Finiteness of mapping degrees and PSL(2, R)–volume on graph manifolds

PIERRE DERBEZ
SHICHENG WANG

For given closed orientable 3–manifolds M and N let $\mathcal{D}(M,N)$ be the set of mapping degrees from M to N. We address the problem: For which N is $\mathcal{D}(M,N)$ finite for all M? The answer is known for prime 3–manifolds unless the target is a nontrivial graph manifold. We prove that for each closed nontrivial graph manifold N, $\mathcal{D}(M,N)$ is finite for any graph manifold M.

The proof uses a recently developed standard form of maps between graph manifolds and the estimation of the $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume for a certain class of graph manifolds.

57M50; 51H20

1 Introduction

Let M and N be two closed oriented 3–dimensional manifolds. Let $\mathcal{D}(M,N)$ be the set of degrees of maps from M to N, that is,

$$\mathcal{D}(M,N)=\{d\in\mathbf{Z}\mid f\colon M\to N,\ \deg(f)=d\}.$$

According to J A Carlson and D Toledo [3], M Gromov considered determining the set $\mathcal{D}(M,N)$ to be a fundamental problem in topology. Indeed the supremum of absolute values of degrees in $\mathcal{D}(M,N)$ was addressed by J Milnor and W Thurston in the 1970's [11]. A basic property of $\mathcal{D}(M,N)$ is reflected in the following:

Question 1 (See also Reznikov [12, Problem A] and Wang [15, Question 1.3].) For which closed orientable 3–manifold N is the set $\mathcal{D}(M,N)$ finite for all closed orientable 3–manifolds M?

This question can be interpreted as a way to detect some new rigidity properties of the geometry and the topology of a manifold. More precisely, when M is fixed, then one can expect that if the geometry-topology of a manifold N is complicated, then the possible degree of maps $f\colon M\to N$ is essentially controlled by the data of N. For geometric 3–manifolds (ie 3–manifolds which admits a locally homogeneous complete Riemannian metric) the answer to this question is summarized in the following:

Published: 2 September 2009 DOI: 10.2140/agt.2009.9.1727

Theorem 1.1 (Thurston [13]; Brooks and Goldman [1; 2]; Wang [15]) *Let N denote a closed orientable geometric 3–manifold.*

(i) *If N supports the hyperbolic or the $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ geometry, then $\mathcal{D}(M,N)$ is finite for any M.*

(ii) *If N admits one of the six remainding geometries, $\mathbf{S}^3$, $\mathbf{S}^2\times\mathbf{R}$, Nil, $\mathbf{R}^3$, $\mathbf{H}^2\times\mathbf{R}$ or Sol then $\mathcal{D}(N,N)$ is infinite.*

To study the set $\mathcal{D}(M,N)$ we need to introduce a special kind of 3–manifold invariant. More precisely, we say that a nonnegative 3–manifold invariant ω has the *degree property* or simply *Property D*, if for any map $f\colon M\to N$, $\omega(M)\geq|\deg(f)|\omega(N)$. We say ω has the *covering property* or simply *Property C*, if for any covering $p\colon M\to N$, $\omega(M)=|\deg(p)|\omega(N)$. The invariants with Property D are important in studying Question 1 due to the following fact (see Lemma 3.1):

Fact ($\star$) *If ω has Property D and if N admits a finite covering $\widetilde{N}$ such that $\omega(\widetilde{N})\neq 0$ then the set $\mathcal{D}(M,N)$ is finite for all M.*

When N is hyperbolic, the finiteness of the set $\mathcal{D}(M,N)$ is essentially controlled by the volume associated to the Riemannian metric with constant negative sectional curvature which satisfies Property D. When N admits a $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ geometry $\mathcal{D}(M,N)$ is essentially controlled by the $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume SV introduced by Brooks and Goldman [2; 1]: it satisfies Property D and it is nonzero for Seifert manifolds supporting a $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ geometry.

To study Question 1 for more general manifolds, M Gromov [7] introduced the simplicial volume $\|N\|$ of a manifold N. This invariant always satisfies Property D. For example, using the simplicial volume and the work of Connell and Farb [4], Lafont and Schmidt [9] generalized point (i) of Theorem 1.1 when the target manifold N is a closed locally symmetric space of noncompact type. However, closed locally symmetric manifolds are a special class of complete locally homogeneous manifolds and thus Question 1 is still open for nongeometric manifolds with zero simplicial volume.

In this paper we focus on closed 3–manifolds. Recall that according to the Perelman Geometrization Theorem, 3–manifolds with zero Gromov simplicial volume are precisely graph manifolds. We call a 3–manifold covered by either a torus bundle or a Seifert manifold a *trivial graph manifold*. Hence for prime 3–manifolds, Question 1 is reduced to:

Question 2 Suppose N is a nontrivial graph manifold. Is $\mathcal{D}(M,N)$ finite for all closed orientable 3–manifolds M?

The main difficulty in studying Question 2 for a nontrivial graph manifold N is to find a 3–manifold invariant satisfying Property D which does not vanish on N. Based on Fact $(\star)$, it is natural to ask:

Question 3 Let N be a closed orientable nontrivial graph manifold. For some finite covering $\tilde{N}$ of N, does $\mathrm{SV}(\tilde{N}) \neq 0$?

The $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume is rather strange and very little is known about it. It can be either zero or nonzero for hyperbolic 3–manifolds [2]; whether it has Property C is still unclear, and it was not addressed for nongeometric 3–manifolds since it was introduced more than 20 years ago.

A main result of this paper is a partial answer of Question 3: we verify that for a family of nongeometric graph manifolds N, they do have finite cover $\tilde{N}$ with $\mathrm{SV}(\tilde{N}) \neq 0$ (Proposition 4.1). Such a partial answer, combined with the standard form of nonzero degree maps developed by Derbez [6], enables us to solve Question 2 when we restrict to graph manifolds.

Theorem 1.2 *For any given closed prime nontrivial graph manifold* N, $\mathcal{D}(M,N)$ *is finite for any graph manifold* M.

Remark Some facts related to Theorem 1.2 were known before: $\mathcal{D}(N,N)$ is finite for any prime nontrivial graph manifold N (see Wang [14] and also Derbez [5]). The covering degrees are uniquely determined by the graph manifolds involved (see Yu and Wang [17]).

This paper is organized as follows.

In Section 2 we define the objects which will be used in the paper: For graph manifolds, we will define their coordinates and gluing matrices, canonical framings, the standard forms of nonzero degree maps, the absolute Euler number and the absolute volume. We also recall $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume and its basic properties.

In Section 3 we state and prove some results on coverings of graph manifolds which will be used in the paper.

Section 4 is devoted to the proof of Proposition 4.1. The strategy is to use a finite sequence of coverings to get a very "large" and "symmetric" covering space which allows some free action of a finite cyclic group so that the quotient can be sent onto a 3–manifold supporting the $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ geometry via a nonzero degree map.

In Section 5 we prove Theorem 1.2. The strategy is to use the standard form of nonzero degree maps between graph manifolds to show that one can reduce the problem to the case where the target is a graph manifold satisfying the hypothesis of Proposition 4.1.

Acknowledgements The first author is very grateful to the School of Mathematical Sciences of the Peking University for hospitality and support during the writing of this paper. The second author wishes to thank the Department of Mathematics, University Aix-Marseille I for offering him an invited professor position in 2007, which made the cooperation effective. He is also supported by grants No. 10631060 of the NNSFC and No. 5171042-055 of the MEC.

2 Notations and known results

From now on all 3–manifolds are irreducible and oriented, and all graph manifolds are nontrivial.

Suppose F (resp. P) is a properly embedded surface (resp. an embedded 3–manifold) in a 3–manifold M. We use $M \setminus F$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along F (resp. removing $\mathrm{int} P$, the interior of P).

