Bordism groups of solutions to differential relations

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In terms of category theory, the *Gromov homotopy principle* for a set valued functor F asserts that the functor F can be induced from a homotopy functor. Similarly, we say that the *bordism principle* for an abelian group valued functor F holds if the functor F can be induced from a (co)homology functor.

We examine the bordism principle in the case of functors given by (co)bordism groups of maps with prescribed singularities. Our main result implies that if a family $\mathcal{J}$ of prescribed singularity types satisfies certain mild conditions, then there exists an infinite loop space $\Omega^{\infty}\mathbf{B}_{\mathcal{J}}$ such that for each smooth manifold W the cobordism group of maps into W with only $\mathcal{J}$–singularities is isomorphic to the group of homotopy classes of maps $[W, \Omega^{\infty}\mathbf{B}_{\mathcal{J}}]$. The spaces $\Omega^{\infty}\mathbf{B}_{\mathcal{J}}$ are relatively simple, which makes explicit computations possible even in the case where the dimension of the source manifold is bigger than the dimension of the target manifold.

1 Introduction

A smooth map $f\colon V \to W$ of manifolds is said to be *singular* at a point $x \in V$ if the rank of the differential of f at x is less than the minimum of dimensions of V and W. If x is a *regular value* of f, ie, if $f^{-1}(x)$ consists only of nonsingular points, then $f^{-1}(x)$ is a smooth manifold of a *formal dimension* $d = \dim V - \dim W$. The integer d is also called the *dimension* of f. In the present paper we are primarily interested in the case $d \geq 0$.

For a set of prescribed singularity types $\mathcal{J}$ of maps of a fixed dimension d, a smooth map f of manifolds is said to be a $\mathcal{J}$–*map* if each singular point of f is of type in $\mathcal{J}$. Similarly, we say that a cobordism of two $\mathcal{J}$–maps is a $\mathcal{J}$–*cobordism* if, as a map, it has only $\mathcal{J}$–singularities. The set of $\mathcal{J}$–cobordism classes of $\mathcal{J}$–maps of closed manifolds into a closed manifold W leads to an abelian group $B(W; \mathcal{J})$; in terms of representatives the group operation is given by taking the disjoint union of maps (see Sections 6, 7).
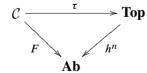
Let us mention that in later sections we will define the group $B(W; \mathcal{J})$ for an arbitrary (not necessarily closed) manifold W. At the moment, however, we consider only maps of closed manifolds, just to simplify the exposition.

Carefully choosing the set $\mathcal{J}$ of singularities, one may derive cobordism groups related to various objects in geometry and topology.

Example 1.1 Since a proper submersion is a smooth fiber bundle, the cobordism group of submersions is closely related to diffeomorphism groups of smooth manifolds. It is also known to be related to various infinite loop spaces, moduli spaces of Riemann surfaces, the cobordism category as in Galatius et al [17]; and, in particular, to the Kahn–Priddy theorem and the standard Mumford conjecture as in Fuks [16], Madsen and Weiss [32] and [17] (see Remark 2.2). Similarly, $\mathcal{J}$–cobordism groups are related to singular cobordism categories by work of the author [42] and a certain version of $\mathcal{J}$–cobordisms is related to the Deligne–Mumford compactification by Eliashberg and Galatius [12]. The cobordism groups of *special generic maps* are related to diffeomorphism groups of spheres and exotic smooth structures on spheres by Saeki [45] and Sadykov [43]. The cobordism groups of *fold maps* are known to be related to stable homotopy groups of spheres and other interesting objects by Chess [8], Ando [2] and Kalmár [25].

A priori $\mathcal{J}$–cobordism groups do not form generalized cohomology theories since, for example, $\mathcal{J}$–cobordism groups are not defined for topological spaces. In the current paper we propose a counterpart of $B(W; \mathcal{J})$ that for a wide range of sets $\mathcal{J}$ can be used to compute $B(W; \mathcal{J})$ in the same way as singular cohomology groups $H^n(W; \mathbb{R})$ can be used to compute De Rham cohomology groups $H^n_{\mathrm{DR}}(W)$.

Definition 1.2 Let F be a contravariant functor from a category $\mathcal{C}$ to the category **Ab** of abelian groups. We say that F satisfies the *bordism principle* (or *b-principle*) if there is a cohomology theory with functors $h^i \colon \textbf{Top} \to \textbf{Ab}$, indexed by $i \in \mathbb{Z}$, on the category **Top** of topological spaces and a covariant factor $\tau \colon \mathcal{C} \to \textbf{Top}$ such that F is naturally equivalent to $h^n \circ \tau$ for some n, ie, there is a noncommutative diagram of functors:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \ \tau\ \ } & \textbf{Top} \\
& {\scriptstyle F} \searrow \quad \swarrow {\scriptstyle h^n} & \\
& \textbf{Ab} &
\end{array}
$$

On the category of smooth manifolds, for example, H^n_{DR} satisfies the b-principle since it is naturally equivalent to $H^n \circ \tau$; here τ takes a smooth manifold onto the underlying topological space.

In general, if holds, the b-principle allows us to replace an a priori incomprehensible functor by one that can be studied by means of the machinery of the cohomology theory.

Remark 1.3 The b-principle is a bordism version of the *homotopy principle*, or *h-principle*. The classical Gromov definition of the h-principle is given in terms of so-called *jet bundles* [19]. There are, however, important h-principle type theorems, eg, Thurston h-principle for foliations [52; 53] (see also Eliashberg and Mishachev [33; 13]) that do not fit the classical jet bundle setting. In general, the h-principle can be formulated in terms of category theory as above by respectively replacing **Top** and **Ab** by the category **Top** $\downarrow$ B of topological spaces over a fixed space B (see Section 3) and a category of sets, and requiring that h^n be a homotopy functor rather than a term of a cohomology theory.

In the present paper we show that for a wide range of sets $\mathcal{J}$ the b-principle holds true for a functor counterpart of $B(*; \mathcal{J})$. More importantly, we construct a cohomology theory extending $B(*; \mathcal{J})$ whose spectrum $\mathbf{B}_{\mathcal{J}}$ is simple enough to make explicit computations possible.

Structure of the paper

The results of the paper are stated in Section 2. In Section 3 we formulate the bordism and weak bordism principles. In Sections 4–5 and 8–10 we recall and develop the language necessary to discuss (co)bordism groups in terms of jet spaces and differential equations and, more generally, differential relations. We will see that each set $\mathcal{J}$ of singularity types is associated with differential relations so that $\mathcal{J}$–maps can be defined to be solutions of associated differential relations. In Sections 6–7 we define bordism and cobordism groups of solutions of differential relations. In Sections 11–13 we prove the Main Theorem and its covariant counterpart. Finally, Sections 14 and 15 are devoted to applications.

2 Results

Definition of B$_\mathcal{J}$ Let $p: EO z_t \to BO_t$ be the universal vector t–bundle. Let $S_t = S_t(\mathcal{J})$ denote the space of Taylor series $T(f)$ at $0 \in \mathbb{R}^{t+d}$ of $\mathcal{J}$–maps $f: \mathbb{R}^{t+d} \to EO_t|b$, with $b \in BO_t$, into the fibers $EO_t|b = p^{-1}(b)$ such that $f(0)$ is $0 \in EO_t|b$. Then the map $\pi: S_t \to BO_t$ that takes $T(f)$ onto b has a structure of a fiber bundle. The desired spectrum **B$_\mathcal{J}$** is defined to be the Thom spectrum with $(t+d)$–th term given by the Thom space $T\pi^*EO_t$ of the bundle π^*EO_t over S_t.

Unless otherwise stated, we will always (tacitly) assume that the set $\mathcal{J}$ of (nonregular) singularity types is *open, $\mathcal{K}$–invariant*. The former means that a map close to a $\mathcal{J}$–map is also a $\mathcal{J}$–map, while the latter means that the set $\mathcal{J}$ has *sufficiently many symmetries* (see Section 4). In addition, unless otherwise stated, we will always (tacitly) assume that if $d \geq 0$, then $\mathcal{J}$ contains folds and the target W of considered maps is of dimension > 1.

Main Theorem   *The cobordism group $B(W; \mathcal{J})$ of $\mathcal{J}$–maps of dimension d to a closed manifold W is isomorphic to the set $[W, \Omega^\infty \mathbf{B}_\mathcal{J}]$ of homotopy classes of maps into the infinite loop space $\Omega^\infty \mathbf{B}_\mathcal{J}$ of $\mathbf{B}_\mathcal{J}$.*

Remark 2.1   Similar definitions and a theorem are valid for bordism groups of $\mathcal{J}$–maps and their functor counterparts (see Section 6 and Theorem 11.1).

Remark 2.2   In the case of maps of a general dimension d with $\mathcal{J} = \varnothing$, ie, in the case where $\mathcal{J}$–maps have only nonsingular points, the statement of the Main Theorem is not true. Let us mention, however, that its version, which we do not consider in the present paper, holds true for $d = 0$ (Kahn–Priddy theorem; see Chess [8] and Fuks [16]) and $d = 2$ (Mumford conjecture; see Madsen and Weiss [32]).

Remark 2.3   The omitted case of maps of dimension $d \geq 0$ into a manifold W of dimension 1 has been considered by Ikegami and Saeki [23], Ikegami [22], Kalmár [26] and Saeki [45]. For this case our approach does not apply because our argument implicitly (but essentially) uses the Eliashberg h-principle [9; 10] for fold maps which requires the condition dim $W > 1$.

The Main Theorem is obviously related to an old question in singularity theory on constructing a classifying space $B_\mathcal{J}$ such that for each closed manifold W there is an isomorphism $B(W; \mathcal{J}) \approx [W, B_\mathcal{J}]$. The first general result on constructing classifying spaces was obtained by Rimányi and Szűcs [38] who constructed a space $B_\mathcal{J}$ for each

finite family $\mathcal{J}$ of *simple stable* singularities of maps of negative dimension. Remarkably, it lead Rimányi to the method of Restriction Equations [37] which prompted a series of explicit computations by Rimányi, Kazarian, Fehér and others (eg, see Rimányi [37], Fehér and Rimányi [15], Kazarian [27] and the references there).

The Rimányi–Szűcs construction uses a technical Jänich theorem (see Jänich [24] and Wall [54]), which a priori has no analogue in the positive dimension case. However, when exists, the Rimányi–Szűcs construction $B_{\mathcal{J}}$ consists of classifying spaces BDiff F_α of diffeomorphism groups of fibers F_α of $\mathcal{J}$–maps, ie, as a set $B_{\mathcal{J}}$ is given by the disjoint union

$$B_{\mathcal{J}} \cong \bigsqcup \text{BDiff}\, F_\alpha.$$

In the case $d > 1$, some of these strata are complicated. For example, the space $B_{\mathcal{J}}$ always contains a stratum BDiff F for each smooth manifold F of dimension d; we recall that computing cohomology groups $H^*(\text{BDiff}\, F; \mathbb{Z})$ is a challenging (open) problem already for a surface F of a high genus.

In contrast, thanks to an extremely helpful observation of Kazarian (eg, see [29]), the proposed spectrum $\mathbf{B}_{\mathcal{J}}$ has a relatively simple structure both in the positive and negative dimension cases. It consists of Thom spaces of vector bundles over $S_t(\mathcal{J})$, while the space $\mathbf{S}(\mathcal{J}) = \lim S_t(\mathcal{J})$ has a natural stratification,

$$\mathbf{S}(\mathcal{J}) \cong \bigsqcup_{\tau \in \mathcal{J}} \text{BDiff}\, \tau,$$

where Diff τ stands for the symmetry group of τ. The spaces BDiff τ are relatively simple. For example, in the case where $\mathcal{J}$ is the empty set of singularities of maps of dimension $d > 0$, the space $\mathbf{S}(\mathcal{J})$ coincides with BO_d (compare with the corresponding monstrosity $\bigsqcup \text{BDiff}\, F_\beta$, where the disjoint union ranges over closed manifolds F_β of dimension d). The simplicity of $\mathbf{B}_{\mathcal{J}}$ makes it possible to carry out explicit computations not only in the case of $d < 0$, but also in the case $d \geq 0$ [51], [44].

