Homotopy nilpotent groups

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We study the connection between the Goodwillie tower of the identity and the lower central series of the loop group on connected spaces. We define homotopy n–nilpotent groups as homotopy algebras over certain simplicial algebraic theories. This notion interpolates between infinite loop spaces and loop spaces, but backwards. We study the relation to ordinary nilpotent groups. We prove that n–excisive functors of the form ΩF factor over the category of homotopy n–nilpotent groups.

1 Introduction

This article arose from the question of what the Goodwillie tower of the identity has to do with the lower central series of Kan's loop group of a connected space. We express the relation with the help of simplicial algebraic theories. We expect the reader to be familiar with the basic notions of homotopical algebra and Goodwillie's calculus of homotopy functors. We will give a short introduction to simplicial algebraic theories and their homotopy algebras in Sections 2 and 3.

We are working simplicially, so for us "space" always means "simplicial set".

Let us introduce some notation valid for the rest of the article. Let $\mathcal{S}_*$ be the category of pointed simplicial sets. Further let $\mathcal{F}$ denote the category of $\mathcal{S}_*$–functors from finite pointed simplicial sets $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{S}_*$. A homotopy functor in $\mathcal{F}$ is a functor that preserves weak equivalences.

For a homotopy functor F in $\mathcal{F}$ Goodwillie [20] constructs a tower of functors

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to P_0 F = F(*),$$

where the n–th stage is the universal n–excisive homotopy functor under F. Here, n–excision is a higher version of excision; a 1–excisive functor is a homotopy functor with a Mayer–Vietoris sequence. For $F = \mathrm{id}$, the Goodwillie tower converges on simply connected spaces to the identity; see Goodwillie [19]. On a connected space K,

however, it converges to the Bousfield–Kan completion proved in [3] by Arone and Kankaanrinta:

$$\operatorname*{holim}_{n} P_n(\mathrm{id})(K) \simeq \mathbb{Z}_\infty K$$

We have $P_1(\mathrm{id}) \simeq \Omega^\infty \Sigma^\infty$, the stable homotopy functor. So the Goodwillie tower interpolates between stable and unstable homotopy. The map $K \to P_n(\mathrm{id})(K)$ is roughly $(n+1)k$–connected, if K is k–connected.

Let $\mathcal{S}_0$ be the category of reduced simplicial sets, ie simplicial sets with exactly one 0–simplex. By a result of Dwyer and Kan [15], Kan's loop group functor G is part of a Quillen equivalence between $\mathcal{S}_0$ and simplicial groups. The homotopy category of $\mathcal{S}_0$ and of path-connected topological spaces are equivalent.

For a reduced simplicial set K, let $GK/\Gamma_n GK$ denote the n–th stage of the lower central series of GK taken degreewise as defined in Definition 6.1. This was studied by Curtis [12] who proved that for simply connected spaces the connectivity of the map

$$GK \to GK/\Gamma_{n+1} GK$$

increases logarithmically with n. The tower $\{GK/\Gamma_{n+1} GK\}_{n \geq 1}$ associated to the filtration converges to the identity on simply connected spaces. As noted by Kan,

$$\pi_{s-1}(GK/\Gamma_2 GK) \cong H_s K \quad \text{for all } s \geq 1.$$

So the lower central series interpolates between singular homology and unstable homotopy. In a similar way to the Goodwillie tower of id, this tower converges to the identity on simply connected spaces and to the Bousfield–Kan completion on connected spaces.

It is not difficult to check that there is a canonical map up to homotopy

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
P_n(\mathrm{id})(K) & \longrightarrow & B\big(GK/\Gamma_{n+1} GK\big) =: \Gamma^n K \\
\downarrow & & \downarrow \\
P_{n-1}(\mathrm{id})(K) & \longrightarrow & B\big(GK/\Gamma_n GK\big) =: \Gamma^{n-1} K \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

from the Goodwillie tower of the identity of K to the classifying space of the lower central series of the loop group of K, obtained from the universal property (up to homotopy) of the first tower. It is well-known to the experts that this map induces an isomorphism on π_1. We prove this statement in Theorem 8.1 and phrase it in terms

of algebraic theories. In fact, the left hand side (looped once) gives rise to certain simplicial algebraic theories that we call $\mathcal{G}_n$ (Definition 5.2). Applying π_0 to this simplicial category gives us the ordinary theory of n–nilpotent groups. The k–ary operations of these theories can be obtained by looping the above towers and evaluating them on a k–fold wedge of circles (Corollary 5.7).

Algebraic theories were introduced by Lawvere [27] to obtain categorical descriptions of algebraic structures like groups, rings, Lie algebras, etc. An algebraic theory is a category T having the natural numbers $k \geq 0$ as objects such that k is the product in T of k copies of 1. The maps from k to 1 are to be thought of as the k–ary operations of T. They can be canonically identified with the free objects on k generators. Algebras over T are product preserving functors from T to sets.

For purposes in homotopy theory we need to consider simplicial algebraic theories where T–algebras have values in simplicial sets. These were first considered by Reedy [30] and more recently by Schwede [31] and Badzioch [5]. It is convenient for us to study pointed versions where algebras are functors from T to pointed simplicial sets and the category T itself will be enriched over $\mathcal{S}_*$. If the theory has only one constant – as in our case – there is no loss in generality. We also need a weaker notion of algebra: homotopy T–algebras were introduced by Badzioch [5]. They are functors from T to $\mathcal{S}_*$ that commute with products up to homotopy.

We define a homotopy n–nilpotent group to be a homotopy $\mathcal{G}_n$–algebra. In other words, these are spaces that admit k–ary operations parametrized by $\Omega P_n(\mathrm{id})\left(\bigvee_k S^1\right)$. Homotopy n–nilpotent groups have not been considered before – only the two extreme cases are classical: Loop spaces and infinite loop spaces can be described as homotopy algebras over our theories $\mathcal{G}_\infty$ and $\mathcal{G}_1$; see Theorem 5.8 and Theorem 5.13. The free objects on k generators of these simplicial theories are given by

$$\Omega\left(\bigvee_{i=1}^{k} S^1\right) \simeq \Omega\Sigma \bigvee_{k} S^0 \quad \text{and} \quad \Omega P_1(\mathrm{id})\left(\bigvee_{i=1}^{k} S^1\right) \simeq \Omega^\infty\Sigma^\infty \bigvee_{k} S^0.$$

Thus, the new notion interpolates between loop spaces and infinite loop space, but "backwards".

