Algebraic & Geometric Topology 9 (2009) 1987–2026

Khovanov homology for signed divides

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The purpose of this paper is to interpret polynomial invariants of strongly invertible links in terms of Khovanov homology theory. To a divide, that is a proper generic immersion of a finite number of copies of the unit interval and circles in a 2–disc, one can associate a strongly invertible link in the 3–sphere. This can be generalized to signed divides: divides with $+$ or $-$ sign assignment to each crossing point. Conversely, to any link L that is strongly invertible for an involution j, one can associate a signed divide. Two strongly invertible links that are isotopic through an isotopy respecting the involution are called strongly equivalent. Such isotopies give rise to moves on divides. In a previous paper [2], the author finds an exhaustive list of moves that preserves strong equivalence, together with a polynomial invariant for these moves, giving therefore an invariant for strong equivalence of the associated strongly invertible links. We prove in this paper that this polynomial can be seen as the graded Euler characteristic of a graded complex of $\mathbb{Z}_2$–vector spaces. Homology of such complexes is invariant for the moves on divides and so is invariant through strong equivalence of strongly invertible links.

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Introduction

A *divide* Γ is the image of a proper generic immersion of a finite number of intervals and circles into the unit 2–disc $\mathbf{D}^2$ of $\mathbb{R}^2$. To a divide Γ, N A'Campo [1] associates a link $\mathcal{L}(\Gamma)$ in the unit 3–sphere of the tangent space $T\mathbf{D}^2$:

$$\mathcal{L}(\Gamma) = \{(p, v) \in T\mathbf{D}^2 : p \in \Gamma,\ v \in T_p\Gamma,\ \|p\|^2 + \|v\|^2 = 1\}.$$

This link has natural orientation and is strongly invertible with respect to the involution $j(p, v) = (p, -v)$.

Couture and Perron [3] give a generalization of divides. Let (x, y) be coordinates in $\mathbf{D}^2$ such that the restriction to Γ of the projection $\pi_1\colon (x, y) \mapsto x$ is a Morse function. A *Morse signed divide* (MS–divide) relative to π_1 stands for such a divide with $+$ or $-$ sign assignment to each double point of Γ. Furthermore, if there exists $a \in]0, 1[$ such that all maxima (resp. minima) of $\pi_{1|\Gamma}$ project on a (resp. $-a$) and all double points in $]-a, a[$, the MS–divide is called *ordered (OMS–divide)*.

Published: 8 October 2009 DOI: 10.2140/agt.2009.9.1987

We also define a link associated with a MS–divide (see Couture and Perron [3]), which is strongly invertible with respect to the involution $j(p,v)=(p,-v)$. If all signs are positive, this link is no more than the link of the divide without signs. The interest of OMS–divides rather than MS–divides is to obtain an immediate braid presentation of the link from the divide.

Strongly invertible links are closely related to OMS–divides. Let L be an oriented link in $\mathbb{S}^3$ and j be an involution of $(\mathbb{S}^3, L)$ with nonempty fixed point set, which preserves the orientation of $\mathbb{S}^3$ and reverses the orientation of L. Then (L, j) is called a strongly invertible link. As we said above, the link of a divide is strongly invertible for the implicit strong inversion $j(p,v)=(p,-v)$. Two strongly invertible links (L, j) and (L', j') are *strongly equivalent* if there exists an isotopy φ_t, $t \in [0,1]$ of $\mathbb{S}^3$ sending L to L' such that $\varphi_1 \circ j = j' \circ \varphi_1$.

Isotopies through MS–divides give rise to strong equivalence of associated links. Also, one can find in Couture [2] an (exhaustive) list of elementary moves on MS–divides that preserve strong equivalence classes of the associated links. As a particular case, given a MS–divide, we can always construct another one using these moves which is an OMS–divide. Besides, we can transpose the moves on MS–divides directly to moves on OMS–divides. Two OMS–divides obtained one from the other by those moves on OMS–divides are called $\mathcal{M}$–equivalent (see the list of moves in Section 1.3). Then as an essential result of [2], we have:

Theorem (Couture [2]) (1) *Every strongly invertible link is strongly equivalent to the link of an OMS–divide.*

(2) *The links of two OMS–divides are strongly equivalent if and only if the OMS–divides are $\mathcal{M}$–equivalent.*

As the Jones polynomial is invariant under Reidemeister moves on links diagrams, there exists a Laurent polynomial for an OMS–divide with integral coefficients (see Couture [2]), which is invariant under $\mathcal{M}$–equivalence and so invariant under strong equivalence of strongly invertible links. Modulo 2, this polynomial coincides with the Jones polynomial of the link. The purpose of this paper is to interpret the polynomial of an OMS–divide as the graded Euler characteristics of a graded complex of $\mathbb{Z}_2$–vector spaces (Proposition 3.16). Besides, if we call the Khovanov homology of an OMS–divide the homology of this complex, then we have a stronger result:

Theorem 3.17 *Khovanov homology of OMS–divides is invariant under $\mathcal{M}$–equivalence.*

Corollary 3.18 *Khovanov homology is an invariant for strong equivalence of strongly invertible links.*

Eventually, Khovanov homology of OMS–divides is a refinement of the polynomial invariant of OMS–divides.

This paper is dedicated to Bernard Perron.

1 Divides and OMS–divides

1.1 Divides and links of divides

A *divide* of the unit 2–disc $\mathbf{D}^2$ of $\mathbb{R}^2(\simeq \mathbb{C})$ is the image Γ of a proper generic immersion γ:

$$\gamma\colon (J,\partial J) \to (\mathbf{D}^2,\partial\mathbf{D}^2), \qquad J = \Big(\bigsqcup_{j=1}^{r} I_j\Big) \sqcup \Big(\bigsqcup_{j=1}^{s} S_j\Big), \tag{1-1}$$

where I_j and S_j are respectively copies of $[0,1]$ and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and generic means that the only singularities of γ are ordinary double points and $\Gamma = \gamma(J)$ intersects $\partial\mathbf{D}^2$ transversally. Every $\gamma(I_j)$ (resp. $\gamma(S_j)$) is called *interval* (resp. *circular*) branch.

Let $S(\mathbf{D}^2) = \{(p,v) \in T\mathbf{D}^2 : |p|^2 + |v|^2 = 1\}$ be the unit sphere of the tangent space $T\mathbf{D}^2 \simeq \mathbf{D}^2 \times \mathbb{C}$. To a divide, A'Campo [1] associates a link $\mathcal{L}(\Gamma)$ in $S(\mathbf{D}^2)$:

$$\mathcal{L}(\Gamma) = \{(p,v) \in S(\mathbf{D}^2) : p \in \Gamma,\ v \in T_p\Gamma\}. \tag{1-2}$$

This link has a natural orientation induced by the two possible orientations of the branches of Γ and is strongly invertible for the involution $j(p,v) = (p,-v)$ of $S(\mathbf{D}^2)$ with axis $\mathrm{Fix}(j) = \partial\mathbf{D}^2 \times \{0\}$ (see Section 1.3 below). Each interval branch of Γ leads to a strongly invertible component of $\mathcal{L}(\Gamma)$ and each circular branch of Γ to two components of $\mathcal{L}(\Gamma)$ interchanged by j.

1.2 OMS–divides

Let Γ be a divide. Let $\rho(x+iy) = x$ be the projection on real axis. Suppose there exists $(a,b) \in]0,1[\times]0,1[$, $a^2 + b^2 < 1$ such that

(1) $\Gamma \subset \{x + iy \in \mathbf{D}^2 : -b < y < b\}$ and the restriction $\rho_{|\Gamma}$ is a Morse function;

(2) all double points of Γ are contained in $]-a,a[\times]-b,b[$;

(3) all maxima (resp. minima) of $\rho_{|\Gamma}$, called *vertical tangent points*, project onto a (resp. $-a$).

Then Γ is called an *ordered Morse divide*. Double points and vertical tangent points will be called *singular points* of Γ. Now let ϵ be a function that associates a $+$ or $-$ sign with each double point of Γ. Then (Γ,ϵ) is called an *ordered Morse signed divide* (OMS–divide) relative to the projection $\rho(x+iy)=x$ (see Couture and Perron [3]).

Let's associate an oriented j–strongly invertible link $\mathcal{L}(\Gamma,\epsilon)$ with an OMS–divide (Γ,ϵ). This link coincides with $\mathcal{L}(\Gamma)$ except in solid tori $TD_p\cap S(\mathbf{D}^2)\simeq D_p\times\mathbb{S}^1$ over small discs D_p around negative double points p of (Γ,ϵ) where we change the two j–symmetric crossings from over to under. More precisely, in such a solid torus $TD_p\cap S(\mathbf{D}^2)$, the link is defined according to Figure 1. If $\epsilon=+$ for all double points then $\mathcal{L}(\Gamma,\epsilon)=\mathcal{L}(\Gamma)$.

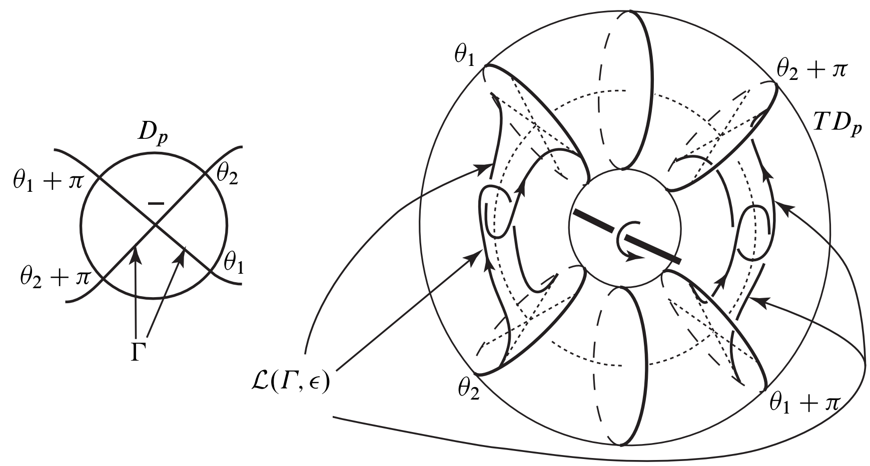

Figure 1: The link $\mathcal{L}(\Gamma,\epsilon)$ over a negative double point

Besides, from a divide Γ one can construct an OMS–divide (Γ',ϵ) by a succession of moves and isotopies, such that $\mathcal{L}(\Gamma',\epsilon)$ and $\mathcal{L}(\Gamma)$ are isotopic [3] by an isotopy that respects the involution j (see Section 1.3 below for a more precise definition).

Remark 1.1 For simplicity, we will only consider an OMS–divide (Γ,ϵ) in $[-a,a]\times[-b,b]$, omitting its trivial part outside this rectangle. After rescaling, we also suppose that $a=b=1$. Since we will often consider diagrams of local parts of OMS–divides (Γ,ϵ), we distinguish *end points* of Γ, ie points of Γ in $\{-1,1\}\times[-1,1]$ without vertical tangent by a big point (see Figure 2).

Moreover, we will simply write Γ instead of (Γ,ϵ) if no ambiguity occurs in the context.

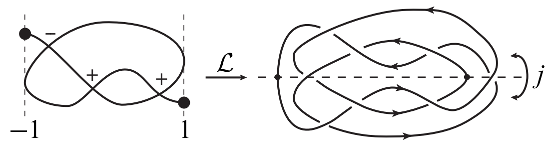

Figure 2: A representative OMS–divide for the strongly invertible knot 5_2

1.3 $\mathcal{M}$–Equivalence for OMS–divides

Two OMS–divides Γ and Γ' are *$\mathcal{M}$–equivalent* if we obtain one from the other by isotopy through OMS–divides and a finite sequence of the moves described in Figure 3 or symmetric situations with respect to horizontal and vertical directions (see Couture [2]). Let j be an orientation preserving involution of $\mathbb{S}^3$ with nonempty fix

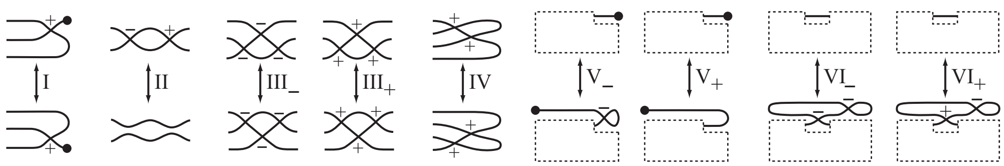

Figure 3: Moves of $\mathcal{M}$–equivalence

point set (ie $\mathrm{Fix}(j)$ is a trivial knot according to the solution of Smith conjecture; see Morgan and Bass [4]). An oriented link $L \subset \mathbb{S}^3$ is j–strongly invertible if j sends L to itself with opposite orientation. The couple (L, j) is called a strongly invertible link. With the link of an OMS–divide, we implicitly associate natural orientation and involution $j(p, v) = (p, -v)$ as in Section 1.1: such a link is strongly invertible.

Two strongly invertible links (L, j) and (L', j') are called *strongly equivalent*[1] if there exists and isotopy φ_t, $t \in [0, 1]$ of $\mathbb{S}^3$ sending L to L' such that $\varphi_1 \circ j = j' \circ \varphi_1$. One can easily prove that $\mathcal{M}$–equivalent OMS–divides give rise to strongly equivalent strongly invertible links. Conversely, let's recall the following crucial theorem relating OMS–divides with strongly invertible links.

Theorem 1.2 (Couture [2]) (1) *Every strongly invertible link is strongly equivalent to the link of an OMS–divide.*

(2) *The links of two OMS–divides are strongly equivalent if and only if the OMS–divides are $\mathcal{M}$–equivalent.*

[1]The same link L may have two strong inversions j and j' such that (L, j) and (L, j') are not strongly equivalent.

2 The polynomial of an OMS–divide

Let's denote by Θ_0 and Θ_1 the local splittings of an OMS–divide (Γ, ϵ) in a neighborhood of a double point or vertical tangent point described in Figure 4 (Θ_0 "smoothes" the OMS–divide whereas Θ_1 introduces horizontal cusps).

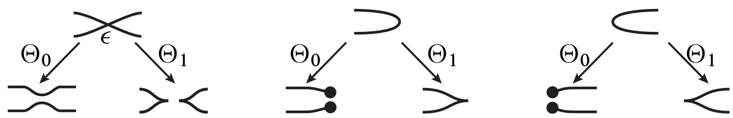

Figure 4: Local splittings

Definition 2.1 (1) We extend the notion of OMS–divide: a *cuspidal divide* $\Gamma := (\Gamma, \epsilon)$ is a signed diagram like an OMS–divide except that it has a finite number of horizontal cusps (as in the result of type Θ_1 splittings). For instance, a partially (or totally) transformed OMS–divide through Θ_0 and Θ_1 is a cuspidal divide.

(2) Let (Γ, ϵ) be an OMS–divide (or more generally a cuspidal divide) with double and vertical tangent points numbered by $p_1, \dots, p_n$. Let $[k]$ be the word $k_1 k_2 \cdots k_n$, $k_i \in \{0, 1\}$. A *state* $(S, \Theta_{[k]})$ of (Γ, ϵ) is the combination of

- a succession of local splittings $\Theta_{[k]} = (\Theta_{k_1}, \dots, \Theta_{k_n})$ at $p_1, \dots, p_n$;
- the cuspidal divide $S = \Theta_{[k]}(\Gamma, \epsilon)$ without double points nor vertical tangent points obtained by transforming Γ through $\Theta_{[k]}$.

For simplification, we will often identify the cuspidal divide S with the state $(S, \Theta_{[k]})$. We denote by $\mathrm{St}(\Gamma, \epsilon)$ the set of all states of (Γ, ϵ).

One can define a j–strongly invertible link $\mathcal{L}(\Gamma, \epsilon)$ associated with a cuspidal divide (Γ, ϵ) exactly in the same way we have done for OMS–divide. However, such a link is generally unoriented precisely because of the introduction of cusps. Each local splitting at a double point of (Γ, ϵ) corresponds to simultaneously smoothing two symmetric crossing points of the corresponding representative closed braid diagram of $\mathcal{L}(\Gamma, \epsilon)$ [3] whereas each local splitting at a vertical tangent point corresponds to smoothing a crossing point through the axis of the inversion j (see Figure 5).

Let $\Gamma := (\Gamma, \epsilon)$ be an OMS–divide. Let $n = n_+ + n_- + n_0$ be the number of singular points of Γ where n_+, n_- are respectively the numbers of positive and negative double points, and n_0 the number of vertical tangent points. Let's call

$$w(\Gamma) = 2n_+ - 2n_- + n_0 \tag{2-1}$$

the *writhe* of Γ (ie the writhe of the representative closed braid diagram of $\mathcal{L}(\Gamma, \epsilon)$ [3] with $2n_+ + 2n_- + n_0$ crossings obtained from (Γ, ϵ)).