Remark 2.4 Note that since both $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ and $B_{\mathcal{J}}$ are classifying spaces, it follows that for each closed manifold W, there is a canonical isomorphism of sets

$$[W, \Omega^\infty \mathbf{B}_{\mathcal{J}}] \cong [W, B_{\mathcal{J}}]$$

of homotopy classes of maps. In a paper in progress we study diffeomorphism groups of manifolds by exploring the relationship between $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ and $B_{\mathcal{J}}$ from geometric point of view.

Remark 2.5 The b-principle is not a direct consequence of the h-principle. In terms of jet spaces, the h-principle for a differential relation $\mathcal{R}$ asserts that if TV and TW

denote the tangent bundles of smooth closed manifolds V and W respectively, then the existence of a *formal solution* $V \to J^k(TV, TW)$ of $\mathcal{R}$ covering a map $f\colon V \to W$ implies the existence of a homotopy of f to a genuine solution of $\mathcal{R}$ (for definitions, see Sections 4, 8 and 10); while the corresponding version of the b-principle implies that for a sufficiently big integer $l \in \mathbb{Z}$ and stabilized tangent bundles $TV \oplus l\varepsilon$ of V and $TW \oplus l\varepsilon$ of W, the existence of a *stable formal solution* $V \to J^k(TV \oplus l\varepsilon, TW \oplus l\varepsilon)$ of $\mathcal{R}$ covering the map $f\colon V \to W$ implies the existence of a cobordism of f to a genuine solution of $\mathcal{R}$ (for definitions, see Sections 5, 8 and 9). Here ε stands for the trivial vector bundle over any topological space.

Remark 2.6 The proof of the Main Theorem in the present paper essentially utilizes the h-principle. The early versions of the h-principle appeared in the works of Nash, Smale, and later Hirsch, Poenaru, Phillips, Feit and others (eg, see Gromov [19] and Eliashberg and Mishachev [14] and the references there). Its general version first appeared in the papers of Gromov [18] and Eliashberg and Gromov [20], and in the present form was first formulated in the foundational book by Gromov [19]. The theory has been extensively developed and several general powerful methods have been discovered (eg, Gromov's [19] methods of convex integration, continuous sheaves and its new version, namely, the method of Eliashberg and Mishachev [14] of holonomic approximations). We will use the h-principle for so-called open $\mathcal{K}$–invariant differential relations which is essentially due to Phillips [34], Eliashberg [9; 10], du Plessis [36] and Ando [1; 5; 6].

Remark 2.7 It is plausible that a weak version of the h-principle is sufficient for proving the Main Theorem. Indeed, for finite families $\mathcal{J}$ of simple singularities in the negative dimension case [51] one only needs the Rourke–Sanderson argument [39]; while in the case of maps of positive dimension a theorem similar to the Main Theorem can be established by using only the Gromov h-principle over open manifolds [42].

Related constructions

The construction of $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ is related to the construction by Eliashberg [11] of classifying spaces for Lagrangian and Legendrian immersions. In fact, our intermediate Theorem 11.1 is a generalization of the Eliashberg theorem.

According to a startling observation of Kazarian, the infinite loop space $\Omega^{\infty+d}\mathbf{MO}$ of the Thom spectrum of the unoriented cobordism groups contains a copy $K_{\mathcal{J}}$ of each space $B_{\mathcal{J}}$ [28; 29] (see also Saeki and Yamamoto [46]). Each space $K_{\mathcal{J}}$ comes with a natural stratification resembling that in the Rimányi–Szűcs construction. In the case $d < 0$, the advantage of the classifying space $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ over $K_{\mathcal{J}}$ is relatively limited,

but in the case $d \geq 0$, the strata of $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ are essentially simpler than those of $K_{\mathcal{J}}$ (see the discussion above).

In the case of immersions, our construction should be compared to that of Wells [55]. In the case $d < 0$ the Main Theorem is also proved by Szűcs [51] by a different line of reasoning. In [6], Ando proposes a homotopy theoretic counterpart of $B(W; \mathcal{J})$ similar to but essentially different from ours and independently proves a theorem similar to but essentially different from our Theorem 11.1. It implies a version of the Main Theorem but only in the case $d < 0$. Ando also suggests an alternative way of deriving the Main Theorem from our Theorem 11.1 (cf our Theorem 13.2 which originally appeared in [41]). Again, the main advantage of our approach is that it applies not only in the case $d < 0$, but also in the case $d \geq 0$ of our primary interest.

For $\mathcal{J} = \varnothing$ and $d = 0$, the space $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ can be identified with a path component of the infinite loop space $\Omega^\infty S^\infty$ and appears in the Kahn–Priddy theorem [16]. In the case where $\mathcal{J} = \varnothing$ and $d = 2$, the space $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ coincides with the space $\Omega^\infty \mathbf{hV}$ in the proof of the Mumford conjecture by Madsen and Weiss [32]. In the case where $\mathcal{J} = \varnothing$ and $d > 1$, the space $\Omega^{\infty-1} \mathbf{B}_{\mathcal{J}}$ is weakly homotopy equivalent to the classifying space BC_d of the cobordism category of manifolds of dimension d of Galatius, Madsen, Tillman and Weiss [17]. In general $\Omega^{\infty-1} \mathbf{B}_{\mathcal{J}}$ is weakly homotopy equivalent to the classifying space $BC_{\mathcal{J}}$ of the corresponding singular cobordism category, provided $d > 1$ [42].

3 Bordism principle

We may say that the homotopy principle for a set valued functor F asserts that F can be induced from a homotopy functor with domain in a homotopy category. Likewise we say that the *bordism principle* for a functor F with values in the category **AG** of abelian groups asserts that F can be induced from a (co)homology functor.

We examine the b-principle in the case of **AG** valued functors $\mathcal{B}$ of $\mathcal{J}$–bordism groups and $\mathcal{C}$ of $\mathcal{J}$–cobordism groups. The functors $\mathcal{B}$ and $\mathcal{C}$ are sometimes confused in the literature; both are defined so that for each closed manifold W of a fixed dimension n, there is a canonical isomorphism

$$\mathcal{B}(W) \cong \mathcal{C}(W) \cong B(W; \mathcal{J}).$$

There is, however, an essential difference between $\mathcal{B}$ and $\mathcal{C}$. The former is covariant, while the latter is contravariant (see Sections 6, 7).

We will show that under the conditions of the Main Theorem the covariant functor $\mathcal{B}$ extends over the category $\mathbf{Top}^2 \downarrow BO_n$ whose object is a vector n–bundle over a pair of

topological spaces, and whose morphism is a fiberwise isomorphism of vector bundles. As we will see, the extended functor satisfies an analogue of the Eilenberg–Steenrod axioms. In other words, for the functor $\mathcal{B}$ of $\mathcal{J}$–bordism groups we will establish a weak version of the b–principle (see Definition 3.2).

On the other hand, we will show that in contrast to the functor of $\mathcal{J}$–bordism groups, under the conditions of the Main Theorem, the functor of $\mathcal{J}$–cobordism groups does satisfy the b-principle, ie, it can be induced from a genuine cohomology functor.

We will need the following definitions.

For a topological space B, let $\mathbf{Top}^2 \downarrow B$ denote the category with objects $(X, A; \varphi)$, each given by a pair of topological spaces (X, A) together with a homotopy class of a map $\varphi\colon X \to B$; and with morphisms $(X_1, A_1; \varphi_1) \to (X_2, A_2; \varphi_2)$, each given by the homotopy class of a continuous map $f\colon (X_1, A_1) \to (X_2, A_2)$ such that $\varphi_1 = \varphi_2 \circ f$. There is a functor $r\colon \mathbf{Top}^2 \downarrow B \to \mathbf{Top}^2 \downarrow B$ defined by $r(X, A; \varphi) = (A, \varnothing; \varphi|A)$.

Definition 3.1 A *homology theory h_* on the category* $\mathbf{Top}^2 \downarrow B$ is a sequence of functors $h_n\colon \mathbf{Top}^2 \downarrow B \to \mathbf{AG}$, with $n \in \mathbb{Z}$, and natural transformations $\partial_n\colon h_n \to h_{n-1} \circ r$ that satisfy the Exactness axiom and the Excision axiom (eg, see Switzer [49, Chapter 7]).

In a similar fashion one may give a definition of a cohomology theory h^* on the category $\mathbf{Top}^2 \downarrow B$.

The category $\mathbf{Top}^2 \downarrow B$ contains a subcategory $\mathbf{Top} \downarrow B$ of topological spaces $(X; \varphi) = (X, \varnothing; \varphi)$ over B and homotopy classes of continuous maps over B.

Definition 3.2 Let F be a contravariant (respectively covariant) functor from a category $\mathcal{C}$ into the category of abelian groups $\mathbf{AG}$. We say that the functor F satisfies the *weak bordism principle* with respect to $\mathbf{Top}^2 \downarrow B$ if there is a cohomology (respectively homology) theory h^* (respectively h_*) on $\mathbf{Top}^2 \downarrow B$ and a functor $\tau\colon \mathcal{C} \to \mathbf{Top} \downarrow B$ such that the functor F is naturally equivalent to the functor $h^n \circ \tau$ (respectively $h_n \circ \tau$) for some $n \in \mathbb{Z}$. We say that F satisfies the *bordism principle* if F satisfies the weak bordism principle in the case where B is a point.

For example, the de Rham cohomology functor defined on the category of smooth manifolds and smooth maps satisfies the bordism principle. Its homotopy analogue is the singular cohomology functor defined on the category $\mathbf{Top}^2 \downarrow \{\mathrm{pt}\}$ of pairs of topological spaces over a point $\{\mathrm{pt}\}$.

Remark 3.3 Since reduced and unreduced (co)homology are essentially equivalent (eg, see Hatcher [21]), Definition 1.2 given in the Introduction is essentially the same as Definition 3.2 of b-principle for contravariant functors.

4 Differential relations

Let $X \to V$ be a smooth fiber bundle. We say that two smooth sections f_1 and f_2, defined in a neighborhood of a point $v \in V$, have a *contact* of order $\geq k$ at v if the values and the partial derivatives of order $\leq k$ of f_1 and f_2 at v are the same. The equivalence class of local sections that have a contact of order k at v with a local section f is called the k–*jet* $[f]_v^k$. The set of k–jets of local sections of X forms the total space $J^k(X)$, called the k–*jet space*, of a smooth fiber bundle over X with projection π_X sending a k–jet $[f]_v^k$ onto $f(v)$. The composition of π_X with the projection $X \to V$ turns $J^k(X)$ into the total space of a smooth fiber bundle over V.

Definition 4.1 A *differential relation* $\mathcal{R}$ of order k over a smooth fiber bundle $X \to V$ is a subset of the k–jet space $J^k(X)$.

Every smooth section f of the bundle $X \to V$ is covered by a smooth map

$$j^k f \colon V \to J^k(X),$$
$$j^k f \colon v \mapsto [f]_v^k,$$

called the k–*jet extension* of f.

Definition 4.2 A *solution* of a differential relation $\mathcal{R} \subset J^k(X)$ is a section f with image $j^k f(V)$ in $\mathcal{R}$.

In other words, a differential relation over a smooth fiber bundle is a relation on the derivatives of smooth sections of the bundle.

If $X \to V$ is a smooth fiber bundle over a manifold of dimension m with fiber of dimension n, then the fiber of the k–jet bundle $J^k(X) \to X$ is isomorphic to the space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ of k–jets of germs

$$(1) \qquad\qquad f \colon (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^n, 0).$$

Occasionally we will identify the space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ with the space of polynomial maps $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ of degree $\leq k$.

The space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ admits a smooth action of the so-called k–*contact group* $\mathcal{K} = \mathcal{K}(k, m, n)$. By definition, the group $\mathcal{K}$ is the subgroup of k–jets of diffeomorphism germs

$$(\mathbb{R}^m \times \mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^m \times \mathbb{R}^n, 0)$$

consisting of those elements which take the horizontal slice $\mathbb{R}^m \times \{0\}$ onto $\mathbb{R}^m \times \{0\}$ and each vertical slice $\{x\} \times \mathbb{R}^n$, with $x \in \mathbb{R}^m$, into a vertical slice $\{y\} \times \mathbb{R}^n$, with

$y \in \mathbb{R}^m$. The action of $\mathcal{K}$ on $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is determined by the action of $\mathcal{K}$ on the graphs $\{(x, f(x)) \in \mathbb{R}^m \times \mathbb{R}^n\}$ of germs of the form (1).