We obtain morphisms of theories

$$\mathcal{G}_\infty \to \cdots \to \mathcal{G}_n \to \mathcal{G}_{n-1} \to \cdots$$

induced by the maps in the Goodwillie tower and functors

$$U_n\colon \mathcal{S}_*^{\mathcal{G}_n} \to \mathcal{S}_*^{\mathcal{G}_\infty}.$$

The functor U_n induces on homotopy categories of the respective Badzioch model structures the forgetful functor from homotopy n–nilpotent groups to loop spaces. It has a left adjoint

$$\Phi_n \colon \mathcal{S}_*^{\mathcal{G}_\infty} \to \mathcal{S}_*^{\mathcal{G}_n}.$$

It is easy to check that (Φ_n, U_n) form a Quillen pair for the Badzioch model structures on each side, and hence their derived functors $(L\Phi_n, RU_n)$ form an adjoint pair of functors on the respective homotopy categories. Left adjoints to forgetful functors are often called free functors. For a connected space X we have $\Omega X \simeq \underset{\sim}{X}(1^+)$ where $\underset{\sim}{X} \colon \mathcal{G}_\infty \to \mathcal{S}_*$ is a homotopy ∞–algebra. In that sense the object $L\Phi_n(\underset{\sim}{X})$ is the free homotopy n–nilpotent group generated by ΩX. Our next result, Theorem 5.16, exhibits the Goodwillie tower of the identity $\Omega P_n(\mathrm{id})(X)$ as the homotopy left Kan extension along Φ_n:

$$\Omega P_n(\mathrm{id})(X) \simeq (L\Phi_n \underset{\sim}{X})(1^+).$$

So the free homotopy n–nilpotent group generated by ΩX is $\Omega P_n(\mathrm{id})(X)$. Equivalently, Theorem 5.16 states that the free homotopy n–nilpotent group generated by a pointed space K is given by $\Omega P_n(\mathrm{id})(\Sigma K)$. This generalizes and interpolates between the fact that $\Omega \Sigma K$ is the free loop space and $\Omega^\infty \Sigma^\infty K$ is the free infinite loop space over K. If the reader prefers the analogy with the lower central series, then the case $n = 1$ corresponds to the abelianization of a group and $\Omega^\infty \Sigma^\infty K$ is the homotopy 1–nilpotent quotient of $\Omega \Sigma K$. Then for $n \geq 1$ the term $\Omega P_n(\mathrm{id})(\Sigma K)$ can be seen as a homotopy version of the quotients in the lower central series.

Corollary 9.3 gives a large source of examples: a space is a homotopy n–nilpotent group if it is the value of a looped n–excisive functor. In work in progress [8] we prove that every homotopy n–nilpotent group arises in this way. An application, Corollary 9.4, is that the category of looped n–excisive functors is naturally enriched in homotopy n–nilpotent groups. This generalizes the fact that the homotopy category of infinite loop spaces is enriched over abelian groups.

In Example 9.8 we give an easy example of homotopy n–nilpotent group: every connected space whose homotopy groups vanish except between dimensions $k + 1$ and $(n + 1)k$ for some $k \geq 1$ is canonically a homotopy n–nilpotent group. We finish with an application, Example 9.10, of the chain rule on Goodwillie's calculus of homotopy functors proved by Arone and Ching in [1] that states that homotopy functors, whose derivatives vanish outside a certain range, factor over the category of homotopy n–nilpotent spaces. This generalizes and simplifies the proof of the corresponding statement for $n = 1$ by Arone, Dwyer and Lesh in [2].

The results in this article are not necessarily deep. This article is meant to formulate the definitions and set the ground for further study of homotopy n–nilpotent spaces. We hope to convince the reader that this subject deserves further attention.

2 Algebraic theories

Algebraic theories were introduced by Lawvere [27] in order to describe algebraic structures, like eg groups or rings, in terms of universal operations. He found that these universal operations are encoded by the homomorphisms between the finitely generated free objects of the respective algebraic structure. Thus the typical example of an algebraic theory is given by the subcategory of some algebraic category formed by the finitely generated free objects. Any object of the algebraic category in question determines and is determined by a functor from this subcategory to sets. Since we prefer to deal with covariant functors we consider the opposite of the subcategory of finitely generated free objects, and since we will deal only with pointed objects we describe here only pointed algebraic theories.

Definition 2.1 For a category $\mathcal{C}$ we will refer to its set of morphisms by $\mathcal{C}(\_,\_)$.

Definition 2.2 Let Γ be the opposite of the category of finite pointed sets. The category Γ has all products and every object is isomorphic to an object of the form

$$k^+ = \{1,\ldots,k\} \cup \{+\}.$$

Here $+$ acts as the base point. For every $1 \le s \le k$ we have maps $i_s^k \colon k^+ \to 1^+$ given by the inclusion of the pointed set 1^+ to k^+ where the non basepoint of 1^+ maps to $s \in k^+$. These maps induce an isomorphism

$$\prod_{s=1}^{k} i_s^k \colon k^+ \cong \prod_{s=1}^{k} 1^+.$$

Definition 2.3 A *pointed algebraic theory* is a category T having the same set of objects as Γ together with a functor $\Gamma \to T$, which is the identity on objects and preserves products. Morphisms of algebraic theories are product preserving functors under Γ.

The category T is usually given as a full subcategory of some other category (see Remark 2.9), so the morphisms in T are often left understood. We will often confuse the objects $k^+ \in \mathrm{Ob}(\Gamma) = \mathrm{Ob}(T)$ with their images under such a full inclusion. If we want to emphasize a particular theory T, we will denote the object k^+ by $T(k^+)$.

