J,*}\otimes\Theta_{K,*}\to\Phi_*(\Delta_n\times\Delta_m)$. Adding these up over all J and K, we get a map $\mu\colon\Phi_{[n],*}\otimes\Phi_{[m],*}\to\Phi_*(\Delta_n\times\Delta_m)$, which is a natural transformation of functors $\mathbf{\Delta}\times\mathbf{\Delta}\to\mathrm{Ch}$. Given simplicial sets X and Y we have functors $(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}}\times\mathbf{\Delta}\times\mathbf{\Delta}\to\mathcal{V}_*$ given by

$$(p,q,n,m)\mapsto\mathbb{Z}\{X_p\times X_q\}\otimes\Phi_{[n],*}\otimes\Phi_{[m],*}$$

and $$(p,q,n,m)\mapsto\mathbb{Z}\{X_p\times X_q\}\otimes\Phi_*(\Delta_n\times\Delta_m).$$

The coend of the first is $\Phi_*(X)\otimes\Phi_*(Y)$, whereas the coend of the second is $\Phi_*(X\times Y)$. The maps μ therefore induce a well-defined map $\Phi_*(X)\otimes\Phi_*(Y)\to\Phi_*(X\times Y)$.

Proposition 5.18 *The maps $\mu\colon\Phi_*(X)\otimes\Phi_*(Y)\to\Phi_*(X\times Y)$ make Φ a symmetric monoidal functor from simplicial sets to graded vector spaces.*

We would also like to know that μ is a chain map, but the proof of that fact is long so we will do it separately in Proposition 5.21.

Proof First, for any (m,n,p)–shuffle (ζ,ξ,θ) we can define

$$\mu_{\zeta\xi\theta}\colon\Theta_{[m],*}\otimes\Theta_{[n],*}\otimes\Theta_{[p],*}\to\Theta_{[m+n+p],*}$$

by the evident analogue of Lemma 5.10. Using this, we define

$$\mu_3\colon\Phi_*(X)\otimes\Phi_*(Y)\otimes\Phi_*(Z)\to\Phi_*(X\times Y\times Z)$$

by $$\mu_3(x\otimes\alpha_1\otimes y\otimes\beta_1\otimes z\otimes\gamma_1)=\sum_{\zeta,\xi,\theta}(\zeta^*(x),\xi^*(y),\theta^*(z))\otimes\mu_{\zeta\xi\theta}(\alpha_1\otimes\beta_1\otimes\gamma_1).$$

Using Lemma A.15 we see that

$$\mu \circ (\mu \otimes 1) = \mu_3 = \mu \circ (1 \otimes \mu)\colon \Phi_*(X) \otimes \Phi_*(Y) \otimes \Phi_*(Z) \to \Phi_*(X \times Y \times Z),$$

so we have made Φ_* into a monoidal functor. It follows from the diagram in Lemma 5.14 that μ is also compatible with the relevant twist maps, so Φ_* is a symmetric monoidal functor. □

Proposition 5.19 *The maps $\mu\colon \Phi_*(X) \otimes \Phi_*(Y) \to \Phi_*(X \times Y)$ and the maps $\wedge\colon \Omega^*(X) \otimes \Omega^*(Y) \to \Omega^*(X \times Y)$ satisfy*

$$(\mu(\alpha \otimes \beta), \omega \wedge \upsilon) = (-1)^{|\beta||\omega|}(\alpha, \omega)(\beta, \upsilon).$$

Proof We may assume that $\alpha = x \otimes \alpha_1$ and $\beta = y \otimes \beta_1$ for some $x \in X_n$, $y \in Y_m$, $\alpha_1 \in \Theta_{[n],d}$ and $\beta_1 \in \Theta_{[m],e}$. For a nonzero result we must then have $\omega \in \Omega^d(X)$ and $\upsilon \in \Omega^e(Y)$, so we can put $\omega_1 = \omega(x) \in \Omega^d_{[n]}$ and $\upsilon_1 = \upsilon(y) \in \Omega^e_{[m]}$. We then put $f = \langle \alpha_1, \omega_1 \rangle \in P_{[n]}$ and $g = \langle \beta_1, \upsilon_1 \rangle \in P_{[m]}$, so that $(\alpha, \omega) = \int_{[n]} f$ and $(\beta, \upsilon) = \int_{[m]} g$.

Using Lemma B.4 we see that

$$(\alpha, \omega)(\beta, \upsilon) = \int_{[n]} f \cdot \int_{[m]} g = \sum_{(\zeta,\xi) \in \Sigma(n,m)} \int_{[n+m]} \zeta^*(f)\xi^*(g).$$

On the other hand, we have

$$\mu(\alpha \otimes \beta) = \sum_{(\zeta,\xi)} (\zeta^*(x), \xi^*(y)) \otimes \mu_{\zeta\xi}(\alpha_1 \otimes \beta_1).$$

Here

$$\begin{aligned}
\langle (\zeta^*(x), \xi^*(y)) \otimes \mu_{\zeta\xi}(\alpha_1 \otimes \beta_1), \omega \wedge \upsilon \rangle &= \langle \mu_{\zeta\xi}(\alpha_1 \otimes \beta_1), \omega(\zeta^*(x)) \wedge \upsilon(\xi^*(y)) \rangle \\
&= \langle \mu_{\zeta\xi}(\alpha_1 \otimes \beta_1), \zeta^*\omega_1 \wedge \xi^*\upsilon_1 \rangle \\
&= (-1)^{|\beta||\omega|} \zeta^*(\langle \alpha_1, \omega_1 \rangle) \xi^*(\langle \beta_1, \upsilon_1 \rangle) \\
&= (-1)^{|\beta||\omega|} \zeta^*(f)\, \xi^*(g).
\end{aligned}$$

□

Proposition 5.20 *The square*

$$\begin{tikzcd}
N_n(X) \otimes N_m(Y) \arrow[r, "\mu"] \arrow[d, "\phi \otimes \phi"'] & N_{n+m}(X \times Y) \arrow[d, "\phi"] \\
\Phi_n(X) \otimes \Phi_m(Y) \arrow[r, "\mu"'] & \Phi_{n+m}(X \times Y)
\end{tikzcd}$$

is commutative.

Proof Suppose we have $x \in X_n$ and $y \in Y_m$. Then

$$\begin{aligned}
\mu(x \otimes y) &= \sum_{\zeta,\xi} \operatorname{sgn}(\zeta,\xi)(\zeta^*(x),\xi^*(y)) \\
\phi\mu(x \otimes y) &= (-1)^{n+m} \sum_{\zeta,\xi} \operatorname{sgn}(\zeta,\xi)(\zeta^*(x),\xi^*(y)) \otimes \theta_{[n+m]} \\
&= (-1)^{n+m} \sum_{\zeta,\xi} (\zeta^*(x),\xi^*(y)) \otimes \mu_{\zeta\xi}(\theta_{[n]} \otimes \theta_{[m]}) \\
&= (-1)^{n+m} \mu((x \otimes \theta_{[n]}) \otimes (y \otimes \theta_{[m]})) \\
&= \mu(\phi \otimes \phi)(x \otimes y).
\end{aligned}$$

□

Proposition 5.21 *The map* $\mu\colon \Phi_*(X) \otimes \Phi_*(Y) \to \Phi_*(X \times Y)$ *is a chain map.*

The proof will follow after a number of preparatory results.

Recall that δ was defined in Definition 3.3 as the sum of two operators δ' and δ''.

Lemma 5.22 *For* $\alpha = f\alpha_0 \in \Theta_{[n],*} \le \Phi_{[n],*} = \Phi_*(\Delta_n)$ *and* $\beta = g\beta_0 \in \Theta_{[m],*} \le \Phi_{[m],*} = \Phi_*(\Delta_m)$ *we have*

$$\delta'(\mu(\alpha \otimes \beta)) = \mu(\delta'(\alpha) \otimes \beta + (-1)^{|\alpha|}\mu(\alpha \otimes \delta'(\beta)) \in \Phi_*(\Delta_n \times \Delta_m).$$

Proof Let (ζ,ξ) be a shuffle. Using Remark 5.12 we see that

$$\begin{aligned}
\delta'\mu_{\zeta\xi}(\alpha \otimes \beta) &= \delta'(\zeta^*(f)\zeta^\bullet(\alpha_0) \wedge \xi^*(g)\xi^\bullet(\beta_0)) \\
&= -d(\zeta^*(f)\xi^*(g)) \vdash (\zeta^\bullet(\alpha_0) \wedge \xi^\bullet(\beta_0)) \\
&= -(\xi^*(g)\zeta^*(df) + \zeta^*(f)\xi^*(dg)) \vdash (\zeta^\bullet(\alpha_0) \wedge \xi^\bullet(\beta_0)) \\
&= -\zeta^\bullet(df \vdash \alpha_0)\xi^\bullet(\beta) - (-1)^{|\alpha|}\zeta^\bullet(\alpha) \wedge \xi^\bullet(dg \vdash \beta_0) \\
&= \mu_{\zeta\xi}(\delta'(\alpha) \otimes \beta + (-1)^{|\alpha|}\mu_{\zeta\xi}(\alpha \otimes \delta'(\beta)).
\end{aligned}$$

Taking the sum over all shuffles (ζ,ξ) gives the claimed result. □

We now start to consider the δ'' terms.

Consider an element $k \in [n+m]$ and an injective map $(\zeta,\xi)\colon [n+m] \setminus \{k\} \to [n] \times [m]$. We say that this pair is *extendable* if there exists a shuffle $(\phi,\psi)\colon [n+m] \to [n] \times [m]$ extending (ζ,ξ). We will need to classify the possible extensions. We first suppose that $0 < k < n+m$. In that case, extendability means precisely that one of the following three things must hold.

(0) For some $(i,j) \in [n]' \times [m]'$ we have $(\zeta,\xi)(k-1) = (i-1, j-1)$ and $(\zeta,\xi)(k+1) = (i,j)$. Here we say that (ζ,ξ) has a *diagonal gap*. There are two possible extensions, given by $(\phi,\psi)(k) = (i-1,j)$ and $(\phi,\psi)(k) = (i,j-1)$.

(1) For some $(i,j) \in \{1,\dots,n-1\} \times [m]$ we have $(\zeta,\xi)(k-1) = (i-1,j)$ and $(\zeta,\xi)(k+1) = (i+1,j)$. Here we say that (ζ,ξ) has a *horizontal gap*. There is only one possible extension, given by $(\phi,\psi)(k) = (i,j)$.