2.1 Coordinated graph manifolds and gluing matrices

Let N be a graph manifold. Denote by $\mathcal{T}_N$ the family of JSJ tori of N, by N^* the set $N \setminus \mathcal{T}_N = \{\Sigma_1, \dots, \Sigma_n\}$ of the JSJ pieces of N, by $\tau\colon \partial N^* \to \partial N^*$ the associated sewing involution defined by Jaco and Shalen [8].

A *dual graph* of N, denoted by Γ_N, is given as follows: each vertex represents a JSJ piece of N; each edge represents a JSJ torus of N; an edge e connects two vertices v_1 and v_2 (may be $v_1 = v_2$) if and only if the corresponding JSJ torus is shared by the corresponding JSJ pieces.

Call a dual graph Γ_N *directed* if each edge of Γ_N is directed, in other words, is endowed with an arrow. Once Γ_N is directed, the sewing involution τ becomes a well defined map, still denoted by $\tau\colon \partial N^* \to \partial N^*$.

Suppose N^* contains no pieces homeomorphic to $I(K)$, the twisted I–bundle over the Klein bottle.

Let Σ be an oriented Seifert manifold which admits a unique Seifert fibration, up to isotopy, and $\partial\Sigma \neq \varnothing$. Denote by h the homotopy class of the regular fiber of Σ, by $\mathcal{O}$ the base 2–orbifold of Σ and by Σ^0 the space obtained from Σ after removing the singular fibers of Σ. Then Σ^0 is a $\mathbf{S}^1$–bundle over a surface $\mathcal{O}^0$ obtained from $\mathcal{O}$ after removing the exceptional points. Then there exists a cross section $s\colon \mathcal{O}^0 \to \Sigma^0$. Call Σ is *coordinated*, if

(1) such a section $s\colon \mathcal{O}^0 \to \Sigma^0$ is chosen,

(2) both $\partial\mathcal{O}^0$ and h are oriented so that their product orientation is matched with the orientation of $\partial\Sigma$ induced by that of Σ.

Once Σ is coordinated, then the orientation on $\partial\mathcal{O}^0$ and the oriented fiber h give a basis of $H_1(T;\mathbf{Z})$ for each component T of $\partial\Sigma$. We also say that Σ is endowed with a (s,h)–basis.

Since N^* has no $I(K)$–components then each component Σ_i of N^* admits a unique Seifert fibration, up to isotopy. Moreover each component Σ_i has the orientation induced from N. Call N is *coordinated*, if each component Σ_i of N^* is coordinated and Γ_N is directed.

Once N is coordinated, then each torus T in $\mathcal{T}_N$ is associated with a unique 2×2–matrix A_T provided by the gluing map $\tau|\colon T_-(s_-,h_-)\to T_+(s_+,h_+)$: where T_-,T_+ are the two components in ∂N^* provided by T, with basis (s_-,h_-) and (s_+,h_+) respectively, and

$$\tau(s_-,h_-)=(s_+,h_+)A_T.$$

Call $\{A_T, T\in\mathcal{T}\}$ the *gluing matrices.*

2.2 Canonical framings and canonical submanifold

Let Σ denote an orientable Seifert manifold with regular fiber h. A *framing* α of Σ is to assign a simple closed essential curve not homotopic to the regular fiber of Σ, for each component T of $\partial\Sigma$. Denote by $\Sigma(\alpha)$ the closed Seifert 3–manifold obtained from Σ after Dehn fillings along the family α and denote by $\pi_\Sigma\colon\Sigma\to\Sigma(\alpha)$ the natural quotient map. Let $p\colon\widetilde{\Sigma}\to\Sigma$ be a finite covering. Assume that Σ and $\widetilde{\Sigma}$ are endowed with a framing α and $\widetilde{\alpha}$. Then we say that $(\widetilde{\Sigma},\widetilde{\alpha})$ covers (Σ,α) if each component of $\widetilde{\alpha}$ is a component of $p^{-1}(\alpha)$. In this case, the map $p\colon(\widetilde{\Sigma},\widetilde{\alpha})\to(\Sigma,\alpha)$ extends to a map $\widehat{p}\colon\widetilde{\Sigma}(\widetilde{\alpha})\to\Sigma(\alpha)$ and the Euler number of $\Sigma(\alpha)$ is nonzero if and only if the Euler number of $\widetilde{\Sigma}(\widetilde{\alpha})$ is nonzero [10]. When N contains no $I(K)$–component in its JSJ decomposition, each Seifert piece Σ of N^* is endowed with a *canonical framing* α_Σ given by the regular fiber of the Seifert pieces of N^* adjacent to Σ. Denote by $\widehat{\Sigma}$ the space $\Sigma(\alpha_\Sigma)$. By minimality of JSJ decomposition, $\widehat{\Sigma}$ admits a unique Seifert fibration extending that of Σ.

Call a submanifold L of a graph manifold N *canonical* if L is a union of some components of $N\setminus\mathcal{T}$, where $\mathcal{T}$ is subfamily of $\mathcal{T}_N$. Similarly call $\alpha_L=\{t_U\subset U\}$ where t_U is the regular fiber of the Seifert piece adjacent to L along the component U, when U runs over the components of ∂L, the canonical framing of L, and denote by $\widehat{L}$ the closed graph manifold obtained from L after Dehn fillings along the family α_L. From the definition we have:

Lemma 2.1 *For a given closed graph manifold M, there are only finitely many canonical framed canonical submanifolds (L,α_L), and thus only finitely many $\widehat{L}$.*

2.3 Standard forms of nonzero degree maps

We recall here two results which are proved in [6] in a more general case. The first result is related to the standard forms of nonzero degree maps.

Proposition 2.2 [6, Lemma 3.4] *For a given closed graph manifold M, there is a finite set $\mathcal{H} = \{M_1, \dots, M_k\}$ of closed graph manifolds satisfying the following property: for any nonzero degree map $g\colon M \to N$ into a closed nontrivial graph manifold N without $I(K)$ piece in N^*, there exists some M_i in $\mathcal{H}$ and a map $f\colon M_i \to N$ such that*

(i) $\deg(f) = \deg(g)$,

(ii) *for each piece Q in N^*, $f^{-1}(Q)$ is a canonical submanifold of M_i.*

The following technical "mapping lemma" will be also useful:

Lemma 2.3 [6, Lemma 4.3] *Suppose $f\colon M \to N$ is a map between closed graph manifolds and N^* contains no $I(K)$ piece. Let S and S' be two components of M^* which are adjacent in M along a subfamily $\mathcal{T}$ of $\mathcal{T}_M$ and satisfy*

(i) $f(S') \subset \mathrm{int}(\Sigma')$ *for some piece Σ' of N^*,*

(ii) $f_*([h_S]) \neq 1$, *where t_S is the regular fiber of S.*

Then there exists a piece Σ of N^* and a homotopy of f supported in a regular neighborhood of S such that $f(S) \subset \mathrm{int}(\Sigma)$. Moreover if $f(h_S)$ is not homotopic to a power of the regular fiber of Σ, then one can choose $\Sigma = \Sigma'$.

[6, Lemma 4.3] was stated for Haken manifolds. Since here we consider only nontrivial graph manifolds instead of Haken manifolds, then we can state [6, Lemma 4.3] in term of the JSJ pieces of N instead of in term of the characteristic Seifert pair of N.

2.4 $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume, absolute volume and absolute Euler number

$\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–volume SV was introduced by Brooks and Goldman [2; 1]. (It is also considered as a special case of volumes of representations; see Reznikov [12] and Wang and Zhou [16]). Two basic properties of SV are reflected in the following:

Lemma 2.4 (i) SV *has Property D* [2].