Definition 2.4 A *strict T–algebra* is a functor $\underset{\sim}{X} \colon T \to \mathrm{Sets}$ that preserves products strictly. This means that the map

$$\prod_{s=1}^{k} \underset{\sim}{X}(i_s^k) \colon \underset{\sim}{X}(k^+) \to \left(\underset{\sim}{X}(1^+)\right)^k$$

is an isomorphism of pointed sets. The category of pointed T–algebras is denoted by $\mathrm{Alg}_{T,*}$ and forms a full subcategory of the category Sets_*^T of all functors from T to pointed sets.

Definition 2.5 The *k–ary operations* of an algebraic theory T are the elements of the set $T(k^+, 1^+)$. The *constants* of T are the 0–ary operations, ie the elements of the set

$$A_0 = T(0^+, 1^+).$$

A *theory with one constant* is an algebraic theory such that $A_0 \cong *$.

Remark 2.6 Let T be an algebraic theory. Then the forgetful functor $u \colon \mathrm{Sets}_* \to \mathrm{Sets}$ induces a functor

$$u_* \colon \mathrm{Sets}_*^T \to \mathrm{Sets}^T.$$

This restricts to a functor

$$u_* \colon \mathrm{Alg}_{T,*} \to \mathrm{Alg}_T,$$

where Alg_T is the category of unpointed T–algebras.

Lemma 2.7 *If T is an algebraic theory with one constant, the functor*

$$u_* \colon \mathrm{Alg}_{T,*} \to \mathrm{Alg}_T$$

is an isomorphism of categories.

Proof Given an unpointed T–algebra X, we can always supply it with a canonical basepoint

$$* \cong X(0^+) \to X(1^+),$$

induced by the unique constant $0^+ \to 1^+$ in T. We obtain an inverse functor for u_*. $\square$

Definition 2.8 The evaluation functor

$$\mathrm{Ev}_{1+} \colon \mathrm{Alg}_*^T \to \mathrm{Sets}_*, \quad \underset{\sim}{X} \mapsto \mathrm{Ev}_{1+}\underset{\sim}{X} = \underset{\sim}{X}(1^+)$$

has a left adjoint denoted by

$$\mathrm{Fr}_T \colon \mathrm{Sets}_* \to \mathrm{Alg}_{T,*}$$

which is called the *free T–algebra functor.*

Remark 2.9 One easily checks that in $\mathrm{Alg}_{T,*}$ there is a canonical isomorphism

$$\mathrm{Fr}_T(k^+) \cong T(k^+, \_).$$

Hence there is an equivalence of categories between T and the opposite category of the full subcategory of $\mathrm{Alg}_{T,*}$ given by the finitely generated free T–algebras.

Remark 2.10 To illustrate the ideas we describe here very briefly the example of the algebraic theory of groups. Let Gr be the category of groups. We let Nil_∞ be the theory of groups.

According to Remark 2.9 the object k^+ of the theory Nil_∞ of groups can be identified with

$$\mathrm{Nil}_\infty(k^+) = F_k,$$

the free group on k letters. Morphisms are given by the set

$$\mathrm{Nil}_\infty(k^+, \ell^+) \cong \mathrm{Gr}(F_\ell, F_k) \cong (F_k)^\ell.$$

One easily checks that composition

$$\mathrm{Nil}_\infty(\ell^+, m^+) \times \mathrm{Nil}_\infty(k^+, \ell^+) \to \mathrm{Nil}_\infty(k^+, m^+)$$

translates to substitution $(F_\ell)^m \times (F_k)^\ell \to (F_k)^m$, where we take ℓ words $w_1, \ldots, w_\ell$ in F_k and substitute them for the generators of F_ℓ.

Now suppose that G is a group. We can define a functor $\underset{\sim}{G} \colon \mathrm{Nil}_\infty \to \mathrm{Sets}$ by setting

$$k^+ \mapsto G^k,$$

and for every element in $(w_1, \ldots, w_k) \in (F_k)^\ell \cong \mathrm{Nil}_\infty(k^+, \ell^+)$ we define a group homomorphism $G^k \to G^\ell$ given by

$$(g_1, \ldots, g_k) \mapsto (h_1, \ldots, h_\ell),$$

where h_j for $1 \le j \le \ell$ is the element in G obtained by substituting g_i for the i–th generator of F_k in the word w_i for $1 \le i \le k$. It is straightforward to check that this is indeed a product-preserving functor, hence a Nil_∞–algebra.

Conversely, given an Nil_∞–algebra $\underset{\sim}{G}$ we obtain a group structure on the set $\underset{\sim}{G}(1^+) =:$ G. A multiplication map

$$G \times G \cong \underset{\sim}{G}(2^+) \to \underset{\sim}{G}(1^+) \cong G$$

is induced by the element

$$f_1 \, f_2 \in F_2 \cong \mathrm{Nil}_\infty(2^+, 1^+).$$

Here we denote by $f_1, \ldots, f_k$ the generators of the free group F_k. This multiplication is associative because of the identity

$$(f_1 \, f_2) \, f_3 = f_1 \, (f_2 \, f_3) \in F_3 \cong \mathrm{Nil}_\infty(3^+, 1^+).$$

The set G is pointed and one easily checks that this basepoint e is the neutral element with respect to the multiplication. The inverse $(\_)^{-1} \colon G \to G$ is induced by the element $f_1^{-1} \in F_1$. Substituting f_1^{-1} for f_2 we obtain the identity $f_1 f_1^{-1} = e$ which proves the inverse as defined is indeed an inverse.

Both construction are mutually inverse and give an equivalence between the category Gr of groups and the category of Nil_∞–algebras. In turn, this illustrates the isomorphism of the opposite of the full subcategory of Gr given by the finitely generated free groups and the theory Nil_∞.

An example of an ordinary algebraic theory that has more than one constant, is the theory of commutative rings with unit. The free object on k generators is given by $\mathbb{Z}[x_1, \ldots, x_k]$, the polynomial ring in k generators. The constants are given by $A_0 \cong \mathbb{Z}$.

3   Simplicial algebraic theories

We denote the category of pointed simplicial sets by $\mathcal{S}_*$.

Definition 3.1 A *pointed simplicial algebraic theory* is an algebraic theory enriched as a category over $\mathcal{S}_*$. We will immediately drop all adjectives and simply speak of a *theory* henceforth.