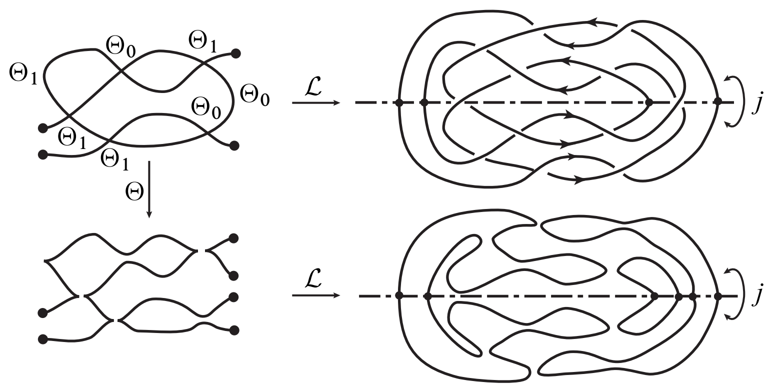

Figure 5: The links $\mathcal{L}(\Gamma,\epsilon)$ and $\mathcal{L}(\Theta(\Gamma,\epsilon))$, $\Theta=\Theta_{1110010}$

For a state $S\in \mathrm{St}(\Gamma,\epsilon)$, let $\mathrm{cl}(S)$ be the number of closed connected components and $\mathrm{op}(S)$ be the number of open connected components (ie with two end points). Let $r_+(S)$, $r_-(S)$ and $r_0(S)$ be the numbers of Θ_1 local splittings (see Figure 4) for positive double points, negative double points and vertical tangent points respectively to obtain S from (Γ,ε). Let's set

$$i(S)=r_+(S)-r_-(S)+r_0(S),\quad k(S)=w(\Gamma)+2i(S)-r_0(S). \tag{2-2}$$

Definition 2.2 (cf [2]) The polynomial of an OMS–divide Γ (and more generally of a cuspidal divide) is the Laurent polynomial (of the variable $\sqrt{t}$) defined by:

$$W_\Gamma(t)=\sum_{S\in \mathrm{St}(\Gamma,\epsilon)}(-1)^{i(S)}(\sqrt{t})^{k(S)}\left(\frac{1}{t}+t\right)^{\mathrm{cl}(S)}\left(\frac{1}{\sqrt{t}}+\sqrt{t}\right)^{\mathrm{op}(S)-1} \tag{2-3}$$

Proposition 2.3 (cf [2]) *The polynomial of an OMS–divide is invariant under $\mathcal{M}$–equivalence of OMS–divides and so is an invariant for strong equivalence of strongly invertible links.*

Definition 2.4 A state S with $+$ or $-$ assignment to each connected component is called an *enhanced state*, and is denoted by $\widetilde{S}$. The set of enhanced states of (Γ,ϵ) is denoted by $\widetilde{\mathrm{St}}(\Gamma,\epsilon)$, and S is called the *underlying state* of $\widetilde{S}$.

Let $\widetilde{S}$ be an enhanced state with underling state S. The numbers $i(\widetilde{S}):=i(S)$ and $k(\widetilde{S}):=k(S)$ in (2-2) do not depend on the signs of the components. The subset of enhanced states $\widetilde{S}$ of (Γ,ϵ) such that $i(\widetilde{S})=i$ is denoted by $\widetilde{\mathrm{St}}_i(\Gamma,\epsilon)$.

Let's denote by $\delta_{\mathrm{cl}}(\widetilde{S})$ (resp. $\delta_{\mathrm{op}}(\widetilde{S})$) the difference between the number of positive and negative closed (resp. open) components of $\widetilde{S}$. Then we define the *degree* $j(\widetilde{S})$ of the enhanced state $\widetilde{S}$, which depends on the signs of the components of S by

$$j(\widetilde{S}) = k(\widetilde{S}) + 2\delta_{\mathrm{cl}}(\widetilde{S}) + \delta_{\mathrm{op}}(\widetilde{S}). \tag{2-4}$$

We can now reformulate the polynomial of an OMS–divide Γ:

$$W_\Gamma(t) = \frac{\sqrt{t}}{1+t} \sum_{\widetilde{S} \in \widetilde{\mathrm{St}}(\Gamma)} (-1)^{i(\widetilde{S})} (\sqrt{t})^{j(\widetilde{S})} \tag{2-5}$$

Remark 2.5 $j(\widetilde{S})$ always has the same parity as half the number of end points of Γ. We also have the inequalities:

$$-n_- \le i(\widetilde{S}) \le n_+ + n_0, \quad 2n_+ - 4n_- + n_0 \le k(\widetilde{S}) \le 4n_+ - 2n_- + 2n_0. \tag{2-6}$$

3 Categorification

3.1 Complex associated with an OMS–divide

In this section, we define a graded complex of $\mathbb{Z}_2$–vector spaces[2] associated with a divide. We follow here Viro's approach of Khovanov homology for links [5], based on the Kauffman state model for the Jones polynomial: the polynomial of a divide also have been defined in [2] by state model.

Let $\Gamma := (\Gamma, \epsilon)$ be an OMS–divide (or a cuspidal divide). For $i \in \mathbb{Z}$, let $[\![\Gamma]\!]_i = \mathbb{Z}_2\{\widetilde{\mathrm{St}}_i(\Gamma)\}$ be the finite dimensional $\mathbb{Z}_2$–vector space generated by enhanced states $\widetilde{S}$ of Γ such that $i(\widetilde{S}) = i$ (if $i < -n_-$ or $i > n_+ + n_0$, $[\![\Gamma]\!]_i = \{0\}$). Degree $j(\widetilde{S})$ defines a grading on $[\![\Gamma]\!]_i$ and we denote

$$[\![\Gamma]\!] = \big([\![\Gamma]\!]_i\big)_{i\in\mathbb{Z}} \qquad [\![\Gamma]\!]_i = \bigoplus_{j\in\mathbb{Z}} [\![\Gamma]\!]_{i,j} \tag{3-1}$$

where $$[\![\Gamma]\!]_{i,j} = \mathbb{Z}_2\{\widetilde{S} \in \mathrm{St}_i(\Gamma) : j(\widetilde{S}) = j\}.$$

Now we define a differential on $[\![\Gamma]\!]$ to obtain a (finite) complex of graded $\mathbb{Z}_2$–vector spaces.

[2] Here we choose $\mathbb{Z}_2$–vector spaces for simplification to avoid signs. One can easily generalize taking for instance $\mathbb{Z}$–modules or $\mathbb{Q}$–vector spaces.

Definition 3.1 Let $\widetilde{S}_1, \widetilde{S}_2 \in \widetilde{\mathrm{St}}(\Gamma, \epsilon)$. We say that $\widetilde{S}_2$ is *adjacent* to $\widetilde{S}_1$ ($\widetilde{S}_1 \rightsquigarrow \widetilde{S}_2$) if:

(1) S_1 and S_2 coincide outside a neighborhood D_p of a singular point p of (Γ, ϵ);

(2) One can pass from $\widetilde{S}_1$ to $\widetilde{S}_2$ by one of the following transformations T in D_p:
- $T = \Theta_1 \circ \Theta_0^{-1}$ if p is a positive double point or a vertical tangent point of (Γ, ϵ);
- $T = \Theta_0 \circ \Theta_1^{-1}$ if p is a negative double point;

(3) Signs rules described in Figures 6, 7, 8 are fulfilled, signs of other components being unchanged. (In these figures, black color is used for open components and gray for closed components, a dotted line means that the points are related in the state outside D_p. Lack of dotted line means that the points are not related outside D_p).

If $\widetilde{S}_2$ is adjacent to $\widetilde{S}_1$ then

$$j(\widetilde{S}_1) = j(\widetilde{S}_2), \quad i(S_1) = i(S_2) - 1. \tag{3-2}$$

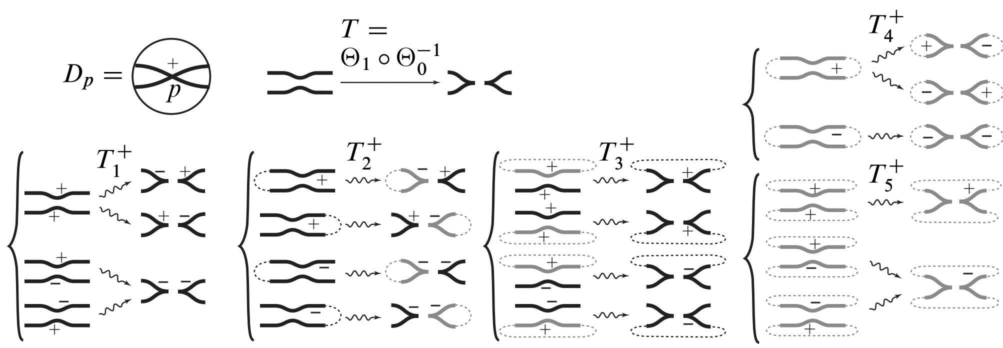

Figure 6: Case of a splitting at a positive double point p ($\epsilon(p) = +$)

The differential $d = (d_i)_{i \in \mathbb{Z}}$ on $[\![\Gamma]\!]$, $d_i \colon [\![\Gamma]\!]_i \to [\![\Gamma]\!]_{i+1}$ is now defined in the following way: the matrix of d_i has coefficients defined by the incidence numbers $(\widetilde{S}_1 \colon \widetilde{S}_2)$, $\widetilde{S}_1 \in \widetilde{\mathrm{St}}_i(\Gamma)$, $\widetilde{S}_2 \in \widetilde{\mathrm{St}}_{i+1}(\Gamma)$:

$$(\widetilde{S}_1 : \widetilde{S}_2) = \begin{cases} 1 & \text{if } \widetilde{S}_1 \rightsquigarrow \widetilde{S}_2, \\ 0 & \text{else.} \end{cases} \tag{3-3}$$

From (3-2), d respects the degree j, ie

$$d_i = \bigoplus_j d_{i,j} \colon \bigoplus_j [\![\Gamma]\!]_{i,j} \to \bigoplus_j [\![\Gamma]\!]_{i+1,j}.$$

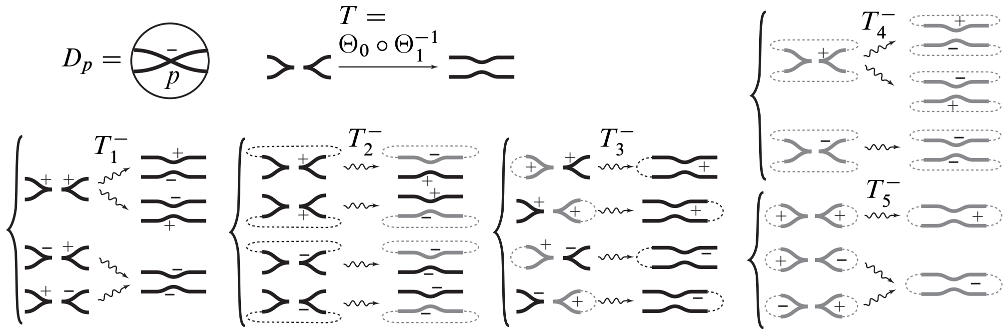

Figure 7: Case of a splitting at a negative double point p $(\epsilon(p) = -)$

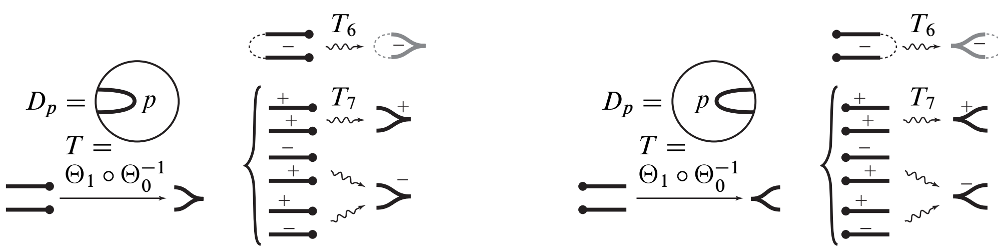

Figure 8: Case of a splitting at a vertical tangent point p

Remark 3.2 We have dual roles for T_1^+ and T_1^-, T_2^+ and T_3^-, T_3^+ and T_2^-, T_4^+ and T_5^-, T_5^+ and T_4^- in Figure 6 and Figure 7. To go further about duality property, we could have introduced "negative tangent points" to interpret dual arrows of T_6 and T_7 in Figure 8. However, we didn't choose this option, since such "negative tangent points" can be replaced by:

Also, we can see that T_i^+ and T_i^-, $i \in \{1, 2, 3, 4\}$ give rise to analogous situations.

Proposition 3.3 *The complex*

$$(\llbracket \Gamma \rrbracket, d) = (\llbracket \Gamma \rrbracket_i, d_i)_{i \in \mathbb{Z}} = \Big(\bigoplus_{j \in \mathbb{Z}} \llbracket \Gamma \rrbracket_{i,j}, \bigoplus_{j \in \mathbb{Z}} d_{i,j} \Big)_{i \in \mathbb{Z}}$$

is a finite complex of graded $\mathbb{Z}_2$–vector spaces (each $\llbracket \Gamma \rrbracket_i$ is finitely graded by degree j).

Proof It suffices to verify all such diagrams

$$
\begin{tikzcd}
\widetilde{S}_1 \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \widetilde{S}_2 \arrow[d, rightsquigarrow] \\
\widetilde{S}'_2 \arrow[r, rightsquigarrow] & \widetilde{S}_3
\end{tikzcd}
\tag{3-4}
$$

corresponding to splitting two singular points are commutative. Since we have $\mathbb{Z}_2$–vector spaces, commutative diagrams induce the relations $d_{i+1}d_i = 0$. Notice that from the previous remark, we can strongly reduce the number of cases to check (see also the proof of Proposition 3.5). ◻

3.2 Alternative point of view

Here we present an alternative (more algebraic) way to see the complex $([\![\Gamma]\!], d)$, in terms of Frobenius algebra: we can link complexes of OMS–divides with $1+1$–TQFT (or more precisely with some $1+1$–TQFT with symmetry property).

Let $\mathcal{A} := \mathbb{Z}_2\{v_-, v_+\}$ be the graded $\mathbb{Z}_2$–vector space generated by two elements v_- and v_+ such that $\deg(v_-) = -1$ and $\deg(v_+) = 1$. We define a commutative product $\mu_1\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, a unit $\eta_1\colon \mathbb{Z}_2 \to \mathcal{A}$ and a nondegenerate symmetric bilinear pairing $\beta_1\colon \mathcal{A}\otimes\mathcal{A}\to\mathbb{Z}_2$ by:

$$
\begin{aligned}
&\mu_1(v_+\otimes v_+) = v_+,\quad \mu_1(v_+\otimes v_-) = \mu_1(v_-\otimes v_+) = v_-,\quad \mu_1(v_-\otimes v_-) = 0\\
&\eta_1(0) = 0,\qquad \eta_1(1) = v_+\\
&\beta_1(v_+\otimes v_+) = 0,\quad \beta_1(v_+\otimes v_-) = \beta_1(v_-\otimes v_+) = 1,\quad \beta_1(v_-\otimes v_-) = 0.
\end{aligned}
\tag{3-5}
$$

The form β_1 induces a duality isomorphism $\mathcal{A} \overset{\simeq}{\longleftrightarrow} \mathcal{A}^*$ and $\mathcal{A}$ is a commutative Frobenius algebra with adjoint coproduct $\delta_1\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ and counit $\varepsilon_1\colon \mathcal{A}\to\mathbb{Z}_2$:

$$
\begin{aligned}
&\delta_1(v_+) = v_+\otimes v_- + v_-\otimes v_+ \qquad && \varepsilon_1(v_+) = 0\\
&\delta_1(v_-) = v_-\otimes v_- && \varepsilon_1(v_-) = 1.
\end{aligned}
\tag{3-6}
$$

Let $\phi_1\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ be the flip morphism: $\phi_1(a\otimes a') = a'\otimes a$, and $A\colon \mathcal{A}\to\mathcal{A}$ the identity morphism. The morphisms $\mu_1, \delta_1, \eta_1, \varepsilon_1$ are homogeneous with respective degrees $-1,-1,1,1$ and satisfy the relations of associativity, commutativity, coassociativity, co-commutativity:

$$\begin{cases}\mu_1\circ\phi_1=\mu_1\\ \mu_1\circ(\mu_1\otimes A)=\mu_1\circ(A\otimes\mu_1)\\ \mu_1\circ(\eta_1\otimes A)=A\end{cases}\qquad\begin{cases}\phi_1\circ\delta_1=\delta_1\\ (\delta_1\otimes A)\circ\delta_1=(A\otimes\delta_1)\circ\delta_1\\ (\varepsilon_1\otimes A)\circ\delta_1=(A\otimes\varepsilon_1)\circ\delta_1=A\end{cases}\tag{3-7}$$

and
$$\delta_1\circ\mu_1=(\mu_1\otimes A)\circ(A\otimes\delta_1).$$

The vector space $\mathcal{A}\otimes\mathcal{A}$ has an induced structure of graded commutative Frobenius algebra with product, coproduct, unit and co-unit:

$$\begin{aligned}\mu_1^{\otimes}&=(\mu_1\otimes\mu_1)\circ(A\otimes\phi\otimes A) &\quad \eta_1^{\otimes}&=\eta_1\otimes\eta_1\\ \delta_1^{\otimes}&=(A\otimes\phi\otimes A)\circ(\delta_1\otimes\delta_1) &\quad \varepsilon_1^{\otimes}&=\varepsilon_1\otimes\varepsilon_1.\end{aligned}\tag{3-8}$$

Let $\mathcal{B}:=\mathbb{Z}_2\{w_-,w_+\}$ be the graded $\mathbb{Z}_2$–vector space generated by two elements w_- and w_+, $\deg(w_-)=-2$, $\deg(w_+)=2$. Let's consider respectively the injection and surjection $\iota\colon\mathcal{B}\to\mathcal{A}\otimes\mathcal{A}$ and $\pi\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{B}$ defined by:

$$\begin{aligned}\iota(w_+)&=v_+\otimes v_+ \\ \iota(w_-)&=v_-\otimes v_-\end{aligned}\qquad\begin{aligned}\pi(v_+\otimes v_+)&=w_+\\ \pi(v_+\otimes v_-)&=\pi(v_-\otimes v_+)=0\\ \pi(v_-\otimes v_-)&=w_-.\end{aligned}\tag{3-9}$$

Then $\mathcal{B}$ canonically inherits from $\mathcal{A}\otimes\mathcal{A}$ of a structure of graded commutative Frobenius algebra with product, coproduct, unit and co-unit $\mu_2,\delta_2,\eta_2,\varepsilon_2$ with respective degrees $-2,-2,2,2$ satisfying

$$\mu_2=\pi\circ\mu_1^{\otimes}\circ(\iota\otimes\iota)\quad \delta_2=(\pi\otimes\pi)\circ\delta_1^{\otimes}\circ\iota\quad \eta_2=\pi\circ\eta_1^{\otimes}\quad \varepsilon_2=\varepsilon_1^{\otimes}\circ\iota.\tag{3-10}$$

The morphisms ι and π are adjoint with degree -2. We denote by $\phi_2\colon\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}\otimes\mathcal{B}$ the flip morphism $\phi_2(b\otimes b')=b'\otimes b$, and $B\colon\mathcal{B}\to\mathcal{B}$ the identity morphism.