Definition 4.3 A *differential relation basis* R of order k is an arbitrary subset of the space $J^k(\mathbb{R}^m, \mathbb{R}^n)$. We say that a basis *has sufficiently many symmetries* or $\mathcal{K}$–*invariant* if it is invariant with respect to the action of $\mathcal{K}$.

A basis invariant with respect to the action of $\mathcal{K}(k, m, n)$, leads to a differential relation over every trivial fiber bundle $\pi_V: X = V \times W \to V$ with base of dimension m and fiber of dimension n. Indeed, over the total space X, there is a canonically defined, principal $\mathcal{K}$–bundle $P_X^k(\mathcal{K}) \to X$ of k–jets of diffeomorphism germs

$$(2) \qquad \alpha: (\mathbb{R}^m \times \mathbb{R}^n, 0) \longrightarrow (X, x), \quad x = (v, w) \in X,$$

such that the restriction $\alpha|(\mathbb{R}^m \times \{0\})$ is a diffeomorphism onto a neighborhood of (v, w) in $V \times \{w\}$, and for each point $y \in \mathbb{R}^m$, the image of the vertical slice $(\{y\} \times \mathbb{R}^n)$ under the map $\pi_V \circ \alpha$ is a single point in V. The projection $P_X^k(\mathcal{K}) \to X$ is given by taking the k–jet represented by a germ (2) onto x. Let us observe that there is a canonical isomorphism of fiber bundles

$$P_X^k(\mathcal{K}) \times_{\mathcal{K}} J^k(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow J^k(X)$$

over X, which, in terms of representatives, takes a pair (α, f) of germs (2) and (1) onto a local section with graph

$$\alpha \circ ([\pi_V \circ \alpha|(\mathbb{R}^m \times \{0\})]^{-1}, \ f \circ [\pi_V \circ \alpha|(\mathbb{R}^m \times \{0\})]^{-1}).$$

If, now, $R \subset J^k(\mathbb{R}^m, \mathbb{R}^n)$ is a $\mathcal{K}$–invariant basis, then the set $P_X^k(\mathcal{K}) \times_{\mathcal{K}} R$ is a well-defined relation over X.

Remark 4.4 The sections of a trivial bundle $X = V \times W \to V$ are in bijective correspondence with the maps $V \to W$. In this case, according to the terminology of singularity theory, the complement $\Sigma = J^k(X) \setminus \mathcal{R}$ to a differential relation $\mathcal{R}$ is a *singularity set*, and a solution to $\mathcal{R}$ is a *map without* Σ–*singularities*. Thus a choice of a $\mathcal{K}$–invariant differential relation basis containing regular jets corresponds to a choice of a (not necessarily open) $\mathcal{K}$–invariant set of prescribed singularity types $\mathcal{J}$ (see Section 1). Namely, the set $\mathcal{J}$ corresponding to $\mathcal{R}$ consists of nonregular singularity types of germs whose jets are in $\mathcal{R}$. We note that if the set of singularity types $\mathcal{J}$ is open, then the corresponding differential relation basis R is open in the topological space $J^k(\mathbb{R}^m, \mathbb{R}^n)$.

5   Sequence of differential relation bases

We have seen that a base $R \subset J^k(\mathbb{R}^m, \mathbb{R}^n)$ with sufficiently many symmetries determines a differential relation on smooth maps $V \to W$ for any pair of manifolds V and W of dimensions m and n respectively. However, to define a bordism group of solutions we also need to consider differential relations on maps of manifolds of higher dimensions, which can be done by introducing a *sequence of differential relation bases*.

For nonnegative integers s, t, with $s < t$, and an integer q, there is an embedding

$$\mathrm{ext}_t^s \colon J^k(\mathbb{R}^s, \mathbb{R}^{q+s}) \longrightarrow J^k(\mathbb{R}^t, \mathbb{R}^{q+t})$$

of k–jet spaces that takes the k–jet of a germ

$$f \colon (\mathbb{R}^s, 0) \to (\mathbb{R}^{q+s}, 0)$$

onto the k–jet of the suspension germ

$$f \times \mathrm{id}_{\mathbb{R}^{t-s}} \colon (\mathbb{R}^t, 0) \to (\mathbb{R}^{q+t}, 0),$$

where $\mathrm{id}_{\mathbb{R}^{t-s}}$ stands for the identity map of the space $\mathbb{R}^{t-s}$. We note that the space $J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ is empty if $q + s < 0$.

Definition 5.1 A *sequence* $\mathbf{R} = \mathbf{R}(q)$ *of differential relation bases*, or a *stable differential relation basis* is a set of bases $R_s(q) \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, one for each $s \geq 0$, such that for nonnegative integers s, t, with $s < t$, the map ext_t^s takes $R_s = R_s(q)$ into R_t and the complement of R_s in $J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ into the complement of R_t in $J^k(\mathbb{R}^t, \mathbb{R}^{q+t})$. We say that a sequence $\mathbf{R} = \{R_s\}$ of bases is *open* if each basis R_s is open in $J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$.

Remark 5.2 For a map $f \colon V \to W$ of a manifold of dimension t to a manifold of dimension $t + q$, the number q is called the *codimension* of f. Thus, if f is of dimension d, then $q = -d$. We tend to use the term "codimension" if $q \geq 0$, eg, in the case of immersions. On the other hand, in the case $q < 0$ we find the term "dimension" more appropriate; for example, the phrase "a map f of dimension 2" means that a nonempty regular fiber of f is a manifold of dimension 2, while the phrase "a map of codimension -2" has no obvious geometric meaning.

We may identify a sequence $\mathbf{R}(q)$ of bases with a subset of the space

$$\mathbf{J} = \mathbf{J}(q) := \operatorname*{colim}_{s \to \infty} J^k(\mathbb{R}^s, \mathbb{R}^{q+s}),$$

where the colimit is taken with respect to the maps ext_{s+1}^s, regarded as inclusions. Indeed, if $\mathbf{R}(q)$ is a subset of $\mathbf{J}(q)$, then the set $\{R_s\}$ of spaces $R_s = \mathbf{R} \cap J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$

is a sequence of differential relation bases. Conversely, if $\{R_s(q)\}$ is a sequence of differential relation bases, then the colimit

$$\mathbf{R} = \mathbf{R}(q) := \operatorname*{colim}_{s \to \infty} R_s(q)$$

taken with respect to the maps $\operatorname{ext}^s_{s+1}$ is a subset of $\mathbf{J}(q)$.

Similarly there are natural inclusions $\mathcal{K}(k, s, q+s) \to \mathcal{K}(k, t, q+t)$, $s < t$, of groups, which allow us to define a stable k–contact group

$$\mathbf{K} = \mathbf{K}(q) := \operatorname*{colim}_{s \to \infty} \mathcal{K}(k, s, q+s).$$

The actions of the groups $\mathcal{K}(k, s, q+s)$ on $J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ induce an action of the group $\mathbf{K}$ on $\mathbf{J}$. It follows that a stable basis $\mathbf{R} \subset \mathbf{J}$ is $\mathbf{K}$–*invariant* if and only if each relation basis R_s, with $s \geq 0$, is invariant with respect to the action of $\mathcal{K}(k, s, q+s)$. In this case we will also say that $\mathbf{R}$ has *sufficiently many symmetries*.

A sequence of bases $\{R_s\}$ is a *suspension* of a (not necessarily $\mathcal{K}$–invariant) differential relation basis R if $R_t \supset R$ for an appropriate t. A suspension is a $\mathbf{K}$–*suspension* if the sequence $\{R_s\}$ is $\mathbf{K}$–invariant. We observe that each basis $R \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ has the minimal $\mathbf{K}$–suspension, defined as the minimal $\mathbf{K}$–invariant subset of $\mathbf{J}$ that contains R; and the maximal $\mathbf{K}$ suspension, defined as the maximal $\mathbf{K}$–invariant subset of $\mathbf{J}$ that does not intersect the set $J^k(\mathbb{R}^s, \mathbb{R}^{q+s}) \setminus R$.

Remark 5.3 If R is a differential relation basis, then its minimal $\mathbf{K}$–suspension exists even if R is not $\mathcal{K}$–invariant. On the other hand the maximal $\mathbf{K}$–suspension of R exists only if R is $\mathcal{K}$–invariant. Indeed, suppose that $\{R_s\}$ is the maximal suspension of R. Since it is a suspension, there is an inclusion $R_t \supset R$ for some t. On the other hand, since the suspension is maximal, there is an inclusion $R_t \subset R$. Consequently, $R_t = R$.

Example 5.4 (Immersions) If $R = R_0, R_1, \ldots$, stand for the bases corresponding to immersions in respectively $J^k(\mathbb{R}^0, \mathbb{R}^q)$, $J^k(\mathbb{R}^1, \mathbb{R}^{q+1})$, ..., then $\{R_s\}$ is the minimal suspension of $R \subset J^k(\mathbb{R}^0, \mathbb{R}^q)$.

Remark 5.5 Due to Mather it is well-known that if $\{R_s\}$ is the minimal suspension of a $\mathcal{K}$–invariant basis $R \subset R_t$, then $R_t = R$. On the other hand, we may define the minimal G–suspension $\{R_s\}$ of R for any subgroup $G < \mathbf{K}$, eg, for the group $\mathcal{A}$ of k–jets of right-left coordinate changes. In this case an $\mathcal{A}$–invariant basis $R \subset R_t$ may not coincide with the t–th space R_t of the minimal $\mathcal{A}$–suspension of R.

6 Bordism groups of solutions

A sequence **R** of bases $R_s \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, $s \geq 0$, with sufficiently many symmetries determines a differential relation $\mathcal{R}$ for mappings from any manifold V of dimension s into any manifold W of dimension $q + s$. To simplify formulation of statements, we will say that a map satisfying $\mathcal{R}$ is a *solution of* **R**, or, simply, an **R**–*map*.

By definition, two **R**–maps $f_i: V_i \to W_i$, with $i = 1, 2$, of closed manifolds are *right-left bordant* if there are

- a compact manifold V with $\partial V = V_1 \sqcup V_2$,
- a compact manifold W with $\partial W = W_1 \sqcup W_2$, and
- an **R**–map $f: V \to W$ such that $f(V_i) \subset W_i$ for $i = 1, 2$, and the restriction of f to collar neighborhoods of V_1 and V_2 in V can be identified with the disjoint union of suspensions of f_1 and f_2.

The right-left bordism classes of **R**–maps constitute a group with addition defined in terms of representatives by taking the disjoint union of maps. Each element of the right-left bordism group is of order 2.

We say that two **R**–maps $f_i: V_i \to W$, with $i = 1, 2$, of closed manifolds V_i into a manifold W are *(right) bordant* if there are

- a compact manifold V with $\partial V = V_1 \sqcup V_2$, and
- an **R**–map $f: V \to W \times [1, 2]$ such that $f(V_i) \subset W \times \{i\}$ for $i = 1, 2$, and the restriction of f to collar neighborhoods of V_1 and V_2 in V can be identified with the disjoint union of suspensions of f_1 and f_2.

Remark 6.1 Note that in contrast to the definition of right-left bordisms, in the definition of right bordisms the target manifold W is not assumed to be closed. For our approach, in the definition of right bordisms it is essential to allow arbitrary (smooth) manifolds (without boundary), not only closed manifolds. Indeed, Theorem 11.1 implies, for example, that the right bordism groups of **R**–maps to nonclosed manifolds $\mathbb{R}^n$, $n \geq 0$, correspond to terms of what in homology theory is called *the coefficient group*.

Finally, two **R**–maps $f_i: V_i \to W_i$, with $i = 1, 2$, of closed manifolds are *left bordant* if $V_1 = V_2$ and there is a right-left bordism (V, W, f) with $V = V_1 \times [0, 1]$.