Definition 3.2 Let T be a theory. A *pointed strict simplicial T–algebra* is an $\mathcal{S}_*$–functor from T to $\mathcal{S}_*$ that preserves products in the sense of Definition 2.4. Again, we will simply speak of a T–*algebra* from now on. The category of T–algebras will be denoted by $\mathrm{Alg}_{T,*}$.

Since we want to do homotopy theory we relax the conditions on T–algebras.

Definition 3.3 Let T be a theory. A *homotopy T–algebra* $\underset{\sim}{X}$ is an $\mathcal{S}_*$–functor from T to $\mathcal{S}_*$ that preserves products up to weak equivalence, ie the map

$$\prod_{s=1}^{k} \underset{\sim}{X}(i_s^k)\colon \underset{\sim}{X}(k^+) \to \left(\underset{\sim}{X}(1^+)\right)^k$$

is a weak equivalence for all $k \geq 1$.

The category $\mathcal{S}_*^T$ can be equipped with a model structure where the objectwise fibrant homotopy T–algebras are exactly the fibrant objects. This model structure was first constructed by Badzioch in [5] and is a localization of the projective model structure on $\mathcal{S}_*^T$.

Definition 3.4 Badzioch in [5] called this model structure the *homotopy algebra model structure*. We will call it the *Badzioch model structure* and denote the category $\mathcal{S}_*^T$ together with this model structure by $(\mathcal{S}_*^T)_{\mathrm{Badz}}$.

It is shown in [5] that there is a Quillen equivalence between strict T–algebras with objectwise weak equivalences and fibrations and homotopy T–algebras. This result tells us that – independently of the theory T – any homotopy T–algebra can be rigidified.

Theorem 3.5 (Badzioch) *Let $F\colon S \to T$ be a morphism of simplicial theories. If F is a weak equivalence of simplicial categories then pulling back along F is the right adjoint of a Quillen equivalence between the associated Badzioch model categories.*

Proof In [16, Theorem 2.1] it is shown that $F^*\colon \mathcal{S}_*^T \to \mathcal{S}_*^S$ is the right adjoint of a Quillen equivalence between the projective model structures. The Badzioch model structures are left Bousfield localizations. This process preserves Quillen equivalences by [22, Theorem 3.3.20]. $\qquad\square$

Remark 3.6 As in the set-valued case the evaluation functor

$$\mathrm{Ev}_{1+}\colon \mathrm{Alg}_{T,*} \to \mathcal{S}_*\,,\ \underset{\sim}{X} \mapsto \mathrm{Ev}_{1+}\,\underset{\sim}{X} = \underset{\sim}{X}(1^+)$$

has an $\mathcal{S}_*$–enriched left adjoint denoted by

$$\mathrm{Fr}_T \colon \mathcal{S}_* \to \mathrm{Alg}_{T,*}$$

which is called the *free T–algebra functor*. One easily checks that $(\mathrm{Fr}_T, \mathrm{Ev}_T)$ form a Quillen pair for the Badzioch model structure and that there is a canonical isomorphism

$$\mathrm{Fr}_T(k^+) \cong T(k^+, \_).$$

4   The *n*–excisive model structure

Definition 4.1 We denote by $\mathcal{F}$ the category of $\mathcal{S}_*$–enriched functors from finite pointed simplicial sets $\mathcal{S}_*^{\mathrm{fin}}$ to pointed simplicial sets $\mathcal{S}_*$.

The category $\mathcal{F}$ is enriched, tensored and cotensored over $\mathcal{S}_*$ where both tensor and cotensor are given objectwise. It carries a projective model structure where weak equivalences and fibrations are given objectwise.

For an introduction to Goodwillie's calculus of homotopy functors and in particular for the notion of n–excisive homotopy functor we refer to Goodwillie [20] and Kuhn [26].

In Biedermann, Chorny and Röndigs [7] and Dwyer [14] the projective model structure on $\mathcal{F}$ was localized to obtain the n–excisive model structure where the fibrant objects are exactly the n–excisive homotopy functors. A map $\underset{\sim}{X} \to \underset{\sim}{Y}$ is an n–excisive weak equivalence if and only if it induces an objectwise weak equivalence

$$P_n \underset{\sim}{X} \to P_n \underset{\sim}{Y}.$$

Here $P_n \underset{\sim}{X}$ denotes the n–th stage in the Goodwillie tower of the functor $\underset{\sim}{X}^h$ which is the functor $\underset{\sim}{X}$ pre- and postcomposed with an objectwise fibrant replacement functor in $\mathcal{S}_*$. However it is more convenient for us to consider the injective model structure on $\mathcal{F}$ constructed by Joyal [25] and Jardine [23] where cofibrations are given by all inclusions. This model structure is also proper and simplicial with the advantage that all objects are cofibrant. The same techniques as in [7] apply to arrive at an n–excisive model structure on $\mathcal{F}$ with the same equivalences and where a map $\underset{\sim}{X} \to \underset{\sim}{Y}$ is an n–excisive fibration if and only if it is an injective fibration such that the square

$$
\begin{array}{ccc}
\underset{\sim}{X} & \longrightarrow & P_n \underset{\sim}{X} \\
\downarrow & & \downarrow \\
\underset{\sim}{Y} & \longrightarrow & P_n \underset{\sim}{Y}
\end{array}
$$

is an objectwise homotopy pullback square. We have:

Theorem 4.2 *The injective n–excisive model structure on $\mathcal{F}$ is a cofibrantly generated proper simplicial model structure. All objects are cofibrant.*

Corollary 4.3 *Let $\underset{\sim}{Y}$ be an injectively fibrant n–excisive homotopy functor. Then for every $\underset{\sim}{X}$ in $\mathcal{F}$ we have a natural weak equivalence*

$$\mathcal{F}(P_n\underset{\sim}{X}, \underset{\sim}{Y}) \simeq \mathcal{F}(\underset{\sim}{X}, \underset{\sim}{Y}).$$

5 Homotopy n–nilpotent groups

We will now describe the simplicial theory of homotopy n–nilpotent groups.

Definition 5.1 In the category $\mathcal{F}$ let $(\_)^{\mathrm{inj}}$ be a fibrant replacement functor with respect to the injective model structure.