(2) For some $(i,j) \in [n] \times \{1,\dots,m-1\}$ we have $(\zeta,\xi)(k-1) = (i,j-1)$ and $(\zeta,\xi)(k+1) = (i,j+1)$. Here we say that (ζ,ξ) has a *vertical gap*. There is only one possible extension, given by $(\phi,\psi)(k) = (i,j)$.

The situation is similar if $k = 0$, but with some slight adjustments. We must have either $(\zeta,\xi)(1) = (1,0)$ or $(\zeta,\xi)(1) = (0,1)$ (otherwise there is not room for (ζ,ξ) to be injective). In these cases we say that (ζ,ξ) has a horizontal (resp. vertical) gap. Either way, there is a unique extension, with $(\phi,\psi)(0) = (0,0)$. Similarly, if $k = n+m$ then we can have only a horizontal or vertical gap, and there is a unique extension given by $(\phi,\psi)(n+m) = (n,m)$.

(This division into three cases is the same as in the well-known proof that the product in Definition 5.16 is a chain map.)

Given an extendable pair (ζ,ξ) and an extension (ϕ,ψ), the expression $\mu(f\alpha_0 \otimes g\beta_0) \in \Phi_*(\Delta_n \times \Delta_n)$ contains a term $(\phi,\psi) \otimes \mu_{\phi\psi}(f\alpha_0 \otimes g\beta_0)$, so $\delta''\mu(f\alpha_0 \otimes g\beta_0)$ contains a term $-(\zeta,\xi) \otimes \rho_{\phi\psi}$, where

$$\begin{aligned}\rho_{\phi\psi} &= \mathrm{res}^{[n+m]}_{[n+m]\setminus\{k\}}(\phi^*(f)\psi^*(g))(dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0)) \\ &= \zeta^*(f)\xi^*(g)(dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0)).\end{aligned}$$

Lemma 5.23 *Suppose that* $(\zeta,\xi)\colon [n+m]\setminus\{k\} \to [n]\times[m]$ *has a diagonal gap between* $(i-1, j-1)$ *and* (i,j), *and let* (ϕ,ψ) *and* $(\overline{\phi},\overline{\psi})$ *be the two shuffles that extend* (ζ,ξ). *Then for any* $\alpha_0 \in \Lambda^*(W^\vee_{[n]}$ *and* $\beta_0 \in W^\vee_{[m]}$ *we have*

$$dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0) \ + \ dt_k \vdash \mu_{\overline{\phi}\overline{\psi}}(\alpha_0 \otimes \beta_0) = 0.$$

Proof Write α_0 as $\alpha_1 + w_i \wedge \alpha_2$, where α_1 and α_2 involve only the generators w_p with $p \neq i$. In particular, this means that $dt_i \vdash \alpha_0 = -\alpha_2$. Write β_0 as $\beta_1 + w_j \wedge \beta_2$ in the same way.

As there is a diagonal gap, we must have $0 < k < n+m$. We have the following table of values:

	ϕ	$\overline{\phi}$	ψ	$\overline{\psi}$
$k-1$	$i-1$	$i-1$	$j-1$	$j-1$
k	i	$i-1$	$j-1$	j
$k+1$	i	i	j	j

Using this we see that $k = \phi^\dagger(i) = \overline{\psi}^\dagger(j)$ and $k+1 = \overline{\phi}^\dagger(i) = \psi^\dagger(j)$. On the other hand, for all $p \neq i$ we have $\phi^\dagger(p) = \overline{\phi}^\dagger p \not\in \{k, k+1\}$, and for all $q \neq j$ we have $\psi^\dagger(j) = \overline{\psi}^\dagger(j) \not\in \{k, k+1\}$, so $\phi^\bullet(\alpha_1) = \overline{\phi}^\bullet(\alpha_1)$ and $\phi^\bullet(\alpha_2) = \overline{\phi}^\bullet(\alpha_2)$. Similarly, $\psi^\bullet(\beta_1) = \overline{\psi}^\bullet(\beta_1)$ and $\psi^\bullet(\beta_2) = \overline{\psi}^\bullet(\beta_2)$. Put

$$\nu = \mu_{\phi\psi}(\alpha_0 \otimes \beta_0) + \mu_{\overline{\phi}\overline{\psi}}(\alpha_0 \otimes \beta_0).$$

We see that

$$\begin{aligned}
\mu_{\phi\psi}(\alpha_0 \otimes \beta_0) &= (\phi^\bullet(\alpha_1) + w_k \wedge \phi^\bullet(\alpha_2)) \wedge (\psi^\bullet(\beta_1) + w_{k+1} \wedge \psi^\bullet(\beta_2)) \\
\mu_{\overline{\phi}\overline{\psi}}(\alpha_0 \otimes \beta_0) &= (\phi^\bullet(\alpha_1) + w_{k+1} \wedge \phi^\bullet(\alpha_2)) \wedge (\psi^\bullet(\beta_1) + w_k \wedge \psi^\bullet(\beta_2)) \\
\nu &= 2\phi^\bullet(\alpha_1) \wedge \psi^\bullet(\beta_1) \\
&\quad + (-1)^{|\alpha|}(w_k + w_{k+1}) \wedge \phi^\bullet(\alpha_1) \wedge \psi^\bullet(\beta_2) \\
&\quad + (w_k + w_{k+1}) \wedge \phi^\bullet(\alpha_2) \wedge \psi^\bullet(\beta_1) \\
s_k \vdash \nu = s_{k+1} \vdash \nu &= (-1)^{|\alpha|+1}\phi^\bullet(\alpha_1) \wedge \psi^\bullet(\beta_2) - \phi^\bullet(\alpha_2) \wedge \psi^\bullet(\beta_1) \\
t_k \vdash \nu &= (s_{k+1} - s_k) \vdash \nu = 0.
\end{aligned}$$ □

Corollary 5.24 *With $\phi, \psi, \overline{\phi}$ and $\overline{\psi}$ as in Lemma 5.23, we have $\rho_{\phi\psi} + \rho_{\overline{\phi}\overline{\psi}} = 0$.*

Proof This follows from the expression

$$\rho_{\phi\psi} = \zeta^*(f)\xi^*(g)(dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0)).$$ □

We next consider the case of a pair $(\zeta, \xi)\colon [n+m] \setminus \{k\} \to [n] \times [m]$ that has a horizontal gap at i, and thus a unique extension (ϕ, ψ). We originally defined shuffles as maps $[p+q] \to [p] \times [q]$ with certain properties, but we can extend the notion in an evident way to cover maps $I \to J \times K$ where I, J and K are any finite, totally ordered sets with $|I| = |J| + |K| - 1$. In this slightly extended sense, we see that $(\zeta, \xi)\colon [n+m] \setminus \{k\} \to ([n] \setminus \{i\}) \times [m]$ is a shuffle, so it gives a map

$$\mu_{\zeta\xi}\colon \Theta_{[n]\setminus\{i\},*} \otimes \Theta_{[m],*} \to \Theta_{[n+m]\setminus\{k\}}.$$

Lemma 5.25 *Suppose we elements $\alpha = f\alpha_0 \in \Theta_{[n],*}$ and $\beta = g\beta_0 \in \Theta_{[m],*}$. Then, in the situation described above we have*

$$\rho_{\phi\psi} = \mu_{\zeta\xi}(\mathrm{res}^{[n]}_{[n]\setminus\{i\}}(f)\,(dt_i \vdash \alpha_0) \otimes g\beta_0).$$

Proof We will cover the case where $0 < k < n+m$, leaving the adjustments for $k = 0$ and $k = n+m$ to the reader. We then have $\phi(k-1) = \zeta(k-1) = i-1$ and $\phi(k) = i$ and $\phi(k+1) = \zeta(k+1) = i+1$. Also, for some j we have $\psi(k-1) = \psi(k) = \psi(k+1) = \xi(k-1) = \xi(k+1) = j$. Using the expression

$$\rho_{\phi\psi} = \zeta^*(f)\xi^*(g)(dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0))$$

we reduce to the case $f = g = 1$, in which case we must prove that

$$dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0) = \mu_{\zeta\xi}((dt_i \vdash \alpha_0) \otimes \beta_0).$$

We write

$$\alpha_0 = \alpha_1 + w_i \wedge \alpha_2 + w_{i+1} \wedge \alpha_3 + w_i \wedge w_{i+1} \wedge \alpha_4,$$

where $\alpha_1, \dots, \alpha_4$ do not involve w_i or w_{i+1}. Put $\overline{\alpha}_t = \phi^\bullet(\alpha_t)$ and $\overline{\beta}_0 = \psi^\bullet(\beta_0)$. Then

$$\mu_{\phi\psi}(\alpha_0 \otimes \beta_0) = (\overline{\alpha}_0 + w_k \wedge \overline{\alpha}_1 + w_{k+1} \wedge \overline{\alpha}_2 + w_k \wedge w_{k+1} \wedge \overline{\alpha}_3) \wedge \overline{\beta}_0,$$

and none of the terms $\overline{\alpha}_t$ or $\overline{\beta}_0$ involves w_k or w_{k+1}. Using this together with the relation $t_k = s_{k+1} - s_k$ we obtain

$$dt_k \vdash \mu_{\phi\psi}(\alpha_0 \otimes \beta_0) = (\overline{\alpha}_2 - \overline{\alpha}_3 + (w_k + w_{k+1}) \wedge \overline{\alpha}_4) \wedge \overline{\beta}_0.$$

We now consider the map $\mu_{\zeta\xi}$ arising from the shuffle

$$(\zeta, \xi)\colon [n+m] \setminus \{k\} \to ([n] \setminus \{i\}) \times [m].$$