Taking the disjoint union of maps leads to a structure of a semigroup on the set of right bordism classes of solutions. We note that the semigroup of bordism classes of solutions may not be a group.

Remark 6.2   We adopt the convention that the empty set $\varnothing$ is a manifold of an arbitrary dimension. In particular, for each manifold W and a sequence $\mathbf{R}$ of relation bases, the map $\varnothing \to W$ is a solution of $\mathbf{R}$.

Example 6.3   Let $\mathbf{R}$ be the sequence of open differential relation bases R_s, with $s \geq 0$, with sufficiently many symmetries corresponding to submersions of dimension $-q > 0$. Then the element of the semigroup of bordism classes of solutions represented by any submersion $V \to W$ of a closed nonempty manifold has no inverse. Consequently, the semigroup of bordism classes of submersions is not a group.

We define the *(right) bordism group* of $\mathbf{R}$–solutions $\mathcal{B}(W)$ as the group given by the Grothendieck construction applied to the semigroup $SB(W)$ of right bordism classes of solutions. In fact, we will see that under the assumptions of the Main Theorem the semigroup of right bordism classes of solutions is already a group (see Corollary 14.1). Furthermore, one may follow the proof of Theorem 11.1 to obtain a description of the inverse element (see Szűcs [51] for the case $q > 0$), but an explicit construction of a representative of the inverse of a given element in $SB(W)$ may not however be a simple task.

Let $\mathbf{Diff}_n$ be the category of smooth manifolds of dimension n without boundary and equidimensional embeddings. We have defined the correspondence

$$\mathcal{B}\colon \mathrm{Obj}(\mathbf{Diff}_n) \longrightarrow \mathrm{Obj}(\mathbf{AG}),$$
$$\mathcal{B}\colon W \;\mapsto\; \mathcal{B}(W),$$

where $\mathrm{Obj}(C)$ stands for the collection of objects of a category C. The collection of morphisms of a category C will be denoted by $\mathrm{Mor}(C)$.

Given an embedding $i\colon W_1 \to W_2$ in $\mathrm{Mor}(\mathbf{Diff}_n)$, and an $\mathbf{R}$–map $f\colon V_1 \to W_1$ representing an element $[f]$ in $\mathcal{B}(W_1)$, the composition $i \circ f\colon V_1 \to W_2$ is an $\mathbf{R}$–map representing an element $[i \circ f]$ in $\mathcal{B}(W_2)$. Furthermore, the element $[i \circ f]$ depends only on the class $[f]$, not on its representative f. Hence, each embedding $i\colon W_1 \to W_2$ in $\mathrm{Obj}(\mathbf{Diff}_n)$ gives rise to a correspondence

$$\mathcal{B}(i)\colon \mathcal{B}(W_1) \longrightarrow \mathcal{B}(W_2),$$
$$\mathcal{B}(i)\colon [f]-[g] \;\mapsto\; [i \circ f]-[i \circ g],$$

which is easily seen to be a group homomorphism. In fact the correspondence $\mathcal{B}$ defines an $\mathbf{AG}$ valued covariant functor on the category $\mathbf{Diff}_n$, called the *functor of the* $\mathbf{R}$*–bordism group*. We will continue to denote the functor by the symbol $\mathcal{B}$.

Remark 6.4 We warn the reader that some authors use the term "cobordism groups" to refer to the values of the functor $\mathcal{B}$ and define "bordism groups" as cobordism groups of Euclidean spaces. Other authors use the term "cobordism groups" to refer both to groups related to the covariant functor $\mathcal{B}$ and to groups related to the contravariant functor $\mathcal{C}$ described in Section 7.

We use the term "bordism groups" in the case of covariant functors and reserve the term "cobordism groups" for contravariant functors (see Section 7). Our choice of terminology is compatible with that in Switzer [49], Rudyak [40], Stong [48], Kochman [30] and other textbooks in algebraic topology.

7 Cobordism groups of solutions

Two proper **R**–maps $f_i\colon V_i \to W$, with $i = 1, 2$, of manifolds are *(right) cobordant* if there are

- a manifold V with $\partial V = V_1 \sqcup V_2$, and
- a proper **R**–map $f\colon V \to W \times [1, 2]$ such that $f(V_i) \subset W \times \{i\}$ for $i = 1, 2$, and the restriction of f to collar neighborhoods of V_1 and V_2 in V can be identified with the disjoint union of suspensions of f_1 and f_2.

Taking the disjoint union of maps leads to a structure of a semigroup on the set of right cobordism classes of solutions.

We recall that $\mathbf{Diff}_n$ denotes the category of smooth manifolds of dimension n without boundary and equidimensional embeddings.

The *(right) cobordism group* of **R**–solutions $\mathcal{C}(W)$ is defined to be the group given by the Grothendieck construction applied to the semigroup of right cobordism classes of solutions. In other words, $\mathcal{C}(\cdot)$ is a correspondence

$$\mathcal{C}\colon \mathrm{Obj}(\mathbf{Diff}_n) \longrightarrow \mathrm{Obj}(\mathbf{AG}),$$
$$\mathcal{C}\colon W \mapsto \mathcal{C}(W).$$

Given an embedding $i\colon W_1 \to W_2$ in $\mathrm{Mor}(\mathbf{Diff}_n)$, and an **R**–map $f\colon V_2 \to W_2$ representing an element $[f]$ in $\mathcal{C}(W_2)$, the pullback

$$i^*f = i^{-1} \circ f\colon V_1 = f^{-1}(i(W_1)) \longrightarrow W_1$$

is an **R**–map representing an element $[i^*f]$ in $\mathcal{C}(W_1)$. The element $[i^*f]$ depends only on the class $[f]$, and therefore, each embedding i in $\mathrm{Mor}(\mathbf{Diff}_n)$ determines a

correspondence

$$\mathcal{C}(i)\colon \mathcal{C}(W_2) \longrightarrow \mathcal{C}(W_1),$$
$$\mathcal{C}(i)\colon [f]-[g] \mapsto [i^*f]-[i^*g],$$

which is a group homomorphism. The correspondence $\mathcal{C}$ defines an **AG** valued contravariant functor on **Diff**$_n$, called the *functor of the* **R**–*cobordism group*. We will continue to denote this functor by $\mathcal{C}$.

Remark 7.1 As has been mentioned in Section 3, it follows from the definitions that if W is a closed manifold, then there is an isomorphism

$$\mathcal{B}(W) \xrightarrow{\cong} \mathcal{C}(W).$$

In fact, in terms of representatives the isomorphism is given by the identity correspondence

(3) $$[f] \mapsto [f].$$

On the other hand, if W is not closed, then the correspondence (3) is not an isomorphism. For example, for any sequence $\mathbf{R}(q)$, with $-q > 0$, the inclusion $f\colon \mathbb{R}^0 \to \mathbb{R}^{-q}$ represents a nontrivial element in $\mathcal{B}(\mathbb{R}^{-q})$ and the trivial element in $\mathcal{C}(\mathbb{R}^{-q})$.

8 Formal jet spaces of vector bundles

Given two vector bundles η and γ of dimensions m and n respectively over a topological space B, let $P^k_{\eta,\gamma}(\mathcal{K})$ denote the principal $\mathcal{K}$–bundle over B whose total space consists of k–jets of diffeomorphism germs

(4) $$f\colon (\mathbb{R}^m \times \mathbb{R}^n, 0) \longrightarrow (\eta|x \oplus \gamma|x, 0), \quad x \in B,$$

where $\eta|x$ and $\gamma|x$ are restrictions to x, such that $f(\mathbb{R}^m \times \{0\}) = \eta|x \oplus 0$ and for each $y \in \mathbb{R}^m$, the projection $\eta \oplus \gamma \to \eta$ maps the image $f(\{y\} \times \mathbb{R}^n)$ onto a point. The projection $P^k_{\eta,\gamma}(\mathcal{K}) \to B$ is defined by sending the germ (4) onto $x \in B$.

Definition 8.1 A *formal differential relation* $\mathcal{R}$ over (η, γ) is an arbitrary subset of the total space of the *formal k–jet bundle*

$$J^k(\eta, \gamma) = P^k_{\eta,\gamma}(\mathcal{K}) \times_{\mathcal{K}} J^k(\mathbb{R}^m, \mathbb{R}^n) \longrightarrow B.$$

Given a $\mathcal{K}$–invariant basis $R \subset J^k(\mathbb{R}^m, \mathbb{R}^n)$, there is an associated differential relation

$$\mathcal{R} = \mathcal{R}(\eta, \gamma) = P^k_{\eta,\gamma}(\mathcal{K}) \times_{\mathcal{K}} R$$

in $J^k(\eta, \gamma)$.

In fact, for a smooth trivial bundle $\pi_V\colon X = V \times W \to V$ where the base and fiber are Riemannian manifolds, the k–jet bundle $J^k(X)$ is canonically isomorphic to a certain formal k–jet bundle. In what follows, the tangent bundle of a manifold M is denoted by TM and the tangent plane of M at x is denoted by $T_x M$. The Riemannian metric on X is given by the product of Riemannian metrics on V and W.

Lemma 8.2 *Let η be the pullback of TV to the bundle over X with respect to π_V and γ the subbundle of TX that consists of vectors tangent to the fibers of the bundle $X \to V$. Then there is a canonical isomorphism $J^k(\eta, \gamma) \to J^k(X)$.*

Proof Let $\mathrm{Exp}_x\colon U_x \to V_x$ denote the exponential diffeomorphism of a neighborhood of the origin in the tangent plane $T_x X$ onto a neighborhood of x in X.

Given a map $\alpha\colon (\mathbb{R}^m \times \mathbb{R}^n, 0) \to (T_x X, 0)$ representing a point in $P^k_{\eta,\gamma}(\mathcal{K})$, the composition $\mathrm{Exp}_x \circ \alpha$ represents an element in $P^k_X(\mathcal{K})$. Hence, the product Riemannian metric on X determines a map $P^k_{\eta,\gamma}(\mathcal{K}) \to P^k_X(\mathcal{K})$ which is easily seen to be an isomorphism of principal $\mathcal{K}$–bundles. The isomorphism of Lemma 8.2 can be produced from the isomorphism of the principal $\mathcal{K}$–bundles by the Borel construction. $\square$

9 Bordism group of formal solutions

Let $\mathbb{R}^\infty$ denote the infinite dimensional vector space, defined as the colimit of inclusions $\mathbb{R}^1 \subset \mathbb{R}^2 \subset ...$, with base $(e_1, e_2, ...)$. We regard the classifying space BO_i as the space of i–subspaces of $\mathbb{R}^\infty$ each of which is contained in a subspace $\mathbb{R}^j$ for some $j < \infty$. Then the fiber of a canonical vector bundle $EO_i \to BO_i$ over a plane $L \in BO_i$ can be interpreted as the space of vectors in L. There is a canonical inclusion $BO_i \to BO_{i+1}$ that takes a plane spanned by vectors $v_1, \ldots, v_i$ onto the plane spanned by $e_1, \theta(v_1), \ldots, \theta(v_i)$, where θ is the shift endomorphism of $\mathbb{R}^\infty$ given by $\theta(e_k) = e_{k+1}$. Similarly there is a canonical inclusion $EO_i \subset EO_{i+1}$ and splitting of $EO_{i+1}|BO_i$ into the sum $\varepsilon \oplus EO_i \cong EO_i \oplus \varepsilon$. Here and in what follows we use the symbol ε to denote the trivial 1–dimensional vector bundle over an arbitrary space.

Let W be a manifold of dimension n. Then a **K**–invariant sequence $\mathbf{R} = \{R_s(q)\}$ of bases in $J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, with $s \geq 0$ and $q + s \geq n$, leads to a sequence of formal differential relations $[R_W]_s$ in the formal k–jet bundles $[J^k_W]_s$ associated to the pair of bundles

$$(\pi_1^* EO_s, \ \pi_2^* TW \oplus t\varepsilon) \quad \text{over} \quad BO_s \times W,$$

where π_1 and π_2 denote the projections of $BO_s \times W$ onto the first and second factors respectively, and $t = q + s - n$.