Definition 5.2 We define a full subcategory $\mathcal{G}_n$ of the category $\mathcal{F}$, which has for each natural number $k \geq 0$ exactly one object given by

$$\mathcal{G}_n(k^+) = \prod_{i=1}^{k} \Omega\big(P_n(\mathrm{id})\big)^{\mathrm{inj}}.$$

We also define for $n = \infty$ the category $\mathcal{G}_\infty$ with objects given by

$$\mathcal{G}_\infty(k^+) = \prod_{i=1}^{k} \Omega(\mathrm{id})^{\mathrm{inj}}.$$

We employ the convention that the empty product is the final object $*$, and so we have for all $1 \leq n \leq \infty$ and $0 \leq k < \infty$:

$$\mathcal{G}_n(k^+) \cong \prod_{i=1}^{k} \mathcal{G}_n(1^+).$$

We let $I_n\colon \mathcal{G}_n \to \mathcal{F}$ be the inclusion functor.

To obtain the status of a theory, $\mathcal{G}_n$ has to live under Γ. So we define a functor $\gamma_n\colon \Gamma \to \mathcal{G}_n$ on objects simply by $k^+ \mapsto \mathcal{G}_n(k^+)$. The space $\mathcal{G}_n(k^+)$ is the space of pointed maps from k^+ to $\mathcal{G}_n(1^+)$. A morphism $f^{\mathrm{op}}\colon \ell^+ \to k^+$ in Γ, which is represented by a map $f\colon k^+ \to \ell^+$ of pointed sets, then induces a map

$$\mathcal{G}_n(\ell^+) \to \mathcal{G}_n(k^+)$$

by pulling back along f.

Remark 5.3 As a full subcategory of $\mathcal{F}$ the category $\mathcal{G}_n$ for $1 \leq n \leq \infty$ is enriched over $\mathcal{S}_*$. The category $\mathcal{G}_n$ constitutes a simplicial theory as discussed in Section 3. Therefore we can consider $\mathcal{G}_n$–algebras and homotopy $\mathcal{G}_n$–algebras. Objectwise fibrant homotopy $\mathcal{G}_n$–algebras are the fibrant objects in $(\mathcal{S}_*^{\mathcal{G}_n})_{\text{Badz}}$.

Definition 5.4 We call a homotopy $\mathcal{G}_n$–algebra a *homotopy n–nilpotent group*. If the context allows, we commit the common abuse of language and refer to the underlying space $\underset{\sim}{X}(1^+)$ (or one canonically weakly equivalent to it) for some homotopy $\mathcal{G}_n$–algebra $\underset{\sim}{X}$ as a homotopy n–nilpotent group.

For $n = 1$ this corresponds to the common abuse of language to refer to an infinite loop space by its underlying space without making the infinite loop space structure explicit. Obviously, a space might not have any infinite loop space structure, and if it has one it might have more than one. Similarly for homotopy n–nilpotent groups.

Lemma 5.5 *For all $n \geq 1$ and $k, \ell \geq 0$ we have canonical weak equivalences*

$$\mathcal{G}_n(k^+, \ell^+) \simeq \prod_\ell \Omega P_n(\text{id}) \left(\bigvee_{i=1}^k S^1 \right)$$

Proof Recall that $\mathcal{F}$ from Definition 4.1 is endowed with the injective n–excisive model structure where all objects are cofibrant. We compute:

$$\mathcal{G}_n(k^+, \ell^+) \cong \mathcal{F}\left(\prod_k \Omega\big(P_n(\text{id})\big)^{\text{inj}}, \prod_\ell \Omega\big(P_n(\text{id})\big)^{\text{inj}} \right)$$

$$\simeq \mathcal{F}\big(\Omega(\text{id})^k, \Omega\big(P_n(\text{id})\big)^{\text{inj}} \big)^\ell$$

$$\cong \mathcal{F}\left(\text{map}_{\mathcal{S}_*}\left(\bigvee_k S^1, \_ \right), \Omega\big(P_n(\text{id})\big)^{\text{inj}} \right)^\ell$$

$$\cong \left(\Omega\big(P_n(\text{id})\big)^{\text{inj}}\left(\bigvee_k S^1 \right) \right)^\ell$$

$$\simeq \left(\Omega P_n(\text{id})\left(\bigvee_k S^1 \right) \right)^\ell$$

The weak equivalence in step 2 comes from Corollary 4.3. We also use the enriched Yoneda lemma and the fact that P_n commutes with Ω. $\square$

Remark 5.6 By Lemma 5.5 the theory $\mathcal{G}_n$ has only one constant. So by Lemma 2.7 we can work in the pointed setting without losing information.

Corollary 5.7 *The k–ary operations of the theory $\mathcal{G}_n$ of homotopy n–nilpotent groups is given by evaluating the free $\mathcal{G}_n$–algebra on k generators at 1^+:*

$$\mathrm{Ev}_{1^+}\big(\mathrm{Fr}_{\mathcal{G}_n}(k^+)\big) \cong \Omega\, P_n(\mathrm{id})\left(\bigvee_{i=1}^{k} S^1\right)$$

Proof The free algebra A_k on k generators in any simplicial theory T can be obtained by the following formula:

$$A_k \cong T(k^+, 1^+) \cong \mathrm{Alg}_{T,*}\big(\mathrm{Fr}_T(1^+), \mathrm{Fr}_T(k^+)\big) \cong \mathrm{Fr}_T(k^+)(1^+) \cong \mathrm{Ev}_{1^+}\big(\mathrm{Fr}_T(k^+)\big).$$

Now the statement follows from Lemma 5.5. □

If F_k is the free group on k generators then we have the following canonical weak equivalences:

$$(1) \qquad\qquad \mathcal{G}_\infty(k^+, 1^+) \simeq \Omega\bigvee_k S^1 \simeq \Omega B F_k \simeq F_k$$

It follows that the theory $\mathcal{G}_\infty$ is weakly equivalent as a simplicial category to the discrete theory of groups.