(ii) *If N supports the $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$ geometry, ie, N admits a Seifert fibration with nonzero Euler number $e(N)$ and whose base 2–orbifold $\mathcal{O}_N$ has a negative Euler characteristic, then* [1]

$$\mathrm{SV}(N) = \left|\frac{\chi^2(\mathcal{O}_N)}{e(N)}\right|.$$

When N is a closed graph manifold with no $I(K)$ piece in N^*, using the notation introduced in Section 2.2, one can define the so-called *absolute volume* $|\,\mathrm{SV}\,|$ by setting

$$|\,\mathrm{SV}\,|(N) = \sum_{\Sigma\in N^*} \mathrm{SV}(\widehat{\Sigma}).$$

In the same way one can define the *absolute Euler number* of N by setting

$$|e|(N) = \sum_{\Sigma\in N^*} |e(\widehat{\Sigma})|.$$

In Section 3.3 we will study the relations between $|e|(N)$ and $|\,\mathrm{SV}\,|(N)$ (see Lemma 3.6).

3 Reduction of complexity via coverings

In this section we state some results on finite coverings of surfaces and 3–manifolds which will be used in the proofs of Proposition 4.1 and Theorem 1.2.

3.1 Two general statements

The first result says that to prove the finiteness of the set $\mathcal{D}(M,N)$ one can replace N by a finite covering of it.

Lemma 3.1 (1) *Let $p\colon N' \to N$ be a finite covering of a closed oriented 3–manifold N. If $\mathcal{D}(P,N')$ is finite for any closed 3–manifold P, then $\mathcal{D}(M,N)$ is finite for any closed 3–manifold M.*

(2) *Let $p\colon N' \to N$ be a finite covering of a closed graph manifold N. If $\mathcal{D}(P,N')$ is finite for any closed graph manifold P, then $\mathcal{D}(M,N)$ is finite for any closed graph manifold M.*

Proof (1) For each nonzero degree map $f\colon M \to N$, let $M(f)$ be the connected covering space of M corresponding to the subgroup $f_*^{-1}(p_*(\pi_1 N'))$ of $\pi_1 M$ which we denote by $r\colon M(f)\to M$. Let $f'\colon M(f)\to N'$ be a lift of f, then $p\circ f' = f\circ r$. We claim that the set

$$\mathcal{C} = \{M(f),\ \text{when } f \text{ runs over the nonzero degree maps from } M \text{ to } N\}$$

is finite. To see this, first note that the index of $f_*^{-1}(p_*(\pi_1 N'))$ in $\pi_1 M$ is bounded by the index of $p_*(\pi_1 N')$ in $\pi_1 N$. Indeed, the homomorphism $f_*\colon \pi_1 M \to \pi_1 N$ descends through an injective map

$$\bar{f}_*\colon \frac{\pi_1 M}{f_*^{-1}(p_*(\pi_1 N'))} \to \frac{\pi_1 N}{p_*(\pi_1 N')}.$$

Since $\pi_1 M$ contains at most finitely many subgroups of a bounded index, it follows that $M(f)$ has only finitely many choices which proves that the set $\mathcal{C}$ is finite. By the construction we have

$$\deg(f) = \frac{\deg(p)}{\deg(r)}\deg(f').$$

By the finiteness of the set $\mathcal{C}$ and assumption on N', the set $\{\deg(f') \mid f\colon M'\to N',\ M'\in\mathcal{C}\}$ is finite. Clearly $\deg(r)$ have only finitely many choices, so the lemma is proved.

(2) If M and N are graph manifolds, then all manifolds $M(f)$, N' in the proof of (1) are graph manifolds. Clearly (2) follows. □

Lemma 3.2 *Let N be a closed 3–manifold with nontrivial JSJ decomposition. Then there exists a 2–fold covering $\widetilde{N}$ of N such that each JSJ–torus of $\widetilde{N}$ is shared by two different pieces of $\widetilde{N}^*$.*

Proof Let $\{T_1,\dots,T_k\}$ be the family of JSJ tori of N such that each T_i is shared by the same piece of N^*. Let $e_1,\dots,e_k$ be the corresponding edges in Γ_N. Then $e_1,\dots,e_k$ are the edges of Γ_N with the two ends of each e_i being at the same vertex. Clearly $H_1(\Gamma_N;\mathbf{Z}) = \langle e_1\rangle\oplus\cdots\oplus\langle e_k\rangle\oplus G$.

Let $r\colon N\to\Gamma_N$ be the retraction. Consider the following epimorphism

$$\phi\colon H_1(N,\mathbf{Z}) \xrightarrow{r_*} H_1(\Gamma_N;\mathbf{Z}) \xrightarrow{q} \langle e_1\rangle\oplus\cdots\oplus\langle e_k\rangle \xrightarrow{\lambda} \mathbf{Z}/2\mathbf{Z}$$

where r_* is induced by r, q is the projection, and λ is defined by $\lambda([e_i]) = \bar{1}$ for $i=1,\dots,k$. Then the double covering $\widetilde{N}$ of N corresponding to ϕ satisfies the conclusion of the lemma, since the double covering of Γ_N corresponding to $\lambda\circ q$, which is the dual graph of $\widetilde{N}$, contains no edge with two ends being at the same vertex. See Figure 1 for the local picture. □

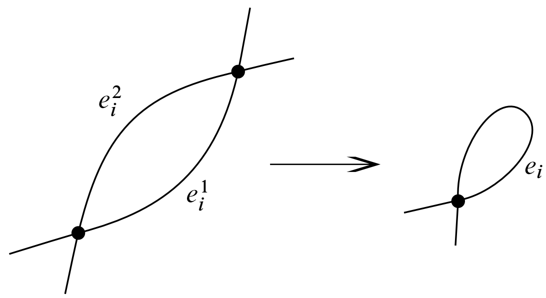

Figure 1

3.2 Separable and characteristic coverings

Let N be a closed graph manifold without $I(K)$ JSJ piece. Let $\mathcal{T}$ be a union of tori and let m be a positive integer. Call a covering $p\colon \widetilde{\mathcal{T}} \to \mathcal{T}$ *m–characteristic* if for each component T of $\mathcal{T}$ and for each component $\widetilde{T}$ of $\widetilde{\mathcal{T}}$ over T, the restriction $p|\colon \widetilde{T} \to T$ is the covering map associated to the characteristic subgroup of index $m \times m$ in $\pi_1 T$. Call a finite covering $\widetilde{N} \to N$ of a graph manifold N m–characteristic if its restriction to $\mathcal{T}_{\widetilde{N}} \to \mathcal{T}_N$ is *m–characteristic*.

Next we define the *separable coverings*. Let Σ be a component of N^* with base 2–orbifold $\mathcal{O}$. Let Σ^0, $\mathcal{O}^0$, and $s\colon \mathcal{O}^0 \to \Sigma^0$ are be given as in Section 2.1. Let $p\colon \widetilde{\Sigma} \to \Sigma$ denote a finite covering. Recall that p is a fiber preserving map.

Recall that the *vertical degree* of p is the integer d_v such that $p_*(\widetilde{h}) = h^{d_v}$, where h and $\widetilde{h}$ denote the homotopy class of the regular fiber in Σ and $\widetilde{\Sigma}$, and the *horizontal degree* d_h is the degree of the induced branched covering $\overline{p}\colon \widetilde{\mathcal{O}} \to \mathcal{O}$, where $\widetilde{\mathcal{O}}$ denotes the base of the bundle $\widetilde{\Sigma}$. We have $\deg(p) = d_v \times d_h$.

On the other hand, p induces a finite covering $p|\colon \widetilde{\Sigma}^0 = p^{-1}(\Sigma^0) \to \Sigma^0$ and a covering $p|_b\colon \widetilde{\mathcal{F}}^0 \to \mathcal{O}^0$, with $\widetilde{\mathcal{F}}^0$ connected. More precisely, $p|_b$ corresponds to the subgroup $s_*^{-1}((p|)_*(\pi_1 \widetilde{\Sigma}^0))$. Note that p and $p|$ have the same degree, same vertical degree and same horizontal degree. If $\deg(p|_b) = d_h$, then we say that the covering p is *separable*. The following result provides two classes of separable coverings which will be used later.