For $s \geq 0$, there is a canonical inclusion $[J_W^k]_s \subset [J_W^k]_{s+1}$ that takes the k–jet $[f]_0^k \in [J_W^k]_s$ over $v \in BO_s \times W$ represented by the germ of a smooth map

$$f \colon (\pi_1|v)^* EO_s \longrightarrow (\pi_2|v)^* \{TW \oplus t\varepsilon\}$$

onto the k–jet $[f \times \mathrm{id}_\varepsilon]_0^k \in [J_W^k]_{s+1}$ of the germ of the map

$$f \times \mathrm{id}_\varepsilon \colon (\pi_1|v)^* (EO_s \oplus \varepsilon) \longrightarrow (\pi_2|v)^* \{TW \oplus t\varepsilon \oplus \varepsilon\},$$

where we make use of the identification of $EO_{s+1}|BO_s$ with $EO_s \oplus \varepsilon$.

Let us observe that, since $\mathbf{R}$ is a $\mathbf{K}$–invariant sequence of bases, the canonical inclusion $[J_W^k]_s \subset [J_W^k]_{s+1}$ takes $[R_W]_s$ into $[R_W]_{s+1}$. In view of these canonical inclusions, we will use the same symbol π to denote any of the projections

$$[J_W^k]_s \longrightarrow BO_s, \qquad s \geq 0,$$
$$[R_W]_s \longrightarrow BO_s, \qquad s \geq 0,$$

defined as the composition of the projection onto $BO_s \times W$ followed by the projection onto the first factor BO_s.

Definition 9.1 Let $\mathbf{R} = \mathbf{R}(q)$ be a $\mathbf{K}$–invariant stable differential relation basis, W a smooth manifold of dimension $n \geq q$, and $m = n - q$. Then, *the bordism group* $\mathbf{MO_R}(W)$ *of stable formal* $\mathbf{R}$*–maps into* W is the m–th bordism group of maps of manifolds with an (R_W, π)–structure in the stable tangent bundle, where (R_W, π) is the (B, f)–sequence (for a definition, see, for example, Stong [48]) given by the commutative diagrams

$$
\begin{array}{ccc}
[R_W]_s & \longrightarrow & [R_W]_{s+1} \\
\pi \downarrow & & \pi \downarrow \\
BO_s & \longrightarrow & BO_{s+1},
\end{array}
$$

indexed by $s \geq m$, with horizontal maps given by the canonical inclusions.

In particular an element of the bordism group of formal solutions is represented by a triple $(V, \alpha, \tilde{\alpha})$ of a closed manifold V of dimension m, a continuous map $\alpha \colon V \to [R_W]_s$ for some $s \geq m$, and a fiberwise isomorphism of vector bundles

$$\tilde{\alpha} \colon TV \oplus (s - m)\varepsilon \longrightarrow \pi^* EO_s$$

covering the map α. In particular, the composition $\pi \circ \alpha$ is a map classifying the stable tangent bundle of V.

By the Pontrjagin–Thom construction, the m–th bordism group of manifolds with (R_W, π)–structure in the stable tangent bundle is isomorphic to the m–th homotopy

group of the Thom spectrum, denoted by $\mathbf{MO} \wedge_{\mathbf{R}} W$, associated with (R_W, π) (see, for example, Stong [48]).

Note that if $\mathbf{R}$ consists of all jets, ie, if any map of codimension q is an $\mathbf{R}$–map, then $\mathbf{MO} \wedge_{\mathbf{R}} W \cong \mathbf{MO} \wedge W$ and Theorem 11.1 below reduces to the classical Pontrjagin–Thom theorem.

In Section 12 we will show that $\mathbf{MO_R}(\cdot)$ can be extended to a covariant functor on the category $\mathbf{Top}^2 \downarrow BO_n$.

10 H-principle

Certainly, a necessary condition for the existence of a solution of a differential relation $\mathcal{R}$ is the existence of a section of the bundle $\pi_V \colon J^k(X) \to V$ with image in $\mathcal{R}$. The homotopy converse statement, which may not be true in general, is referred to as the *homotopy principle in the existence level*, or, simply, *h-principle*.

Definition 10.1 (H-principle) Every section $s \colon V \to \mathcal{R} \subset J^k(X)$ is homotopic to the k–jet extension of a section $V \to X$ by homotopy of sections $V \to \mathcal{R}$.

In particular, if a differential relation satisfies the h-principle, and there is a section $s \colon V \to \mathcal{R}$, then the differential relation has a solution.

By the Gromov theorem [19], an open so-called Diff–invariant differential relation over a fiber bundle $X \to V$ always satisfies the homotopy principle if V is an open manifold. We refer the reader to Gromov [19], Eliashberg and Mishachev [14] and Spring [47] for further examples of differential relations satisfying the h-principle.

A section $V \to J^k(X)$ is *holonomic* if it extends a section $V \to X$.

We will also need the relative version of the h-principle.

Definition 10.2 (Relative h-principle) Suppose that a section $s \colon V \to \mathcal{R}$ restricted to a neighborhood of a closed subset $V_0 \subset V$ is holonomic. Then s is homotopic to a holonomic section through sections $V \to \mathcal{R}$ constant in a neighborhood of V_0.

Under the conditions of the Main Theorem, the relative h-principle for open $\mathcal{K}$–invariant differential relations imposed on maps of manifolds always holds true. It is essentially due to Phillips [34], Eliashberg [9; 10], du Plessis [36] and Ando [1; 5; 6].

We say that a $\mathbf{K}$–invariant sequence $\mathbf{R} = \{R_s\}$ of differential relation bases $R_s \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, $s \geq 0$, *satisfies the (relative) h-principle* if for every trivial bundle $X \to V$ over an s–dimensional manifold with a $(q+s)$–dimensional fiber, the corresponding differential relation in $J^k(X)$ satisfies the (relative) h-principle.
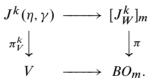
11   Destabilization argument

Let $\mathbf{R}$ be a sequence of differential relation bases $R_s \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, $s \geq 0$, with sufficiently many symmetries. Then, for any smooth manifold W of dimension n, there is a semigroup homomorphism from the semigroup $SB(W)$ of right bordism classes of $\mathbf{R}$–maps into W to the bordism group $\mathbf{MO_R}(W)$ of stable formal $\mathbf{R}$–maps into W.

Indeed, suppose that V is a manifold of dimension $m = n - q$ and $f: V \to W$ is a smooth map representing an element in the semigroup $SB(W)$. Let η and γ be the vector bundles over $X = V \times W$ defined as the pullbacks of the tangent bundles of V and W with respect to the projections of $V \times W$ onto the first and second factors respectively. Then, in view of [Lemma 8.2](#), by choosing Riemannian metrics on V and W, we may identify $J^k(\eta, \gamma)$ with the k–jet space $J^k(X)$ of the trivial bundle $X \to V$. In particular, we may say that the map f gives rise to a section $j^k f$ of the fiber bundle $\pi^k_V: \mathcal{R}_m(\eta, \gamma) \to V$.

Let $\tau: TV \to EO_m$ be a bundle map classifying the tangent bundle of V. In several occasions we will implicitly use a well-known fact that the homotopy class of τ is unique.

The bundle map τ gives rise to a fiberwise isomorphism of bundles

$$
\begin{array}{ccc}
J^k(\eta, \gamma) & \longrightarrow & [J^k_W]_m \\
\pi^k_V \downarrow & & \downarrow \pi \\
V & \longrightarrow & BO_m.
\end{array}
$$

Let $\tilde{\alpha}: TV \to \pi^* EO_m$ be the canonical lift of τ covering the composition α of $j^k f$ and the fiberwise isomorphism $J^k(\eta, \gamma) \to [J^k_W]_m$. Then the triple $(V, \alpha, \tilde{\alpha})$ represents an element in $\mathbf{MO_R}(W)$, which, as it is easy to see, does not depend on τ and the choice of Riemannian metrics on V and W. Furthermore, if $f_i: V_i \to W$, $i = 1, 2$, are two $\mathbf{R}$–maps representing the same element in $SB(W)$, ie, if there exists a bordism between f_1 and f_2 satisfying $\mathbf{R}$, then, by a similar argument, the triples corresponding to f_1 and f_2 determine the same element in $\mathbf{MO_R}(W)$. Thus, there is a well-defined map

$$
\psi: SB(W) \longrightarrow \mathbf{MO_R}(W),
$$

which is easily seen to be a semigroup homomorphism.

Theorem 11.1   *The homomorphism ψ is an isomorphism for any (not necessarily closed) manifold W of dimension n. In particular, the bordism semigroup $SB(W)$ of $\mathbf{R}$–maps into W is isomorphic to the group $\pi_m(\mathbf{MO} \wedge_\mathbf{R} W)$ where $m = n - q$.*

Proof The Pontrjagin–Thom construction establishes an isomorphism of the bordism group $\mathbf{MO_R}(W)$ of manifolds with (R_W, π)–structure in the stable tangent bundle and the homotopy group $\pi_m(\mathbf{MO} \wedge_{\mathbf{R}} W)$. Hence to prove the theorem it suffices to show that the homomorphism ψ is an isomorphism.

In what follows, in the notation of pullbacks of vector bundles we will occasionally suppress the symbols of maps that induce the bundles. For example,

$$J^k(EO_m, TW) \longrightarrow BO_m \times W$$

will denote the formal k–jet bundle associated with the pullbacks of EO_m and TW with respect to the projections of $BO_m \times W$ onto the first and second factors respectively.

Lemma 11.2 *The homomorphism ψ is surjective.*

Proof Given an element of $\mathbf{MO_R}(W)$, let an (R_W, π)–manifold V_0 be its representative. Suppose that the (R_W, π)–structure of V_0 is represented by a continuous map $\alpha \colon V_0 \to [R_W]_{m+r}$ and a fiberwise isomorphism of vector bundles

$$\tilde{\alpha} \colon TV_0 \oplus r\varepsilon \longrightarrow \pi^* EO_{m+r}$$

that covers α producing a commutative diagram

$$
\begin{array}{ccc}
TV_0 \oplus r\varepsilon & \xrightarrow{\ \tilde{\alpha}\ } & \pi^* EO_{m+r} \\
\downarrow & & \downarrow \\
V_0 & \xrightarrow{\ \alpha\ } & [R_W]_{m+r}.
\end{array}
$$

We claim that the (R_W, π)–structure in the stable tangent bundle of V_0 determines an $(R_{W \times L}, \pi_{W \times L})$–structure in the stable tangent bundle of $V_0 \times L$, where L is a closed parallelizable manifold of dimension r with a fixed trivialization of the tangent bundle. Indeed, the bundle

$$[J^k_{W \times L}]_{m+r} := J^k(EO_{m+r}, T(W \times L)) \longrightarrow BO_{m+r} \times (W \times L)$$

is canonically isomorphic to the bundle

$$[J^k_W]_{m+r} \times L := J^k(EO_{m+r}, TW \oplus r\varepsilon) \times L \longrightarrow (BO_{m+r} \times W) \times L$$

under an isomorphism that takes the fiber

$$J^k(EO_{m+r}|b, T(W \times L)|(w, l)) \quad \text{over } (b, (w, l)) \in BO_{m+r} \times (W \times L)$$

onto the fiber

$$J^k(EO_{m+r}|b, (TW \oplus r\varepsilon)|w) \times \{l\} \quad \text{over } ((b, w), l) \in (BO_{m+r} \times W) \times L.$$

Consequently, since the basis of the differential relation for $[R_{W \times L}]_{m+r}$ coincides with the basis of the differential relation for $[R_W]_{m+r}$, the bundle

$$[R_{W \times L}]_{m+r} := R(EO_{m+r}, T(W \times L)) \longrightarrow BO_{m+r} \times (W \times L)$$

and the bundle

$$[R_W]_{m+r} \times L := R(EO_{m+r}, TW \oplus r\varepsilon) \times L \longrightarrow (BO_{m+r} \times W) \times L$$

are canonically isomorphic. Furthermore, there is a canonical isomorphism of vector bundles

$$
\begin{array}{ccc}
\pi^* EO_{m+r} \times L & \longrightarrow & \pi^*_{W \times L} EO_{m+r} \\
\downarrow & & \downarrow \\
[R_W]_{m+r} \times L & \longrightarrow & [R_{W \times L}]_{m+r},
\end{array}
$$

which we precompose with the composition

$$T(V_0 \times L) \longrightarrow (TV_0 \oplus r\varepsilon) \times L \longrightarrow \pi^* EO_{m+r} \times L$$

of a canonical isomorphism and the bundle map $\tilde{\alpha} \times \mathrm{id}_L$, where id_L is the identity map of L, in order to obtain a desired $(R_{W \times L}, \pi_{W \times L})$–structure

$$
\begin{array}{ccc}
T(V_0 \times L) & \longrightarrow & \pi^*_{W \times L} EO_{m+r} \\
\downarrow & & \downarrow \\
V_0 \times L & \longrightarrow & [R_{W \times L}]_{m+r}
\end{array}
$$

on the stable tangent bundle of $V_0 \times L$.