Theorem 5.8 *The category $(\mathcal{S}_*^{\mathcal{G}_\infty})_{\mathrm{Badz}}$ is Quillen equivalent to the category of simplicial groups. In particular, the homotopy category of homotopy ∞–nilpotent groups is equivalent to the homotopy category of loop spaces.*

Proof This follows readily from the equivalences (1) and Theorem 3.5. □

In [6] Badzioch, Chung and Voronov define a theory $\mathcal{T}_n$ such that its homotopy algebras are exactly n–fold loop spaces.

Definition 5.9 Set $\mathcal{T}_n(k^+) = \bigvee_k S^n$ and take as morphisms the derived mapping space

$$\mathcal{T}_n(k^+, \ell^+) = \mathrm{map}^{\mathrm{der}}_{\mathcal{S}_*}\left(\bigvee_\ell S^n, \bigvee_k S^n\right).$$

So $\mathcal{T}_n$ is the opposite of the full subcategory of $\mathcal{S}_*$ given by the finite wedges of n–spheres. The suspension functor induces a morphism $t_n\colon \mathcal{T}_n \to \mathcal{T}_{n+1}$ of simplicial theories:

$$\mathcal{T}_n(k^+, \ell^+) = \left(\Omega^n \bigvee_k S^n\right)^\ell \to \left(\Omega^{n+1} \bigvee_k S^{n+1}\right)^\ell \cong \mathcal{T}_{n+1}(k^+, \ell^+).$$

Theorem 5.10 [6, Theorem 1.1] *A pointed space X is an n–fold loop space if and only if there exists a homotopy $\mathcal{T}_n$–algebra $\underset{\sim}{X}$ with $\underset{\sim}{X}(1^+) \simeq X$.*

Definition 5.11 We obtain a morphism of theories $\vartheta_n \colon \mathcal{T}_n \to \mathcal{G}_1$ induced by the maps

$$\mathcal{T}_n(k^+, \ell^+) = \left(\Omega^n \bigvee_k S^n \right)^\ell \to \left(\underset{s}{\mathrm{colim}}\, \Omega^{n+s} \bigvee_k S^{n+s} \right)^\ell \simeq \mathcal{G}_1(k^+, \ell^+)$$

together with the equation $\vartheta_{n+1} t_n = \vartheta_n$.

Remark 5.12 The theory $\mathcal{G}_1$ is canonically weakly equivalent to the colimit of the sequence

$$\cdots \longrightarrow \mathcal{T}_n \xrightarrow{\ t_n\ } \mathcal{T}_{n+1} \xrightarrow{\ t_{n+1}\ } \cdots$$

in the category of simplicial categories (with fixed set of objects) and hence in the category of simplicial theories.

Theorem 5.13 *A pointed space X is an infinite loop space if and only if there exists a homotopy $\mathcal{G}_1$–algebra $\underset{\sim}{X}$ with $\underset{\sim}{X}(1^+) \simeq X$. The homotopy category of homotopy 1–nilpotent groups is equivalent to the homotopy category of infinite loop spaces.*

Proof Recall Theorem 5.10. By Theorem 3.5 and Remark 5.12 a space X is an infinite loop space if and only if the associated functor

$$\underset{\sim}{X} \colon \Gamma \to \mathcal{S}_*, k^+ \mapsto X^k$$

extends via the maps t_n to a product-preserving functor from $\mathcal{T}_n$ for all $n \geq 0$. The existence of the morphism ϑ_n shows that $\underset{\sim}{X}$ is a homotopy $\mathcal{G}_1$–algebra if and only if $\underset{\sim}{X}$ restricts to a homotopy $\mathcal{T}_n$–algebra for each $n \geq 0$. The equivalence of homotopy categories now also follows. □

Definition 5.14 For $1 \leq n \leq \infty$ the maps $\mathrm{id} \to P_n(\mathrm{id})$ induce morphisms of theories

$$\mathcal{G}_\infty \to \mathcal{G}_n$$

that by pulling back along them induce forgetful functors

$$U_n \colon \mathcal{S}_*^{\mathcal{G}_n} \to \mathcal{S}_*^{\mathcal{G}_\infty}.$$

Denote by RU_n its right derived functor. These forgetful functors possess left adjoints

$$\Phi_n \colon \mathcal{S}_*^{\mathcal{G}_\infty} \to \mathcal{S}_*^{\mathcal{G}_n}.$$

We let
$$L\Phi_n \colon \mathcal{S}_*^{\mathcal{G}_\infty} \to \mathcal{S}_*^{\mathcal{G}_n}$$

be the enriched homotopy left Kan extension, which is obtained by precomposing Φ_n with a projective cofibrant replacement functor on $\mathcal{S}_*^{\mathcal{G}_\infty}$. It is the left derived functor of Φ_n.

Remark 5.15 (1) One can easily prove that the pair (Φ_n, U_n) forms a Quillen pair for the Badzioch model structures on both sides.

(2) Note that $L\Phi_n \underset{\sim}{X}$ depends only on the homotopy ∞–nilpotent group structure even when $\underset{\sim}{X}$ has a homotopy m–nilpotent group structure for some $m \leq \infty$. For $1 \leq n \leq m \leq \infty$ there exist left adjoint functors

$$\Phi_n^m \colon \mathcal{S}_*^{\mathcal{G}_m} \to \mathcal{S}_*^{\mathcal{G}_n}$$

that we will consider in future work. It is not true that $L\Phi_n R U_m \underset{\sim}{X}$ is equivalent to $L\Phi_n^m \underset{\sim}{X}$.

(3) Note that:
$$\mathrm{Fr}_{\mathcal{G}_n} \cong \Phi_n \mathrm{Fr}_{\mathcal{G}_\infty}$$
The object $\mathrm{Fr}_{\mathcal{G}_\infty}(k^+) \cong \mathcal{G}_\infty(k^+, \_)$ is projectively cofibrant. So Corollary 5.7 says:

$$L\Phi_n \mathrm{Fr}_{\mathcal{G}_\infty}(k^+) \simeq \Phi_n \mathrm{Fr}_{\mathcal{G}_\infty}(k^+) \simeq \Omega P_n(\mathrm{id})\Sigma(k^+) \simeq \Omega P_n(\mathrm{id})\left(\bigvee_k S^1\right)$$

The next result, Theorem 5.16, can be interpreted as a generalization of this fact.