Lemma 3.3 *Let $p\colon \widetilde{\Sigma} \to \Sigma$ be an oriented Seifert manifold finite covering.*

(i) *If p has fiber degree one, then p is a separable covering.*

(ii) *If $\Sigma = F \times \mathbf{S}^1$ and p is a regular covering corresponding to an epimorphism $\phi\colon \pi_1 \Sigma = \pi_1 F \times \mathbf{Z} \to G = G_1 \times G_2$ satisfying $\phi(\pi_1 F \times \{1\}) = G_1$ and $\phi(\{1\} \times \mathbf{Z}) = G_2$ then p is separable.*

Proof Using the same notation as above it is easy to see that the map $p|_b\colon \widetilde{\mathcal{F}}^0 \to \mathcal{O}^0$ factors through covering maps $q\colon \widetilde{\mathcal{F}}^0 \to \widetilde{\mathcal{O}}^0$ and $\overline{p}\colon \widetilde{\mathcal{O}}^0 \to \mathcal{O}^0$ where $\widetilde{\mathcal{O}}^0$ denotes the base of the bundle $\widetilde{\Sigma}^0$. Then we get

$$\deg(p|_b) = d_h \times \deg(q) = \deg(p) \times \deg(q)$$

since p has vertical degree one. On the other hand, since $\deg(p|_b) \leq \deg(p)$ then $\deg(q) = 1$. This proves (i).

If Σ is homeomorphic to a product $F \times \mathbf{S}^1$ then we have the following commutative diagram

$$\begin{tikzcd}
\widetilde{F} \arrow[r, "\widetilde{s}"] \arrow[d, "p|_b"'] & \widetilde{\Sigma} \arrow[d, "p"] \\
F \arrow[r, "s"] & \Sigma
\end{tikzcd}$$

where $\widetilde{F}$ is connected. Since $\phi(\pi_1 F \times \{1\}) = G_1$ then $p^{-1}(s(F))$ has $|G_2|$ components and thus $\deg(p) = \deg(p|_b) \times |G_2|$. Since $\deg(p) = |G_1| \times |G_2|$ then $\deg(p|_b) = |G_1| = d_h$. This proves (ii). □

3.3 Lifting of coordinates and gluing matrices

From now on we assume the graph manifold N is coordinated. Let $p\colon \widetilde{N} \to N$ be a finite covering of graph manifolds. Then obviously $\Gamma_{\widetilde{N}}$ can be directed in a unique way such that the induced map $p_\#\colon \Gamma_{\widetilde{N}} \to \Gamma_N$ preserves the directions of the edges. Below we also assume that $\Gamma_{\widetilde{N}}$ is directed in such a way.

Let $p\colon \widetilde{N} \to N$ be a finite covering of graph manifolds. Call p is *separable* if the restriction $p|\colon \widetilde{\Sigma} \to \Sigma$ on connected Seifert pieces is separable for all possible $\widetilde{\Sigma}$ and Σ. Call a coordinate on $\widetilde{N}$ a *lift* of the coordinate of N, if for each possible covering $p|\colon \widetilde{\Sigma} \to \Sigma$ on connected Seifert pieces, the (s,h)–basis of $\widetilde{\Sigma}$ is lifted from the (s,h)–basis of Σ.

Lemma 3.4 (i) *Let $p\colon \widetilde{N} \to N$ be a separable finite covering of graph manifolds. Then the coordinate of N can be lifted on $\widetilde{N}$.*

(ii) *Moreover, if the covering p is characteristic, then for each component T of $\mathcal{T}_N$ and for each component $\widetilde{T}$ over T we have $A_T = A_{\widetilde{T}}$, where the coordinate of $\widetilde{N}$ is lifted from N.*

Proof To prove (i), one need only to show that for a separable finite covering $p\colon \widetilde{\Sigma} \to \Sigma$ of a connected Seifert piece, then any (s,h)–basis of Σ lifts to a (s,h)–basis of $\widetilde{\Sigma}$.

Using the same notation as in the proof of Lemma 3.3 we have

$$\deg(p|_b) = d_h \times \deg(q).$$

Since we assume $\deg(p|_b) = d_h$, we have $\deg(q) = 1$ and thus $\widetilde{s}$ is a cross section. This proves (i).

Once N is coordinated, then each torus T in $\mathcal{T}_N$ is associated with a unique 2×2–matrix A_T provided by the gluing map $\tau|\colon T_-(s_-, h_-) \to T_+(s_+, h_+)$ such that $\tau(s_-, h_-) = (s_+, h_+)A_T$.

Similarly with lifted coordinate on $\widetilde{N}$ we have $\widetilde{\tau}|\colon \widetilde{T}_-(\widetilde{s}_-, \widetilde{h}_-) \to \widetilde{T}_+(\widetilde{s}_+, \widetilde{h}_+)$ and $\tau(\widetilde{s}_-, \widetilde{h}_-) = (\widetilde{s}_+, \widetilde{h}_+)A_{\widetilde{T}}$.

Since the coordinate of $\widetilde{N}$ are lifted from N, and p is m–characteristic for some m, we have the following commutative diagram:

$$\begin{tikzcd}
(\widetilde{s}_-, \widetilde{h}_-) \arrow[r, "A_{\widetilde{T}}"] \arrow[d, "\times m"'] & (\widetilde{s}_+, \widetilde{h}_+) \arrow[d, "\times m"] \\
(s_-, h_-) \arrow[r, "A_T"] & (s_+, h_+)
\end{tikzcd}$$

Then one verifies directly that $A_{\widetilde{T}} = A_T$. This proves (ii). □

3.4 The absolute volume and the absolute Euler number

We end this section with a result (see Lemma 3.6) which states the relation between the absolute volume and the absolute Euler number of a graph manifold. First we begin with a technical result.

Lemma 3.5 *Suppose N is a closed graph manifold without $I(K)$ JSJ piece.*

(i) *For any finite covering $\widetilde{N} \to N$, $|e|(\widetilde{N}) = 0$ if and only if $|e|(N) = 0$.*

(ii) *There is a finite covering $p\colon \widetilde{N} \to N$ which is separable and characteristic, and each Seifert piece of $\widetilde{N}$ is the product of a surface of genus at least 2 and the circle. Moreover $\widetilde{N}$ may be chosen so that $\Gamma_{\widetilde{N}}$ has two vertices if Γ_N has two vertices.*

Proof (i) follows from the definition and [10, Proposition 2.3].

(ii) It has been proved in [10, Proposition 4.4], that there is a characteristic finite covering $p\colon \widetilde{N} \to N$ whose each piece is the product of a surface and the circle. By checking the proof, it is easy to see that the condition "genus at least 2" can be satisfied;

moreover the restriction $p|\colon \widetilde{\Sigma} \to \Sigma$ on connected JSJ pieces is a composition of separable coverings described in Lemma 3.3, which is still separable.