For each (nonenhanced) state S of Γ, let's number all open components with $p\in\{1,\dots,\mathrm{op}(S)\}$ and all closed components with $q\in\{1,\dots,\mathrm{cl}(S)\}$. Then for an enhanced state $\widetilde{S}$ with underlying state S,we define the tensor product

$$t(\widetilde{S})=\bigotimes_{p=1}^{\mathrm{op}(S)}v_{\pm_{\mathrm{op}}(p)}\otimes\bigotimes_{q=1}^{\mathrm{cl}(S)}w_{\pm_{\mathrm{cl}}(q)}\in\mathcal{A}^{\otimes\mathrm{op}(S)}\otimes\mathcal{B}^{\otimes\mathrm{cl}(S)}\tag{3-11}$$

where $\pm_{\mathrm{op}}(p)$ and $\pm_{\mathrm{cl}}(q)$ are the $+$ or $-$ signs of the p–th open and the q–th closed components of $\widetilde{S}$ respectively. The degree of $t(\widetilde{S})$ does not correspond to the degree $j(\widetilde{S})$ of $\widetilde{S}$:

$$\deg(t(\widetilde{S}))=\delta_{\mathrm{op}}(\widetilde{S})+2\delta_{\mathrm{cl}}(\widetilde{S})=j(\widetilde{S})-k(\widetilde{S})=j(\widetilde{S})-k(S).\tag{3-12}$$

So we introduce the following definition:

Definition 3.4 (Translation of the degree of a graded vector space) Let $\mathcal{V} = \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_j$ be a graded $\mathbb{Z}_2$–vector space. The translated graded $\mathbb{Z}_2$–vector space $\mathcal{V}\{\ell\}$ is defined by $\mathcal{V}\{\ell\}_j = \mathcal{V}_{j-\ell}$.

Now we translate the degree of $t(\widetilde{S})$ by $k(S)$ and we define

$$\mathcal{C}(\Gamma) = (\mathcal{C}_i(\Gamma))_{i \in \mathbb{Z}} \quad \text{where } \mathcal{C}_i(\Gamma) = \bigoplus_{\substack{S \in \mathrm{St}(\Gamma) \\ i(S)=i}} (\mathcal{A}^{\otimes \mathrm{op}(S)} \otimes \mathcal{B}^{\otimes \mathrm{cl}(S)})\{k(S)\}. \tag{3-13}$$

Proposition 3.5 *The map $t\colon \widetilde{\mathrm{St}}(\Gamma) \to \mathcal{C}(\Gamma)$ defined by (3-11) extends to an isomorphism of complexes: $t\colon [\![\Gamma]\!] \to \mathcal{C}(\Gamma)$.*

Proof The incidence relations $T_i^{\pm}$, $1 \le i \le 5$ of Figures 6, 7 induce morphisms of $\mathbb{Z}_2$–vector spaces denoted by T_i, which have degree -2:

$$\begin{array}{lll} T_1 = \delta_1 \circ \mu_1 & T_3 = \mu_1 \circ (A \otimes \mu_1) \circ (A \otimes \iota) & T_5 = \mu_2 \\ T_2 = \big(A \otimes \pi\big) \circ (A \otimes \delta_1) \circ \delta_1 & T_4 = \delta_2. & \end{array} \tag{3-14}$$

The incidence relations T_6, T_7 of Figure 8 induce morphisms denoted by the same letters T_6, T_7, which have degree -1:

$$T_6 = \pi \circ \delta_1 \qquad T_7 = \mu_1. \tag{3-15}$$

More precisely, we have:

$$\begin{array}{l} T_1\colon\ \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A} \\ \quad v_+ \otimes v_+ \mapsto \begin{cases} v_+ \otimes v_- \\ +v_- \otimes v_+ \end{cases} \\ \quad \left.\begin{array}{l} v_+ \otimes v_- \\ v_- \otimes v_+ \end{array}\right\} \mapsto v_- \otimes v_- \\ \quad v_- \otimes v_- \mapsto 0 \end{array}
\qquad
\begin{array}{l} T_3\colon\ \mathcal{A}\otimes\mathcal{B} \to \mathcal{A} \\ \quad v_+ \otimes w_+ \mapsto v_+ \\ \quad v_- \otimes w_+ \mapsto v_- \\ \quad \left.\begin{array}{l} v_+ \otimes w_- \\ v_- \otimes w_- \end{array}\right\} \mapsto 0 \end{array}
\qquad
\begin{array}{l} T_5\colon\ \mathcal{B}\otimes\mathcal{B} \to \mathcal{B} \\ \quad w_+ \otimes w_+ \mapsto w_+ \\ \quad \left.\begin{array}{l} w_+ \otimes w_- \\ w_- \otimes w_+ \end{array}\right\} \mapsto w_- \\ \quad w_- \otimes w_- \mapsto 0 \end{array}$$

$$\begin{array}{l} T_2\colon\ \mathcal{A} \to \mathcal{A}\otimes\mathcal{B} \\ \quad v_+ \mapsto v_+ \otimes w_- \\ \quad v_- \mapsto v_- \otimes w_- \end{array}
\qquad
\begin{array}{l} T_4\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{B} \\ \quad w_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+ \\ \quad w_- \mapsto w_- \otimes w_- \end{array} \tag{3-16}$$

$$\begin{array}{l} T_6\colon\ \mathcal{A} \to \mathcal{B} \\ \quad v_+ \mapsto 0 \\ \quad v_- \mapsto w_- \end{array}
\qquad
\begin{array}{l} T_7\colon\ \mathcal{A}\otimes\mathcal{A} \to \mathcal{A} \\ \quad v_+ \otimes v_+ \mapsto v_+ \\ \quad \left.\begin{array}{l} v_+ \otimes v_- \\ v_- \otimes v_+ \end{array}\right\} \mapsto v_- \\ \quad v_- \otimes v_- \mapsto 0 \end{array}$$

Using these morphisms, we transfer the differential on $[\![\Gamma]\!]$ to a differential on $\mathcal{C}(\Gamma)$. Notice according to Remark 3.2 that T_1 is self-adjoint and that T_2 and T_3 (resp. T_4 and T_5) are adjoint. Moreover, T_2 and T_4 are injective whereas T_3 and T_5 are surjective. Also T_7 is surjective. The relation $d \circ d = 0$ (induced by commutative diagrams (3-4) in the proof of Proposition 3.3) is recovered using the following relations:

- Symmetry properties:

$$\begin{aligned} &T_1 \circ \phi_1 = \phi_1 \circ T_1 = T_1 && T_4 = \phi_2 \circ T_4 \\ &(A \otimes T_2) \circ \phi_1 = (\phi_1 \otimes B) \circ (A \otimes T_2) && T_5 = T_5 \circ \phi_2 \\ &\phi_1 \circ (A \otimes T_3) = (A \otimes T_3) \circ (\phi_1 \otimes B) && T_7 = T_7 \circ \phi_1 \end{aligned}$$

- Commutativity properties corresponding to the splitting of two double points:

$$\begin{aligned} &(T_1 \otimes A) \circ (A \otimes T_1) = (A \otimes T_1) \circ (T_1 \otimes A) && (T_4 \otimes B) \circ T_4 = (B \otimes T_4) \circ T_4 \\ &(A \otimes T_2) \circ T_1 = (T_1 \otimes B) \circ (A \otimes T_2) && T_5 \circ (T_5 \otimes B) = T_5 \circ (B \otimes T_5) \\ &T_1 \circ (A \otimes T_3) = (A \otimes T_3) \circ (T_1 \otimes B) && T_4 \circ T_5 = (B \otimes T_5) \circ (T_4 \otimes B) \end{aligned}$$

$$T_1 \circ T_1 = (A \otimes T_3) \circ (\phi_1 \otimes B) \circ (A \otimes T_2) = 0$$
$$(T_2 \otimes B) \circ T_2 = (A \otimes \phi_2) \circ (T_2 \otimes B) \circ T_2 = (A \otimes T_4) \circ T_2$$
$$T_3 \circ (T_3 \otimes B) = T_3 \circ (T_3 \otimes B) \circ (A \otimes \phi_2) = T_3 \circ (A \otimes T_5)$$
$$\begin{aligned} T_2 \circ T_3 &= (T_3 \otimes B) \circ (A \otimes \phi_2) \circ (T_2 \otimes B) \\ &= (T_3 \otimes B) \circ (A \otimes T_4) = (A \otimes T_5) \circ (T_2 \otimes B) \end{aligned}$$

- Commutativity properties corresponding to the splitting of a double point and a vertical tangent point:

$$\begin{aligned} &T_4 \circ T_6 = (T_6 \otimes B) \circ T_2 && T_7 \circ (A \otimes T_3) = T_3 \circ (T_7 \otimes B) \\ &T_6 \circ T_3 = T_5 \circ (T_6 \otimes B) && T_2 \circ T_7 = (T_7 \otimes B) \circ (A \otimes T_2) \\ &T_2 \circ T_7 = (A \otimes T_6) \circ T_1 && T_7 \circ T_1 = 0 = T_3 \circ (A \otimes T_6) \end{aligned}$$

- Commutativity properties corresponding to the splitting of two vertical tangent points:

$$T_1 \circ (A \otimes T_7) = (A \otimes T_7) \circ (T_1 \otimes A). \qquad \square$$

Remark 3.6 The units and counits $\eta_1, \eta_2, \varepsilon_1, \varepsilon_2$ of $\mathcal{A}$ and $\mathcal{B}$ correspond respectively to the creation of a positive open component, the creation of a positive closed component, the destruction of a negative open component and the destruction of a negative closed component. Besides, $A \otimes \varepsilon_2$ and $B \otimes \varepsilon_2$ are left inverses of T_2 and T_4 whereas

$A \otimes \eta_2$, $B \otimes \eta_2$ and $A \otimes \eta_1$ are right inverses of T_3, T_5 and T_7. In Section 4, we will often refer to these morphisms and to the following ones:

$$\begin{array}{rcl} \overline{\eta}_1\colon \mathbb{Z}_2 & \to & \mathcal{A} \\ 1 & \mapsto & v_- \end{array} \qquad \begin{array}{rcl} \overline{\eta}_2\colon \mathbb{Z}_2 & \to & \mathcal{B} \\ 1 & \mapsto & w_- \end{array} \qquad \begin{array}{rcl} \overline{\varepsilon}_1\colon \mathcal{A} & \to & \mathbb{Z}_2 \\ v_+ & \mapsto & 1 \\ v_- & \mapsto & 0 \end{array} \qquad \begin{array}{rcl} \overline{\varepsilon}_2\colon \mathcal{B} & \to & \mathbb{Z}_2 \\ w_+ & \mapsto & 1 \\ w_- & \mapsto & 0 \end{array}$$

which correspond respectively to the creation of a negative open component, the creation of a negative closed component, the destruction of a positive open component and the destruction of a positive closed component, and to the following composed morphisms:

$$\begin{array}{rcl} \tau = \eta_1\varepsilon_1\colon \mathcal{A} & \to & \mathcal{A} \\ v_+ & \mapsto & 0 \\ v_- & \mapsto & v_+ \end{array} \qquad \begin{array}{rcl} \sigma = \overline{\eta}_1\varepsilon_2\colon \mathcal{B} & \to & \mathcal{A} \\ w_+ & \mapsto & 0 \\ w_- & \mapsto & v_- \end{array}$$

3.3 Review of basic facts about complexes

Let $\mathcal{C} := (\mathcal{C}, d) = (\mathcal{C}_i, d_i)_{i\in\mathbb{Z}}$ be a complex of $\mathbb{Z}_2$–vector spaces. We denote by $\mathcal{H}(\mathcal{C})$ its homology

$$\mathcal{H}(\mathcal{C}) = (H_i)_{i\in\mathbb{Z}}, \quad H_i = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}. \tag{3-17}$$

A complex is *acyclic* if its homology is null.

Definition 3.7 (Shift of the grading of a complex) Let $(\mathcal{C}, d) = (\mathcal{C}_i, d_i)_{i\in\mathbb{Z}}$ be a complex of $\mathbb{Z}_2$–vector spaces. We define the complex

$$(\mathcal{C}, d)[k] = (\mathcal{C}[k], d[k]) \quad \text{by} \quad \mathcal{C}[k]_i = \mathcal{C}_{i-k} \quad \text{and} \quad d[k]_i = d_{i-k}.$$

(If $(\mathcal{C}, d) = \big(\bigoplus_{j\in\mathbb{Z}} \mathcal{C}_{i,j}, \bigoplus_{j\in\mathbb{Z}} d_{i,j}\big)_{i\in\mathbb{Z}}$ is a complex of graded $\mathbb{Z}_2$–vector spaces, then we can translate both the grading of the complex and the degree of the vector spaces:

$$(\mathcal{C}, d)[k]\{\ell\} = (\mathcal{C}, d)\{\ell\}[k]$$

is defined by $\quad \mathcal{C}[k]\{\ell\}_{i,j} = \mathcal{C}_{i-k,j-\ell}, \quad d[k]\{\ell\}_{i,j} = d_{i-k,j-\ell}$).

A morphism of complexes of $\mathbb{Z}_2$–vector spaces $f\colon (\mathcal{C}^0, d^0) \to (\mathcal{C}^1, d^1)$ is a sequence $f = (f_i)_{i\in\mathbb{Z}}$ of linear maps $f_i\colon \mathcal{C}^0_i \to \mathcal{C}^1_i$ such that[3]: $f\, d^0 = d^1 f$ (ie for all i, $d^1_i\, f_i = f_{i+1}\, d^0_i$).

[3] Since we are working with field $\mathbb{Z}_2$, commutativity and anticommutativity coincide so that we have equivalently $d^1 f + f d^0 = 0$.

Definition 3.8 For a morphism of complexes $f\colon (\mathcal{C}^0, d^0) \to (\mathcal{C}^1, d^1)$, the *cone* of f is the complex denoted by $\mathrm{Cone}(f) = (\mathcal{C}_i, D_i)_{i\in\mathbb{Z}}$ and defined by

$$\mathcal{C}_i = \mathcal{C}_i^0 \oplus \mathcal{C}_{i-1}^1 = \mathcal{C}_i^0 \oplus (\mathcal{C}^1[1])_i, \quad D_i = \begin{pmatrix} d_i^0 & 0 \\ f_i & d_{i-1}^1 \end{pmatrix}. \tag{3-18}$$

(Notice that $(\mathcal{C}^0, d^0)$ is a quotient-complex and $(\mathcal{C}^1, d^1)[1]$ a subcomplex of $(\mathcal{C}, D)$).

A morphism of complexes $f\colon (\mathcal{C}^0, d^0) \to (\mathcal{C}^1, d^1)$ induces an isomorphism in homology if and only if $\mathrm{Cone}(f)$ is acyclic. This is the case if f is a *homotopy equivalence* of complexes, ie there exist a morphism of complexes $g\colon (\mathcal{C}^1, d^1) \to (\mathcal{C}^0, d^0)$ and sequences $h^0 = (h_i^0)_{i\in\mathbb{Z}}, h^1 = (h_i^1)_{i\in\mathbb{Z}}$ of linear maps (homotopies) $h_i^0\colon \mathcal{C}_{i+1}^0 \to \mathcal{C}_i^0$ and $h_i^1\colon \mathcal{C}_{i+1}^1 \to \mathcal{C}_i^1$ such that

$$gf = \mathrm{id} + h^0 d^0 + d^0 h^0 \qquad \text{and} \qquad fg = \mathrm{id} + h^1 d^1 + d^1 h^1. \tag{3-19}$$
$$(\text{ie } \forall i\ g_i f_i = \mathrm{id} + h_i^0 d_i^0 + d_{i-1}^0 h_{i-1}^0 \quad \text{and} \quad f_i g_i = \mathrm{id} + h_i^1 d_i^1 + d_{i-1}^1 h_{i-1}^1).$$

Remark 3.9 As a particular case, if $h^0 = 0$, the complex $(\mathcal{C}^0, d^0)$ is called a *strong deformation retract* of $(\mathcal{C}^1, d^1)$, with inclusion map f, retraction g and homotopy map h^1. Besides, up to changing h^1 to a new homotopy h, we can always suppose that $hh = 0$, $hf = 0$ and $gh = 0$. We will assume these properties are always satisfied in the definition of strong deformation retraction.

Proposition 3.10 *Let $(\bar{\mathcal{C}}^1, \bar{d}^1)$ be a strong deformation retract of $(\mathcal{C}^1, d^1)$ with retraction r, inclusion j and homotopy map h such that $hh = 0$, $rh = 0$, $hj = 0$. Let $f\colon (\mathcal{C}^0, d^0) \to (\mathcal{C}^1, d^1)$ be a morphism of complexes. Then $\mathrm{Cone}(rf)$ is a strong deformation retract of $\mathrm{Cone}(f)$ with*

$$\textit{retraction} \quad R = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & r \end{pmatrix},$$
$$\textit{inclusion} \quad J = \begin{pmatrix} \mathrm{id} & 0 \\ hf & j \end{pmatrix},$$
$$\textit{homotopy} \quad H = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}$$

such that $HH = 0$, $RH = 0$ and $HJ = 0$.