Similarly, the triple $(V_0, \alpha, \tilde{\alpha})$ determines a fiberwise isomorphism of bundles

$$(5) \qquad J^k(T(V_0 \times L), T(W \times L)) \longrightarrow [J^k_{W \times L}]_{m+r}.$$

Let $\hat{X} \to V_0 \times L$ be the trivial bundle with fiber $W \times L$. Again, by choosing Riemannian metrics on $V_0 \times L$ and $W \times L$, we fix an isomorphism

$$(6) \qquad J^k(\hat{X}) \longrightarrow J^k(T(V_0 \times L), T(W \times L))$$

and denote its composition with the bundle map (5) by

$$\mathrm{jet} \colon J^k(\hat{X}) \longrightarrow [J^k_{W \times L}]_{m+r}.$$

By the h-principle, there is a homotopy $H = H_\tau$, with $\tau \in [0, 1]$, of the map

$$H_0 = \alpha \times \mathrm{id}_L \colon V_0 \times L \longrightarrow [R_W]_{m+r} \times L = [R_{W \times L}]_{m+r}$$

to a section H_1 such that $H_1 = \text{jet} \circ j^k h$ for some genuine smooth **R**–map

$$h\colon V_0 \times L \longrightarrow W \times L.$$

By Sard's Lemma applied to the composition of h and the projection of $W \times L$ onto the second factor, there is a value $\text{pt} \in L$ for which the map h is transversal to the copy $W \times \{\text{pt}\}$ of W. In particular the subset $V_1 := h^{-1}(W \times \{\text{pt}\})$ is a compact submanifold of $V_0 \times L$. Let $h_1\colon V_1 \to W$ denote the restriction $h|V_1$ composed with the identification $W \times \{\text{pt}\} \to W$. We will show that h_1 satisfies the differential relation **R**, and $\psi([h_1])$ belongs to the class represented by $(V_0, \alpha, \widetilde{\alpha})$.

Lemma 11.3 *The map $h_1\colon V_1 \to W$ is a solution to* **R**.

Proof For a point $x \in V_1$, by the Inverse Function Theorem, there are neighborhoods $U(x)$ of x in $V_0 \times L$ and $U(h(x))$ of $h(x)$ in $W \times L$ with coordinates $(s_1, \ldots, s_r, v_1, \ldots, v_m)$ in $U(x)$ and $(\widetilde{s}_1, \ldots, \widetilde{s}_r, w_1, \ldots, w_n)$ in $U(h(x))$ such that $(0, \ldots, 0, v_1, \ldots, v_m)$ are coordinates in $V_1 \cap U(x)$, $(0, \ldots, 0, w_1, \ldots, w_n)$ are coordinates in $(W \times \{\text{pt}\}) \cap U(h(x))$, and the mapping $h|U(x)$ has the form

$$
\begin{aligned}
\widetilde{s}_i &= s_i, & i &= 1, \ldots, r, \\
w_j &= w_j(s_1, \ldots, s_r, v_1, \ldots, v_m), & j &= 1, \ldots, n.
\end{aligned}
$$

We note that in the chosen coordinates the mapping $h|V_1 \cap U(x)$ has the form

$$w_j = w_j(0, \ldots, 0, v_1, \ldots, v_m), \qquad j = 1, \ldots, n.$$

Hence the local ring of the germ h at x is isomorphic to the local ring of the germ $h|V_1$ at x. If $h|V_1$ is not a solution to a **K**–invariant relation **R** in a neighborhood of x, then, since

$$\text{ext}^m_{m+r}(J^k(\mathbb{R}^m, \mathbb{R}^n) \setminus R_m) \subset J^k(\mathbb{R}^{m+r}, \mathbb{R}^{n+r}) \setminus R_{m+r},$$

h is not an **R**–map, which is a contradiction. Thus $h|V_1$ is a solution to **R**. $\square$

It remains to show that $\psi([h_1])$ is the class represented by $(V_0, \alpha, \widetilde{\alpha})$.

To simplify the notation we will occasionally use the same symbol both for a map and its restrictions. In Lemma 11.4 we will chase the diagram

$$
\begin{array}{ccccccc}
[R_W]_m & \xrightarrow{\subset} & [R_W]_{m+r} & \xrightarrow{i_L} & [R_W]_{m+r} \times L & \xrightarrow{\cong} & [R_{W \times L}]_{m+r} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
BO_m & \xrightarrow{\subset} & BO_{m+r} & \longrightarrow & BO_{m+r} & \longrightarrow & BO_{m+r},
\end{array}
$$

where the horizontal map i_L is the inclusion of the slice $[R_W]_{m+r} \times \{\text{pt}\}$.

Lemma 11.4 *The bordism class $\psi([h_1])$ is represented by an (R_W, π)–manifold $(V_1, \beta, \widetilde{\beta})$, where $\beta = H_1|V_1$ and $\widetilde{\beta}$ is a bundle map covering β.*

Proof Suppose that the construction in the definition of ψ applied to h_1 yields an (R_W, π)–manifold $(V_1, \gamma_h, \widetilde{\gamma}_h)$ representing $\psi([h_1])$, with

$$\gamma_h\colon V_1 \longrightarrow [R_W]_m.$$

We recall that the triple $(V_1, \gamma_h, \widetilde{\gamma}_h)$ is completely determined by the choices of Riemannian metrics on V_1 and W and a bundle map

$$\tau\colon TV_1 \longrightarrow EO_m.$$

We pass to the r–th suspension of $(V_1, \gamma_h, \widetilde{\gamma}_h)$ and denote it by the same symbols so that now for example

$$\gamma_h\colon V_1 \longrightarrow [R_W]_{m+r}.$$

On the other hand, the map

$$\beta = H_1|V_1\colon V_1 \longrightarrow [R_{W\times L}]_{m+r}$$

factors through a map

$$\gamma_H\colon V_1 \longrightarrow [R_W]_{m+r},$$

while the map $\widetilde{\beta}$ factors through a map $\widetilde{\gamma}_H$ covering γ_H. We need to show that the (R_W, π)–manifolds $(V_1, \gamma_h, \widetilde{\gamma}_h)$ and $(V_1, \gamma_H, \widetilde{\gamma}_H)$ represent the same bordism class. In fact we will construct a homotopy of $(\gamma_H, \widetilde{\gamma}_H)$ to $(\gamma_h, \widetilde{\gamma}_h)$.

In view of the Riemannian metric on L and a fixed trivialization of TL, we may identify a neighborhood U of 0 in the vector space $\mathbb{R}^r$ with a neighborhood of pt in L. Furthermore, we may identify a tubular neighborhood of V_1 in $V_0 \times L$ with $V_1 \times U$ so that the restriction of h to $V_1 \times U$ is of the form

$$V_1 \times U \longrightarrow W \times U,$$
$$(v, u) \mapsto (\bar{h}_1(v, u), u)$$

for some map $\bar{h}_1$. This map is fiberwise (over U) homotopic to the map $h_1 \times \mathrm{id}_U$ through **R**–maps. Consequently, we may assume that $h|V_1 \times U$ coincides with $h_1 \times \mathrm{id}_U$.

Let $\widehat{Y} \to V_1 \times U$ denote the trivial bundle with fiber $W \times U$. Then there is an obvious inclusion $\widehat{Y} \subset \widehat{X}$ which in its turn determines an inclusion $J^k(\widehat{Y}) \subset J^k(\widehat{X})$. Let us recall that the restriction $H_1|V_1 \times U$ is given by the composition of

$$j^k h\colon V_1 \times U \longrightarrow J^k(\widehat{Y}) \subset J^k(\widehat{X})$$

and $\qquad$ jet: $J^k(\widehat{Y}) \xrightarrow{\text{(6)}} J^k(T(V_1 \times U), T(W \times U)) \xrightarrow{\text{(5)}} [J^k_{W\times U}]_{m+r}.$

To begin with we modify the Riemannian metrics on $V_1 \times U$ and $W \times U$ by homotopy to Riemannian metrics given by products of those already fixed on V_1, W and U. This determines a homotopy of the map (6), and therefore a homotopy of $H_1|V_1 \times U$ and $(\gamma_H, \tilde{\gamma}_H)$.

Next we recall that the map (5) is determined by a bundle map

$$T(V_1 \times U) = TV_1 \times TU \longrightarrow EO_{m+r}.$$

We modify it by homotopy to a bundle map given by the composition

$$
\begin{array}{ccccccc}
TV_1 \times TU & \longrightarrow & TV_1 \oplus \varepsilon^r & \longrightarrow & EO_m \oplus \varepsilon^r & \longrightarrow & EO_{m+r} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V_1 \times U & \longrightarrow & V_1 & \longrightarrow & BO_m & \longrightarrow & BO_{m+r},
\end{array}
$$

where the left horizontal maps are obvious projections along the factor U, the middle horizontal maps are determined by τ, and the right horizontal maps are inclusions. This determines a homotopy of (5) and a further homotopy of $H_1|V_1 \times U$ and $(\gamma_H, \tilde{\gamma}_H)$.

Let us observe now that the obtained pair $(\gamma_H, \tilde{\gamma}_H)$ coincides with $(\gamma_h, \tilde{\gamma}_h)$. □

We may perturb the homotopy H relative to H_0 and H_1 so that H becomes transversal to the submanifold

(7) $\qquad [R_W]_{m+r} \times \{pt\} \times [0, 1] \quad \text{in} \quad [R_W]_{m+r} \times L \times [0, 1].$

Then the inverse image V of $[R_W]_{m+r} \times \{pt\} \times [0, 1]$ under the map H is a compact submanifold of $V_0 \times L \times [0, 1]$ with boundary ∂V that consists of two parts

$$V_0 \times \{pt\} \times \{0\} \quad \text{and} \quad V_1 \times \{1\}.$$

Now it suffices to prove the assertion that the composition

$$
\begin{aligned}
V &\xrightarrow{H|V} [R_W]_{m+r} \times \{pt\} \times [0, 1] \\
&\xrightarrow{\subset} [R_W]_{m+r+1} \times \{pt\} \times [0, 1] \\
&\longrightarrow [R_W]_{m+r+1}
\end{aligned}
$$

can be covered by a bundle map

$$TV \oplus r\varepsilon \longrightarrow \pi^* EO_{m+r+1}$$

that leads to an (R_W, π)–bordism between the 1–step stabilizations of $(V_0, \alpha, \tilde{\alpha})$ and $(V_1, \beta, \tilde{\beta})$.

To verify the assertion, let us observe that the 1–step stabilization of the bundle map $\tilde{\alpha}$ extends to a bundle map $\tilde{H}$,

$$
\begin{array}{ccc}
T(V_0 \times L \times [0,1]) & \xrightarrow{\tilde{H}} & (\pi_{W \times L} \circ \pi_{1,2})^* EO_{m+r+1} \\
\downarrow & & \downarrow \\
V_0 \times L \times [0,1] & \longrightarrow & [R_W]_{m+r+1} \times L \times [0,1],
\end{array}
$$

covering the composition

$$
H' \colon V_0 \times L \times [0,1] \xrightarrow{H} [R_W]_{m+r} \times L \times [0,1] \xrightarrow{\subset} [R_W]_{m+r+1} \times L \times [0,1],
$$

where $\pi_{1,2}$ is the projection of $[R_W]_{m+r+1} \times L \times [0,1]$ onto the product of the first two factors.