If moreover Γ_N has exactly two vertices Σ_1 and Σ_2, then for $i = 1, 2$, denote by $p_i\colon \Sigma_i' \to \Sigma_i$ the m–characteristic separable finite covering such that Σ_i' is the product of a surface of genus at least 2 and the circle. There exists a 1–characteristic finite covering $q_i\colon \widetilde{\Sigma}_i \to \Sigma_i'$ such that $\partial\widetilde{\Sigma}_1$ and $\partial\widetilde{\Sigma}_2$ have the same number of components. Next one can glue $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ by the lift of the sewing involution of N to get a characteristic and separable finite covering $p\colon \widetilde{N} \to N$ whose dual graph has two vertices. This completes the proof of the lemma. □

Lemma 3.6 *Let N be a closed graph manifold without $I(K)$ JSJ pieces.*

(i) *If $|e|(N) \neq 0$ then N admits a finite covering $\widetilde{N}$ with $|\operatorname{SV}|(\widetilde{N}) \neq 0$.*

(ii) *If $|e|(N) = 0$ then N admits a finite covering $\widetilde{N}$ which can be coordinated such that each gluing matrix is in the form*

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof By Lemma 3.5 (ii), let $p\colon \widetilde{N} \to N$ be a finite covering which is separable and characteristic and each piece of $\widetilde{N}^*$ is a product $F \times S^1$ with $g(F) \geq 2$.

By Lemma 3.5 (i), $|e|(N) \neq 0$ implies $|e|(\widetilde{N}) \neq 0$. By definition of $|e|$, $e(\widehat{\Sigma}) \neq 0$ for some $\Sigma = F \times S^1 \in \widetilde{N}^*$. Since $g(F) \geq 2$, $\operatorname{SV}(\widehat{\Sigma}) \neq 0$, and hence $|\operatorname{SV}|(\widetilde{N}) \neq 0$ by definition in Section 2.4. This proves (i).

Denote by $\Sigma_1, \dots, \Sigma_n$ the components of N^*. For each $i = 1, \dots, n$, denote by (Σ_i, α_i) the Seifert piece Σ_i of N^* endowed with its canonical framing. Since $e(N) = 0$ then $e(\Sigma_i(\alpha_i)) = e(\widehat{\Sigma}_i) = 0$ and thus there exists a finite covering of $\widehat{\Sigma}_i$, with fiber degree one, homeomorphic to a product. By pulling back this covering via the quotient map $\pi\colon \Sigma_i \to \widehat{\Sigma}_i$ we get a covering $\widetilde{\Sigma}_i$ of Σ_i such that the framing $(\widetilde{\Sigma}_i, \widetilde{\alpha}_i)$ satisfies the following condition: there exists a properly embedded incompressible surface F_i in $\widetilde{\Sigma}_i$ such that $\widetilde{\Sigma}_i \simeq F_i \times \mathbf{S}^1$ and $\partial F_i = \widetilde{\alpha}_i$.

Suppose T is a component of $\partial\Sigma_i$ and T' is a component of $\partial\Sigma_j$ such that T is identified to T' then the sewing involution $\tau|T\colon T \to T'$ lifts to a sewing involution $\widetilde{\tau}\colon \widetilde{T} \to \widetilde{T}'$, where $\widetilde{T}$, resp. $\widetilde{T}'$, denotes a component of $\partial\widetilde{\Sigma}_i$, resp. a component of $\partial\widetilde{\Sigma}_j$, over T, resp. T'. Indeed by our construction the induced coverings $\widetilde{T} \to T$ and $\widetilde{T}' \to T'$ correspond exactly to the subgroup of $\pi_1 T$, resp. of $\pi_1 T'$, generated by h and h', where h is the fiber of Σ_i represented in T and h' is the fiber of Σ_j represented in T', hence the gluing map lifts by the lifting criterion.

Denote by η_i the degree of the covering map $\widetilde{\Sigma}_i \to \Sigma_i$. Let

$$\eta = \mathrm{lcm}\{\eta_1, \dots, \eta_n\}.$$

For each $i = 1, \dots, n$, take $t_i = \eta/\eta_i$ copies of $\widetilde{\Sigma}_i$ and glue the components of

$$\coprod_{i=1}^{n} (t_i \text{ copies of } \widetilde{\Sigma}_i)$$

together via lifts of the sewing involution τ of N to get a separable finite covering $p\colon \widetilde{N} \to N$. By coordinating each piece $\widetilde{\Sigma}_i$ of $\widetilde{N}^*$ with such a section F_i and its regular fiber, $\widetilde{N}$ is coordinated. Clearly each component of ∂F_i is identified with the regular fiber of its adjacent piece and vice versa. Therefore each gluing matrix should be in the form

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Since the determinant should be -1, therefore the gluing matrix is in the form

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This proves point (ii). □

4 $\widetilde{\mathrm{PSL}}(2, \mathbf{R})$–volume of graph manifolds

Let N be a closed graph manifold which consists of two JSJ pieces Σ_1 and Σ_2 and n JSJ tori $\{T_1, \dots, T_n\}$, moreover no Σ_i is $I(K)$ and each T_i is shared by both Σ_1 and Σ_2. Call such a manifold *n–multiple edges graph manifold*, whose dual graph Γ_N is shown in Figure 2. We assume Γ_N is also directed as in Figure 2. In this section we use A_i for A_{T_i} for short.

Proposition 4.1 *Let N be a n–multiple edges graph manifold which is coordinated. Assume that the gluing matrices of N satisfy the condition $A_1 = \pm A_2 = \cdots = \pm A_n$. Then N admits a finite covering space $\widetilde{N}$ such that* $\mathrm{SV}(\widetilde{N}) \neq 0$.

Corollary 4.2 *Suppose N is a closed graph manifold whose dual graph has two vertices and one edge. Then $\mathcal{D}(M, N)$ is finite for any 3–manifold M.*

Proof We may suppose that N contains no $I(K)$ piece. Otherwise N is doubly covered by a nontrivial graph manifold which contains no $I(K)$ piece and whose dual graph still has two vertices and one edge (since we assume that N is nontrivial graph manifold). In any case N has a finite cover $\widetilde{N}$ such that $\mathrm{SV}(\widetilde{N}) \neq 0$ by Proposition 4.1. Then by Lemmas 2.4 and 3.1, $\mathcal{D}(M, N)$ is finite for any 3–manifold M. □

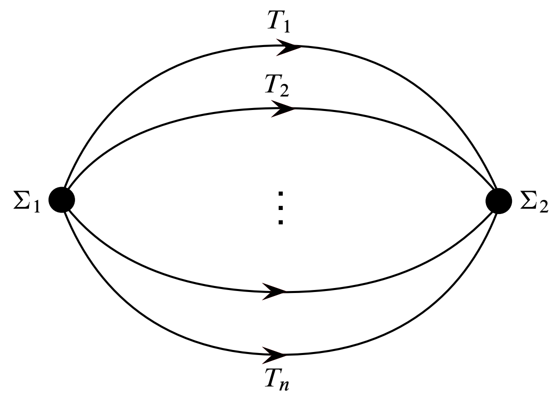

Figure 2: Multiple edges graph manifold

The proof of Proposition 4.1 follows from the sequence of lemmas below.

Let N be a n–multiple edges coordinated graph manifold. We define Property I for N as follows:

Property *I* (1) The JSJ piece Σ_i is homeomorphic to a product $F_i \times \mathbf{S}^1$ where F_i is an oriented surface with genus ≥ 2, for $i = 1, 2$.

(2) $A_1 = A_2 = \cdots = A_n$.

Lemma 4.3 *Let N be a n–multiple edges graph manifold satisfying the assumption of Proposition 4.1. Then there exist separable and characteristic finite coverings $p_1\colon N_1 \to N$ and $p_2\colon N_1 \to N_2$ such that N_2 satisfies Property I.*

Proof By Lemma 3.5 (ii) and Lemma 3.4, we may assume that N is a n–multiple edges graph manifold satisfying the assumption of Proposition 4.1, and moreover Σ_i is homeomorphic to a product $F_i \times \mathbf{S}^1$ where F_i is an oriented surface with genus ≥ 2.

We may assume that $A_1 = \cdots = A_k = -A$ and $A_{k+1} = \cdots = A_n = A$, $0 < k < n$, shown as in the right of Figure 3.