Proof Immediate. □

A *double complex* $(\mathcal{C}, d, \partial)$ is a sequence of complexes $(\mathcal{C}^k, d^k)_{k\in\mathbb{Z}}$ of $\mathbb{Z}_2$–vector spaces and morphisms of complexes $(\partial^k)_{k\in\mathbb{Z}}$:

$$\cdots \xrightarrow{\partial^{k-1}} (\mathcal{C}^k, d^k) \xrightarrow{\partial^k} (\mathcal{C}^{k+1}, d^{k+1}) \xrightarrow{\partial^{k+1}} (\mathcal{C}^{k+2}, d^{k+2}) \xrightarrow{\partial^{k+2}} \cdots$$

such that for all $k \in \mathbb{Z}$, $\partial^{k+1}\partial^k = 0$. A *morphism of two double complexes* is a sequence of morphisms of complexes $f = (f^k)_{k\in\mathbb{Z}}$:

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\overline{\partial}^{k-1}} & (\overline{\mathcal{C}}^k, \overline{d}^k) & \xrightarrow{\overline{\partial}^k} & (\overline{\mathcal{C}}^{k+1}, \overline{d}^{k+1}) & \xrightarrow{\overline{\partial}^{k+1}} & (\overline{\mathcal{C}}^{k+2}, \overline{d}^{k+2}) & \xrightarrow{\overline{\partial}^{k+2}} \cdots \\
& \Big\downarrow{\scriptstyle f^k} & & \Big\downarrow{\scriptstyle f^{k+1}} & & \Big\downarrow{\scriptstyle f^{k+2}} & \\
\cdots \xrightarrow{\partial^{k-1}} & (\mathcal{C}^k, d^k) & \xrightarrow{\partial^k} & (\overline{\mathcal{C}}^{k+1}, d^{k+1}) & \xrightarrow{\partial^{k+1}} & (\mathcal{C}^{k+2}, d^{k+2}) & \xrightarrow{\partial^{n-1}} \cdots
\end{array}$$

such that for all $k \in \mathbb{Z}$, $f^{k+1}\overline{\partial}^k = \partial^k f^k$. We also have notions of homotopy equivalence and strong deformation retraction for double complex. A morphism of double complexes $f\colon (\overline{\mathcal{C}}, \overline{d}, \overline{\partial}) \to (\mathcal{C}, d, \partial)$ is a *homotopy equivalence (of double complexes)* if there exists a morphism of double complexes $g\colon (\mathcal{C}, d, \partial) \to (\overline{\mathcal{C}}, \overline{d}, \overline{\partial})$ and homotopy maps $\overline{h} = (\overline{h}^k)_{k\in\mathbb{Z}}$, $h = (h^k)_{k\in\mathbb{Z}}$ (sequences of morphisms of complexes) $\overline{h}^k\colon (\overline{\mathcal{C}}^{k+1}, \overline{d}^{k+1}) \to (\overline{\mathcal{C}}^k, \overline{d}^k)$ and $\overline{h}^k\colon (\mathcal{C}^{k+1}, d^{k+1}) \to (\mathcal{C}^k, d^k)$ such that for all k

$$g^k f^k = \mathrm{id} + \overline{h}^k\overline{\partial}^k + \overline{\partial}^{k-1}\overline{h}^{k-1} \quad \text{and} \quad f^k g^k = \mathrm{id} + h^k\partial^k + \partial^{k-1}h^{k-1}.$$

If $\overline{h} = 0$, $(\overline{\mathcal{C}}, \overline{d}, \overline{\partial})$ is called a *strong deformation retract* of the double complex $(\mathcal{C}, d, \partial)$ with inclusion f and retraction g. Again, up to changing the homotopy h, we assume that it satisfies: $hh = 0$, $hf = 0$ and $gh = 0$.

Now we extend the definition of cone to a finite sequence of morphisms of complexes. A double complex $(\mathcal{C}, d, \partial)$ is ∂–finite if $(\mathcal{C}^k, d^k)$ is trivial except for a finite number of values of k.

Definition 3.11 Let

$$(\mathcal{C}^0, d^0) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} (\mathcal{C}^n, d^n)$$

be a ∂–finite double complex. Let's denote

$$\begin{array}{rccl}
\widetilde{\partial}^0_i\colon & \mathcal{C}^0_i & \to & \mathcal{C}^1_i \oplus \mathcal{C}^2_{i-1} \oplus \cdots \oplus \mathcal{C}^n_{i-n+1} \\
& u & \mapsto & (\partial^0_i(u), 0, \ldots, 0).
\end{array}$$

Then the *cone* of $(\partial^0, \ldots, \partial^{n-1})$ is the complex defined by

$$\mathrm{Cone}(\partial^0, \ldots, \partial^{n-1}) = \mathrm{Cone}(\widetilde{\partial}^0, \mathrm{Cone}(\partial^1, \ldots, \partial^{n-1})). \tag{3-20}$$

Suppose that $f=(f^k)_{0\le k\le n}$ is a morphism from a $\overline{\partial}$–finite double complexes $(\overline{\mathcal{C}},\overline{d},\overline{\partial})$ to a ∂–finite $(\mathcal{C},d,\partial)$:

$$(\overline{\mathcal{C}}^0,\overline{d}^0)\xrightarrow{\overline{\partial}^0}\cdots\xrightarrow{\overline{\partial}^{n-1}}(\overline{\mathcal{C}}^n,\overline{d}^n)\quad\text{and}\quad(\mathcal{C}^0,d^0)\xrightarrow{\partial^0}\cdots\xrightarrow{\partial^{n-1}}(\mathcal{C}^n,d^n).$$

Let's set $F_i=f_i^0\oplus f_{i-1}^1\oplus\cdots\oplus f_{i-n}^n$. Then f induces a morphism of complexes

$$C(f)=(F_i)_{i\in\mathbb{Z}}\colon \operatorname{Cone}(\overline{\partial}^0,\dots,\overline{\partial}^{n-1})\longrightarrow\operatorname{Cone}(\partial^0,\dots,\partial^{n-1}).$$

If f^k are isomorphisms, $C(f)$ is also an isomorphism.

Proposition 3.12 *If $(\overline{\mathcal{C}},\overline{d},\overline{\partial})$ is a $\overline{\partial}$–finite double complex, $(\mathcal{C},d,\partial)$ a ∂–finite double complex and $f\colon(\overline{\mathcal{C}},\overline{d},\overline{\partial})\to(\mathcal{C},d,\partial)$ a homotopy equivalence with inverse g, then*

$$C(f)\colon\operatorname{Cone}(\overline{\partial}^0,\dots,\overline{\partial}^{n-1})\longrightarrow\operatorname{Cone}(\partial^0,\dots,\partial^{n-1})$$

is a homotopy equivalence of complexes with inverse $C(g)$. So $C(f)$ induces an isomorphism in homology.

Furthermore if $(\overline{\mathcal{C}},\overline{d},\overline{\partial})$ is a strong deformation retract of $(\mathcal{C},d,\partial)$ with inclusion map f and retraction g then $\operatorname{Cone}(\overline{\partial}^0,\dots,\overline{\partial}^{n-1})$ is a strong deformation retract of $\operatorname{Cone}(\partial^0,\dots,\partial^{n-1})$ with inclusion map $C(f)$ and retraction $C(g)$, and so $C(f)$ induces an isomorphism in homology.

Proof Let $\overline{h}=(\overline{h}^k)_{1\le k<n}$ and $h=(h^k)_{0\le k<n}$ be homotopies associated with f and g. Then we have:

$$\begin{array}{ll}\overline{h}_{i+1}^{k-1}\overline{d}_i^k=\overline{d}_i^{k-1}\overline{h}_i^{k-1} & \overline{h}_i^k\overline{\partial}_i^k+\overline{\partial}_i^{k-1}\overline{h}_i^{k-1}+\mathrm{id}=g_i^kf_i^k\\ h_{i+1}^{k-1}d_i^k=d_i^{k-1}h_i^{k-1} & h_i^k\partial_i^k+\partial_i^{k-1}h_i^{k-1}+\mathrm{id}=f_i^kg_i^k\end{array}$$

Let $H=(H_i)_{i\in\mathbb{Z}}$ be the sequence of linear maps defined by

$$\begin{aligned}H_i\colon \mathcal{C}_{i+1}^0\oplus\mathcal{C}_i^1\oplus\cdots\oplus\mathcal{C}_{i-n+1}^n&\longrightarrow\mathcal{C}_i^0\oplus\mathcal{C}_{i-1}^1\oplus\cdots\oplus\mathcal{C}_{i-n}^n\\(x_0,x_1,\dots,x_n)&\longmapsto(h_i^0(x_1),h_{i-1}^1(x_2),\dots,h_{i-n+1}^{n-1}(x_n),0)\end{aligned}$$

and $\overline{H}=(\overline{H}_i)_{i\in\mathbb{Z}}$ defined analogously on $(\overline{\mathcal{C}},\overline{d},\overline{\partial})$. Then if $\overline{D}$ and D are the differentials of $\operatorname{Cone}(\overline{\partial}^0,\dots,\overline{\partial}^{n-1})$ and $\operatorname{Cone}(\partial^0,\dots,\partial^{n-1})$, we have

$$FG=\mathrm{id}+HD+DH\quad\text{and}\quad GF=\mathrm{id}+\overline{H}\,\overline{D}+\overline{D}\,\overline{H}.\qquad\square$$

3.4 Fundamental splitting lemmas

Let $(\varGamma, \epsilon)$ be an OMS–divide or a cuspidal divide. Let p be a double point or a vertical tangent point of $\varGamma$. Let $\varGamma^0$ and $\varGamma^1$ be the cuspidal divides obtained from $\varGamma$ by applying Θ_0 and Θ_1 at p respectively. Then each enhanced state of $\varGamma$ can be identified with either an enhanced state of $\varGamma^0$ or $\varGamma^1$ ie

$$\widetilde{\mathrm{St}}(\varGamma) \overset{1-1}{\simeq} \widetilde{\mathrm{St}}(\varGamma^0) \sqcup \widetilde{\mathrm{St}}(\varGamma^1). \tag{3-21}$$

Consequently, $[\![\varGamma^0]\!]$ and $[\![\varGamma^1]\!]$ can be seen as sub–vector spaces of $[\![\varGamma]\!]$ up to translations of the grading i and the degree j. More precisely, we have:

Lemma 3.13 *Let d be the differential of $[\![\varGamma]\!]$.*

(1) *If p is a positive double point then d induces the differentials d^0 and d^1 of $[\![\varGamma^0]\!]\{2\}$ and $[\![\varGamma^1]\!]\{4\}$ and a morphism*

$$[\![\varGamma^0]\!]\{2\} \xrightarrow{d^\bullet} [\![\varGamma^1]\!]\{4\}$$

such that $[\![\varGamma]\!] = \mathrm{Cone}(d^\bullet)$.

(2) *If p is a negative double point then d induces the differentials d^0 and d^1 of $[\![\varGamma^0]\!]\{-2\}$ and $[\![\varGamma^1]\!]\{-4\}$ and a morphism*

$$[\![\varGamma^1]\!]\{-4\} \xrightarrow{d^\bullet} [\![\varGamma^0]\!]\{-2\}$$

such that $[\![\varGamma]\!] = \mathrm{Cone}(d^\bullet)[-1]$.

(3) *If p is vertical tangent point, then d induces the differentials d^0 and d^1 of $[\![\varGamma^0]\!]\{1\}$ and $[\![\varGamma^1]\!]\{2\}$ and a morphism*

$$[\![\varGamma^0]\!]\{1\} \xrightarrow{d^\bullet} [\![\varGamma^1]\!]\{2\}$$

such that $[\![\varGamma]\!] = \mathrm{Cone}(d^\bullet)$.

Proof Suppose that p is a positive double point of $\varGamma$. Then $\varGamma^0$ and $\varGamma^1$ have one positive double point less than $\varGamma$ so that the writhes of $\varGamma$, $\varGamma^0$ and $\varGamma^1$ are related by

$$w(\varGamma) = w(\varGamma^0) + 2 = w(\varGamma^1) + 2$$

Let $\widetilde{S}$ be an enhanced state of $\widetilde{\mathrm{St}}_i(\varGamma)$ with degree $j = j(\widetilde{S})$. If $\widetilde{S}$ is obtained from $\varGamma$ using Θ_0 (resp. Θ_1) at p, then $\widetilde{S}$ can be seen as an enhanced state of $\widetilde{\mathrm{St}}_i(\varGamma^0)$ with degree $j-2$ (resp. of $\widetilde{\mathrm{St}}_{i-1}(\varGamma^1)$ with degree $j-4$). Besides, if $\widetilde{S}_0 \in \widetilde{\mathrm{St}}_i(\varGamma)$ and $\widetilde{S}_1 \in \widetilde{\mathrm{St}}_{i+1}(\varGamma)$ are adjacent enhanced states of degrees j then it involves three cases:

- Either $\widetilde{S}_0 \in \widetilde{\mathrm{St}}_i(\Gamma^0)$ and $\widetilde{S}_1 \in \widetilde{\mathrm{St}}_{i+1}(\Gamma^0)$ are adjacent enhanced states of Γ^0 with degrees $j-2$, so the differential d^0 of $[\![\Gamma^0]\!]\{2\}$ coincides with the restriction of d to $[\![\Gamma^0]\!]\{2\}$; or
- $\widetilde{S}_0 \in \widetilde{\mathrm{St}}_{i-1}(\Gamma^1)$ and $\widetilde{S}_1 \in \widetilde{\mathrm{St}}_i(\Gamma^1)$ are adjacent enhanced states of Γ^1 with degrees $j-4$, so the differential d^1 of $[\![\Gamma^1]\!]\{4\}[1]$ coincides with the restriction of d to $[\![\Gamma]\!]\{4\}[1]$; or
- $\widetilde{S}_0 \in \widetilde{\mathrm{St}}_i(\Gamma^0)$ with degree $j-2$ and $\widetilde{S}_1 \in \widetilde{\mathrm{St}}_i(\Gamma^1)$ with degree $j-4$, then d induces a map $d^\bullet\colon [\![\Gamma^0]\!]\{2\} \to [\![\Gamma^1]\!]\{4\}$ which is a morphism of complexes since from the proof of Proposition 3.3 $dd=0$ implies $d^\bullet d^0 = d^1 d^\bullet$.

Hence $[\![\Gamma]\!] = \mathrm{Cone}(d^\bullet)$. Similar arguments hold for the two other cases. □

More generally, consider $k = k_+ + k_- + k_0$ double vertical tangent points $p_1, \dots, p_k$ such that the k_+ first ones are positive double points, the next k_- ones negative double points and the last k_0 ones vertical tangent points. For each words $[a] = a_1 a_2 \cdots a_{k_+}$, $a_i \in \{0,1\}$, $[b] = b_1 b_2 \cdots b_{k_-}$, $b_i \in \{0,1\}$, $[c] = c_1 c_2 \cdots c_{k_0}$, $c_i \in \{0,1\}$, let $[a][b][c]$ be the word obtained by concatenation of $[a],[b],[c]$ and denote by $(\Gamma^{[a][b][c]}, \epsilon^{[a][b][c]})$ the cuspidal divide obtained from (Γ,ϵ) by performing:

$$\begin{cases} \Theta_{a_i} \text{ splitting at } p_i \text{ for } 1 \le i \le k_+ \\ \Theta_{b_i} \text{ splitting at } p_i \text{ for } k_+ < i \le k_+ + k_- \\ \Theta_{c_i} \text{ splitting at } p_i \text{ for } k_+ + k_- < i \le k = k_+ + k_- + k_0. \end{cases}$$

Let $1_{[a]}$, $1_{[b]}$ and $1_{[c]}$ be the numbers of occurrences of 1 in $[a]$, $[b]$ and $[c]$ and $\mathrm{gr}([a][b][c]) = 1_{[a]} - 1_{[b]} + 1_{[c]}$. By restriction, the differential d of $[\![\Gamma]\!]$ coincides with the differential $d^{[a][b][c]}$ of $[\![\Gamma^{[a][b][c]}]\!]$. By iterating Lemma 3.13, using same arguments, just following the incidence relations, we have:

Lemma 3.14 *For each ℓ, $-k_- \le \ell \le k_+ + k_0$, we can identify the complex*

$$(\mathcal{C}^\ell, D^\ell) = \bigoplus_{\mathrm{gr}([a][b][c])=\ell} [\![\Gamma^{[a][b][c]}]\!]\big\{2(1_{[a]} - 1_{[b]} + k_+ - k_-) + 1_{[c]} + k_0\big\}$$

with a subquotient-complex of $[\![\Gamma]\!]$, with differential

$$D^\ell = \bigoplus_{\mathrm{gr}([a][b][c])=\ell} d^{[a][b][c]}.$$

The differential d *induces a structure of double complex*

$$\big(\mathcal{C}^{-k_-}, D^{-k_-}\big) \xrightarrow{\Delta^{-k_-}} \big(\mathcal{C}^{-k_-+1}, D^{-k_-+1}\big) \xrightarrow{\Delta^{-k_-+1}} \cdots \xrightarrow{\Delta^{k_++k_0-1}} \big(\mathcal{C}^{k_++k_0}, D^{k_++k_0}\big)$$

such that $[\![\Gamma]\!] = \mathrm{Cone}\,\big(\Delta^{-k_-}, \dots, \Delta^{k_++k_0-1}\big)[-k_-]$.