The normal bundle ν of V in $V_0 \times L \times [0,1]$ is trivial as it is isomorphic to the pullback via H of the trivial normal bundle of the submanifold (7). Consequently the restriction of the bundle map $\tilde{H}$ to V,
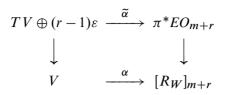
$$
\begin{array}{ccccc}
TV \oplus \nu & \xrightarrow{\tilde{H}|V} & (\pi \circ \pi_{1,2})^* EO_{m+r+1} & \xrightarrow{=} & \pi_1^* EO_{m+r+1} \\
\downarrow & & \downarrow & & \downarrow \\
V & \xrightarrow{H'} & [R_W]_{m+r+1} \times \{\text{pt}\} \times [0,1] & \xrightarrow{=} & [R_W]_{m+r+1} \times [0,1],
\end{array}
$$

where π_1 is the projection of $[R_W]_{m+r} \times [0,1]$ onto the first factor, leads to an (R_W, π)–bordism of the 1–step stabilizations of the (R_W, π)–manifolds $(V_0, \alpha, \tilde{\alpha})$ and $(V_1, \beta, \tilde{\beta})$. $\qquad\square$

Lemma 11.5 *The homomorphism ψ is injective.*

Proof Let $f \colon V_0 \to W_0$ be a map representing an element $[f] \in SB(W_0)$ in the kernel of the homomorphism ψ. Then f determines an (R_{W_0}, π)–structure in the stable tangent bundle of V_0 such that for a sufficiently big r, the structure map $\tilde{\alpha}_0$ in the commutative diagram

$$
\begin{array}{ccc}
TV_0 \oplus r\varepsilon & \xrightarrow{\tilde{\alpha}_0} & \pi^* EO_{m+r} \\
\downarrow & & \downarrow \\
V_0 & \xrightarrow{\alpha_0} & [R_{W_0}]_{m+r}
\end{array}
$$

extends to a structure map $\widetilde{\alpha}$ in the commutative diagram

$$
\begin{array}{ccc}
TV \oplus (r-1)\varepsilon & \xrightarrow{\ \widetilde{\alpha}\ } & \pi^*EO_{m+r} \\
\downarrow & & \downarrow \\
V & \xrightarrow{\ \alpha\ } & [R_W]_{m+r}
\end{array}
$$

where $W = W_0 \times [0,1]$, and V is a compact $(m+1)$–dimensional manifold with boundary $\partial V = V_0$.

Let $\widehat{X} \to V \times L'$ be the trivial bundle with fiber $W \times L'$, where L' is a parallelizable closed manifold of dimension $r-1$ with a fixed trivialization of the tangent bundle. As in the proof of Lemma 11.2, we use the map $\widetilde{\alpha}$ to construct a commutative diagram

$$
\begin{array}{ccc}
T(V \times L') & \xrightarrow{\ \widetilde{\delta}\ } & \pi^*_{W \times L'}EO_{m+r} \\
\downarrow & & \downarrow \\
V \times L' & \xrightarrow{\ \delta\ } & [R_{W \times L'}]_{m+r}
\end{array}
$$

that defines an $(R_{W \times L'}, \pi_{W \times L'})$–structure in the stable tangent bundle of $V \times L'$. Also, as in the proof of Lemma 11.2, we define a mapping

$$
\text{jet}: J^k(\widehat{X}) \longrightarrow [J^k_{W \times L'}]_{m+r}.
$$

Let $C \approx V_0 \times [0,a)$ for some $0 < a \ll 1$ be a small collar neighborhood of V_0 in V. We may assume that in $C \times L'$ the structure map δ coincides with $\text{jet} \circ j^k[f \times \text{id}_{[0,a)} \times \text{id}_{L'}]$. Then, by the relative h-principle applied to the pair $(V \times L', C \times L')$, there is a homotopy of δ to a map $V \times L' \to [R_{W \times L'}]_{m+r}$ given by the composition $\text{jet} \circ j^k h$ for some genuine solution $h \colon V \times L' \to W \times L'$ of $\mathbf{R}$.

Again, as in the proof of Lemma 11.2, there is a regular value pt of the composition of h and the projection of $W \times L'$ onto the second factor. For a compact submanifold V_1 defined as $h^{-1}(W \times \{\text{pt}\})$ we can show that

$$
h|V_1 \colon V_1 \longrightarrow W \times \{\text{pt}\},
$$

is a solution of $\mathbf{R}$. Since $h|\partial V_1$ can be identified with f, this implies that the map represents the trivial element in $SB(W)$. $\qquad\square$

The proof of Theorem 11.1 is complete. $\qquad\square$

12   Proof of the weak bordism principle

Let $\mathbf{Diff}_n$ denote the category of smooth (possibly noncompact) n–manifolds without boundary and smooth (possibly nonproper) embeddings. In this section we consider the covariant functor

$$\mathcal{B}(\,\cdot\,)\colon \mathbf{Diff}_n \longrightarrow \mathbf{AG}$$

(see Section 6) that takes a manifold W of dimension n onto the bordism group $\mathcal{B}(W)$ of $\mathbf{R}$–maps of $(n-q)$–dimensional manifolds into W, and prove the weak bordism principle (see Definition 3.2) for $\mathcal{B}(\,\cdot\,)$ with respect to the category $\mathbf{Top}^2 \downarrow BO_n$ of pairs of topological spaces over the classifying space BO_n of vector n–bundles.

To begin with, let us observe that the definition of the bordism group $\mathbf{MO_R}(W)$ of stable formal $\mathbf{R}$–maps into an n–manifold W can be easily extended to that of the bordism group of stable formal $\mathbf{R}$–maps into a pair $(X, A; \varphi)$ of topological spaces over BO_n.

Indeed, a $\mathbf{K}$–invariant sequence $\mathbf{R} = \{R_s(q)\}$ of differential relation basis $R_s \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, with $s \geq 0$ and $s \geq n-q$, leads to a sequence of formal differential relations $[R_X]_s$ in the formal k–jet bundles $[J_X]_s$ associated to the pair of bundles

$$(\pi_1^* EO_s, (\varphi \circ \pi_2)^* EO_n \oplus t\varepsilon) \qquad \text{over} \qquad BO_s \times X,$$

where π_1 and π_2 denote the projections of $BO_s \times X$ onto the first and second factors respectively, and $t = q + s - n$. Following Definition 9.1, we define the bordism group $\mathbf{MO_R}(X; \varphi)$ of stable formal $\mathbf{R}$–maps into $(X; \varphi)$ as the m–th bordism group of maps of manifolds with (R_X, π)–structure in the stable tangent bundle, where $m = n - q$, and (R_X, π) is the (B, f)–sequence given by commutative diagrams as in Definition 9.1. In other words, $\mathbf{MO_R}(X; \varphi)$ is the m–th homotopy group of the Thom spectrum $\mathbf{MO} \wedge_\mathbf{R}(X; \varphi)$ associated to the tangent (R_X, π)–structures. Finally we define the group $\mathbf{MO_R}(X, A; \varphi)$ as the relative m–th homotopy group of the pair of $\mathbf{MO} \wedge_\mathbf{R}(X; \varphi)$ and its subspectrum $\mathbf{MO} \wedge_\mathbf{R}(A; \varphi|A)$.

More generally we define the group $\mathbf{MO_R}_l(X, A; \varphi)$ as the relative l–th homotopy group of the pair of $\mathbf{MO} \wedge_\mathbf{R}(X; \varphi)$ and $\mathbf{MO} \wedge_\mathbf{R}(A; \varphi|A)$. Furthermore, for each $l \geq 0$, we define a functor $\mathbf{MO_R}_l$ on the category $\mathbf{Top}^2 \downarrow BO_n$ so that

$$\mathbf{MO_R}_l\colon \mathbf{Top}^2 \downarrow BO_n \longrightarrow \mathbf{AG}$$

is given by

$$\mathbf{MO_R}_l\colon (X, A, \varphi) \mapsto \mathbf{MO_R}_l(X, A, \varphi)$$

and for each morphism

$$i\colon (X_1, A_1, \varphi_1) \longrightarrow (X_2, A_2, \varphi_2)$$

of pairs of topological spaces over BO_n, the homomorphism

$$\mathbf{MO_R}\mathit{l}(i)\colon \mathbf{MO_R}\mathit{l}(X_1, A_1, \varphi_1) \longrightarrow \mathbf{MO_R}\mathit{l}(X_2, A_2, \varphi_2)$$

is induced by the obvious map of pairs of spectra

$$(\mathbf{MO}{\wedge}_{\mathbf{R}}(X_1; \varphi_1), \mathbf{MO}{\wedge}_{\mathbf{R}}(A_1; \varphi_1)) \longrightarrow (\mathbf{MO}{\wedge}_{\mathbf{R}}(X_2; \varphi_2), \mathbf{MO}{\wedge}_{\mathbf{R}}(A_2; \varphi_2)).$$

In view of the Blakers–Massey theorem, it is easily verified that $\mathbf{MO^*_R}(\cdot)$ determines a homology theory on the category of pairs of topological spaces over BO_n.

To complete the proof of the weak bordism principle for the functor $\mathcal{B}(\cdot)$, it suffices to observe that Theorem 11.1 implies that $\mathcal{B}(\cdot)$ is naturally equivalent to the functor $\mathbf{MO_R}(\cdot)$ precomposed with the functor

$$G\colon \mathbf{Diff}_n \longrightarrow \mathbf{Top}^2 \downarrow BO_n$$

such that

- G takes a manifold W of dimension n onto the pair $(X; \varphi)$ of the topological space X underlying W and a map $\varphi\colon X \to BO_n$ classifying the vector bundle inherited from TW; and

- G takes a smooth embedding

$$i\colon W_1 \longrightarrow W_2$$

in $\mathbf{Diff}_n$ onto the morphism

$$G(i)\colon (X_1, \varphi_1) \longrightarrow (X_2, \varphi_2),$$

where $(X_i, \varphi_i) := G(W_i)$ for $i = 1, 2$, that as a map of topological spaces is given by the composition

$$X_1 \xrightarrow{\cong} W_1 \xrightarrow{i} W_2 \xrightarrow{\cong} X_2.$$

Theorem 12.1 *Under the assumptions of the Main Theorem, the covariant functor $\mathcal{B}$ satisfies the weak bordism principle.*

13 A contravariant functor dual to $\mathbf{MO_R}(\cdot)$

As it has been shown in the previous section, the functor $\mathbf{MO_R}(\cdot)$ associated to the functor

$$\mathcal{B}(\cdot)\colon \mathbf{Diff}_n \longrightarrow \mathbf{AG}$$

of bordism groups of maps satisfying a $\mathbf{K}$–invariant relation $\mathbf{R}$ is defined only on the category $\mathbf{Top}^2 \downarrow BO_n$ of pairs of topological spaces over BO_n. In particular a number of effective methods of homotopy theory are not available for $\mathbf{MO_R}(\cdot)$. We will show, however, that there is a contravariant functor $H_{\mathbf{R}}^0(\cdot)$ dual to $\mathbf{MO_R}(\cdot)$ such that $H_{\mathbf{R}}^0(\cdot)$ is a cohomology functor in the classical sense, and there is a duality isomorphism
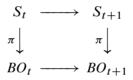
$$H_{\mathbf{R}}^0(W) \cong \mathbf{MO_R}(W)$$

for any closed manifold W of dimension n.

To define $H_{\mathbf{R}}^*(\cdot)$, let us recall that $EO_t \to BO_t$, with $t \geq 0$, denotes the universal vector t–bundle. There is a fiber bundle $\pi\colon S_t \to BO_t$ with the total space given by the space of k–jets of $\mathbf{R}$–maps

(8) $(\mathbb{R}^{t-q}, 0) \longrightarrow (EO_t|b, 0), \quad \text{with } b \in BO_t,$

and with projection π taking the k–jet of a germ (8) onto the point $b \in BO_t$. A construction similar to that in Definition 9.1 provides us with a (B, f)–sequence (S, π) given by the commutative diagrams

$$
\begin{array}{ccc}
S_t & \longrightarrow & S_{t+1} \\
\pi \downarrow & & \pi \downarrow \\
BO_t & \longrightarrow & BO_{t+1}
\end{array}
$$

where the horizontal maps are the canonical inclusions as in Definition 9.1 (cf the definition of $\mathbf{B}_{\mathcal{J}}$ in Section 2 where $k = \infty$).