Denote by $c_{i,j}$ the loops of Γ_N corresponding to the "composition" $T_i \,.\, (-T_j)$, note that here T_i represents an oriented edge. Then $c_{i,k+1}$ for $i = 1, \dots, k$ and $c_{j,n}$ for $j = k+1, \dots, n-1$ form a basis of $H_1(\Gamma_N)$ and we have

$$H_1(\Gamma_N) = \Big(\bigoplus_{i=1}^{k} \langle c_{i,k+1}\rangle\Big) \oplus \Big(\bigoplus_{j=k+1}^{n-1} \langle c_{j,n}\rangle\Big).$$

Next we define an epimorphism

$$\phi\colon H_1(N,Z) \xrightarrow{r_*} H_1(\Gamma_N;Z) \xrightarrow{q} \bigoplus_{i=1}^{k}\langle c_{i,k+1}\rangle \xrightarrow{\lambda} \mathbf{Z}/2\mathbf{Z}$$

where r_* is induced by the retraction $r\colon N \to \Gamma_N$, q is the projection and λ is defined by $\lambda(c_{i,k+1}) = \overline{1}$ for any $i \in \{1,\dots,k\}$. Denote by $p_1\colon N_1 \to N$ the 2–fold covering corresponding to ϕ, and by μ the deck transformation of this covering.

It is easy to see that this covering is separable and 1–characteristic. Moreover with the lifted coordinates of N, the directed graph Γ_{N_1} with gluing matrices $\pm A$, as well as the two lifts Σ_i^1 and Σ_i^2 of Σ_i, $i = 1,2$, are shown in the left of Figure 3.

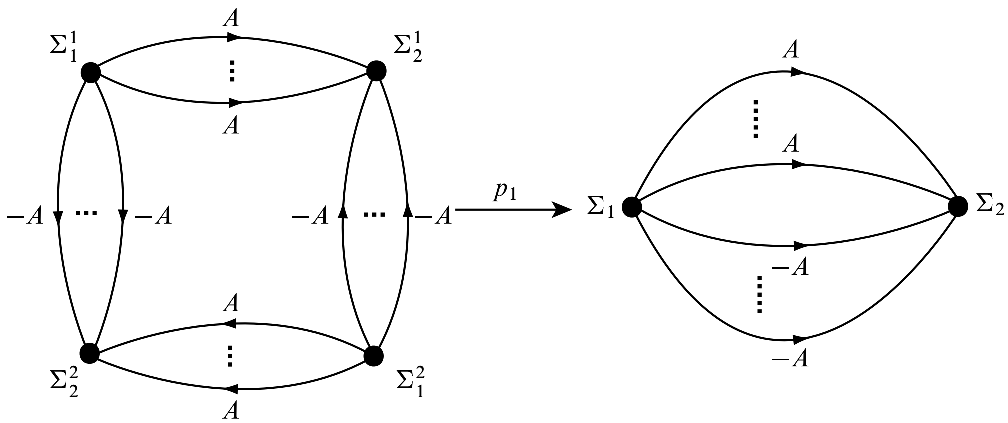

Figure 3

Let $\Sigma_i^j = F_i^j \times S^1$. It is not difficult to see that there is an orientation preserving involution η_i^j on Σ_i^j satisfying the following:

(1) η_i^j reverses both the orientation of F_i^j and S^1.

(2) For each coordinated component $(T,(s,h))$ of $\partial\Sigma_i^j$,

$$\eta_i^j((T,(s,h))) = (T,(-s,-h)).$$

Then all those η_i^j, $i,j = 1,2$ match together to get an involution η on N_1.

Keep the coordinate of Σ_i^1 for $i = 1,2$, and re-coordinate Σ_i^2 for $i = 1,2$ by $(T,(-s,-h))$ for each component of $\partial\Sigma_i^2$ for $i = 1,2$, and denote the new coordinated graph manifold by N_1' (N_1' is N_1 if we forget their coordinates). Then all gluing matrices of N_1' are A.

Now consider the composition $\eta\circ\mu$, it is easy to see that

(1) $\eta \circ \mu$ is free involution on N_1',

(2) for each JSJ piece Σ of N_1', $\eta \circ \mu$ sends the coordinate systems of Σ to the coordinate systems of $\eta \circ \mu(\Sigma)$.

Now consider the double covering $p_2\colon N_1 = N_1' \to N_2 = N_1'/\eta \circ \mu$. Since the coordinates of N_1' are invariant under $\eta \circ \mu$, and all gluing matrices of N_1' are A, we conclude that N_2 has Property I. □

Lemma 4.4 *Let N be a d–multiple edges graph manifold satisfying Property I. Then there exists a finite separable d–characteristic covering $p\colon N_1 \to N$ such that N_1 is a d–multiple edges graph manifold satisfying Property I and each JSJ piece Σ_i^1 is the product $F_i^1 \times \mathbf{S}^1$ with $g(F_i^1) = a_i d + b_i \geq 2$ for some positive integers a_i, b_i.*

Proof Denote by F_i the orbit space of Σ_i, by h_i its fiber, and by $c_1^i, \dots, c_d^i$ the components of ∂F_i and consider the homomorphism

$$\varepsilon_i\colon \pi_1 \Sigma_i = \pi_1 F_i \times \langle [h_i] \rangle \to \mathbf{Z}/d\mathbf{Z} \times \mathbf{Z}/d\mathbf{Z}$$

defined by $\varepsilon_i(a_l) = (\overline{0}, \overline{0})$ for $l \geq 1$, $\varepsilon_i(b_j) = (\overline{0}, \overline{0})$ for $j \geq 1$, $\varepsilon_i(c_1^i) = \cdots = \varepsilon_i(c_{d-1}^i) = (\overline{1}, \overline{0})$ and $\varepsilon_i(h_i) = (\overline{0}, \overline{1})$, where $\pi_1 F_i$ has a presentation:

$$\langle a_1, b_1, \dots, a_{g_i}, b_{g_i}, c_1^i, \dots, c_d^i \mid [a_1, b_1] \cdots [a_{g_i}, b_{g_i}] c_1^i \cdots c_d^i = 1 \rangle,$$

where $g_i = g(F_i)$. Since $c_1^i + \cdots + c_{d-1}^i + c_d^i = 0$ in $H_1(F_i; \mathbf{Z})$ and since $\overline{d-1}$ is invertible in $\mathbf{Z}/d\mathbf{Z}$ then $\varepsilon_i(c_l^i)$ is of order d in $\mathbf{Z}/d\mathbf{Z}$ for $l = 1, \dots, d$. Denote by $p_i^1\colon \Sigma_i^1 \to \Sigma_i$ the associated covering, then the number of components of $\partial \Sigma_i^1$ is d by the construction. Denote by $p\colon N_1 \to N$ the d^2–fold covering of N obtained by gluing Σ_1^1 with Σ_2^1. This is possible since the p_i^1 induce the d–characteristic covering on the boundary for $i = 1, 2$. This defines a finite separable d–characteristic covering by construction. Since p_i^1 has horizontal degree d then $\chi(F_i^1) = d\chi(F_i)$, where F_i^1 denotes the orbit space of Σ_i^1. This implies that

$$2g(F_i^1) + d - 2 = d(2g(F_i) + d - 2).$$

Hence we get

$$g(F_i^1) = d(g(F_i) - 1) + \left(\frac{d(d-1)}{2} + 1 \right).$$

This proves the lemma. □

Call a proper degree one map $p\colon F' \to F$ between compact surfaces a *pinch* if there is a disc D in $\mathrm{int} F$ such that $p|\colon p^{-1}(V) \to V$ is a homeomorphism, where $V = F - \mathrm{int}(D)$. We call a proper degree one map $f\colon F' \times S^1 \to F \times S^1$ a *vertical pinch* if $f = p \times \mathrm{id}$, where p is a pinch.