Here the natural basis to use for $W^\vee_{[n]\setminus\{i\}}$ is the list

$$e_1 - e_0, \dots, e_{i-1} - e_{i-2}, e_{i+1} - e_{i-1}, e_{i+2} - e_{i+1}, \dots, e_n - e_{n-1},$$

or in other words

$$w_1, \dots, w_{i-1}, w_i + w_{i+1}, w_{i+2}, \dots, w_n.$$

Similarly, the natural basis for $W^\vee_{[n+m]\setminus\{k\}}$ is

$$w_1, \dots, w_{k-1}, w_k + w_{k+1}, w_{k+2}, \dots, w_{n+m}.$$

We see that $\zeta^\bullet(w_p)=w_{\zeta^\dagger(p)}=w_{\phi^\dagger(p)}$ for $p\neq i+1$ and $\zeta^\bullet(w_i+w_{i+1})=w_k+w_{k+1}$. Also, we have

$$\begin{aligned} dt_i\vdash\alpha_0 &= (ds_{i+1}-ds_i)\vdash(\alpha_1+w_i\wedge\alpha_2+w_{i+1}\wedge\alpha_3+w_i\wedge w_{i+1}\wedge\alpha_4)\\ &= \alpha_2-\alpha_3+(w_i+w_{i+1})\wedge\alpha_4, \end{aligned}$$

so $$\mu_{\zeta\xi}((dt_i\vdash\alpha_0)\otimes\beta_0)=(\overline{\alpha}_2-\overline{\alpha}_3+(w_k+w_{k+1})\wedge\overline{\alpha}_4)\wedge\overline{\beta}_0,$$

as required. □

Lemma 5.26 *If $(\zeta,\xi)\colon [n+m]\setminus\{k\}\to[n]\times[m]$ has a vertical gap at j and (ϕ,ψ) is the unique extension of (ζ,ξ) then*

$$\rho_{\phi\psi}=(-1)^{|\alpha|}\mu_{\zeta\xi}(f\alpha_0,\operatorname{res}^{[m]}_{[m]\setminus\{j\}}(g)(dt_j\vdash\beta_0)).$$

Proof This follows from Lemma 5.25 by applying suitable twist maps. □

Corollary 5.27 *In $\Phi_*(\Delta_n\times\Delta_m)$ we have*

$$\delta(\mu(\alpha\otimes\beta))=\mu(\delta(\alpha)\otimes\beta+(-1)^{|\alpha|}\alpha\otimes\delta(\beta)).$$

Proof Lemma 5.22 tells us that this holds when δ is replaced by δ', so we need only prove the corresponding formula for δ''. We have seen that $\delta''(\mu(\alpha\otimes\beta))$ is a sum of terms $-(\zeta,\xi)\otimes\rho_{\phi\psi}$, one for each extendable pair (ζ,ξ) and each extension (ϕ,ψ). The terms where (ζ,ξ) has a diagonal gap all cancel out in pairs, by Lemma 5.23. Those where (ζ,ξ) has a horizontal gap add up to give $\mu(\delta'(\alpha)\otimes\beta)$, as we see from Lemma 5.25. The remaining terms give $(-1)^{|\alpha|}\mu(\alpha\otimes\delta(\beta))$, by Lemma 5.26. □

Proof of Proposition 5.21 The group $\Phi_*(X)\otimes\Phi_*(Y)$ is generated by terms of the form $(x\otimes\alpha)\otimes(y\otimes\beta)$ with $x\in X_n$ and $y\in Y_m$ and $\alpha\in\Theta_{[n],*}$ and $\beta\in\Theta_{[m],*}$. We then have

$$\begin{aligned} \mu((x\otimes\alpha)\otimes(y\otimes\beta)) &= (x,y)\otimes\mu(\alpha\otimes\beta)\\ \delta(\mu((x\otimes\alpha)\otimes(y\otimes\beta))) &= (x,y)\otimes\delta(\mu(\alpha\otimes\beta))\\ &= (x,y)\otimes\mu(\delta(\alpha)\otimes\beta+(-1)^{|\alpha|}\alpha\otimes\delta(\beta))\\ &= \mu((x\otimes\delta(\alpha))\otimes(y\otimes\beta))\\ &\quad+(-1)^{|\alpha|}\mu((x\otimes\alpha)\otimes(y\otimes\delta(\beta))). \end{aligned}$$ □

6 The colimit description

In this section, we explain and prove Theorem 2.10, which asserts that $\Phi_*(X)$ can be written as a colimit of the groups $\mathrm{Hom}(\widetilde{H}_*(S^A), \widetilde{N}_*(S^A \wedge X_+))$, as A runs over the category of finite sets and injective maps.

Definition 6.1 Given a finite set A, we put $S^A = \bigwedge_{a \in A} S^1$, where $S^1 = \Delta_1/\partial\Delta_1$. More explicitly, we define $BA = \prod_{a\in A} \Delta_1$, so that $(BA)_n = \mathrm{Map}(A, \mathbf{\Delta}([n],[1]))$. We then put

$$(\partial BA)_n = \mathrm{Map}(A, \mathbf{\Delta}([n],[1])) \setminus \mathrm{Map}(A, \mathbb{E}([n],[1])),$$

which defines a subcomplex ∂BA. Finally, we have $S^A = BA/\partial BA$.

It is clear that if $|A| = n$ then S^A is a model of the sphere S^n, so that $\widetilde{H}_*(S^A)$ is a copy of $\mathbb{Z}$, concentrated in degree n. However, there is no natural choice of generator for this group. Instead, the best thing to say is that there is a natural isomorphism $\Lambda^n\mathbb{Z}\{A\} \to \widetilde{H}_n(S^A)$.

Definition 6.2 Given a set A with $|A| = m$ and a simplex $\alpha \in (BA)_d$ we define

$$z(\alpha) \in \widetilde{H}_m(S^A) \otimes \Phi_{[d],d-m} = \widetilde{H}_m(S^A) \otimes \Phi_{d-m}(\Delta_d)$$

as follows. First, we note that $\mathrm{Map}(A,[1])$ can be regarded as a partially ordered set using the pointwise order, and

$$(BA)_d = \mathrm{Poset}([d], \mathrm{Map}(A,[1])) = \prod_{a\in A} \mathbf{\Delta}([d],[1]).$$

Thus α gives a system of maps $\alpha_a \colon [d] \to [1]$.

(a) If any α_a is constant (or equivalently, not surjective) we put $z(\alpha) = 0$.

(b) Otherwise, we define $f \colon A \to [d]'$ by $f(a) = \alpha_a^\dagger(1)$. If f is not injective, we again put $z(\alpha) = 0$.

(c) Otherwise, we put

$$\begin{aligned} U &= \mathbb{K}\{w_{f(a)} \mid a \in A\} \\ V &= \ker(\alpha_* \colon W^\vee_{[d]} \to W^\vee_{\mathrm{Map}(A,[1])}) \\ &= \mathbb{K}\{w_i \mid \alpha(i) = \alpha(i-1)\} = \mathbb{K}\{w_i \mid i \notin f(A)\}. \end{aligned}$$

(Here we are using the notation of Definition 5.8.) We find that $W^\vee_{[d]} = U \oplus V$, so there is a natural isomorphism

$$\Lambda^m(U) \otimes \Lambda^{d-m}(V) \to \Lambda^d(W^\vee_{[d]}) = \mathbb{K}\theta_{[d]}.$$

Moreover, the map $a \mapsto w_{f(a)}$ induces an isomorphism $\widetilde{H}_m(S^A) = \Lambda^m \mathbb{K}\{A\} \to \Lambda^m(U)$, and there are natural inclusions

$$\Lambda^{d-m}(V) \le \Lambda^{d-m}(W_{[d]}^\vee) \le \Theta_{[d],d-m} \le \Phi_{d-m}(\Delta_d).$$

By putting these together, we get a map $\Lambda^d(W_{[d]}^\vee) \to \widetilde{H}_m(S^A) \otimes \Phi_{d-m}(\Delta_d)$. We write $z(\alpha)$ for the image of $\theta_{[d]}$ under this map.

Remark 6.3 For some purposes it is useful to be more explicit. Suppose that we are in case (c) of the definition, so that $f\colon A \to [d]'$ is injective. We can then list the elements of A as $\{a_1, \dots, a_m\}$, ordered in such a way that $f(a_1) < \cdots < f(a_m)$. Similarly, we list the elements of $[d]' \setminus f(A)$ as $\{j_1 < j_2 < \cdots < j_{d-m}\}$. There is then a number $\epsilon(\alpha) \in \{\pm 1\}$ such that

$$\theta_{[d]} = \epsilon(\alpha)\, w_{f(a_1)} \wedge \cdots \wedge w_{f(a_m)} \wedge w_{j_1} \wedge \cdots \wedge w_{j_{d-m}}.$$

Put
$$\begin{aligned} u(\alpha) &= w_{f(a_1)} \wedge \cdots \wedge w_{f(a_m)} \in \Lambda^m(U) \\ z'(\alpha) &= a_1 \wedge \cdots \wedge a_m \in \Lambda^m(\mathbb{K}\{A\}) = \widetilde{H}_m(S^{|A|}) \\ z''(\alpha) &= w_{j_1} \wedge \cdots \wedge w_{j_{d-m}} \in \Lambda^{d-m}(W_{[d]}^\vee). \end{aligned}$$

In this notation, the defining property of $\epsilon(\alpha)$ is that $\theta_{[d]} = \epsilon(\alpha) u(\alpha) \wedge z''(\alpha)$. We find that $z(\alpha) = \epsilon(\alpha)\, z'(\alpha) \otimes z''(\alpha)$.

Definition 6.4 For any simplicial set X we define

$$\phi\colon C_d(BA \times X) \to \widetilde{H}_m(S^A) \otimes \Phi_{d-m}(X)$$

as follows. Any d–simplex in $BA \times X$ has the form (α, x) where $x \in X_d$ and α is as in Definition 6.2. The simplex x corresponds to a map $\hat{x}\colon \Delta_d \to X$. We put

$$\phi(\alpha, x) = (1 \otimes \hat{x}_*)(z(\alpha)) = \epsilon(\alpha) z'(\alpha) \otimes (x \otimes z''(\alpha)).$$

Remark 6.5 Clause (a) in Definition 6.2 tells us that the map ϕ factors through $\widetilde{C}_*(S^A \wedge X_+)$, and similarly $\phi^\#$ induces a map

$$\operatorname{Hom}(\widetilde{H}_*(S^A), \widetilde{C}_*(S^A \wedge X_+)) \to \Phi_*(X).$$

Lemma 6.6 *If the simplex $(\alpha, x) \in (BA \times X)_d$ is degenerate then $\phi(\alpha, x) = 0$.*

Proof As (α, x) is degenerate, there must exist a surjection $\sigma\colon [d] \to [e]$ (with $e < d$) and a map $\beta\colon [a] \to \operatorname{Map}(A, [1])$ and a simplex $y \in X_e$ such that $\alpha = \beta\sigma$ and $x = \sigma^*(y)$. As $e < d$ we must have $\sigma(i-1) = \sigma(i)$ for some $i > 0$. As $\alpha = \beta\sigma$ this means that

$\alpha(i) = \alpha(i-1)$, so $w_i \in V$. Clearly $\sigma_*(w_i) = \sigma_*(e_i - e_{i-1}) = e_{\sigma(i)} - e_{\sigma(i-1)} = 0$, so $\sigma_* = 0$ on $\Lambda^{d-m}(V)$, so $(1 \otimes \sigma_*)(z(\alpha)) = 0$. By definition we have

$$\phi(\alpha, x) = (1 \otimes \hat{x}_*)(z(\alpha)) = (1 \otimes \hat{y}_*)(1 \otimes \sigma_*)(z(\alpha)) = 0. \qquad \square$$

Corollary 6.7 *There are induced maps $\phi\colon \widetilde{N}_*(S^A \wedge X_+) \to \widetilde{H}_*(S^A) \otimes \Phi_*(X)$.* $\square$

Definition 6.8 Put

$$U_*(A, X) = \mathrm{Hom}(\widetilde{H}_*(S^A), \widetilde{N}_*(S^A \wedge X_+)).$$

As $\widetilde{H}_*(S^A)$ is invertible under the tensor product, the map ϕ gives rise to an adjoint map $U_*(A, X) \to \Phi_*(X)$, which we denote by $\phi^\#$.