Theorem 5.16 *For any space K in $\mathcal{S}_*$ we have a natural weak equivalence*

$$L\Phi_n \mathrm{Fr}_{\mathcal{G}_\infty} K \simeq \Omega P_n(\mathrm{id})(\Sigma K).$$

Equivalently, let $\underset{\sim}{X}$ be a homotopy $\mathcal{G}_\infty$–algebra with $\underset{\sim}{X}(1^+) \simeq \Omega X$ for some reduced simplicial set X. Then there is a natural weak equivalence

$$(L\Phi_n \underset{\sim}{X})(1^+) \simeq \Omega P_n(\mathrm{id})(X).$$

Proof By observing that $\mathrm{Fr}_{\mathcal{G}_\infty}(K) = \Omega\Sigma K$ and setting $X = \Sigma K$ the first claim translates into the second one.

In the case $X = \bigvee_k S^1$ the associated homotopy $\mathcal{G}_\infty$–algebra $\underset{\sim}{X}$ is given by

$$\underset{\sim}{X}(\ell^+) \cong \left(\Omega \bigvee_k S^1\right)^\ell \simeq \mathcal{G}_\infty(k^+, \ell^+),$$

while
$$\mathcal{G}_n(k^+, 1^+) \simeq \Omega P_n(\mathrm{id})\left(\bigvee_k S^1\right).$$

Representable functors are projectively cofibrant and (homotopy) left Kan extension preserves representable functors. So we have an equivalence:

$$L\Phi_n \mathcal{G}_\infty(k^+, \_) \simeq \mathcal{G}_n(k^+, \_)$$

This proves the case $X = \bigvee_k S^1$. Now we observe that every reduced finite simplicial set X is weakly equivalent to the realization of a bisimplicial set $X_\bullet$, which consists degreewise of a finite wedge of copies of the circle S^1: for $k \geq 0$ let X_k be the set of k–simplices of X, and consider X_k as a discrete simplicial set. Set $Y_k = X_{k+1} \wedge S^1$ and recall $X_0 = *$. Then we have canonical equivalences:

$$X \simeq |k \mapsto X_k| \simeq |k \mapsto Y_k|$$

Here $|\_|$ denotes the realization of a bisimplicial set. The statement now follows from Theorem A.2 applied with $F = \Omega P_n(\mathrm{id})$ and $\rho = q = 1$. $\square$

6   The lower central series of the loop group

Definition 6.1   Let G be a group. For subgroups H and K of G let $[H, K]$ denote the normal subgroup generated by elements of the form $h^{-1}k^{-1}hk$ where $h \in H$ and $k \in K$. The *lower central filtration* for G is defined in the following inductive way: Let

$$\Gamma_1 G = G \text{ and } \Gamma_{n+1}G = [G, \Gamma_n G].$$

We obtain a filtration of G by normal subgroups with an associated tower:

$$
\begin{array}{ccccccc}
G/\Gamma_2 G & \longleftarrow & G/\Gamma_3 G & \longleftarrow & G/\Gamma_4 G & \longleftarrow & \cdots \\
\| & & \uparrow & & \uparrow & & \\
G/[G,G] & & \Gamma_2 G/\Gamma_3 G & & \Gamma_3 G/\Gamma_4 G & &
\end{array}
$$

This is the *lower central series* of G. A group G is called n–nilpotent if $\Gamma_{n+1}G = 0$.

Definition 6.2   For an abelian group A let

$$\mathrm{Lie}_* A = \bigoplus_{n \geq 1} \mathrm{Lie}_n A$$

be the free graded Lie algebra on A.

Remark 6.3   The Poincaré–Birkhoff–Witt theorem [32, I.4.3] says that there is a natural isomorphism of abelian groups

$$\mathrm{Lie}_n(G/[G,G]) \cong \Gamma_n G/\Gamma_{n+1}G$$

for every free group G. In fact, this group is free abelian on generators given by a Hall basis of basic commutators of weight n over the generators of G [21].

Remember that $\mathcal{S}_0$ denotes the category of reduced simplicial sets and $s\mathrm{Gr}$ the category of simplicial groups. Let $G\colon \mathcal{S}_0 \to s\mathrm{Gr}$ be Kan's loop group functor. We can apply the functors from the lower central series degreewise.

Definition 6.4 Let $\Gamma^n\colon \mathcal{S}_0 \to \mathcal{S}_*$ be the functor given by

$$\Gamma^n X = B\left(GX/\Gamma_{n+1}GX\right).$$

The functor $\bar{\Gamma}^n\colon \mathcal{S}_0 \to \mathcal{S}_*$ will be given by

$$\bar{\Gamma}^n X = B\left(\Gamma_n GX/\Gamma_{n+1}GX\right).$$

Remark 6.5 The loop group is a free simplicial group. It follows from a theorem by Dold [13] that both functors Γ^n and $\bar{\Gamma}^n$ preserve weak equivalences. Moreover with Remark 6.3 we have a formula:

$$\bar{\Gamma}^n X \simeq B\mathrm{Lie}_n(\widetilde{\mathbb{Z}}X)$$

Here $\widetilde{\mathbb{Z}}X = \mathbb{Z}X/\mathbb{Z}*$ is the reduced free simplicial abelian group on X.

Remark 6.6 It is proved by Curtis [12] that for a simply connected space X the map

$$X \simeq BGX \to B(GX/\Gamma_n GX) = \Gamma^{n-1}X$$

is $\{\log_2 n\}$–connected where $\{a\}$ is the least integer $\geq a$. If X is merely connected, the tower $\{\Gamma^n(X)\}_{n\in\mathbb{N}}$ converges to the Bousfield–Kan completion $\mathbb{Z}_\infty(X)$. Compare Remark 7.1 about the Goodwillie tower of the identity.

For $n = 1$ we have $\bar{\Gamma}^1 X = B\widetilde{\mathbb{Z}}X$. This functor is linear, because we have for all $s \geq 0$:

$$\pi_s B\widetilde{\mathbb{Z}}X \cong \widetilde{H}_s X$$

Here $\widetilde{H}_* X$ is the reduced singular homology of the reduced space X. More generally, there is the following lemma.