Lemma 4.5 *Let N be a d–multiple edges graph manifold satisfying Property I and assume that $g(F_i) = a_i d + b_i$ for some positive integers a_i, b_i and for $i = 1, 2$. Then N dominates a $\widetilde{\mathrm{PSL}}(2, \mathbf{R})$–manifold.*

Proof First note that after performing a vertical pinch on $\Sigma_1 = F_1 \times \langle h_1 \rangle$ and on $\Sigma_2 = F_2 \times \langle h_2 \rangle$ we may assume that $g_1 = g_2 = ad + 1$ for some $a \in \mathbf{Z}_+$. Note there is a cyclic d–fold covering $p_i'\colon F_i \to {F_i}'$ with $g(F_i') = a + 1$, $\partial F_i'$ connected, and the restriction of p_i' is trivial on each component of ∂F_i. This covering is given by a rotation of angle $2\pi/d$ on F_i whose axis does not meet F_i (see Figure 4).

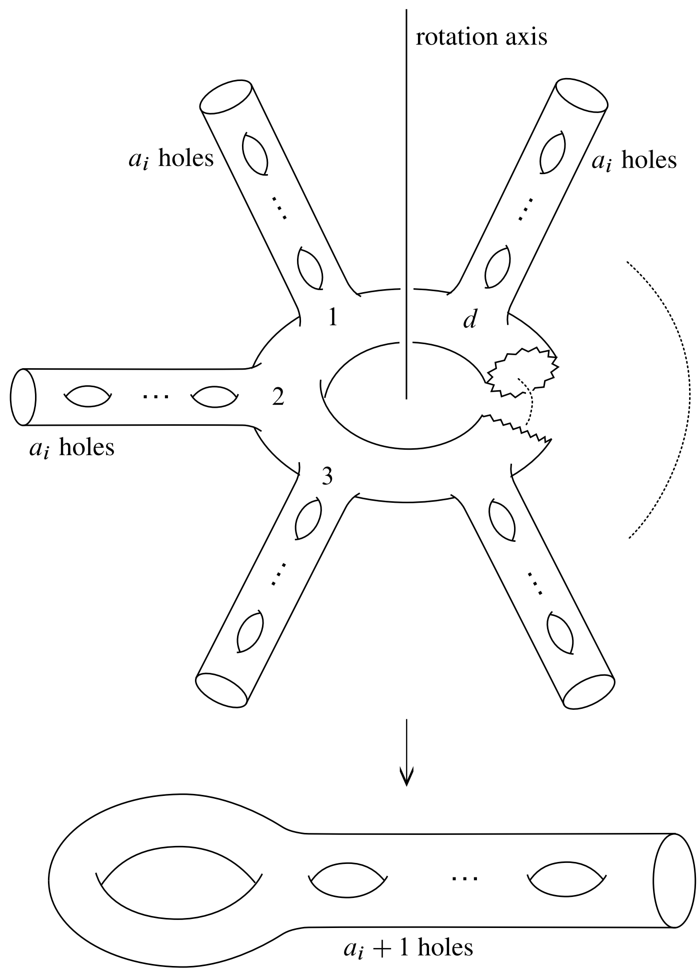

Figure 4: Fixed point free action of $\mathbf{Z}/d\mathbf{Z}$

The coverings $p_i'\colon F_i \to F_i'$ trivially extend to coverings $p_i'\colon \Sigma_i = F_i \times \langle h_i\rangle \to \Sigma_i' = F_i' \times \langle h_i'\rangle$ by setting $p_i'(h_i) = h_i'$ for $i = 1, 2$. Since all the gluing matrices of N are A by Property I, then the coverings $p_i'\colon \Sigma_i \to \Sigma_i'$ extend to a covering $p'\colon N \to N'$, where the graph manifold N' consists of the Seifert pieces Σ_1' and Σ_2' and the gluing matrix A under obvious basis.

We fix some notation. For $i = 1, 2$, denote by $\partial F_i' = s_i$ and $\tau'\colon \partial\Sigma_1' = s_1 \times h_1' \to \partial\Sigma_2' = s_2 \times h_2'$ the induced sewing map satisfying $\tau'(s_1, h_1') = (s_2, h_2')A$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$$

with $ad - bc = -1$. Moreover $b \neq 0$ by the basic properties of JSJ decomposition.

Note that $\tau_*'(s_1) = as_2 + ch_2'$ and $\tau_*'^{-1}(s_2) = -ds_1 + ch_1'$. If $ac \neq 0$, then first pinch $\Sigma_1' = F_1' \times h_1'$ into a solid torus $V_1 = D^2 \times h_1'$ by pinching s_1. This pinch provides a degree one map $\pi\colon N' \to \widehat{\Sigma}_2'$ where $\widehat{\Sigma}_2'$ is the closed 3–manifold obtained from Σ_2' by Dehn filling along the curve $as_2 + ch_2'$. Since $ac \neq 0$ then $\widehat{\Sigma}_2'$ is a $\widetilde{\mathrm{PSL}}(2, \mathbf{R})$–manifold. If $dc \neq 0$ similarly one can perform the same construction with Σ_1'. This proves the lemma when $ac \neq 0$ or $dc \neq 0$.

Let us assume now that $ac = dc = 0$. Then either $c = 0$ or $a = d = 0$. Since $ad - bc = -1$, then either

$$A = \pm\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \text{ with } b \neq 0 \quad\text{or}\quad A = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Keeping the coordinate on Σ_1' and re-coordinating Σ_2' by $(-s_2, -h_2')$ if needed, we may assume that

$$A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \quad\text{or}\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose first

$$A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}.$$

Denote $F_1' \simeq F_2'$ by F. Denote by

$$\pi\colon F_1' \times h_1' \coprod F_2' \times h_2' \to \Sigma = F \times h$$

the trivial 2–fold covering map, where $\pi(h_i') = h$ and $\pi(s_i) = s = \partial F$. Denote by $\rho\colon \Sigma \to \widehat{\Sigma}$ the quotient map associated with the Dehn filling on $\partial\Sigma$ along the curve

$$\frac{b}{(2,b)}s - \frac{2}{(2,b)}h$$

where $(2,b)$ denotes the greatest common divisor of 2 and b. Note that, since $b \neq 0$, then $\widehat{\Sigma}$ is a $\widetilde{\mathrm{PSL}}(2,\mathbf{R})$–manifold.

One can verify routinely that in the π_1 level the relations provided by gluing Σ_1' and Σ_2' via τ' are sent to the relation provided by Dehn filling on Σ via

$$\frac{b}{(2,b)}s - \frac{2}{(2,b)}h$$

under ρ, hence the map $\rho\circ\pi\colon \Sigma_1' \coprod \Sigma_2' \to \widehat{\Sigma}$ factors through

$$\frac{\Sigma_1' \coprod \Sigma_2'}{\tau'} \simeq N'$$

which is sent into $\widehat{\Sigma}$ by a degree 2 map, since the sewing involution τ' is orientation reversing so that N' inherits compatible orientations from the pieces Σ_1' and Σ_2'.

In the case

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we can perform the same construction as above, replacing the filling curve $(b/(2,b))s - (2/(2,b))h$ by the curve $s-h$. This proves Lemma 4.5. $\square$

By Lemmas 4.3, 4.4 and 4.5 and their proofs, we have the following diagram

$$\begin{tikzcd} N_1 \arrow[d, "p_1"'] \arrow[dr, "p_2"] & N_3 \arrow[d, "p_3"'] \arrow[dr, "p_4"] & \\ N & N_2 & N_4 \end{tikzcd}$$

where p_1 and p_2 are coverings provided by Lemma 4.3, p_3 is the coverings provided by Lemma 4.4, and the nonzero degree map p_4 is provided by Lemma 4.5, where $\mathrm{SV}(N_4) \neq 0$. Since SV has Property D, $\mathrm{SV}(N_3) \neq 0$.