Now consider $\alpha\colon [d] \to \mathrm{Map}(A, [1])$ and $i \in [d]$, giving a map $\alpha\delta_i\colon [d-1] \to \mathrm{Map}(A, [1])$ and an element $z(\alpha\delta_i) \in \widetilde{H}_m(S^{|A|}) \otimes \Theta_{[d-1],d-m-1}$. We can regard δ_i as a bijection $[d-1] \to [d] \setminus \{i\}$, so we get an element $(1 \otimes (\delta_i)_*)z(\alpha\delta_i) \in \widetilde{H}_m(S^{|A|}) \otimes \Theta_{[d]\setminus\{i\},d-m-1}$. We also have a map $\tau_i\colon \Theta_{[d],*} \to \Theta_{[d]\setminus\{i\},*}$ given by $\tau_i(\zeta) = dt_i \vdash \zeta$.

Lemma 6.9 $(-1)^i(1 \otimes (\delta_i)_*)z(\alpha\delta_i) = (-1)^{m+1}(1 \otimes \tau_i)(z(\alpha))$.

Proof We will consider the case $0 < i < d$; small adjustments for the end cases are left to the reader. Note that $\alpha_a\colon [d] \to [1]$ is surjective if and only if ($\alpha_a(0) = 0$ and $\alpha_a(d) = 1$) if and only if $\alpha\delta_i$ is surjective. We may assume that this holds for all a, otherwise both sides of the claimed identity are zero. Next, put $f(a) = \alpha_a^\dagger(1)$ as before, and $g(a) = (\alpha_a\delta_i)^\dagger(1)$. By a check of the various possible cases, we see that

$$g(a) = \sigma_i(f(a)) = \begin{cases} f(a) & \text{if } f(a) \le i, \\ f(a) - 1 & \text{if } i < f(a). \end{cases}$$

It follows that g is injective unless $\{i, i+1\} \subseteq f(A)$.

Suppose that $\{i, i+1\} \subseteq f(A)$, so g is not injective, so $z(\alpha\delta_i) = 0$. In this case $z''(\alpha)$ does not involve w_i or w_{i+1}, so $dt_i \vdash z''(\alpha) = (ds_{i+1} - ds_i) \vdash z''(\alpha) = 0$, and we see that both sides of the claimed identity are again zero.

Suppose instead that $\{i, i+1\} \not\subseteq f(A)$. One checks that $z'(\alpha) = z'(\alpha\delta_i)$. Let w' be the wedge of all the factors w_{j_t} in $z''(\alpha)$ with $j_t \in \{i, i+1\}$, and let w'' be the wedge of the remaining factors, so

$$z''(\alpha) = \epsilon' w' \wedge w''$$

for some $\epsilon' \in \{\pm 1\}$. Because $\{i, i+1\} \not\subseteq f(A)$ we must have $w' = w_i$ or $w' = w_{i+1}$ or $w' = w_i \wedge w_{i+1}$. In computing $(\delta_i)_* z''(\alpha\delta_i)$, we use that $(\delta_i)_* w_j = (\delta_i)_*(e_i - e_{i-1}) =$

$w_{\delta_i(j)}$ except in the case $j = i$, in which case we have $(\delta_i)_*(w_i) = w_i + w_{i+1}$. There are three cases to consider.

(a) If $w' = w_i$ (so $i \notin f(A)$ but $i+1 \in f(A)$) we find that $ds_i \vdash z''(\alpha) = -\epsilon' w''$ and $ds_{i+1} \vdash z''(\alpha) = 0$ so $dt_i \vdash z''(\alpha) = \epsilon' w''$. On the other hand, as $i \notin f(A)$ we have $\delta_i(g(a)) = \delta_i(\sigma_i(f(a))) = f(a)$ for all a, so $\delta_i([d-1]' \setminus g(A)) = ([d]' \setminus f(A)) \setminus \{i\}$, so $(\delta_i)_* z''(\alpha\delta_i) = w''$. We next need to understand $\epsilon(\alpha\delta_i)$. By definition we have

$$\epsilon(\alpha\delta_i) u(\alpha\delta_i) \wedge z''(\alpha\delta_i) = \theta_{[d-1]}.$$

As $\delta_i f = g$ and $(\delta_i)_* z''(\alpha\delta_i) = w''$ we see that $u(\alpha) = u(\alpha\delta_i)$ and

$$\epsilon(\alpha\delta_i) u(\alpha) \wedge w'' = \theta_{[d]\setminus\{i\}}.$$

We then multiply both sides on the left by w_i to get

$$(-1)^m \epsilon(\alpha\delta_i) u(\alpha) \wedge w_i \wedge w'' = (-1)^{i-1} \theta_{[d]}.$$

On the other hand, by the definitions of $\epsilon(\alpha)$ and ϵ' we have

$$\epsilon' \epsilon(\alpha) u(\alpha) \wedge w_i \wedge w'' = \theta_{[d]}.$$

It follows that $\epsilon(\alpha\delta_i) = (-1)^{m+i+1} \epsilon' \epsilon(\alpha)$. This gives

$$\begin{aligned}
(-1)^i (1 \otimes (\delta_i)_*) z(\alpha\delta_i) &= (-1)^i \epsilon(\alpha\delta_i) z'(\alpha\delta_i) \otimes (\delta_i)_* z''(\alpha\delta_i) \\
&= (-1)^i (-1)^{m+i+1} \epsilon' \epsilon(\alpha) z'(\alpha) w'' \\
&= (-1)^{m+1} \epsilon(\alpha) z'(\alpha) (dt_i \vdash z''(\alpha)) \\
&= (-1)^{m+1} (1 \otimes \tau_i)(z(\alpha))
\end{aligned}$$

as required.

(b) Now suppose instead that $w' = w_{i+1}$, so that $i \in f(A)$ but $i+1 \notin f(A)$. We find that $ds_i \vdash z''(\alpha) = 0$ and $s_{i+1} \vdash z''(\alpha) = -w''$, so $dt_i \vdash z''(\alpha) = -w''$. On the other hand, we find that

$$\delta_i(g(a)) = \delta_i \sigma_i(f(a)) = \begin{cases} f(a) & \text{if } f(a) \neq i, \\ i+1 & \text{if } f(a) = i. \end{cases}$$

From this we see that $\delta_i([d-1]' \setminus g(A)) = ([d]' \setminus f(A)) \setminus \{i+1\}$, and thus that $(\delta_i)_* z''(\alpha\delta_i) = w''$. We next need to understand $\epsilon(\alpha\delta_i)$. From the definitions we have

$$\epsilon(\alpha\delta_i) w_{\delta_i g(a_1)} \wedge \cdots \wedge w_{\delta_i g(a_m)} \wedge (\delta_i)_* z''(\alpha\delta_i) = \theta_{[d]\setminus\{i\}}.$$

Let r be such that $f(a_r)=i$, and let v be the wedge of the terms $w_{f(a_p)}$ for $p\neq r$. The above equation can then be written as

$$(-1)^r\epsilon(\alpha\delta_i)w_{i+1}\wedge v\wedge w''=\theta_{[d]\setminus\{i\}}.$$

We now multiply both sides on the left by w_i to get

$$(-1)^r\epsilon(\alpha\delta_i)w_i\wedge w_{i+1}\wedge v\wedge w''=(-1)^{i+1}\theta_{[d]}.$$

On the other hand, by the definitions of $\epsilon(\alpha)$ and ϵ' we have

$$(-1)^r\epsilon'\epsilon(\alpha)w_i\wedge v\wedge w_{i+1}\wedge w''=\theta_{[d]}.$$

It follows that $\epsilon(\alpha\delta_i)=(-1)^{m+i}\epsilon'\epsilon(\alpha)$. This gives

$$\begin{aligned}(-1)^i(1\otimes(\delta_i)_*)z(\alpha\delta_i)&=(-1)^i\epsilon(\alpha\delta_i)z'(\alpha\delta_i)\otimes(\delta_i)_*z''(\alpha\delta_i)\\&=(-1)^i(-1)^{m+i}\epsilon'\epsilon(\alpha)z'(\alpha)w''\\&=(-1)^m\epsilon(\alpha)z'(\alpha)(-dt_i\vdash z''(\alpha))=(-1)^{m+1}(1\otimes\tau_i)(z(\alpha))\end{aligned}$$

as required.