In the sequel, such a double complex will be called a *splitting diagram* of $[\![\Gamma]\!]$.

3.5 Khovanov homology of OMS–divides

Definition 3.15 We call *Khovanov homology* $\mathcal{H}(\Gamma)$ of an OMS–divide (or a cuspidal divide) $\Gamma = (\Gamma, \epsilon)$ the homology of the complex $[\![\Gamma]\!] = \big([\![\Gamma]\!]_i\big)_{i\in\mathbb{Z}}$:

$$\begin{gathered}\mathcal{H}(\Gamma) = \big(\mathcal{H}_i(\Gamma)\big)_{i\in\mathbb{Z}}, \quad \mathcal{H}_i(\Gamma) = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}_{i,j}(\Gamma), \\ \mathcal{H}_{i,j}(\Gamma) = \mathrm{Ker}\, d_{i,j} \,/\, \mathrm{Im}\, d_{i-1,j}.\end{gathered} \tag{3-22}$$

Proposition 3.16 *If* $\Gamma = (\Gamma, \epsilon)$ *is an OMS–divide, then the polynomial* W_Γ *and the graded Euler characteristics of* $\mathcal{H}(\Gamma)$ *are related by*

$$W_\Gamma(t^2) = \frac{t}{1+t^2}\, \chi_{\mathrm{gr}}(\mathcal{H}(\Gamma)) = \frac{t}{1+t^2} \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathrm{gr}} \mathcal{H}_i(\Gamma) \tag{3-23}$$

where the graded dimension is $\dim_{\mathrm{gr}} \mathcal{H}_i(\Gamma) = \sum_{j\in\mathbb{Z}} t^j \dim_{\mathbb{Z}_2} \mathcal{H}_{i,j}(\Gamma)$.

Proof Immediate from formula (2-5). □

We can now formulate our main theorem:

Theorem 3.17 *Khovanov homology of OMS–divides is invariant under* $\mathcal{M}$*–equivalence.*

Combined with Theorem 1.2, we obtain:

Corollary 3.18 *Khovanov homology of OMS–divides is an invariant for strong equivalence of strongly invertible links.*

Section 4 is devoted to the proof of this theorem. Notice that from Proposition 3.16, Theorem 3.17 is a refinement of Proposition 2.3.

3.6 Examples

(1) Figure 9 shows a divide for the link 3_1 and its splitting diagram.

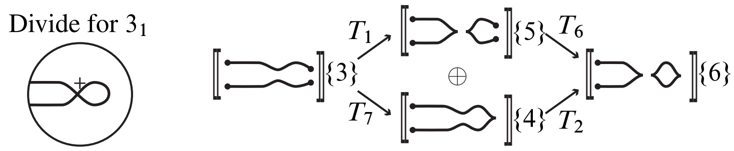

Figure 9

The associated complex and homology entries are:

$$(\mathcal{A}\otimes\mathcal{A})\{3\}\xrightarrow{d^0}(\mathcal{A}\otimes\mathcal{A})\{5\}\oplus\mathcal{A}\{4\}\xrightarrow{d^1}(\mathcal{A}\otimes\mathcal{B})\{6\}$$

$i \setminus j$	1	3	5	7	9
0	$\mathbb{Z}_2$	$\mathbb{Z}_2$			
1			$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2				$\mathbb{Z}_2$	$\mathbb{Z}_2$

(2) Figure 10 shows a divide for the link 4_1 and its splitting diagram.

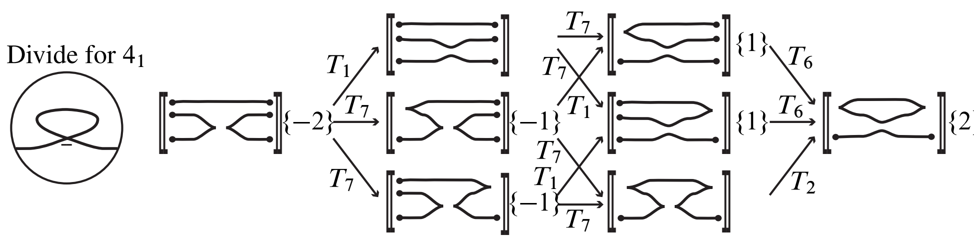

Figure 10

The associated complex and homology entries are:

$$(\mathcal{A}^{\otimes 3})\{-2\}\xrightarrow{d^{-1}}(\mathcal{A}^{\otimes 3})\oplus(\mathcal{A}^{\otimes 2}\oplus\mathcal{A}^{\otimes 2})\{-1\}\xrightarrow{d^0}(\mathcal{A}^{\otimes 2}\oplus\mathcal{A}^{\otimes 2})\{1\}\oplus\mathcal{A}\xrightarrow{d^1}(\mathcal{A}\otimes\mathcal{B})\{2\}$$

$i \setminus j$	−5	−3	−1	1	3	5
−1	$\mathbb{Z}_2$	$\mathbb{Z}_2$				
0		$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2$		
1				$\mathbb{Z}_2$	$\mathbb{Z}_2$	
2					$\mathbb{Z}_2$	$\mathbb{Z}_2$

4 Invariance under $\mathcal{M}$–equivalence

4.1 Invariance under type I moves

Let Γ and $\widetilde{\Gamma}$ be OMS–divides which differ only by a type I move (see Figure 11).

Figure 11: Type I move

Proposition 4.1 *The complexes* $[\![\Gamma]\!]$ *and* $[\![\widetilde{\Gamma}]\!]$ *have the same homology.*

Let's denote by Γ^{st} $\big($resp. $\widetilde{\Gamma}^{st}\big)$, $s,t \in \{0,1\}$ the cuspidal divides obtained by performing Θ_s, Θ_t splittings respectively at the $+$ double point and the vertical tangent point of Γ $\big($resp. of $\widetilde{\Gamma}\big)$ in Figure 11, without changing any other singular point of these divides. From Lemma 3.14, we have splitting diagrams given in Figure 12.

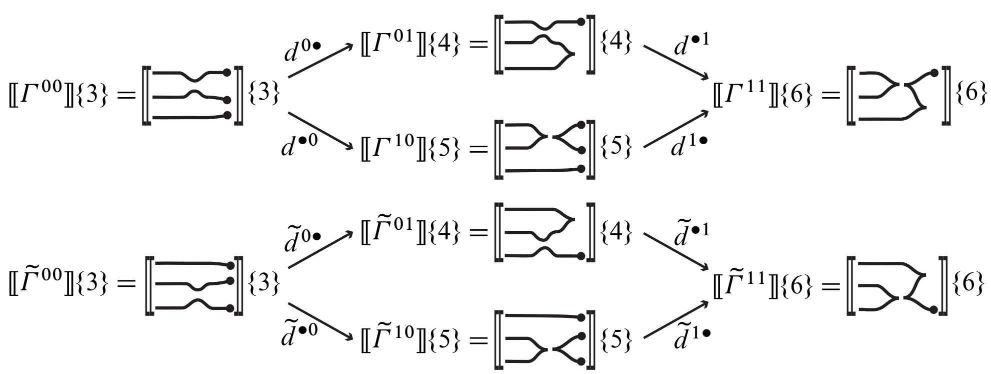

Figure 12: Splitting diagram for type I move

Notice that $[\![\Gamma^{00}]\!] = [\![\widetilde{\Gamma}^{00}]\!]$. In other words we have:

Lemma 4.2 *Let's denote*

$$\Delta^0 = \begin{pmatrix} d^{0\bullet} \\ d^{\bullet 0} \end{pmatrix}, \quad \Delta^1 = \begin{pmatrix} d^{\bullet 1} & d^{1\bullet} \end{pmatrix}, \quad \widetilde{\Delta}^0 = \begin{pmatrix} \widetilde{d}^{0\bullet} \\ \widetilde{d}^{\bullet 0} \end{pmatrix} \quad \text{and} \quad \widetilde{\Delta}^1 = \begin{pmatrix} \widetilde{d}^{\bullet 1} & \widetilde{d}^{1\bullet} \end{pmatrix}.$$

Then $[\![\Gamma]\!] = \mathrm{Cone}(\Delta^0, \Delta^1)$ *and* $[\![\widetilde{\Gamma}]\!] = \mathrm{Cone}(\widetilde{\Delta}^0, \widetilde{\Delta}^1)$*:*

$$\begin{array}{c} [\![\Gamma^{00}]\!]\{3\} \xrightarrow{\Delta^0} [\![\Gamma^{01}]\!]\{4\} \oplus [\![\Gamma^{10}]\!]\{5\} \xrightarrow{\Delta^1} [\![\Gamma^{11}]\!]\{6\} \\ \parallel \\ [\![\widetilde{\Gamma}^{00}]\!]\{3\} \xrightarrow{\widetilde{\Delta}^0} [\![\widetilde{\Gamma}^{01}]\!]\{4\} \oplus [\![\widetilde{\Gamma}^{10}]\!]\{5\} \xrightarrow{\widetilde{\Delta}^1} [\![\widetilde{\Gamma}^{11}]\!]\{6\} \end{array} \tag{4-1}$$

Let's consider the "creation and destruction" morphisms (see Remark 3.6)

$$\begin{array}{lll} [\![\Gamma^{11}]\!]\{6\} \xrightarrow{\eta_1} [\![\Gamma^{10}]\!]\{5\} & [\![\Gamma^{01}]\!]\{4\} \xrightarrow{\overline{\eta}_1} [\![\widetilde{\Gamma}^{10}]\!]\{5\} & [\![\widetilde{\Gamma}^{10}]\!]\{5\} \xrightarrow{\widetilde{\varepsilon}_1} [\![\Gamma^{01}]\!]\{4\} \\ [\![\widetilde{\Gamma}^{11}]\!]\{6\} \xrightarrow{\widetilde{\eta}_1} [\![\widetilde{\Gamma}^{10}]\!]\{5\} & [\![\widetilde{\Gamma}^{01}]\!]\{4\} \xrightarrow{\overline{\overline{\eta}}_1} [\![\Gamma^{10}]\!]\{5\} & [\![\Gamma^{10}]\!]\{5\} \xrightarrow{\varepsilon_1} [\![\widetilde{\Gamma}^{01}]\!]\{4\} \end{array}$$

defined by Figure 13.

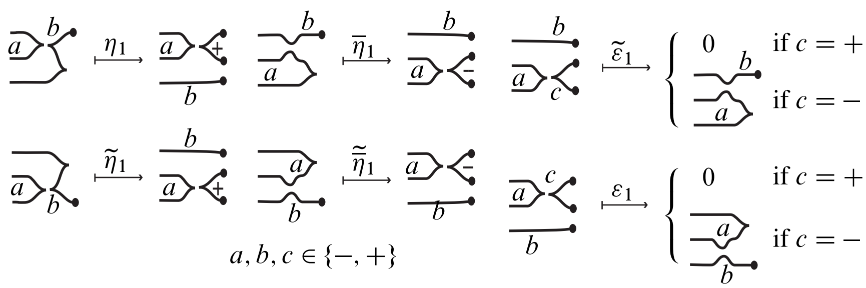

Figure 13

Lemma 4.3 *The two sequences*

$$\begin{array}{c} 0 \longrightarrow [\![\Gamma^{11}]\!]\{6\} \xrightarrow{\eta_1} [\![\Gamma^{10}]\!]\{5\} \xrightarrow{\varepsilon_1} [\![\widetilde{\Gamma}^{01}]\!]\{4\} \longrightarrow 0 \\ 0 \longrightarrow [\![\widetilde{\Gamma}^{11}]\!]\{6\} \xrightarrow{\widetilde{\eta}_1} [\![\widetilde{\Gamma}^{10}]\!]\{5\} \xrightarrow{\widetilde{\varepsilon}_1} [\![\Gamma^{01}]\!]\{4\} \longrightarrow 0 \end{array} \tag{4-2}$$

are exact and $d^{1\bullet}$, $\overline{\overline{\eta}}_1$, $\widetilde{d}^{1\bullet}$ *and* $\overline{\eta}_1$ *are respectively splitting morphisms of* η_1, ε_1, $\widetilde{\eta}_1$ *and* $\widetilde{\varepsilon}_1$*:*

$$\begin{array}{lll} d^{1\bullet}\eta_1 = \mathrm{id}, & \varepsilon_1\overline{\overline{\eta}}_1 = \mathrm{id}, & \eta_1 d^{1\bullet} + \overline{\overline{\eta}}_1\varepsilon_1 = \mathrm{id} + \eta_1 d^{1\bullet}\overline{\overline{\eta}}_1\varepsilon_1 \\ \widetilde{d}^{1\bullet}\widetilde{\eta}_1 = \mathrm{id}, & \widetilde{\varepsilon}_1\overline{\eta}_1 = \mathrm{id}, & \widetilde{\eta}_1\widetilde{d}^{1\bullet} + \overline{\eta}_1\widetilde{\varepsilon}_1 = \mathrm{id} + \widetilde{\eta}_1\widetilde{d}^{1\bullet}\overline{\eta}_1\widetilde{\varepsilon}_1 \end{array} \tag{4-3}$$

Moreover

$$\varepsilon_1 d^{\bullet 0} = \widetilde{d}^{0\bullet} \quad \text{and} \quad \widetilde{\varepsilon}_1\widetilde{d}^{\bullet 0} = d^{0\bullet}. \tag{4-4}$$

Proof The morphisms $d^{1\bullet}$ and $\widetilde{d}^{1\bullet}$ correspond to T_7, $d^{\bullet 0}$ and $\widetilde{d}^{\bullet 0}$ to T_1 or T_2, $d^{0\bullet}$ and $\widetilde{d}^{0\bullet}$ to T_6 or T_4 (see (3-16)). Then the result is an consequence of the comments and the definitions made in Remark 3.6 (see also Figure 6 and Figure 8). Indeed, with the notation of this remark, relations (4-3) correspond to

$$T_7 \circ (A \otimes \eta_1) = \mathrm{id} \qquad \varepsilon_1 \circ \overline{\eta}_1 = \mathrm{id}$$
$$(A \otimes \eta_1) \circ T_7 + (A \otimes (\overline{\eta}_1 \varepsilon_1)) = \mathrm{id} + (A \otimes \eta_1) \circ T_7 \circ (A \otimes (\overline{\eta}_1 \varepsilon_1))$$

and relations (4-4) to $(\varepsilon_1 \otimes A) \circ T_1 = T_6$ or $(\varepsilon_1 \otimes B) \circ T_2 = T_4$. $\square$

Proof of Proposition 4.1 Consider the diagram

$$\begin{tikzcd}
\llbracket \Gamma^{00} \rrbracket\{3\} \arrow[r, "\Delta^0"] \arrow[d, leftrightarrow, "\mathrm{id}"] & \llbracket \Gamma^{01} \rrbracket\{4\} \oplus \llbracket \Gamma^{10} \rrbracket\{5\} \arrow[r, "\Delta^1"] \arrow[d, shift right, "F"'] & \llbracket \Gamma^{11} \rrbracket\{6\} \arrow[l, bend left, "H"] \arrow[d, leftrightarrow, "0"] \\
\llbracket \widetilde{\Gamma}^{00} \rrbracket\{3\} \arrow[r, "\widetilde{\Delta}^0"] & \llbracket \widetilde{\Gamma}^{01} \rrbracket\{4\} \oplus \llbracket \widetilde{\Gamma}^{10} \rrbracket\{5\} \arrow[r, "\widetilde{\Delta}^1"] \arrow[u, shift right, "\widetilde{F}"'] & \llbracket \widetilde{\Gamma}^{11} \rrbracket\{6\} \arrow[l, bend left, "\widetilde{H}"]
\end{tikzcd}$$

where $H = \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix}$ $\qquad$ $\widetilde{H} = \begin{pmatrix} 0 \\ \widetilde{\eta}_1 \end{pmatrix}$

$$F = \begin{pmatrix} 0 & \varepsilon_1 \\ \overline{\eta}_1 + \widetilde{\eta}_1 \widetilde{d}^{1\bullet} \overline{\eta}_1 & \widetilde{\eta}_1 \widetilde{d}^{\bullet 1} \varepsilon_1 \end{pmatrix} \qquad \widetilde{F} = \begin{pmatrix} 0 & \widetilde{\varepsilon}_1 \\ \overline{\widetilde{\eta}}_1 + \eta_1 d^{1\bullet} \overline{\widetilde{\eta}}_1 & \eta_1 d^{\bullet 1} \widetilde{\varepsilon}_1 \end{pmatrix}.$$

From Lemmas 4.2 and 4.3, we have

$$F\Delta^0 = \widetilde{\Delta}^0 \quad \widetilde{\Delta}^1 F = 0 \quad \widetilde{F} F = \mathrm{id} + H\Delta^1 \quad \Delta^1 H = \mathrm{id}$$
$$\widetilde{F}\widetilde{\Delta}^0 = \Delta^0 \quad \Delta^1 \widetilde{F} = 0 \quad F\widetilde{F} = \mathrm{id} + \widetilde{H}\widetilde{\Delta}^1 \quad \widetilde{\Delta}^1 \widetilde{H} = \mathrm{id}.$$

Hence vertical arrows define a homotopy equivalence. From Proposition 3.12, $\llbracket \Gamma \rrbracket = \mathrm{Cone}(\Delta^0, \Delta^1)$ and $\llbracket \widetilde{\Gamma} \rrbracket = \mathrm{Cone}(\widetilde{\Delta}^0, \widetilde{\Delta}^1)$ have the same homology. $\square$

4.2 Invariance under type II move

Let Γ and Γ_0 be OMS–divides which differ only by a type II move (see Figure 14).