Definition 13.1 The Thom spectrum associated to the normal (S, π)–structures defines a generalized cohomology theory (in the classical sense). Its contravariant functors are denoted by $\{H_{\mathbf{R}}^*(\cdot)\}$.

For a manifold W of dimension n', the group $H_{\mathbf{R}}^r(W)$ is the r–th cobordism group of maps into W of manifolds with an (S, π)–structure in the stable normal bundle. In particular, every element of $H_{\mathbf{R}}^r(W)$ is represented by a 4–tuple $(V, i, \alpha, \widetilde{\alpha})$ of

- a manifold V of dimension $m = n' - r - q$,
- an embedding $i\colon V \to \mathbb{R}^{t-r-q} \times W$ for sufficiently big t such that the composition of i and the projection $\mathbb{R}^{t-r-q} \times W \to W$ onto the second factor is proper,
- a continuous map $\alpha\colon V \to S_t$, and
- a fiberwise isomorphism $\widetilde{\alpha}\colon \nu \to \pi^*EO_t$ covering α, where ν is the normal vector bundle of V induced by the embedding i.

Theorem 13.2 *Under the assumptions of the* Main Theorem *there is a canonical isomorphism* $H^0_{\mathbf{R}}(W) \to \mathcal{C}(W)$ *for any closed manifold W of dimension n.*

Proof Let $(V, i, \alpha, \tilde{\alpha})$ be a 4–tuple representing a given element of $H^0_{\mathbf{R}}(W)$. We note that since W is closed, the manifold V is closed as well. Let

$$(9) \qquad\qquad J^k((t-q)\varepsilon, \nu) \longrightarrow V$$

be the formal k–jet bundle over V associated to the pair of bundles over V, namely, $(t-q)\varepsilon$ and the normal vector t–bundle ν induced by i. We define a section s of the bundle (9) by taking a point $v \in V$ onto the k–jet represented by the composition of a map germ (8) representing $\alpha(v)$ and the composition of two isomorphisms

$$EO_t|b \xrightarrow{(\pi|\alpha(v))^*} ((\pi|\alpha(v))^*)(EO_t|b) \xrightarrow{(\tilde{\alpha}|v)^{-1}} \nu|v,$$

where $b = \pi(\alpha(v))$. There is a canonical inclusion

$$(10) \qquad\qquad J^k((t-q)\varepsilon, \nu) \longrightarrow J^k(TV \oplus (t-q)\varepsilon, TV \oplus \nu)$$

of total spaces of k–jet bundles over V that takes the k–jet at $v \in V$ represented by a germ f onto the k–jet at v represented by the germ $\mathrm{id}_{TV|v} \times f$. Let

$$\tilde{s}: V \longrightarrow J^k(TV \oplus (t-q)\varepsilon, TV \oplus \nu)$$

be the composition of s and the canonical inclusion (10). Let us observe that $TV \oplus \nu$ is canonically isomorphic to the pullback of $(t-q)\varepsilon \oplus TW$ with respect to the composition

$$V \longrightarrow \mathbb{R}^{t-q} \times W \longrightarrow W$$

of the embedding i and the projection of $\mathbb{R}^{t-q} \times W$ onto the second factor. Consequently, $\tilde{s}$ determines an (R_W, π)–structure on the stable tangent bundle of V, which in its turn, by Theorem 11.1, determines an element of $\mathcal{C}(W) \cong \mathcal{B}(W)$. It is easily verified that the obtained element in $\mathcal{C}(W)$ does not depend on the choice of the representative of $H^0_{\mathbf{R}}(W)$. Thus, there is a canonical map

$$\xi_W : H^0_{\mathbf{R}}(W) \longrightarrow \mathcal{C}(W)$$

which is, in fact, a homomorphism. A similar construction provides us with a homomorphism

$$\xi_W^{-1} : \mathcal{C}(W) \longrightarrow H^0_{\mathbf{R}}(W)$$

inverse to ξ_W. Thus the homomorphism ξ_W is an isomorphism. $\qquad\qquad\square$

It is easily verified that the homomorphisms ξ_W^{-1} in Theorem 13.2 determine a natural transformation of functors on the category of closed manifolds of dimension n.

Now the Main Theorem readily follows from Theorem 13.2.

Theorem 13.3 *Under the assumptions of the Main Theorem, the contravariant functor $\mathcal{C}$ restricted to the full subcategory of closed manifolds of dimension n satisfies the b-principle.*

14 Applications

As has been mentioned, for a sequence **R** of $\mathcal{K}$–invariant bases, the set of right bordism classes of **R**–maps into a manifold W forms a semigroup, which is a priori not a group. On the other hand, since $\mathbf{MO_R}(W)$ is always a group, Theorem 11.1 implies that this semigroup is often a group.

Corollary 14.1 *If a sequence **R** of bases satisfies the conditions of Theorem 11.1, then the right bordism classes of **R**–maps into a manifold W form a group.*

We note that the assumptions in Corollary 14.1 are necessary as, for example, the right bordism classes of submersions into a manifold W do not form a group. Corollary 14.1 was also independently observed by Szűcs [51] in the negative dimension case and by Ando [6] in the general case.

Furthermore, the weak b-principle proved for the functor $\mathcal{B}$ allows us to use the machinery of cohomology theory, notably spectral sequences and theorems on homology of infinite loop spaces, in order to carry out explicit computations [44] (see also Szűcs [51]).

The class of differential relations for which our theorem applies is large. For example, as we will see next, it contains almost all Thom–Boardman differential relations.

14.1 Maps with prescribed Thom–Boardman singularities

Thom–Boardman singularities $\Sigma_{\mathcal{I}}$ are singularities of smooth maps, each indexed by a sequence $\mathcal{I} = (i_1, \ldots, i_l)$ of integers. For a definition and properties of Thom–Boardman singularities we refer the reader to the original paper of Boardman [7].

Definition 14.2 Given a sequence $\mathcal{I} = (i_1, \ldots, i_l)$ of integers, the *Thom–Boardman basis* $R_s^{\mathcal{I}} \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ is defined to be the complement to the set $\bigcup \Sigma_{\mathcal{I}'}$ where the union is taken over $\mathcal{I}' > \mathcal{I}$ with respect to the lexicographic order. It is well-known that the relations $R_s^{\mathcal{I}}$, $s \geq 0$, form a sequence, which we will denote by $\mathbf{R}^{\mathcal{I}}$, of open $\mathcal{K}$–invariant bases.

It follows that solutions of differential relations associated with the basis $R_s^{\mathcal{I}}$ are the maps without Thom–Boardman singularities $\Sigma_{\mathcal{I}'}$ with index $\mathcal{I}' > \mathcal{I}$.

The homotopy principle for Thom–Boardman bases $R_s^{\mathcal{I}} \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, $q \leq 0$, was proved by du Plessis [35] and a version of the h-principle in the case of the symbol $(1 - q, 0)$ was proved by Eliashberg [9; 10] (see also Ando [3]), and then, in the case of an arbitrary symbol $\mathcal{I}$ with $\mathcal{I} > (1 - q, 0)$, using the Eliashberg h-principle, by Ando [4]. Consequently, we derive the bordism principle for Thom–Boardman differential relations.

Corollary 14.3 *Suppose that either $q > 0$ or $q \leq 0$ and $\mathcal{I} \geq (1 - q, 0)$. Then for the Thom–Boardman sequence of bases $R_s^{\mathcal{I}} \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$ the weak bordism principle holds for the corresponding functor $\mathcal{C}$ on the category of closed manifolds and the corresponding functor $\mathcal{B}$ on the category of manifolds.*

14.2 Maps with additional structure

It is easy to extend the results to the case of bordism groups of manifolds with an additional structure, eg, oriented manifolds, manifolds with almost complex structure or spin manifolds. We refer the reader to the paper [11] of Eliashberg for the necessary adjustments in the case of Lagrangian and Legendrian immersions.

14.3 Kazarian conjecture

Under the assumptions of the Main Theorem, for oriented $\mathcal{J}$–cobordism groups we can deduce the Kazarian conjecture [51], which relates rational cohomology groups $H^*(B_{\mathcal{J}}; \mathbb{Q})$ of the classifying space $B_{\mathcal{J}}$ of $\mathcal{J}$–cobordism groups with the cohomology groups of $\mathbf{S}(\mathcal{J}) = \lim S_t(\mathcal{J})$, provided of course that $B_{\mathcal{J}}$ exists. Namely it was conjectured that the classifying space of $\mathcal{J}$–cobordism groups is rationally equivalent to the space

$$\lim_{t \to \infty} \Omega^{t+d} S^{t+d} S_t(\mathcal{J}).$$

On the other hand, the latter space is rationally equivalent to $\Omega^\infty \mathbf{B}_{\mathcal{J}}$. Since $\Omega^\infty \mathbf{B}_{\mathcal{J}}$ is itself a classifying space of $\mathcal{J}$–cobordism groups (by the Main Theorem), we establish the Kazarian conjecture. In the negative dimension case the Kazarian conjecture is proved by Szűcs [51].

14.4 Prospective applications

In view of the Rimányi-Szűcs construction $B_{\mathcal{J}}$ and the Kazarian construction $K_{\mathcal{J}}$ of classifying spaces, the Main Theorem can be used to study diffeomorphism groups of smooth manifolds (see Remark 2.4 and the discussion before it).

15 Obstruction spectrum

Let **R** be a sequence of differential relation bases $R_s \subset J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, with $s \geq 0$, for which Theorem 13.2 holds. In this section we define[1] a complete obstruction to the existence of a cobordism of a given smooth map to an **R**–map.

Let $\mathbf{R}' = \{R'_s\}$ be the sequence of full differential relation bases $R'_s = J^k(\mathbb{R}^s, \mathbb{R}^{q+s})$, with $s \geq 0$. Then the inclusions $R_s \subset R'_s$ lead to an inclusion i of the Thom spectrum of $H^*_{\mathbf{R}}$ into the Thom spectrum of the cobordism group $H^{*-q}_{\mathbf{R}'} = \mathcal{N}^*$. Let $\mathcal{O}^*_{\mathbf{R}}$ denote the cofiber of i, the *obstruction spectrum*. Then for a manifold W there is a long exact sequence of groups

$$\cdots \longrightarrow H^{*-q}_{\mathbf{R}}(W) \longrightarrow \mathcal{N}^*(W) \overset{j}{\longrightarrow} \mathcal{O}^*_{\mathbf{R}}(W) \longrightarrow H^{*+1-q}_{\mathbf{R}}(W) \longrightarrow \cdots .$$

A proper smooth map $f\colon V \to W$ of dimension $-q$ represents an element $[f]$ in the cobordism group $\mathcal{N}^q(W)$. We define the obstruction class $o(f, \mathbf{R}) \in \mathcal{O}^q_{\mathbf{R}}(W)$ as the image of $[f]$ with respect to the homomorphism j.

Theorem 15.1 *Let $f\colon V \to W$ be a map of dimension $-q$ of closed manifolds. Then $o(f, \mathbf{R}) \in \mathcal{O}^q_{\mathbf{R}}(W)$ is a complete obstruction to the existence of a cobordism of f to an* **R**–*map.*

In his Habilitation thesis [27] (see also Szűcs [50]), Kazarian presented obstructions to the existence of a cobordism of a given map to a map without certain multi-singularities as cohomology operations with values in singular cohomology groups. Similarly, we may view the map $j\colon \mathcal{N}^*(W) \to \mathcal{O}^*_{\mathbf{R}}(W)$ as a cohomology operation with values in an extraordinary cohomology theory. Thus if **R** satisfies the assumptions of Theorem 15.1, then the complete obstruction to the existence of a bordism to an **R**–map is given by a cohomology operation with values in an extraordinary cohomology theory.

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