(c) Finally, suppose that neither i nor $i+1$ is in $f(A)$, so $w'=w_i\wedge w_{i+1}$. As this has even degree we have $\epsilon'=1$ and $z''(\alpha)=w'\wedge w''$. We then have $ds_i\vdash z''(\alpha)=-w_{i+1}\wedge w''$ and $ds_{i+1}\vdash z''(\alpha)=w_i\wedge w''$ so $dt_i\vdash z''(\alpha)=(w_i+w_{i+1})\wedge w''$. On the other hand, as in case (a) we see that $f=\delta_i g$ and $\delta_i([d-1]'\setminus g(A))=([d]'\setminus f(A))\setminus\{i\}$. Suppose that i occurs as the r–th element in $[d-1]'\setminus g(A)$, so $i+1$ occurs as the r–th element in $\delta_i([d-1]'\setminus g(A))$. Then

$$(\delta_i)_*z''(\alpha\delta_i)=(-1)^{r-1}(\delta_i)_*(w_i)\wedge w''=(-1)^{r-1}(w_i+w_{i+1})\wedge w''.$$

We next need to understand $\epsilon(\alpha\delta_i)$. By definition we have

$$\epsilon(\alpha\delta_i)u(\alpha\delta_i)\wedge z''(\alpha\delta_i)=\theta_{[d-1]}.$$

As $\delta_i f=g$ and $(\delta_i)_*z''(\alpha\delta_i)=(-1)^{r-1}(w_i+w_{i+1})\wedge w''$ we have $u(\alpha\delta_i)=u(\alpha)$ and

$$(-1)^{r-1}\epsilon(\alpha\delta_i)u(\alpha)\wedge(w_i+w_{i+1})\wedge w''=\theta_{[d]\setminus\{i\}}.$$

We then multiply both sides on the left by w_i to get

$$(-1)^{m+r-1}\epsilon(\alpha\delta_i)u(\alpha)\wedge w_i\wedge w_{i+1}\wedge w''=(-1)^{i-1}\theta_{[d]}.$$

After comparing this with the definition of $\epsilon(\alpha)$, we see that $\epsilon(\alpha\delta_i) = (-1)^{m+r+i}\epsilon(\alpha)$. This gives

$$\begin{aligned}(-1)^i(1\otimes(\delta_i)_*)z(\alpha\delta_i) &= (-1)^i(-1)^{m+r+i}\epsilon(\alpha)z'(\alpha)\otimes(-1)^{r-1}(w_i+w_{i+1})\wedge w''\\ &= (-1)^{m+1}\epsilon(\alpha)z'(\alpha)(dt_i\vdash z''(\alpha))\\ &= (-1)^{m+1}(1\otimes\tau_i)(z(\alpha))\end{aligned}$$

as required. □

Corollary 6.10 *The maps*

$$\begin{aligned}\phi\colon \widetilde{N}_*(S^A\wedge X_+) &\to \widetilde{H}_*(S^A)\otimes\Phi_*(X),\\ \phi^\#\colon U_*(A,X) &\to \Phi_*(X)\end{aligned}$$

are chain maps.

Proof Lemma 6.9 is the universal example. In more detail, we first note that $z(\alpha)$ involves only the exterior generators dt_i so $(1\otimes\delta')(z(\alpha))=0$ and

$$\begin{aligned}(1\otimes\delta)(z(\alpha)) &= (1\otimes\delta'')(z(\alpha))\\ &= -\sum_j i_{[d]\setminus\{j\}}(1\otimes\tau_j)(z(\alpha))\\ &= (-1)^m\sum_j(-1)^j(1\otimes(\delta_j)_*)(z(\alpha\delta_j)).\end{aligned}$$

Next, we will also write δ for the standard differential on $\widetilde{H}_*(S^A)\otimes\Phi_*(X)$, which is $\delta(a\otimes b)=(-1)^{|a|}a\otimes\delta(b)$. This gives

$$\delta(z(\alpha)) = (-1)^m(1\otimes\delta)(z(\alpha)) = \sum_j(-1)^j(1\otimes(\delta_j)_*)(z(\alpha\delta_j)).$$

Now consider an element $x\in X_d$, giving a map $\hat{x}\colon \Delta_d\to X$ and thus a map $\hat{x}_*\colon \Phi_{[d],*}=\Phi_*(\Delta_d)\to\Phi_*(X)$. If we apply the map $1\otimes\hat{x}_*$ to the above equation and use the naturality of δ, the left hand side becomes $\delta(\phi(x,\alpha))$. The right hand side becomes $\sum_j(-1)^j(1\otimes\widehat{d_jx}_*)(z(\alpha\delta_j))$, which is $\phi(d(x,\alpha))$. This shows that ϕ is a chain map, and it follows adjointly that the same is true for $\phi^\#$. □

Definition 6.11 We define $\nu\colon U_*(A,X)\otimes U_*(B,Y)\to U_*(A\amalg B, X\times Y)$ by applying the functor $\mathrm{Hom}(\widetilde{H}_*(S^{A\amalg B}),-)$ to the composite

$$\widetilde{N}_*(S^A\wedge X_+)\otimes\widetilde{N}_*(S^B\wedge Y_+)\xrightarrow{\mu}\widetilde{N}_*(S^A\wedge X_+\wedge S^B\wedge Y_+)\xrightarrow{(1\wedge\tau\wedge1)_*}\widetilde{N}_*(S^{A\amalg B}\wedge(X\times Y)_+)$$

and using the isomorphism $\widetilde{H}_*(S^A)\otimes\widetilde{H}_*(S^B)\to\widetilde{H}_*(S^{A\amalg B})$.

Lemma 6.12 *Suppose we have a shuffle $(\zeta,\xi)\colon[d+e]\to[d]\times[e]$ and maps $\alpha\colon[d]\to\mathrm{Map}(A,[1])$ and $\beta\colon[e]\to\mathrm{Map}(B,[1])$ (with $|A|=m$ and $|B|=n$). Define $\gamma\colon[d+e]\to\mathrm{Map}(A\amalg B,[1])$ by $\gamma_a(k)=\alpha_a(\zeta(k))$ for $a\in A$ and $\gamma_b(k)=\beta_b(\xi(k))$ for $b\in B$. Let λ denote the map*

$$\widetilde{H}_*(S^A)\otimes\Theta_{[d],*}\otimes\widetilde{H}_*(S^B)\otimes\Theta_{[e],*}\xrightarrow{1\otimes\tau\otimes1}\widetilde{H}_*(S^A)\otimes\widetilde{H}_*(S^B)\otimes\Theta_{[d],*}\otimes\Theta_{[e],*}\xrightarrow{\mu\otimes\mu_{\zeta\xi}}\widetilde{H}_*(S^{A\amalg B})\otimes\Theta_{[d+e],*}.$$

Then $z(\gamma)=\mathrm{sgn}(\zeta,\xi)\lambda(z(\alpha)\otimes z(\beta))$.

Proof If any α_a or β_b fails to be surjective then so does the corresponding map γ_a or γ_b, so both sides of the claimed equality are zero. We ignore this case from now on.

Put $f(a)=\alpha_a^\dagger(1)$ and $g(b)=\beta_b^\dagger(1)$ and $h(c)=\gamma_c^\dagger(1)$. As (ζ,ξ) is a shuffle we know that the maps

$$[d]'\xrightarrow{\zeta^\dagger}[d+e]'\xleftarrow{\xi^\dagger}[e]'$$

give a coproduct decomposition, and from the definitions we have $h(a)=\zeta^\dagger(f(a))$ and $h(b)=\xi^\dagger(g(b))$. It follows that h is injective if and only if both f and g are injective, and we may assume that this is the case as otherwise both sides of the claimed equality are zero.

From our description of h, we have

$$[d+e]'\setminus h(A\amalg B)=\zeta^\dagger([d]'\setminus f(A))\amalg\xi^\dagger([e]'\setminus g(B)),$$

so $z''(\gamma)=\pm\zeta^\bullet(z''(\alpha))\wedge\xi^\bullet(z''(\beta))$. By a similar argument we have $z'(\gamma)=\pm\mu(z'(\alpha)\otimes z'(\beta))$ and so $z(\gamma)=\pm\lambda(z(\alpha)\otimes z(\beta))$. The real issue is just to control the signs more precisely. For this we note that

$$\theta_{[d]}=\epsilon(\alpha)u(\alpha)\wedge z''(\alpha),\quad \theta_{[e]}=\epsilon(\beta)u(\beta)\wedge z''(\beta),$$

so $$\zeta^\bullet\theta_{[d]}\wedge\xi^\bullet\theta_{[e]}=\epsilon(\alpha)\epsilon(\beta)\zeta^\bullet u(\alpha)\wedge\zeta^\bullet z''(\alpha)\wedge\xi^\bullet u(\beta)\wedge\xi^\bullet z''(\beta).$$

After using Definition 5.15 and reordering the factors, this gives

$$\theta_{[d+e]} = (-1)^{n(d-m)} \operatorname{sgn}(\zeta,\xi)\epsilon(\alpha)\epsilon(\beta)\zeta^\bullet u(\alpha) \wedge \xi^\bullet u(\beta) \wedge \zeta^\bullet z''(\alpha) \wedge \xi^\bullet z''(\beta).$$

Now put

$$\begin{aligned} U &= \mathbb{K}\{w_{h(c)} \mid c \in A \amalg B\} \\ V &= \mathbb{K}\{w_i \mid \gamma(i) = \gamma(i-1)\} = \mathbb{K}\{w_i \mid i \notin h(A \amalg B)\} \end{aligned}$$

as in Definition 6.2. We find that

$$\begin{aligned} \zeta^\bullet u(\alpha) \wedge \xi^\bullet u(\beta) &\in U \\ \zeta^\bullet z''(\alpha) \wedge \xi^\bullet z''(\beta) &\in V, \end{aligned}$$

so the above expression for $\theta_{[d+e]}$ can be used (together with the isomorphism $\widetilde{H}_{m+n}(S^{A\amalg B}) \simeq \Lambda^{m+n}(U)$ induced by h) to calculate $z(\gamma)$. The result is

$$z(\gamma) = (-1)^{n(d-m)} \operatorname{sgn}(\zeta,\xi)\epsilon(\alpha)\epsilon(\beta)\mu(z'(\alpha) \otimes z'(\beta)) \otimes \zeta^\bullet z''(\alpha) \wedge \xi^\bullet z''(\beta),$$

and this is the same as $\lambda(z(\alpha) \otimes z(\beta))$. □

Proposition 6.13 *The following diagram commutes:*

$$\begin{tikzcd} U_*(A,X) \otimes U_*(B,Y) \arrow[r, "\nu"] \arrow[d, "\phi^\# \otimes \phi^\#"'] & U_*(A \amalg B, X \times Y) \arrow[d, "\phi"] \\ \Phi_*(X) \otimes \Phi_*(Y) \arrow[r, "\mu"'] & \Phi_*(X \times Y). \end{tikzcd}$$

Proof It will be enough to check commutativity of the adjoint diagram

$$\begin{tikzcd} \begin{matrix}\widetilde{N}_*(S^A \wedge X_+) \\ \otimes \widetilde{N}_*(S^B \wedge Y_+)\end{matrix} \arrow[r, "\mu"] \arrow[d, "\phi\otimes\phi"'] & \widetilde{N}_*(S^A \wedge X_+ \wedge S^B \wedge Y_+) \arrow[r, "(1\wedge\tau\wedge 1)_*"] & \widetilde{N}_*(S^{A\amalg B} \wedge (X \times Y)_+) \arrow[d, "\phi"] \\ \begin{matrix}\widetilde{H}_*(S^A) \otimes \Phi_*(X) \\ \otimes \widetilde{H}_*(S^B) \otimes \Phi_*(Y)\end{matrix} \arrow[r, "1\otimes\tau\otimes 1"'] & \begin{matrix}\widetilde{H}_*(S^A) \otimes \widetilde{H}_*(S^B) \\ \otimes \Phi_*(X) \otimes \Phi_*(Y)\end{matrix} \arrow[r, "\mu\otimes\mu"'] & \widetilde{H}_*(S^{A\amalg B}) \otimes \Phi_*(X \times Y). \end{tikzcd}$$