$\Gamma =$ $\qquad$ $\Gamma_0 =$

Figure 14: Type II move

Proposition 4.4 *The complexes* $[\![\Gamma]\!]$ *and* $[\![\Gamma_0]\!]$ *have the same homology.*

Let Γ^{st}, $s,t \in \{0,1\}$ be the cuspidal divides obtained by performing Θ_s and Θ_t splittings respectively at the $+$ and the $-$ double points of Γ in Figure 14 without changing any other singular point. From Lemma 3.14 we have a splitting diagram given by Figure 15.

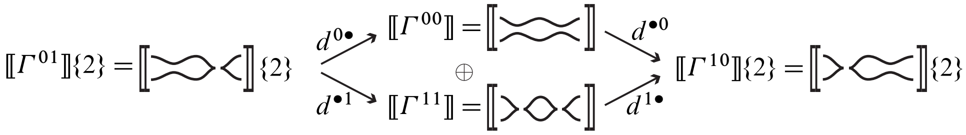

Figure 15: Splitting diagram for type II move

We remark that $[\![\Gamma^{00}]\!] = [\![\Gamma_0]\!]$ and we have the following lemma:

Lemma 4.5 $[\![\Gamma]\!] = \mathrm{Cone}(\Delta^0, \Delta^1)[-1]$ *where*

$$[\![\Gamma^{01}]\!]\{-2\} \xrightarrow{\Delta^0} [\![\Gamma^{00}]\!] \oplus [\![\Gamma^{11}]\!] \xrightarrow{\Delta^1} [\![\Gamma^{10}]\!]\{2\} \qquad \Delta^0 = \begin{pmatrix} d^{0\bullet} \\ d^{\bullet 1} \end{pmatrix} \quad \Delta^1 = \begin{pmatrix} d^{\bullet 0} & d^{1\bullet} \end{pmatrix}.$$

Let's consider the "destruction and creation" morphisms of complexes (see Remark 3.6) defined by Figure 16.

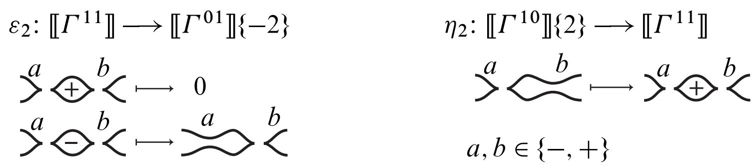

Figure 16

Lemma 4.6 *The sequence*

$$0 \longrightarrow [\![\Gamma^{10}]\!]\{2\} \xrightarrow{\eta_2} [\![\Gamma^{11}]\!] \xrightarrow{\varepsilon_2} [\![\Gamma^{01}]\!]\{-2\} \longrightarrow 0$$

is exact and $d^{1\bullet}$, $d^{\bullet 1}$ *are respectively splitting morphisms of* η_2, ε_2*:*

$$\varepsilon_2 d^{\bullet 1} = \mathrm{id}, \quad d^{1\bullet}\eta_2 = \mathrm{id}, \quad d^{\bullet 1}\varepsilon_2 + \eta_2 d^{1\bullet} = \mathrm{id} + \eta_2 d^{1\bullet} d^{\bullet 1}\varepsilon_2. \tag{4-5}$$

Proof The morphism $d^{\bullet 1}$ corresponds to T_2 or T_4 and the morphism $d^{1\bullet}$ to T_3 or T_5 in (3-16). The lemma is a direct consequence of the comments and the definitions made in Remark 3.6 (see also Figure 6 and Figure 7). □

Proof of Proposition 4.4 Consider the diagram

$$\begin{tikzcd}
 & {[\![\Gamma^{00}]\!] = [\![\Gamma_0]\!]} \arrow[d, shift left, "J"] & \\
{[\![\Gamma^{01}]\!]\{-2\}} \arrow[r, "\Delta^0"] & {[\![\Gamma^{00}]\!] \oplus [\![\Gamma^{11}]\!]} \arrow[l, bend left, "H^0"] \arrow[r, "\Delta^1"] \arrow[u, shift left, "R"] & {[\![\Gamma^{10}]\!]\{2\}} \arrow[l, bend left, "H^1"]
\end{tikzcd}$$

where
$$J = \begin{pmatrix} \mathrm{id} \\ \eta_2 d^{\bullet 0} \end{pmatrix} \qquad R = \begin{pmatrix} \mathrm{id} & d^{0\bullet}\varepsilon_2 \end{pmatrix}$$
$$H^0 = \begin{pmatrix} 0 & \varepsilon_2 \end{pmatrix} \qquad H^1 = \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}$$

From the previous two lemmas, $[\![\Gamma_0]\!]$ is a strong deformation retract of $[\![\Gamma]\!] = \mathrm{Cone}(\Delta^0, \Delta^1)[-1]$ with retraction R, inclusion J, and homotopy (H^0, H^1):

$$R\Delta^0 = 0,\ \Delta^1 J = 0,\ H^0\Delta^0 = \mathrm{id},\ RJ = \mathrm{id},\ JR = \mathrm{id} + \Delta^0 H^0 + H^1\Delta^1,\ \Delta^1 H^1 = \mathrm{id}.$$

Hence from Proposition 3.12 they have the same homology. □

4.3 Invariance under type III move

In this section, we only consider the case of move III_+. The case of III_- can be checked in a similar way: we have dual situations as is said in Remark 3.2 and in the proof of Proposition 3.5.

Let Γ_1 and Γ_2 be OMS–divides which differ only by a type III_+ move (see Figure 17).

Figure 17: Type III_+ move

Proposition 4.7 *The complexes $[\![\Gamma_1]\!]$ and $[\![\Gamma_2]\!]$ have the same homology.*

Let Γ_1^{stu}, $s, t, u \in \{0, 1\}$ be the cuspidal divides obtained by performing Θ_s, Θ_t and Θ_u splittings at the double points shown on the figure of Γ_1 (see Figure 17). From Lemma 3.14, we have the splitting diagram of $[\![\Gamma_1]\!]$ shown in Figure 18.

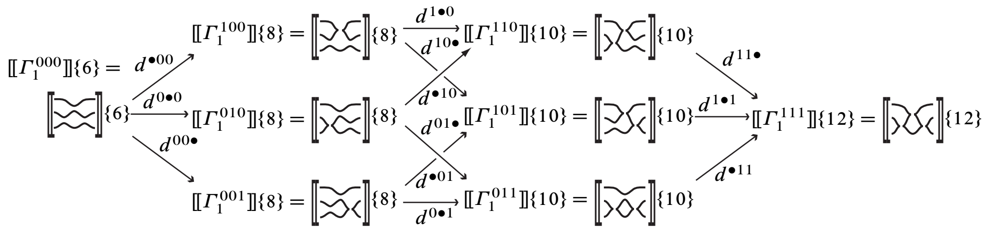

Figure 18: Splitting diagram for type III_+ move

Let's denote $\mathcal{C}^0 = [\![\Gamma_1^{000}]\!]\{6\}$, $\mathcal{C}^3 = [\![\Gamma_1^{111}]\!]\{12\}$, $\mathcal{C}^1 = \hat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1$ and $\mathcal{C}^2 = \hat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2$

where
$$\hat{\mathcal{C}}^1 = [\![\Gamma_1^{100}]\!]\{8\} \oplus [\![\Gamma_1^{010}]\!]\{8\} \qquad \check{\mathcal{C}}^1 = [\![\Gamma_1^{001}]\!]\{8\}$$
$$\hat{\mathcal{C}}^2 = [\![\Gamma_1^{110}]\!]\{10\} \oplus [\![\Gamma_1^{101}]\!]\{10\} \qquad \check{\mathcal{C}}^2 = [\![\Gamma_1^{011}]\!]\{10\}.$$

Lemma 4.8 $[\![\Gamma_1]\!] = \mathrm{Cone}(\Delta^0, \Delta^1, \Delta^2)$ *where*

$$\mathcal{C}^0 \xrightarrow{\Delta^0} \mathcal{C}^1 \xrightarrow{\Delta^1} \mathcal{C}^2 \xrightarrow{\Delta^2} \mathcal{C}^3$$

and $$\Delta^0 = \begin{pmatrix} d^{\bullet 00} \\ d^{0\bullet 0} \\ \hdashline d^{00\bullet} \end{pmatrix} \quad \Delta^1 = \left(\begin{array}{cc:c} d^{1\bullet 0} & d^{\bullet 10} & 0 \\ d^{10\bullet} & 0 & d^{\bullet 01} \\ \hdashline 0 & d^{01\bullet} & d^{0\bullet 1} \end{array}\right) \quad \Delta^2 = \left(\begin{array}{cc:c} d^{11\bullet} & d^{1\bullet 1} & d^{\bullet 11} \end{array}\right).$$

Now we can use the same arguments as in Section 4.2. The morphism $d^{0\bullet 1}$ (corresponding to T_2 and T_4) is injective and $d^{\bullet 11}$ (corresponding to T_3 and T_5) is surjective. Let $\varepsilon_2\colon \check{\mathcal{C}}^2 \to \check{\mathcal{C}}^1$ be the "destruction" morphism of complexes and $\eta_2\colon \mathcal{C}^3 \to \check{\mathcal{C}}^2$ be the "creation" morphism of complexes (see Figure 19).

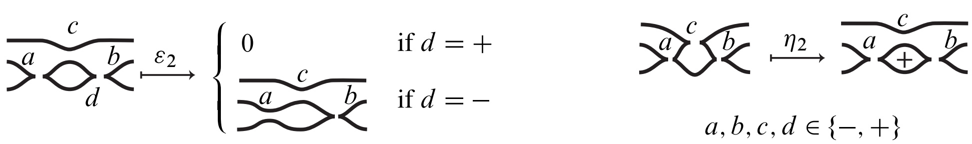

Figure 19

Lemma 4.9 *The sequence*

$$0 \longrightarrow \mathcal{C}^3 \xrightarrow{\eta_2} \check{\mathcal{C}}^2 \xrightarrow{\varepsilon_2} \check{\mathcal{C}}^1 \longrightarrow 0$$

is exact and $d^{\bullet 11}$, $d^{0\bullet 1}$ *are respectively splitting morphisms of* η_2 *and* ε_2*:*

$$d^{\bullet 11}\eta_2 = \mathrm{id}, \quad \varepsilon_2 d^{0\bullet 1} = \mathrm{id}, \quad d^{0\bullet 1}\varepsilon_2 + \eta_2 d^{\bullet 11} = \mathrm{id} + \eta_2 d^{\bullet 11} d^{0\bullet 1}\varepsilon_2. \tag{4-6}$$

Proof We have a similar situation as in Lemma 4.6: $d^{0\bullet 1}$ corresponds to T_2 or T_4 and $d^{\bullet 11}$ corresponds to T_3 or T_5 in (3-16). The lemma is a direct consequence of the comments and the definitions made in Remark 3.6 (see also Figure 6). $\square$

Lemma 4.10 *Let* $\delta = d^{\bullet 01}\varepsilon_2 d^{01\bullet}\colon [\![\Gamma_1^{010}]\!]\{8\} \to [\![\Gamma_1^{101}]\!]\{10\}$.
The following sequence is a double-complex:

$$(\mathcal{C}^0, d^0) \xrightarrow{\widehat{\Delta}^0} (\widehat{\mathcal{C}}^1, \widehat{d}^1) \xrightarrow{\widehat{\Delta}^1} (\widehat{\mathcal{C}}^2, \widehat{d}^2) \quad \text{where} \quad \widehat{\Delta}^0 = \begin{pmatrix} d^{\bullet 00} \\ d^{0\bullet 0} \end{pmatrix} \quad \widehat{\Delta}^1 = \begin{pmatrix} d^{1\bullet 0} & d^{\bullet 10} \\ d^{10\bullet} & \delta \end{pmatrix}.$$

Proof Since $\Delta^1\Delta^0 = 0$, we obtain from Lemma 4.9

$$\widehat{\Delta}^1\,\widehat{\Delta}^0 = \begin{pmatrix} d^{1\bullet 0}d^{\bullet 00} + d^{\bullet 10}d^{0\bullet 0} \\ d^{10\bullet}d^{\bullet 00} + \delta d^{0\bullet 0} \end{pmatrix} = \begin{pmatrix} 0 \\ d^{10\bullet}d^{\bullet 00} + d^{\bullet 01}\varepsilon_2 d^{01\bullet}d^{0\bullet 0} \end{pmatrix},$$

$$d^{10\bullet}d^{\bullet 00} + d^{\bullet 01}\varepsilon_2 d^{01\bullet}d^{0\bullet 0} = d^{\bullet 01}d^{00\bullet} + d^{\bullet 01}\varepsilon_2 d^{0\bullet 1}d^{00\bullet} = 0. \qquad \square$$

Lemma 4.11 *Let* $\widehat{\mathcal{C}} = \operatorname{Cone}(\widehat{\Delta}^0, \widehat{\Delta}^1)$. *Then the complex* $\widehat{\mathcal{C}}$ *is a strong deformation retract of* $[\![\Gamma_1]\!]$ *and so they have the same homology.*

Proof Consider the diagram

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "\widehat{\Delta}^0"] \arrow[d, shift left, "\mathrm{id}"] & \widehat{\mathcal{C}}^1 \arrow[r, "\widehat{\Delta}^1"] \arrow[d, shift left, "J^1"] & \widehat{\mathcal{C}}^2 \arrow[d, shift left, "J^2"] & \\
\mathcal{C}^0 \arrow[r, "\Delta^0"] \arrow[u, shift left] & \widehat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1 \arrow[r, "\Delta^1"] \arrow[u, shift left, "R^1"] & \widehat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2 \arrow[r, "\Delta^2"] \arrow[u, shift left, "R^2"] \arrow[l, bend left, "H^1"] & \mathcal{C}^3 \arrow[l, bend left, "H^2"]
\end{tikzcd}$$

where
$$J^1 = \left(\begin{array}{cc} \mathrm{id} & 0 \\ 0 & \mathrm{id} \\ \hline 0 & \varepsilon_2 d^{01\bullet} \end{array}\right) \qquad R^1 = \left(\begin{array}{cc|c} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \end{array}\right) \qquad H^1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \varepsilon_2 \end{array}\right)$$

$$J^2 = \left(\begin{array}{cc} \mathrm{id} & 0 \\ 0 & \mathrm{id} \\ \hline \eta_2 d^{11\bullet} & \eta_2 d^{1\bullet 1} \end{array}\right) \quad R^2 = \left(\begin{array}{cc|c} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & d^{\bullet 01}\varepsilon_2 \end{array}\right) \quad H^2 = \left(\begin{array}{c} 0 \\ 0 \\ \hline \eta_2 \end{array}\right).$$

From Lemma 4.9, we easily verify the relations:

$$J^1\widehat{\Delta}^0 = \Delta^0, \quad J^2\widehat{\Delta}^1 = \Delta^1 J^1, \quad \Delta^2 J^2 = 0, \quad R^1\Delta^0 = \widehat{\Delta}^0, \quad R^2\Delta^1 = \widehat{\Delta}^1 R^1,$$
$$R^1 J^1 = \mathrm{id}, \quad R^2 J^2 = \mathrm{id}, J^1 R^1 = \mathrm{id} + H^1\Delta^1, \quad J^2 R^2 = \mathrm{id} + H^2\Delta^2 + \Delta^1 H^1,$$
$$0 = \mathrm{id} + \Delta^2 H^2.$$

So downward arrows define an inclusion map, upward ones a retraction and H_1, H_2 homotopy maps of double complexes. We can apply Proposition 3.12: $\widehat{\mathcal{C}}$ is a deformation retract of $[\![\Gamma_1]\!]$ so they have the same homology. $\square$

Proof of Proposition 4.7 Up to isotopy, the drawn part of Γ_1^{000} on Figure 18 is symmetric with respect to horizontal direction. Also the drawn part of Γ_1^{100} is symmetric to the one of Γ_1^{010} as well as the drawn part of Γ_1^{110} is symmetric to the one of Γ_1^{101}. The morphisms $d^{\bullet 00}$ and $d^{0\bullet 0}$ (resp. $d^{10\bullet}$ and $d^{\bullet 10}$) clearly play symmetric roles. Besides, the morphisms $\delta\colon [\![\Gamma_1^{010}]\!]\{8\} \to [\![\Gamma_1^{101}]\!]\{10\}$ and $d^{1\bullet 0}\colon [\![\Gamma_1^{100}]\!]\{8\} \to [\![\Gamma_1^{110}]\!]\{10\}$ also play symmetric roles. Since the drawn parts of Γ_1 and Γ_2 in Figure 17 also are symmetric with respect to horizontal direction, we deduce that the complexes $[\![\Gamma_1]\!]$ and $[\![\Gamma_2]\!]$ both have the same homology as the complex $\widehat{\mathcal{C}}$. $\square$

4.4 Invariance under type IV moves

Let Γ_1 and Γ_2 be OMS–divides which differ only by a type IV move (see Figure 20).

Figure 20: Type IV move

Proposition 4.12 *The complexes $[\![\Gamma_1]\!]$ and $[\![\Gamma_2]\!]$ have the same homology.*

Before proving this proposition, we first introduce the following intermediate result.

Lemma 4.13 *Let Γ and $\widetilde{\Gamma}$ be two cuspidal divides which differ only in the way depicted in Figure 21 (or the symmetric situation with respect to horizontal direction).*

Figure 21

Then $[\![\widetilde{\Gamma}]\!]\{6\}[1]$ is a strong deformation retract of $[\![\Gamma]\!]$.

Proof With the same arguments as in the previous sections, using Lemma 3.14 we have a splitting diagram of $[\![\Gamma]\!]$ as in Figure 22.