Consider the covering $\widetilde{N}$ corresponding to the finite index subgroup $p_{2*}(\pi_1 N_1) \cap p_{3*}(\pi_1 N_3)$ in $\pi_1 N_2$. Then $\widetilde{N}$ covers both N_1 (and thus N) and N_3, and $\mathrm{SV}(\widetilde{N}) \neq 0$. Then the proof of Proposition 4.1 is complete.

5 Proof of Theorem 1.2

5.1 Simplifications

Let N be a closed nontrivial graph manifold. We are going to show that $|\mathcal{D}(M,N)|$ is finite for any given graph manifold M.

(1) First we simplify N: By Lemma 3.2 and Lemma 3.5, there is a finite covering $\widetilde{N}$ of N satisfying the following condition:

$$(*) \qquad \begin{cases} \text{Each JSJ piece of } \widetilde{N} \text{ is a product of an oriented surface with genus} \geq 2 \\ \text{and the circle, and each JSJ torus is shared by two different JSJ pieces.} \end{cases}$$

By Lemma 3.1, if $|\mathcal{D}(M,N)|$ is not finite for some graph manifold M, then $|\mathcal{D}(P,\widetilde{N})|$ is not finite for some graph manifold P. So we may assume N already satisfies condition $(*)$.

(2) Then we simplify M: For given M, let $\mathcal{H} = \{M_1, \dots, M_k\}$ be the finite set of graph manifolds provided by Proposition 2.2. By Proposition 2.2 (i), if $|\mathcal{D}(M,N)|$ is not finite, then $|\mathcal{D}(M_i,N)|$ is not finite for some $M_i \in \mathcal{H}$. So may assume

$$(**) \qquad M = M_i \in \mathcal{H} \text{ for some } i \in \{1, \dots, k\}.$$

5.2 Proof of Theorem 1.2 when $|e|(N) \neq 0$

Suppose $|e|(N) \neq 0$. By Lemma 3.6 and $(*)$, we may assume that $|\operatorname{SV}|(N) \neq 0$. Then there exists a Seifert piece Q of N^* such that $\operatorname{SV}(\widehat{Q}) \neq 0$. By $(**)$ and Proposition 2.2 (ii), we may assume that $L_Q(f) = f^{-1}(Q)$ is a canonical submanifold of M. Below we denote $L_Q(f)$ as L_Q for short.

Lemma 5.1 *L_Q can be chosen so that any component T of ∂L_Q is shared by two distinct Seifert pieces of M: one in L_Q and another in $M \setminus L_Q$.*

Proof Indeed if not, then there exist two distinct components T and T' of ∂L_Q which are identified by the sewing involution τ_M of M and such that T and T' are sent by f into the same component of ∂Q. Denote by $\bar{L}_Q$ the canonical submanifold of M obtained by identifying T and T' via τ_M. Since each component of ∂Q is shared by two distinct Seifert pieces of N by assumption $(**)$, f induces a proper map $\bar{f}\colon \bar{L}_Q \to Q$. After finitely many such operations, we reach a new L_Q satisfying the requirement of Lemma 5.1 □

Below we assume that L_Q satisfies the requirement of Lemma 5.1. Now we choose L_Q to be maximal in the sense that for any Seifert piece S in $M \setminus L_Q$ adjacent to L_Q, S is not able to be added into L_Q by homotopy on f. Then $f(S) \subset B_S$, where B_S is a Seifert piece of N, distinct from Q and adjacent to Q.

Since L_Q is maximal, by Lemma 2.3, we deduce that for any Seifert piece S adjacent to L_Q along a component of ∂L_Q, $f|S\colon S \to B_S$ is fiber preserving. Hence the proper map $f|L_Q\colon L_Q \to Q$ preserves the canonical framings, and it induces a map

$\widehat{f}\colon \widehat{L}_Q \to \widehat{Q}$ between the closed manifolds obtained after Dehn filling along the canonical framings. By Lemma 2.4 we have

$$\mathrm{SV}(\widehat{L}_Q) \geq |\deg(\widehat{f})|\,\mathrm{SV}(\widehat{Q}).$$

Since $\deg(f) = \deg(f|L_Q) = \deg(\widehat{f})$, we get

$$|\deg(f)| \leq \frac{\mathrm{SV}(\widehat{L}_Q)}{\mathrm{SV}(\widehat{Q})}.$$

Therefore

$$|\deg(f)| \leq \max\left\{ \frac{\mathrm{SV}(\widehat{L})}{\mathrm{SV}(\widehat{Q})} \;\middle|\; Q \in N^*, \mathrm{SV}(\widehat{Q}) \neq 0; L \text{ is canonical in } M \right\}.$$

By Lemma 2.1 there are only finitely many $\widehat{Q}$ and only finitely many $\widehat{L}$. So the right side of the above inequality is finite. This completes the proof of Theorem 1.2 when $|e|(N) \neq 0$.

5.3 Proof of Theorem 1.2 when $|e|(N) = 0$

By Lemma 3.1 and Lemma 3.6 we can assume each gluing matrix of N is equal to

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Choose two distinct adjacent Seifert pieces S_1 and S_2 in N, denote by $\mathcal{T} = \partial S_1 \cap \partial S_2$ and by Q the connected graph manifold $S_1 \cup_{\mathcal{T}} S_2$ (such Seifert pieces exist by Section 5.1). By Proposition 2.2, we may assume that $f^{-1}(Q) = L_Q$ is a canonical submanifold of M.

Since each JSJ torus of N is shared by two different JSJ pieces, by the same arguments as in Section 5.2, we may assume that each component of ∂L_Q is shared by two distinct Seifert pieces of M one in L_Q and another in $M \setminus L_Q$. Furthermore we can arrange L_Q to be maximal in the sense of Section 5.2, then by Lemma 2.3, we deduce that any Seifert piece S' of M adjacent to L_Q is sent by f to a Seifert piece B' adjacent to Q such that $f|S'\colon S' \to B'$ is fiber preserving.

As in Section 5.2, it follows that the proper map $f|L_Q\colon L_Q \to Q$ induces a map $\widehat{f}\colon \widehat{L}_Q \to \widehat{Q}$ between closed graph manifolds obtained by Dehn filling along the canonical framings. Moreover, as in Section 5.2 we have $\deg(f) = \deg(f|L_Q) = \deg(\widehat{f})$ and thus

$$|\mathcal{D}(M,N)| \leq \max\left\{ |\mathcal{D}(\widehat{L}, \widehat{Q})| L \text{ is canonical in } M \right\}.$$

Note that $\widehat{Q} = \widehat{S}_1 \cup_{\mathcal{T}} \widehat{S}_2$, where $\widehat{S}_i$ is obtained by Dehn filling along the canonical framings on $\partial S_i \setminus \mathcal{T}$, $i = 1, 2$. It follows that $\widehat{Q}$ satisfies the hypothesis of Proposition 4.1. Then $\widehat{Q}$ has a finite covering $\widetilde{Q}$ with $\mathrm{SV}(\widetilde{Q}) \neq 0$ by Proposition 4.1. Hence by Lemma 3.1, the set $|\mathcal{D}(\widehat{L}, \widehat{Q})|$ is finite for any $\widehat{L}$. Since by Lemma 2.1 there are only finitely many $\widehat{L}$, this completes the proof of Theorem 1.2 when $|e|(N) = 0$. Hence Theorem 1.2 is proved.

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CMI, Technopôle Château-Gombert
39 rue Joliot Curie, 13453 Marseille Cedex 13, France

Department of Mathematics, Peking University
Beijing, 100871, China

derbez@cmi.univ-mrs.fr, wangsc@math.pku.edu.cn

Received: 23 March 2009 Revised: 17 July 2009