Consider elements $\alpha \in (BA)_d$ and $x \in X_d$ and $\beta \in (BB)_e$ and $y \in Y_e$. The generator $(\alpha, x) \otimes (\beta, y)$ maps to

$$\sum_{\zeta,\xi} \operatorname{sgn}(\zeta,\xi)(\alpha\zeta, \beta\xi, \zeta^*(x), \xi^*(y)) \in \widetilde{N}_{d+e}(S^{A\amalg B} \wedge (X \times Y)_+).$$

The term indexed by the shuffle (ζ,ξ) then maps to $\operatorname{sgn}(\zeta,\xi)(\zeta^*(x), \xi^*(y)) \otimes z(\alpha\zeta, \beta\xi)$ in $\widetilde{H}_*(S^{A\amalg B}) \otimes \Phi_*(X \times Y)$. It follows from Lemma 6.12 that the other route around the diagram yields the same result. □

Definition 6.14 Suppose we have a set A with $|A| = m$. We note that when $k > m$ we have $\mathrm{ND}_k(S^A) = \varnothing$ and so $\widetilde{N}_k(S^A) = 0$; this means that

$$\widetilde{H}_m(S^A) = \ker(d\colon \widetilde{N}_m(S^A) \to \widetilde{N}_{m-1}(S^A)) \leq \widetilde{N}_m(S^A).$$

The inclusion $\widetilde{H}_m(S^A) \to \widetilde{N}_m(S^A)$ gives a cycle in

$$U_0(A, 1) = \mathrm{Hom}(\widetilde{H}_m(S^A), \widetilde{N}_m(S^A)),$$

which we denote by η_A.

Definition 6.15 Given an injective map $\lambda\colon A \to B$, we define

$$\lambda_*\colon \mathrm{Hom}(\widetilde{H}_*(S^A), \widetilde{N}_*(S^A \wedge X_+)) \to \mathrm{Hom}(\widetilde{H}_*(S^B), \widetilde{N}_*(S^B \wedge X_+))$$

as follows. Firstly, if λ is a bijection then we just transport the structure in the obvious way. Next, suppose that λ is just the inclusion of a subset, so $B = A \amalg Z$ for some Z. We then have a map

$$\nu\colon U_*(A, X) \otimes U_*(Z, 1) \to U_*(A \amalg Z, X \times 1) = U_*(B, X)$$

and we put $\lambda_*(u) = \nu(u \otimes \eta_Z)$. Finally, an arbitrary monomorphism can be written uniquely as $\lambda = \lambda_1\lambda_0$, where λ_1 is a subset inclusion and λ_0 is a bijection. We then put $\lambda_* = (\lambda_1)_*(\lambda_0)_*$.

Lemma 6.16 *λ_* is a chain map and is functorial.*

Proof Left to the reader. □

Lemma 6.17 *For any monomorphism $\lambda\colon A \to B$, the diagram*

$$\begin{tikzcd} U_*(A, X) \arrow[rr, "\lambda_*"] \arrow[dr, "\phi^\#"'] & & U_*(B, X) \arrow[dl, "\phi^\#"] \\ & \Phi_*(X) & \end{tikzcd}$$

commutes.

Proof This is clear if λ is an isomorphism, and is a special case of Proposition 6.13 if λ is a subset inclusion. The general case follows from these special cases. □

Definition 6.18 We write $U_*(X)$ for the colimit of the complexes $U_*(A, X)$ as A runs over the category of finite sets an injective maps. We let $\psi\colon U_*(X) \to \Phi_*(X)$ denote the map induced by the maps $\phi^\#\colon U_*(A, X) \to \Phi_*(X)$ (which exists by Lemma 6.17).

Theorem 6.19 *The map $\psi\colon U_*(X)\to\Phi_*(X)$ is an isomorphism.*

The proof will be given in several stages. Firstly, the construction given below immediately implies that ψ is surjective.

Construction 6.20 Suppose we have $x\in\mathrm{ND}_n(X)$ and $\nu\colon [n]\to\mathbb{N}$ and $J\subseteq[n]'$, say $J=\{j_1<\dots<j_r\}$. Put $w_J=w_{j_1}\wedge\dots\wedge w_{j_r}\in\Theta_{[n],r}$. We will construct an element $\zeta(x,\nu,J)\in U_*(X)$ with $\psi(\zeta(x,\nu,J))=x\otimes t^{[\nu]}w_J\in\Phi_*(X)$.

First put $d=-1+\sum_{i=0}^n(\nu_i+1)$, and let $\sigma\colon [d]\to[n]$ be the unique nondecreasing surjective map such that $|\sigma^{-1}(i)|=\nu_i+1$ for all i. Put $A=[d]'\setminus\sigma^\dagger(J)$ and $m=|A|$ and let $f\colon A\to[d]'$ be the inclusion. Define $\alpha\colon [d]\to\mathrm{Map}(A,[1])$ by

$$\alpha_a(i)=\begin{cases}0 & \text{if } i<f(a),\\ 1 & \text{if } f(a)\le i.\end{cases}$$

We find that $z''(\alpha)=w_{\sigma^\dagger(J)}$ and so (using Definition 4.1) we have $\sigma_*(z''(\alpha))=t^{[\nu]}w_J$. Now let

$$\zeta_1(x,\nu,J)\colon \widetilde{H}_m(S^A)\to\widetilde{N}_d(S^A\wedge X)$$

be the map that sends the generator $\epsilon(\alpha)z'(\alpha)$ to (α,x). Then $\zeta_1(x,\nu,J)\in U_*(A,X)$ and $\phi^\#\zeta_1(x,\nu,J)=x\otimes t^{[\nu]}w_J$. We also write $\zeta(x,\nu,J)$ for the image of $\zeta_1(x,\nu,J)$ in $U_*(X)$, so that $\psi(\zeta(x,\nu,J))$.

We next need the counterpart in $U_*(X)$ of the relation $\sum_i t_i=1$.

Lemma 6.21 *In the notation of Construction 6.20 we have*

$$\sum_{i=0}^n(\nu_i+1)\zeta(x,\nu+\delta_i,J)=\zeta(x,\nu,J).$$

Proof We will freely use the notation of the above construction.

Put $A_+=A\amalg\{\infty\}$ so we have a class $\xi=\mu(\zeta_1(x,\nu,J)\otimes\eta_{\{\infty\}})\in U_*(A_+,X)$ which represents $\zeta(x,\nu,J)$. Now ξ can be written as a sum of terms, one for each shuffle $(\lambda,\rho)\colon [d+1]\to[d]\times[1]$. Such a shuffle is determined by the number $k=\rho^\dagger(1)\in[d+1]'$; indeed, λ is forced to be the unique map in $\mathbb{E}([d+1],[d])$ that takes the value $k-1$ twice. Define $\nu(k)\colon [n]\to\mathbb{N}$ by $\nu(k)_i=|(\sigma\lambda)^{-1}\{i\}|-1$. We find that the k–th term in the product $\mu(\zeta_1(x,\nu,J)\otimes\eta_{\{\infty\}})$ represents $\zeta(x,\nu(k),J)$, and that there are ν_i+1 different values of k for which $\nu(k)=\nu+\delta_i$. The claim follows. $\square$

Corollary 6.22 *There is a well-defined map*

$$\zeta'\colon \Phi_*(X) = \bigoplus_{x \in \mathrm{ND}_n(X)} \Theta_{[n],*} \to U_*(X)$$

given by $\zeta'(x \otimes t^{[\nu]} w_J) = \zeta(x, \nu, J)$*. Moreover, we have* $\psi\zeta' = 1\colon \Phi_*(X) \to \Phi_*(X)$*.*

Proof The stated formula certainly defines a map

$$\bigoplus_{x \in \mathrm{ND}_n(X)} \widetilde{P}_{[n]} \otimes \Lambda^*(W_{[n]}^\vee) \to U_*(X).$$

We simply need to pass from $\widetilde{P}_{[n]}$ to $P_{[n]} = \widetilde{P}_{[n]}/(1 - \sum_i t_i)$, and this is precisely what we get from Lemma 6.21. □

Lemma 6.23 *The composite* $U_*(A, X) \xrightarrow{\phi^\#} \Phi_*(X) \xrightarrow{\zeta'} U_*(X)$ *is just the colimit inclusion map.*

Proof Put $m = |A|$ and fix a generator $u \in \widetilde{H}_m(S^A)$. Given $v \in \widetilde{N}_d(S^A \wedge X_+)$ we write $u^{-1}v$ for the element of $U_*(A, X)$ given by $u \mapsto v$. The group $U_d(A, X)$ is generated by elements $u^{-1}(\alpha, x)$ where $\alpha\colon [d] \to \mathrm{Map}(A, [1])$ and $x \in X_d$ and the pair (α, x) is nondegenerate. To avoid trivial cases, we may assume that each $\alpha_a\colon [d] \to [1]$ is surjective, so we can define $f(a) = \alpha_a^\dagger(1)$ as usual.

If f is not injective, it is built into the definitions that $z(\alpha) = 0$ and so $\phi^\#(\alpha, x) = 0$, so we must show that $u^{-1}(\alpha, x)$ also maps to zero in the colimit. We can choose $a \neq a'$ with $f(a) = f(a')$, and let τ denote the transposition that exchanges a and a'. We find that $\tau_*(u) = -u$ but $\tau_*(\alpha, x) = (\alpha, x)$, so $\tau_*(u^{-1}(\alpha, x)) = -u^{-1}(\alpha, x)$, which gives the required vanishing.