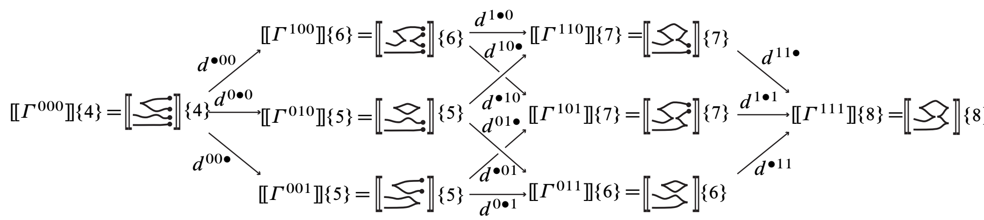

Figure 22: Splitting diagram of $[\![\Gamma]\!]$

Let's denote:

$$\mathcal{C}^0 = [\![\Gamma^{000}]\!]\{4\} \qquad \check{\mathcal{C}}^1 = [\![\Gamma^{001}]\!]\{5\} \qquad \check{\mathcal{C}}^2 = [\![\Gamma^{011}]\!]\{6\} \qquad \mathcal{C}^3 = [\![\Gamma^{111}]\!]\{8\}$$
$$\widehat{\mathcal{C}}^1 = [\![\Gamma^{100}]\!]\{6\} \oplus [\![\Gamma_1^{010}]\!]\{5\} \quad \widehat{\mathcal{C}}^2 = [\![\Gamma^{110}]\!]\{7\} \oplus [\![\Gamma_1^{101}]\!]\{7\}.$$

Then $[\![\Gamma]\!] = \mathrm{Cone}(\Delta^0, \Delta^1, \Delta^2)$ where

$$\mathcal{C}^0 \xrightarrow{\Delta^0} \widehat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1 \xrightarrow{\Delta^1} \widehat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2 \xrightarrow{\Delta^2} \mathcal{C}^3$$

$$\Delta^0 = \begin{pmatrix} d^{\bullet 00} \\ d^{0\bullet 0} \\ \hdashline d^{00\bullet} \end{pmatrix} = \begin{pmatrix} \widehat{\Delta}^0 \\ \hdashline d^{00\bullet} \end{pmatrix} \qquad \Delta^1 = \left(\begin{array}{cc:c} d^{1\bullet 0} & d^{\bullet 10} & 0 \\ d^{10\bullet} & 0 & d^{\bullet 01} \\ \hdashline 0 & d^{01\bullet} & d^{0\bullet 1} \end{array}\right) = \left(\begin{array}{c:c} \widehat{\Delta}^1 & U \\ \hdashline L & d^{0\bullet 1} \end{array}\right)$$

and
$$\Delta^2 = \left(\begin{array}{cc:c} d^{11\bullet} & d^{1\bullet 1} & d^{\bullet 11} \end{array}\right) = \left(\begin{array}{c:c} \widehat{\Delta}^2 & d^{\bullet 11} \end{array}\right).$$

Consider the creation/destruction morphisms of complexes (see Figure 23):

$$[\![\Gamma^{100}]\!]\{6\} \xrightarrow{\tau} [\![\Gamma^{000}]\!]\{4\} \quad [\![\Gamma^{110}]\!]\{7\} \xrightarrow{\eta_2} [\![\Gamma^{010}]\!]\{5\} \quad [\![\Gamma^{101}]\!]\{8\} \xrightarrow{\eta_2} [\![\Gamma^{011}]\!]\{5\}$$
$$[\![\Gamma^{010}]\!]\{5\} \xrightarrow{\sigma} [\![\Gamma^{000}]\!]\{4\} \quad [\![\Gamma^{101}]\!]\{7\} \xrightarrow{\eta_1} [\![\Gamma^{100}]\!]\{5\} \quad [\![\widetilde{\Gamma}]\!]\{6\} \xrightarrow{\eta_1} [\![\Gamma^{001}]\!]\{5\}$$
$$[\![\Gamma^{101}]\!]\{7\} \xrightarrow{\eta_2} [\![\Gamma^{010}]\!]\{5\} \quad [\![\Gamma^{001}]\!]\{5\} \xrightarrow{\overline{\varepsilon}_1} [\![\widetilde{\Gamma}]\!]\{6\}$$

Let's define $\sigma' = (\mathrm{id} + \tau d^{\bullet 00})\sigma \colon [\![\Gamma^{010}]\!]\{5\} \to [\![\Gamma^{000}]\!]\{4\}$ and

$$\widehat{H}^1 = \begin{pmatrix} 0 & \eta_1 \\ \eta_2 & \eta_2 \end{pmatrix} \qquad \widehat{H}^0 = \begin{pmatrix} \tau & \sigma' \end{pmatrix}.$$

Then we have a short exact sequence

$$0 \longrightarrow \widehat{\mathcal{C}}^2 \xrightarrow{\widehat{H}^1} \widehat{\mathcal{C}}^1 \xrightarrow{\widehat{H}^0} \mathcal{C}^0 \longrightarrow 0$$

such that $\widehat{\Delta}^1$ and $\widehat{\Delta}^0$ are splitting morphisms of $\widehat{H}^1$ and $\widehat{H}^0$:

(4-7) $$\widehat{\Delta}^1 \widehat{H}^1 = \mathrm{id}, \quad \widehat{H}^0 \widehat{\Delta}^0 = \mathrm{id}, \quad \widehat{\Delta}^0 \widehat{H}^0 + \widehat{H}^1 \widehat{\Delta}^1 = \mathrm{id} + \widehat{H}^1 \widehat{\Delta}^1 \widehat{\Delta}^0 \widehat{H}^0.$$

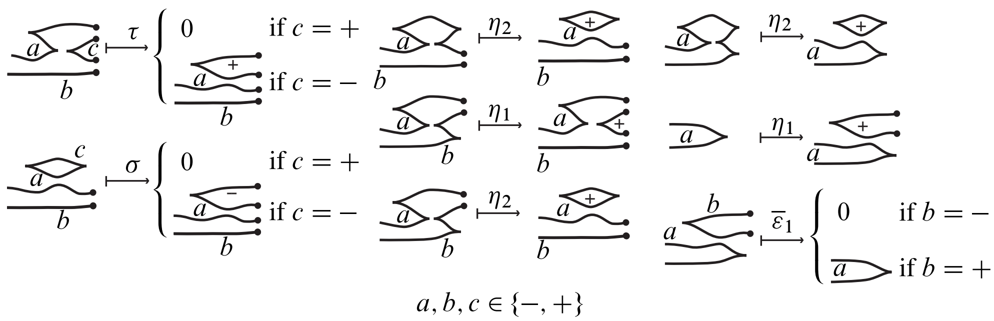

Figure 23

Moreover, for the compositions

$$[\![\widetilde{\Gamma}]\!]\{6\} \xrightarrow{\eta_1} \check{\mathcal{C}}^1 \xrightarrow{d^{0\bullet 1}} \check{\mathcal{C}}^2 \quad \text{and} \quad \check{\mathcal{C}}^1 \xrightarrow{U} \widehat{\mathcal{C}}^2 \xrightarrow{\widehat{H}^1} \widehat{\mathcal{C}}^1 \xrightarrow{L} \check{\mathcal{C}}^2$$

we have

$$d^{0\bullet 1}\eta_1 = 0 \qquad L\widehat{H}^1 U = d^{01\bullet}\eta_2 d^{\bullet 01} = 0. \tag{4-8}$$

Let's define homotopies $H^0\colon \widehat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1 \to \mathcal{C}^0$, $H^1\colon \widehat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2 \to \widehat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1$ and $H^2\colon \mathcal{C}^3 \to \widehat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2$ by

$$H^0 = \left(\widehat{H}^0 \;\vdots\; 0\right) \qquad H^1 = \left(\begin{array}{c:c} \widehat{H}^1 & \begin{matrix}0\\0\end{matrix} \\ \hdashline 0\;\;0 & 0 \end{array}\right) \qquad H^2 = \left(\begin{array}{c} 0 \\ 0 \\ \hdashline \eta_2 \end{array}\right)$$

and consider the injection J and retraction R

$$\begin{tikzcd} & {[\![\widetilde{\Gamma}]\!]\{6\}} \arrow[d, shift left, "J"] & & \\ \mathcal{C}^0 \arrow[r, "\Delta^0"] & \widehat{\mathcal{C}}^1 \oplus \check{\mathcal{C}}^1 \arrow[u, shift left, "R"] \arrow[l, bend left, "H^0"] \arrow[r, "\Delta^1"] & \widehat{\mathcal{C}}^2 \oplus \check{\mathcal{C}}^2 \arrow[l, bend left, "H^1"] \arrow[r, "\Delta^2"] & \mathcal{C}^3 \arrow[l, bend left, "H^2"] \end{tikzcd}$$

where $$R = \left(0\;\; 0 \;\vdots\; \overline{\varepsilon}_1\right)(\mathrm{id} + \Delta^0 H^0) = \left(\overline{\varepsilon}_1 d^{00\bullet}\tau \;\; \overline{\varepsilon}_1 d^{00\bullet}\sigma' \;\vdots\; \overline{\varepsilon}_1\right)$$

$$J = (\mathrm{id} + H^1\Delta^1)\left(\begin{array}{c} 0 \\ 0 \\ \hdashline \eta_1 \end{array}\right) = \left(\begin{array}{c} \eta_1 d^{\bullet 01}\eta_1 \\ \eta_2 d^{\bullet 01}\eta_1 \\ \hdashline \eta_1 \end{array}\right).$$

(see Figure 24 for the retraction R).

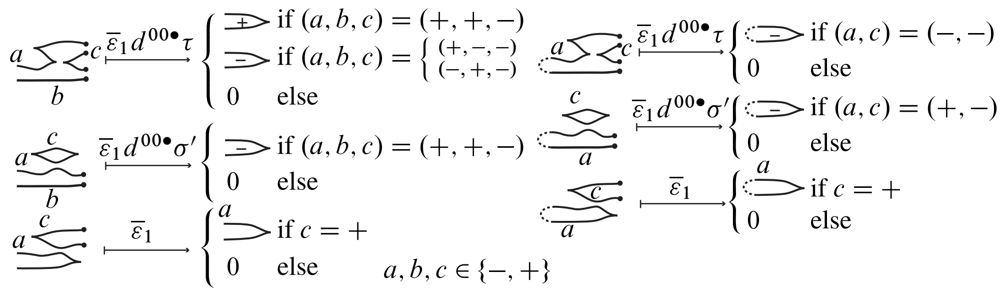

Figure 24: The retraction R

From (4-7) and (4-8), we easily verify the relations:

$$R\Delta^0=0,\quad \Delta^1 J=0,\quad H^0 J=0,\quad RH^1=0,\quad H^0H^1=0,\quad H^1H^2=0$$
$$RJ=\mathrm{id},\ JR=\mathrm{id}+\Delta^0H^0+H^1\Delta^1,\ \mathrm{id}+\Delta^1H^1+H^2\Delta^2=0,\ \mathrm{id}+\Delta^2H^2=0.$$

Then $[\![\widetilde{\Gamma}]\!]\{6\}[1]$ is a strong deformation retract of $[\![\Gamma]\!]=\mathrm{Cone}(\Delta^0,\Delta^1,\Delta^2)$ from Proposition 3.12. □

Proof of Proposition 4.12 Let Γ_1^s (resp. Γ_2^s), $s\in\{0,1\}$ be the cuspidal divides obtained by performing Θ_s splittings at the left hand $+$ double point of Γ_1 (resp. Γ_2) in Figure 20. Let's also denote, according to Lemma 4.13, the cuspidal divides $\widetilde{\Gamma}_1^1$ (resp. $\widetilde{\Gamma}_2^1$) obtained by "retracting" Γ_1^1 (resp. Γ_2^1) (see Figure 25). Notice that

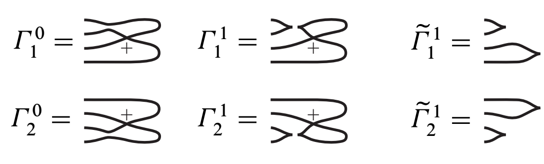

Figure 25

$[\![\Gamma_1^0]\!]=[\![\Gamma_2^0]\!]$ and $[\![\widetilde{\Gamma}_1^1]\!]=[\![\widetilde{\Gamma}_2^1]\!]$. From Lemma 3.14, the differential d_1 on $[\![\Gamma_1]\!]$ (resp. d_2 on $[\![\Gamma_2]\!]$) gives us the following cone:

$$[\![\Gamma_1]\!]=\mathrm{Cone}\left([\![\Gamma_1^0]\!]\{2\}\xrightarrow{d_1^\bullet}[\![\Gamma_1^1]\!]\{4\}\right)\quad\left(\text{resp. } [\![\Gamma_2]\!]=\mathrm{Cone}\left([\![\Gamma_2^0]\!]\{2\}\xrightarrow{d_2^\bullet}[\![\Gamma_2^1]\!]\{4\}\right)\right).$$

From Lemma 4.13, there exist strong deformation retractions

$$[\![\Gamma_1^1]\!]\ \underset{J_1}{\overset{R_1}{\rightleftarrows}}\ [\![\widetilde{\Gamma}_1^1]\!]\{6\}[1]\qquad [\![\Gamma_2^1]\!]\ \underset{J_2}{\overset{R_2}{\rightleftarrows}}\ [\![\widetilde{\Gamma}_2^1]\!]\{6\}[1].$$

So from Proposition 3.10, $\mathrm{Cone}(R_1 d_1^\bullet)$ (resp. $\mathrm{Cone}(R_2 d_2^\bullet)$) is a strong deformation retract of $[\![\Gamma_1]\!]$ $\big($resp. $[\![\Gamma_2]\!]\big)$. Hence it suffices to show that $R_1 d_1^\bullet = R_2 d_2^\bullet$. Let's consider the splitting diagram of $[\![\Gamma_1^0]\!] = [\![\Gamma_2^0]\!]$ as $\mathrm{Cone}(D^0, D^1, D^2)$ (see Figure 26).

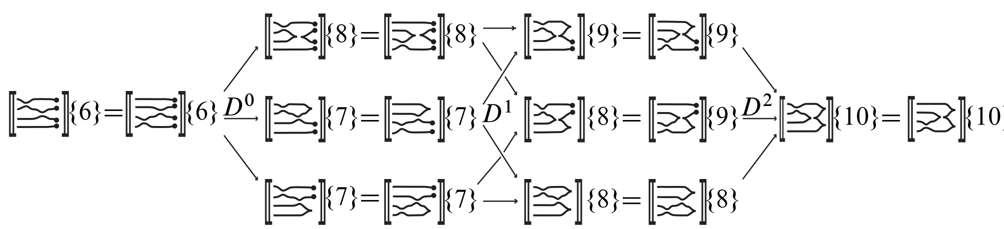

Figure 26

Now $R_1 d_1^\bullet$ and $R_2 d_2^\bullet$ corresponds to the diagram of Figure 27.

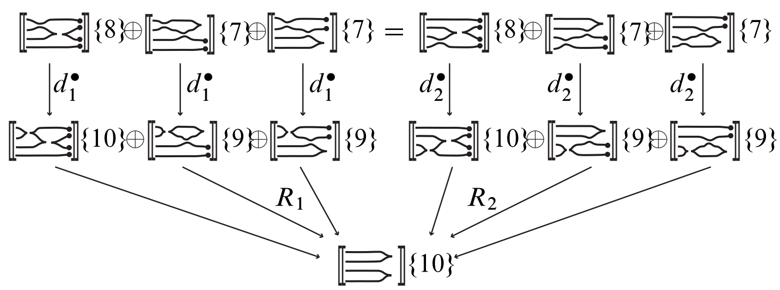

Figure 27

By combining Figure 24 with Figure 6 we easily verify that $R_1 d_1^\bullet = R_2 d_2^\bullet$. $\square$

4.5 Invariance under type V moves

Let Γ_0 and Γ_+ (resp. Γ_-) be OMS–divides which differ only by a type $\mathrm{V}^\cdot +$ (resp. type $\mathrm{V}^\cdot -$) move (see Figure 28).

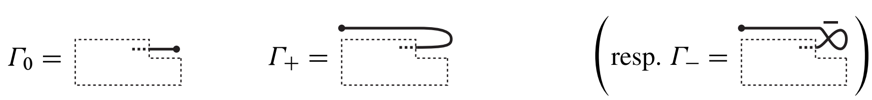

Figure 28: Type V moves

Proposition 4.14 *The complexes $[\![\Gamma_0]\!]$, $[\![\Gamma_+]\!]$ and $[\![\Gamma_-]\!]$ have the same homology.*

Proof From Lemma 3.14, we can see $[\![\Gamma_+]\!]$ as the cone of the surjective morphism as in Figure 29.

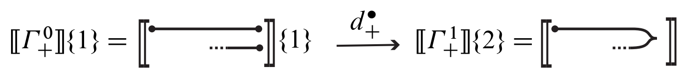

Figure 29: Splitting diagram for type V_+ move

Let's consider the creation/destruction morphisms $\eta_1\colon [\![\Gamma_+^1]\!]\{2\} \to [\![\Gamma_+^0]\!]\{1\}$ (right inverse of $d_+^\bullet$), $\overline{\eta}_1\colon [\![\Gamma]\!] \to [\![\Gamma_+^0]\!]\{1\}$ and $\varepsilon_1\colon [\![\Gamma_+^0]\!]\{1\} \to [\![\Gamma]\!]$ defined in Figure 30.