From now on we assume that f is injective. As in Lemma A.10 we can write $x = \sigma^*(y)$ for some nondegenerate simplex $y \in X_n$ and some surjective map $\sigma\colon [d] \to [n]$. To avoid further trivial cases, we may assume that the pair (α, x) is nondegenerate, which is equivalent to the condition $[d]' = f(A) \cup \sigma^\dagger([n]')$. Define $\nu\colon [n] \to \mathbb{N}$ by $\nu_i = |\sigma^{-1}\{i\}| - 1$, so that $\sigma_*(1) = t^{[\nu]}$. Put $J' = [d]' \setminus f(A)$, so that $z''(\alpha) = w_{J'}$. As $[d]' = f(A) \cup \sigma^\dagger([n]')$ we must have $J' = \sigma^\dagger(J)$ for some $J \subseteq [n]'$, and this implies that $J = \sigma(J')$ and so $\sigma_*(z''(\alpha)) = t^{[\nu]} w_J$. It follows that $\phi^\#(u^{-1}(\alpha, x)) = \epsilon' x \otimes t^{[\nu]} w_J$, where the sign $\epsilon' \in \{\pm 1\}$ is determined by the relation $z(\alpha) = \epsilon' u \otimes z''(\alpha)$. Now put $A' = f(A)$, so f gives a bijection $A \to A'$ and thus an isomorphism $U_*(A, X) \to U_*(A', X)$. From the definitions we see that $\zeta'\phi^\#(u^{-1}(\alpha, x))$ is represented by $\epsilon' \zeta_1(x, \nu, J) \in U_*(A', X)$, which is just the image of $u^{-1}(x, \alpha)$ under this isomorphism. The claim follows. □

Proof of Theorem 6.19 Corollary 6.22 tells us that $\psi\zeta' = 1$, and Lemma 6.23 implies that $\zeta'\psi = 1$. □

Appendix A Recollections on the simplicial category

In this section we recall some facts about the simplicial category. Most of them are fairly standard but we will need to use the details so it is convenient to give a self-contained account here. Many of these facts were first proved by Eilenberg and Zilber [2] or Gabriel and Zisman [4]; the more recent book Fritsch and Piccinini [3] is also a useful reference.

Definition A.1 As usual, we let $\mathbf{\Delta}$ denote the category whose objects are the finite ordered sets $[n]=\{0,\dots,n\}$, and whose morphisms are the nondecreasing maps. All maps mentioned in this section are implicitly assumed to be nondecreasing. We also write $\mathbb{E}([n],[m])$ for the subset of $\mathbf{\Delta}([n],[m])$ consisting of surjective maps.

Definition A.2 Given a surjective map $\alpha\colon [n]\to[m]$, we define $\alpha^\dagger\colon [m]\to[n]$ by $\alpha^\dagger(j)=\min\{i \mid \alpha(i)=j\}$. We also write $[n]'=[n]\setminus\{0\}=\{1,\dots,n\}$ and note that $\alpha^\dagger(0)=0$ and $\alpha^\dagger([m]')\subseteq[n]'$.

Lemma A.3 *The map $\alpha^\dagger$ is injective, and $\alpha\alpha^\dagger=1$. Moreover, if $\beta\colon[n]\to[p]$ is another surjection then $(\beta\alpha)^\dagger=\alpha^\dagger\beta^\dagger$.*

Proof Left to the reader. □

Definition A.4 We say that a subset $A\subseteq[n]$ is *pointed* if $0\in A$. Given a pointed subset $A\subseteq[n]$ with $|A|=m+1$, we let $\sigma_A\colon[m]\to[n]$ be the unique injection with $\sigma_A([m])=A$, and we define $\pi_A\colon[n]\to[m]$ by $\pi_A(i)=\max\{j \mid \sigma_A(j)\le i\}$. We also define $\epsilon_A=\sigma_A\pi_A\colon[n]\to[n]$, so $\epsilon_A(i)=\max\{j\in A \mid j\le i\}$ and $\epsilon_A^2=\epsilon_A$.

Lemma A.5 (a) *Any injective map $\beta\colon[m]\to[n]$ with $\beta(0)=0$ has the form $\beta=\sigma_A$ for some (unique) pointed set A, namely $A=\beta([m])$.*

(b) *Any surjective map $\alpha\colon[n]\to[m]$ has the form $\alpha=\pi_A$ for some (unique) pointed set A, namely $A=\{0\}\cup\{i>0 \mid \alpha(i)>\alpha(i-1)\}$.*

(c) *Let $\gamma\colon[n]\to[n]$ be a map with $i\ge\gamma(i)=\gamma^2(i)$ for all i. Then $\gamma=\epsilon_A$ for some (unique) pointed set A, namely $A=\{i \mid \gamma(i)=i\}$.*

Proof Left to the reader. □

Lemma A.6 *Suppose we have pointed sets $A\subseteq B\subseteq[n]$ with $|A|=m+1$ and $|B|=p+1$. Put $\alpha=\pi_A\sigma_B\colon[p]\to[m]$. Then α is surjective and fits into a commutative* *diagram as follows.*

$$\begin{tikzcd}
{[m]} \arrow[rr, two heads, "\alpha"] \arrow[dr, tail, "\sigma_B"] \arrow[dd, "1"'] & & {[p]} \\
& {[n]} \arrow[ur, two heads, "\pi_A"] \arrow[dl, two heads, "\pi_B"] & \\
{[m]} & & {[p]} \arrow[ll, tail, "\alpha^\dagger"] \arrow[ul, tail, "\sigma_A"'] \arrow[uu, "1"']
\end{tikzcd}$$

Moreover, α is bijective if and only if $\alpha = 1$ if and only if $A = B$.

Proof Left to the reader. □

Lemma A.7 *If A and B are pointed subsets of $[n]$ then $(\epsilon_A \epsilon_B)^N = \epsilon_{A \cap B}$ for $N \gg 0$.*

Proof For any $i \in [n]$ we have a decreasing sequence

$$i \geq \epsilon_B(i) \geq \epsilon_A \epsilon_B(i) \geq \epsilon_B \epsilon_A \epsilon_B(i) \geq \cdots \geq 0.$$

Let $\gamma(i)$ denote the eventual value of this sequence. We find that for $N \gg 0$ we have $\gamma = (\epsilon_A \epsilon_B)^N = \epsilon_B(\epsilon_A \epsilon_B)^N$, from which it follows that $\gamma = \epsilon_A \gamma = \epsilon_B \gamma = \gamma^2$ and $i \geq \gamma(i)$. It follows that $\gamma = \epsilon_C$, where $C = \mathrm{image}(\gamma) = \{i \mid \gamma(i) = i\}$. As $\gamma = \epsilon_A \gamma = \epsilon_B \gamma$ we see that $C = \mathrm{image}(\gamma) \subseteq \mathrm{image}(\epsilon_A) \cap \mathrm{image}(\epsilon_B) = A \cap B$, but it is clear that γ is the identity on $A \cap B$ so $C = A \cap B$. □

A.1 Degeneracy

Lemma A.8 *Let X be a simplicial set, and let x be an n–simplex of X. Then the following conditions are equivalent.*

(1) $x = \alpha^*(y)$ *for some noninjective map* $\alpha\colon [n] \to [m]$ *and some* $y \in X_m$.

(2) $x = \beta^*(z)$ *for some surjective map* $\beta\colon [n] \to [p]$ *(with* $p < n$*) and some* $z \in X_p$.

(3) $x = \pi_A^*(y)$ *for some proper pointed subset* $A \subset [n]$ *and some* $y \in X_{|A|-1}$.

(4) $x = \epsilon_A^*(x)$ *for some proper pointed subset* $A \subset [n]$.

Proof It is clear that (2) implies (1), and we can prove the converse by factoring α as a surjection followed by an injection. Lemma A.5(b) tells us that (2) is equivalent to (3). Using the facts that $\epsilon_A = \sigma_A \pi_A$ and $\pi_A \sigma_A = 1$ we see that (3) is equivalent to (4). □

Definition A.9 We say that x is *degenerate* if the above conditions hold. We write $\mathrm{ND}_n(X)$ for the set of nondegenerate n–simplices.

The next result is known as the Eilenberg-Zilber lemma.

Lemma A.10 *There is a canonical bijection* $\psi\colon \coprod_m \mathbb{E}([n],[m]) \times \mathrm{ND}_m(X) \to X_n$ *given by* $\psi(\alpha, y) = \alpha^*(y)$.

Proof Given $x \in X_n$, let $\mathcal{A}$ denote the collection of pointed subsets $A \subseteq [n]$ such that $x = \epsilon_A^*(x)$. Using $\epsilon_A = \sigma_A \pi_A$ and $\pi_A \sigma_A = 1$ we see that $\mathcal{A} = \{A \mid x \in \mathrm{image}(\pi_A^*)\}$. It is clear that $[n] \in \mathcal{A}$, and Lemma A.7 implies that $\mathcal{A}$ is closed under intersections, so $\mathcal{A}$ has a smallest element, say A. Put $m = |A| - 1$ and $y = \sigma_A^*(x) \in X_m$ and note that $x = \pi_A^*(y)$.

Suppose that $y = \beta^*(z)$ for some surjection $\beta\colon [m] \to [p]$. Then $\beta \pi_A = \pi_B$ for some $B \subseteq A$, but $x = \pi_B^*(Z)$ so $B \in \mathcal{A}$ so $A \subseteq B$. It follows that $A = B$ and $p = m$ and $\beta = 1$ so $y = z$. Using this we see that y is nondegenerate.

More generally, suppose we also have $x = \pi_B^*(z)$ for some B (*a priori* unrelated to A) and $z \in X_p$ (*a priori* unrelated to y). Then again $B \in \mathcal{A}$ so $A \subseteq B$ so we can apply Lemma A.6: the map $\alpha = \pi_A \sigma_B\colon [p] \to [m]$ is surjective and satisfies $\alpha \pi_B = \pi_A$. As $x = \pi_B^*(z)$ we have $z = \sigma_B^*(x) = \sigma_B^* \pi_A^*(y) = \alpha^*(y)$. If z is nondegenerate it follows that we must have $p = m$ and α must be the identity so $A = B$ and $y = z$. Using this we see that ψ is a bijection. $\square$

A.2 Shuffles

We now recall some theory of shuffles.

Definition A.11 Given a sequence $\underline{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ with $\sum_i n_i = n$, an $\underline{n}$–*shuffle* means a system of surjective maps $\zeta_i\colon [n] \to [n_i]$ such that the combined map $\zeta\colon [n] \to \prod_i [n_i]$ is injective. We write $\Sigma(\underline{n})$ for the set of all $\underline{n}$–shuffles.

Remark A.12 We will most often need the case $r = 2$. An (n, m)–shuffle is then a pair of surjections
$$[n] \xleftarrow{\zeta} [n+m] \xrightarrow{\xi} [m]$$
such that the map $(\zeta, \xi)\colon [n+m] \to [n] \times [m]$ is injective.

Lemma A.13 *Let* $\underline{n}$ *and* n *be as above, and suppose we have sets* $A_1, \dots, A_r \subseteq [n]' = \{1, \dots, n\}$ *with* $|A_i| = n_i$ *and we put* $\zeta_i = \pi_{A_i \cup \{0\}}\colon [n] \to [n_i]$. *Then the list* $\underline{\zeta}$ *is an* $\underline{n}$*–shuffle if and only if* $[n]' = \coprod_i A_i$.

Proof From the definition of $\pi_{A_i\cup\{0\}}$ we see that $A_i=\{s\in[n]'\mid\zeta_i(s)>\zeta_i(s-1)\}$, so that $\bigcup_i A_i=\{s\in[n]'\mid\zeta(s)\neq\zeta(s-1)\}$. Thus $\underline{\zeta}$ is a shuffle if and only if ζ is injective if and only if $\bigcup_i A_i=[n]'$, and if this happens then the union is automatically disjoint by counting. □

Corollary A.14 *We have*

$$|\Sigma(\underline{n})|=n!/\prod_i n_i!\,.$$

In particular,

$$|\Sigma(n,m)|=(n+m)!/n!m!\,. \qquad \square$$

Lemma A.15 *There are natural bijections*

$$\Sigma(m+n,p)\times\Sigma(m,n)\xrightarrow{L}\Sigma(m,n,p)\xleftarrow{R}\Sigma(n,m+p)\times\Sigma(n,p)$$

given by $L(\zeta,\xi;\phi,\psi)=(\phi\zeta,\psi\zeta,\xi)$ *and* $R(\zeta,\xi;\phi,\psi)=(\zeta,\xi\phi,\xi\psi)$*.*

Proof We will only discuss L; the case of R is similar.

Suppose that $(\zeta,\xi)\in\Sigma(m+n,p)$ and $(\phi,\psi)\in\Sigma(m,n)$. Then ζ, ξ, ϕ and ψ are all surjective, so the same is true of $\phi\zeta$ and $\psi\zeta$. The maps $(\phi,\psi)\times 1\colon[m+n]\times[p]\to[m]\times[n]\times[p]$ and $(\zeta,\xi)\colon[m+n+p]\to[m+n]\times[p]$ are injective, so the same is true of their composite, so $L(\zeta,\xi;\phi,\psi)\in\Sigma(m,n,p)$. Next, observe that to give a three-piece splitting $[m+n+p]'=A\amalg B\amalg C$ (with $|A|=m$ and $|B|=n$ and $|C|=p$) is the same as to give a splitting $[m+n+p]'=U\amalg C$ (with $|U|=m+n$ and $|C|=p$) together with a splitting $U=A\amalg B$ (with $|A|=m$ and $|B|=n$). Using this together with the correspondence $T\leftrightarrow\pi_T$ we obtain a bijection $L'\colon\Sigma(m+n,p)\times\Sigma(m,n)\to\Sigma(m,n,p)$. We leave it to the reader to check that $L=L'$. □

Appendix B Integration over simplices

Recall that the map $\int_I\colon\widetilde{P}_I\to\mathbb{K}$ is defined by

$$\int_I t^\nu=\Big(\prod_i\nu_i!\Big)\Big/\Big(n+\sum_i\nu_i\Big)!,$$

(where $n=|I|-1$) or equivalently $\int_I t^{[\nu]}=1/(n+|\nu|)!$.

Lemma B.1 *The map* $\int_I\colon\widetilde{P}_I\to\mathbb{K}$ *factors through* P_I*.*

Proof We must show that $\int_I$ annihilates the ideal generated by $1-\sum_i t_i$, or equivalently that $\int_I t^{[\nu]} = \sum_i \int_I t_i t^{[\nu]}$. We have $t_i t^{[\nu]} = (1+\nu_i)t^{[\delta_i+\nu]}$, where $\delta_i\colon I \to \mathbb{N}$ is the Kronecker delta, so $|\delta_i+\nu| = 1+|\nu|$. We thus have

$$\begin{aligned}\sum_i \int_I t_i t^{[\nu]} &= \sum_i (1+\nu_i)\int_I t^{[\nu+\delta_i]} = \frac{1}{(n+1+|\nu|)!}\sum_i (1+\nu_i)\\ &= \frac{n+1+|\nu|}{(n+1+|\nu|)!} = \frac{1}{(n+|\nu|)!} = \int_I t^{[\nu]}.\end{aligned}$$ □

Lemma B.2 *If $\mathbb{K}=\mathbb{R}$ then $\int_I f$ is just the integral of f over the simplex $\Delta_I = \{x\colon I \to \mathbb{R}_+ \mid \sum_i x_i = 1\}$, with respect to the usual Lebesgue measure normalised so that $\mu(\Delta_I) = 1/(|I|-1)!$.*

Proof We may assume that $I=\{0,\dots,n\}$ and work by induction on n. We can identify Δ_n by projection with $\Delta'_I = \{x\in\mathbb{R}^n \mid \sum_{i=1}^n x_i \le 1\}$. Define

$$\int_I' f = \int_{\Delta'_I} f\left(1-\sum_{i=1}^n x_i, x_1, x_2, \dots, x_n\right) dx_1\cdots dx_n.$$

We will show that $\int_I t^{[\nu]} = \int_I' t^{[\nu]}$ for any multiindex ν with $\nu_0 = 0$. This will suffice because $P_I = \mathbb{R}[t_1,\dots,t_n]$. When $n=0$ the claim is trivial. When $n=1$, the claim says that $\int_{t=0}^1 t^{[n]} = 1/(1+n)!$, which is also trivial. This implies that $\int = \int'$ even on polynomials that are not in our preferred basis, which gives

$$\int_{t=0}^1 (1-t)^{[i]}t^{[j]} = 1/(1+i+j)!\,.$$

This will be useful later.

For $n>0$ we define a map $\phi\colon \Delta'_{n-1}\times[0,1]\to\Delta'_n$ by $\phi(t,s) = (st, 1-s)$. This is bijective away from a set of measure zero, and the Jacobian is s^{n-1}. Given a multiindex $\nu = (0,\nu_1,\dots,\nu_n)$, write ν' for the truncated sequence $(0,\nu_1,\dots,\nu_{n-1})$. We then have

$$\phi(t,s)^{[\nu]} = (1-s)^{[\nu_n]}(ts)^{[\nu']} = (1-s)^{[\nu_n]}s^{|\nu'|}t^{[\nu']}.$$

We may assume inductively that

$$\int'_{[n-1]} t^{[\nu']} = \frac{1}{(n-1+|\nu'|)!},$$

so
$$\begin{aligned}\int'_{[n]} t^{[\nu]} &= \int_{s=0}^{1}\int'_{[n-1]} \phi(x,s)^{[\nu]} s^{n-1}\,ds \\ &= \int_{s=0}^{1} (1-s)^{[\nu_n]} s^{n-1+|\nu'|} ds \int'_{[n-1]} x^{[\nu']} \\ &= \int_{s=0}^{1} (1-s)^{[\nu_n]} s^{[n-1+|\nu'|]} ds = \frac{1}{(1+\nu_n+n-1+|\nu'|)!} = \frac{1}{(n+|\nu|)!},\end{aligned}$$

as required.

Now $\int'_I f$ is certainly the integral of f over Δ_I with respect to some normalisation of Lebesgue measure. To determine the normalisation, note that $\int'_I 1 = \int_I t^{[0]} = 1/n!$ as required. □

Lemma B.3 *Take $I = [n] = \{0,1,\dots,n\}$, use the parameters $s_k = \sum_{j<k} t_j$ for $k = 1,\dots,n$. Consider a monomial $s^\nu = \prod_{k=1}^n s_k^{\nu_k}$. Put $\mu_k = \sum_{j\le k}(\nu_j+1)$ and $\mu = \prod_i \mu_i$. Then $\int_{[n]} s^\nu = 1/\mu$.*

Proof It will suffice to prove this when $\mathbb{K} = \mathbb{R}$, in which case we have $\int_I s^\nu = \int'_I s^\nu$. By a straightforward change of variables this becomes

$$\int_I s^\nu = \int_{0\le s_1\le\cdots\le s_n\le 1} s^\nu ds_1\cdots ds_n.$$

Suppose that the lemma holds for some n. Using the change of variables $s_i \mapsto r s_i$ (which has Jacobian r^n) we see that

$$\int_{0\le s_1\le\cdots\le s_n\le r} s^\nu\, ds_1\cdots ds_n = r^{n+\sum_i \nu_i}\int_{0\le s_1\le\cdots\le s_n\le r} s^\nu\, ds_1\cdots ds_n = r^{\mu_n}/\mu.$$

Now multiply by r^m and integrate from $r=0$ to $r=1$; the right hand side becomes $1/((m+1+\mu_n)\mu)$. Now change notation, replacing r by s_{n+1} and m by ν_{n+1}; this gives the case $n+1$ of the lemma. □

Lemma B.4 *Suppose that $f \in P_{[n]}$ and $g \in P_{[m]}$. Then*

$$\int_{[n]} f\cdot\int_{[m]} g = \sum_{(\alpha,\beta\in\Sigma(n,m))}\int_{[n+m]} \alpha^*(f)\beta^*(g).$$

(Here $\Sigma(n,m)$ is the set of (n,m)–shuffles, as in Definition A.11.)

Proof Put $\Delta''_n = \{s \in \mathbb{R}^n \mid 0 \leq s_1 \leq \cdots \leq s_n \leq 1\}$. As implicitly used in the proof of the previous lemma, there is a homeomorphism $\Delta_{[n]} \to \Delta''_n$ given by

$$t \mapsto \left(t_0, t_0 + t_1, \ldots, \sum_{i<n} t_i\right).$$

Now consider a shuffle $(\alpha, \beta) \in \Sigma(n, m)$, and the corresponding maps

$$\{1, \ldots, n\} \xrightarrow{\phi} \{1, \ldots, n+m\} \xleftarrow{\psi} \{1, \ldots, m\}$$

given by $\phi(j) = \min\{i \mid \alpha(i) = j\}$ and $\psi(k) = \min\{i \mid \beta(i) = k\}$. These give a map $(\alpha_*, \beta_*)\colon \Delta''_{n+m} \to \Delta''_n \times \Delta''_m$, with $\alpha_*(s)_i = s_{\phi(i)}$ and $\beta_*(s)_j = s_{\psi(j)}$. Let $X_{\alpha\beta}$ be the image of this map. It is standard that these are the top-dimensional simplices in a triangulation of $\Delta''_n \times \Delta''_m$, so

$$\int_{[n]} f \cdot \int_{[m]} g = \sum_{\alpha,\beta} \int_{X_{\alpha\beta}} f \otimes g.$$

Moreover, from the form of the maps α_* and β_* it is clear that the Jacobian of $(\alpha_*, \beta_*)\colon \Delta''_{n+m} \to \Delta''_n \times \Delta''_m$ is one. The lemma follows. □

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