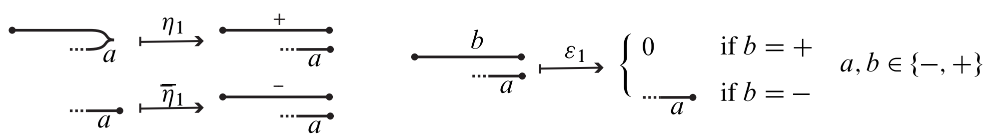

Figure 30

Let $j = (\mathrm{id} + \eta_1 d^\bullet)\overline{\eta}_1$ and $r = \varepsilon_1$. Then from the following diagram

$$\begin{tikzcd}
{[\![\Gamma_0]\!]} \arrow[d, shift right, "j"'] \\
{[\![\Gamma_+^0]\!]\{1\}} \arrow[u, shift right, "r"'] \arrow[r, "d_+^\bullet"] & {[\![\Gamma_+^1]\!]\{2\}} \arrow[l, bend left, "\eta_1"]
\end{tikzcd}$$

where
$$d_+^\bullet j = 0, \qquad rj = \mathrm{id},$$
$$jr = \mathrm{id} + \eta_1 d_+^\bullet, \qquad d_+^\bullet \eta_1 = \mathrm{id}$$

we deduce using Proposition 3.12 that $[\![\Gamma_0]\!]$ is a strong deformation retract of $[\![\Gamma_+]\!] = \mathrm{Cone}(d_+^\bullet)$. So they have the same homology.

On the other hand, from Lemma 3.14, $[\![\Gamma_-]\!] = \mathrm{Cone}(\Delta^0, \Delta^1)[-1]$:

$$[\![\Gamma_-^{10}]\!]\{-3\} \xrightarrow{\Delta^0} [\![\Gamma_-^{00}]\!]\{-1\} \oplus [\![\Gamma_-^{11}]\!]\{-2\} \xrightarrow{\Delta^1} [\![\Gamma_-^{01}]\!]\{3\} \quad \Delta^0 = \begin{pmatrix} d_-^{\bullet 0} \\ d_-^{1\bullet} \end{pmatrix} \quad \Delta^1 = \begin{pmatrix} d_-^{0\bullet} & d_-^{\bullet 1} \end{pmatrix}$$

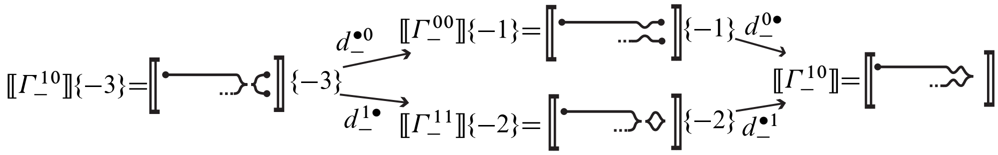

Figure 31: Splitting diagram for type V_- move

Let's consider the morphisms

$$[\![\Gamma_-^{00}]\!]\{-1\} \xrightarrow{\tau} [\![\Gamma_-^{10}]\!]\{-3\}, \qquad [\![\Gamma_-^{11}]\!]\{-2\} \xrightarrow{\sigma} [\![\Gamma_-^{10}]\!]\{-3\},$$
$$[\![\Gamma_-^{01}]\!] \xrightarrow{\eta_1} [\![\Gamma_-^{00}]\!]\{-1\}, \qquad [\![\Gamma_-^{01}]\!] \xrightarrow{\eta_2} [\![\Gamma_-^{11}]\!]\{-2\}$$

defined in Figure 32.

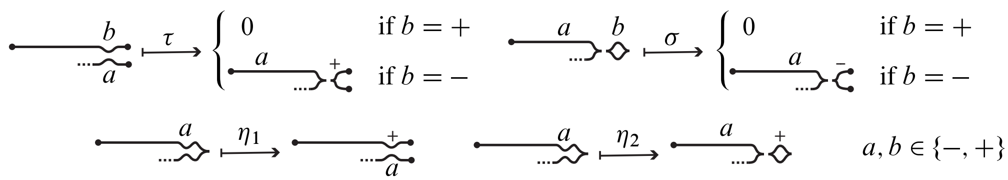

Figure 32

Then from the diagram

$$\begin{tikzcd}
 & {[\![\Gamma_0]\!]} \arrow[d, shift right, "J"'] & \\
{[\![\Gamma_-^{10}]\!]\{-3\}} \arrow[r, "\Delta^0"] & {[\![\Gamma_-^{00}]\!]\{-1\} \oplus [\![\Gamma_-^{11}]\!]\{-2\}} \arrow[u, shift right, "R"'] \arrow[l, bend left, "H^0"] \arrow[r, "\Delta^1"] & {[\![\Gamma_-^{01}]\!]} \arrow[l, bend left, "H^1"]
\end{tikzcd}$$

where $J = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $\quad R = \begin{pmatrix} 0 & \overline{\varepsilon}_2 \end{pmatrix}$, $\quad H^0 = \begin{pmatrix} \tau & (\mathrm{id} + \tau d^{\bullet 0})\sigma \end{pmatrix}$ and $H^1 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$

we deduce that $[\![\Gamma_0]\!]$ is a strong deformation retract of $[\![\Gamma_-]\!] = \mathrm{Cone}(\Delta^0, \Delta^1)[-1]$. They have the same homology. $\square$

4.6 Invariance under type VI moves

Let Γ and Γ_+ (resp. Γ_-) be OMS–divides which differ only by a type VI_+ (resp. type VI_-) move (see Figure 33).

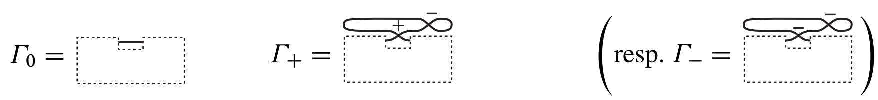

Figure 33: Type VI moves

Proposition 4.15 *The complexes $[\![\Gamma_0]\!]$, $[\![\Gamma_+]\!]$ and $[\![\Gamma_-]\!]$ have the same homology.*

We will break down the proof in two steps: the result is an immediate consequence of the following two lemmas.

Lemma 4.16 *Let Γ and $\widetilde{\Gamma}$ be cuspidal divides as depicted in Figure 34.*

Figure 34

Then $[\![\widetilde{\Gamma}]\!]$ is a strong deformation retract of $[\![\Gamma]\!]$.

Proof Let's apply Lemma 3.14 to $[\![\Gamma]\!]$. We have a splitting diagram as depicted in Figure 35.

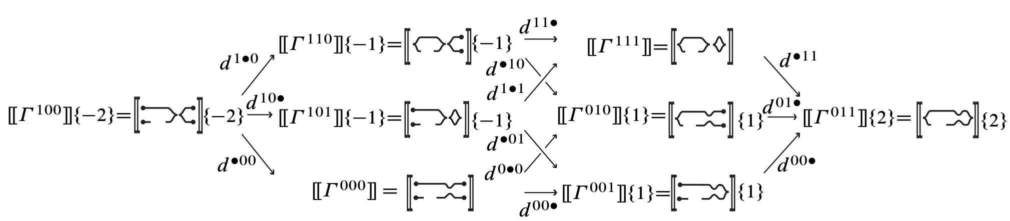

Figure 35: Splitting diagram of $[\![\Gamma]\!]$

Let's denote:

$$\mathcal{C}^0 = [\![\Gamma^{100}]\!]\{-2\} \qquad \mathcal{C}^1 = [\![\Gamma^{110}]\!]\{-1\} \oplus \big([\![\Gamma^{101}]\!]\{-1\} \oplus [\![\Gamma^{000}]\!]\big)$$

$$\mathcal{C}^2 = [\![\Gamma^{111}]\!] \oplus \big([\![\Gamma^{010}]\!]\{1\} \oplus [\![\Gamma^{001}]\!]\{1\}\big) \qquad \mathcal{C}^3 = [\![\Gamma^{011}]\!]\{2\}$$

$$\Delta^0 = \begin{pmatrix} d^{1\bullet 0} \\ \hdashline d^{10\bullet} \\ d^{\bullet 00} \end{pmatrix} \qquad \Delta^1 = \left(\begin{array}{c:cc} d^{11\bullet} & d^{1\bullet 1} & 0 \\ \hdashline d^{\bullet 10} & 0 & d^{0\bullet 0} \\ 0 & d^{\bullet 01} & d^{00\bullet} \end{array}\right) \qquad \Delta^2 = \left(\begin{array}{c:cc} d^{\bullet 11} & d^{01\bullet} & d^{0\bullet 1} \end{array}\right).$$

Then $[\![\Gamma]\!] = \mathrm{Cone}(\Delta^0, \Delta^1, \Delta^2)[-1]$ where

$$\mathcal{C}^0 \xrightarrow{\Delta^0} \mathcal{C}^1 \xrightarrow{\Delta^1} \mathcal{C}^2 \xrightarrow{\Delta^2} \mathcal{C}^3.$$

Let's consider the creation / destruction morphisms

$$[\![\Gamma^{101}]\!]\{-1\} \xrightarrow{\sigma} [\![\Gamma^{100}]\!]\{-2\} \quad [\![\Gamma^{111}]\!] \xrightarrow{\sigma} [\![\Gamma^{110}]\!]\{-1\} \quad [\![\Gamma^{011}]\!]\{2\} \xrightarrow{\eta_2} [\![\Gamma^{111}]\!]$$
$$[\![\Gamma^{000}]\!] \xrightarrow{\tau} [\![\Gamma^{100}]\!]\{-2\} \quad [\![\Gamma^{010}]\!]\{1\} \xrightarrow{\eta_1} [\![\Gamma^{000}]\!] \quad [\![\widetilde{\Gamma}]\!] \xrightarrow{\eta_1} [\![\Gamma^{110}]\!]\{-1\}$$
$$[\![\Gamma^{001}]\!]\{1\} \xrightarrow{\eta_2} [\![\Gamma^{101}]\!]\{-1\} \quad [\![\Gamma^{110}]\!]\{-1\} \xrightarrow{\overline{\varepsilon}_1} [\![\widetilde{\Gamma}]\!]$$

as depicted in Figure 36.

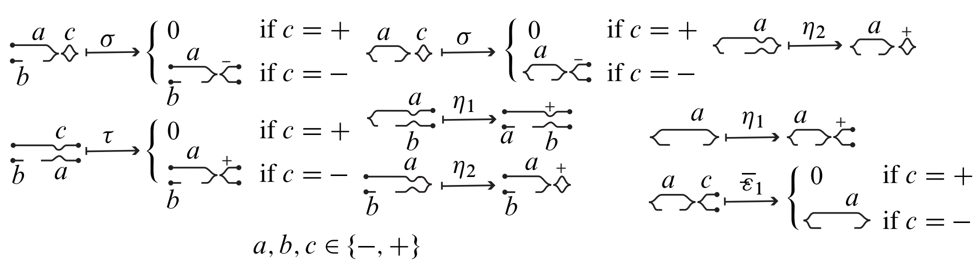

Figure 36

We define homotopies $H^0\colon \mathcal{C}^1 \to \mathcal{C}^0$, $H^1\colon \mathcal{C}^2 \to \mathcal{C}^1$ and $H^2\colon \mathcal{C}^3 \to \mathcal{C}^2$:

$$H^0 = \begin{pmatrix} 0 & (1+\tau d^{\bullet 00})\sigma & \tau \end{pmatrix}, \ H^1 = \begin{pmatrix} \sigma & 0 & 0 \\ \eta_2 d^{00\bullet}\eta_1 d^{\bullet 10}\sigma & \eta_2 d^{00\bullet}\eta_1 & \eta_2 \\ \eta_1 d^{\bullet 10}\sigma & \eta_1 & 0 \end{pmatrix}, \ H^2 = \begin{pmatrix} \eta_2 \\ 0 \\ 0 \end{pmatrix}$$

together with retraction and inclusion maps $R\colon \mathcal{C}^1 \to [\![\widetilde{\Gamma}]\!]$ and $J\colon [\![\widetilde{\Gamma}]\!] \to \mathcal{C}^1$:

$$R = \begin{pmatrix} \overline{\varepsilon}_1 & 0 & 0 \end{pmatrix} (\mathrm{id} + \Delta^0 H^0) = \begin{pmatrix} \overline{\varepsilon}_1 & \overline{\varepsilon}_1 d^{1\bullet 0}\tau d^{\bullet 00}\sigma & \overline{\varepsilon}_1 d^{1\bullet 0}\tau \end{pmatrix}$$
$$J = (\mathrm{id} + H^1\Delta^1) \begin{pmatrix} \eta_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 d^{00\bullet}\eta_1 d^{\bullet 10}\eta_1 \\ \eta_1 d^{\bullet 10}\eta_1 \end{pmatrix}$$

such that we have the diagram:

$$
\begin{tikzcd}
 & {[\![\widetilde{\Gamma}]\!]} \arrow[d, shift right, "J"'] & & \\
\mathcal{C}^0 \arrow[r, "\Delta^0"] & \mathcal{C}^1 \arrow[u, shift right, "R"'] \arrow[l, bend left, "H^0"] \arrow[r, "\Delta^1"] & \mathcal{C}^2 \arrow[l, bend left, "H^1"] \arrow[r, "\Delta^2"] & \mathcal{C}^3 \arrow[l, bend left, "H^2"]
\end{tikzcd}
$$

We easily verify that

$$R\Delta^0 = 0, \quad \Delta^1 J = 0, \quad H^0 J = 0, \quad RH^1 = 0, \quad H^0 H^1 = 0, \quad H^1 H^2 = 0,$$

$$RJ = \mathrm{id}, \; JR = \mathrm{id} + \Delta^0 H^0 + H^1 \Delta^1, \; \mathrm{id} + \Delta^1 H^1 + H^2 \Delta^2 = 0, \; \mathrm{id} + \Delta^2 H^2 = 0.$$

Then from Proposition 3.12, $[\![\widetilde{\Gamma}]\!]$ is a strong deformation retract of $[\![\Gamma]\!]$. □

Lemma 4.17 *We have strong deformation retractions r_+ (resp. r_-) with injection j_+ (resp. j_-) pictured in Figure 37.*

Figure 37

Proof From Lemma 3.14, $[\![\widetilde{\Gamma}_+]\!] = \mathrm{Cone}(d_+)$ and $[\![\widetilde{\Gamma}_-]\!] = \mathrm{Cone}(d_-)[-1]$ where d_+ and d_- are shown in Figure 38.

Figure 38

For $[\![\Gamma_+]\!]$, using Proposition 3.12, we deduce the strong deformation retraction from the following diagram of Figure 39 since $d_+\eta_2 = \mathrm{id}$, $r_+\eta_2 = 0$, $d_+ j_+ = 0$, $r_+ j_+ = \mathrm{id}$ and $j_+ r_+ = \mathrm{id} + \eta_2 d_+$.

Similarly for $[\![\Gamma_-]\!]$ we deduce the strong deformation retraction from the diagram of Figure 40 since $\varepsilon_2 d_- = \mathrm{id}$, $r_- d_- = 0$, $\varepsilon_2 j_- = 0$, $r_- j_- = \mathrm{id}$ and $j_- r_- = \mathrm{id} + d_- \varepsilon_2$. □

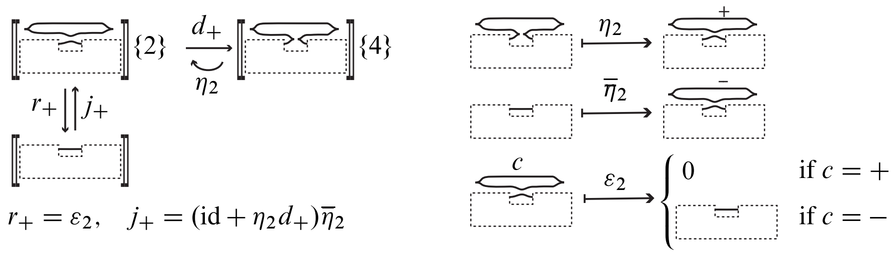

Figure 39

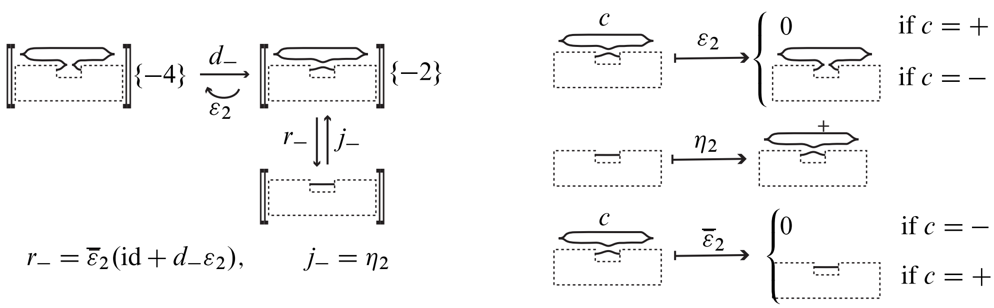

Figure 40

References

[1] **N A'Campo**, *Real deformations and complex topology of plane curve singularities*, Ann. Fac. Sci. Toulouse Math. (6) 8 (1999) 5–23 MR1721511

[2] **O Couture**, *Strongly invertible links and divides*, Topology 47 (2008) 316–350 MR2422530

[3] **O Couture**, **B Perron**, *Representative braids for links associated to plane immersed curves*, J. Knot Theory Ramifications 9 (2000) 1–30 MR1749500

[4] **J W Morgan**, **H Bass** (editors), *The Smith conjecture*, Pure and Applied Math. 112, Academic Press, Orlando, FL (1984) MR758459 Papers presented at the symposium held at Columbia University, New York, 1979

[5] **O Viro**, *Remarks on the definition of Khovanov homology* `arXiv:math.GT/0202199`

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Received: 17 February 2009 Revised: 27 August 2009