Lemma 6.7 *The functor $\bar{\Gamma}^n$ is n–excisive.*

Proof Consider for free abelian groups $A_1,\ldots,A_{n+1}$ the cubical diagram

$$P(\underline{n+1}) \to \mathrm{FrAb}, \quad S \mapsto \bigoplus_{i\in\underline{n+1}-S} A_i,$$

where the maps are induced by collapsing summands. The $(n+1)$–st cross effect of the functor $\mathrm{Lie}_n\colon \mathrm{FrAb} \to \mathrm{Ab}$ vanishes because the associated complex

$$k \mapsto \bigoplus_{|S|=k} \mathrm{Lie}_n\left(\bigoplus_{i \in \underline{n+1}-S} A_i \right) =: L_k.$$

is exact. It follows that the composition

$$\bar{\Gamma}^n X = B\mathrm{Lie}_n(\tilde{\mathbb{Z}} X)$$

is n–excisive. □

Remark 6.8 It is not true though that the functor $\bar{\Gamma}^n$ is n–homogeneous. By Curtis' result Remark 6.6 the tower $\{\Gamma^n\}_{n\geq 0}$ converges to the identity on simply connected spaces. This shows that the layers of the tower have to contribute something to the linear part given by reduced homology $\tilde{H}_*$ in order to make it up to the first derivative of the identity given by stable homotopy π_*^{st}.

Corollary 6.9 *The functor Γ^n is n–excisive.*

Proof There is a homotopy fiber sequence

$$\Gamma^n \to \Gamma^{n-1} \to B\bar{\Gamma}^n$$

of functors. By induction the statement follows from Lemma 6.7. □

7 The Goodwillie tower of the identity

We need to gather some remarks on the Goodwillie tower of the identity.

Remark 7.1 The identity functor is 1–analytic [19], which shows that its Goodwillie tower converges on simply connected spaces to the identity. However, on connected spaces X it converges to the Bousfield–Kan completion of X:

(2) $$\mathrm{holim}_n P_n(\mathrm{id})(X) \simeq \mathbb{Z}_\infty X$$

This is proved on the last page of Arone and Kankaanrinta [3].

Definition 7.2 Let $D_n(\mathrm{id})\left(\bigvee_k S^1\right)$ be the homotopy fiber of the map

$$P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to P_{n-1}(\mathrm{id})\left(\bigvee_k S^1\right)$$

in the Goodwillie tower. Then $D_n(\mathrm{id})(\bigvee_k S^1) = \Omega^\infty E_n$ for the following spectrum

$$\left(\partial^n(\mathrm{id}) \wedge \left(\bigvee_k S^1\right)^{\wedge n}\right)_{h\Sigma_n} =: E_n,$$

where $\partial^n(\mathrm{id})$ is the n–th derivative of the identity.

Lemma 7.3 *The rational homology of E_n is concentrated in degree 1.*

Proof There is a Serre spectral sequence:

$$H_i\left(\Sigma_n, H\mathbb{Q}_j\left(\partial^n(\mathrm{id}) \wedge \left(\bigvee_k S^1\right)^{\wedge n}\right)\right) \implies H\mathbb{Q}_{i+j}(E_n)$$

The homology of Σ_n with coefficients in a rational vector space vanishes for $i > 0$ and gives the formula:

$$H\mathbb{Q}_* E_n \cong H\mathbb{Q}_*(\partial^n(\mathrm{id})) \otimes_{\mathbb{Q}[\Sigma_n]} H\mathbb{Q}_*\left(\left(\bigvee_k S^1\right)^{\wedge n}\right).$$

The homology of $(\bigvee_k S^1)^{\wedge n}$ is concentrated in degree n. By the work of Johnson [24] and Arone and Mahowald [4] we know that the spectrum $\partial^n(\mathrm{id})$ is nonequivariantly equivalent to $\bigvee_{(n-1)!} S^{1-n}$ and so has homology concentrated in degree $1-n$. So $1-n+n = 1$, and the statement follows. $\qquad\square$

Lemma 7.4 *The group $\pi_1 P_n(\mathrm{id})(\bigvee_k S^1)$ is nilpotent of degree n.*

Proof We can settle the case $n = 1$ right away:

$$\pi_0 \Omega P_1(\mathrm{id})\left(\bigvee_k S^1\right) \cong \pi_1\left(\Omega^\infty \Sigma^\infty \bigvee_k S^1\right) \cong \mathbb{Z}^k \cong F_k/\Gamma_2 F_k$$

In particular, this group is nilpotent of degree 1. We proceed by induction on n. From the Goodwillie tower we have for each $n \geq 1$ the following exact sequence of groups:

(3)
$$\pi_1 D_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \pi_1 P_{n-1}(\mathrm{id})\left(\bigvee_k S^1\right)$$
$$\to \pi_0 D_n(\mathrm{id})\left(\bigvee_k S^1\right) \cong 0$$

The last group vanishes because of Lemma 7.3. By Goodwillie's results [20] the spaces $D_n(\mathrm{id})(\bigvee_k S^1)$ are infinite loop spaces and the map in the Goodwillie tower is a principal fibration, ie there is a homotopy pullback square

$$
\begin{array}{ccc}
P_n(\mathrm{id})\left(\bigvee_k S^1\right) & \longrightarrow & * \\
\downarrow & & \downarrow \\
P_{n-1}(\mathrm{id})\left(\bigvee_k S^1\right) & \longrightarrow & BD_n(\mathrm{id})\left(\bigvee_k S^1\right)
\end{array}
$$

where $BD_n(\mathrm{id})(\bigvee_k S^1)$ is a delooping of $D_n(\mathrm{id})(\bigvee_k S^1)$ and therefore simply connected. Let

$$
K_n = \mathrm{im}\left(\pi_1 D_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right)\right).
$$

Then the short exact sequence

$$
(4) \qquad 0 \to K_n \to \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \pi_1 P_{n-1}(\mathrm{id})\left(\bigvee_k S^1\right) \to 0
$$

is a central extension. It follows inductively that the group $\pi_1 P_n(\mathrm{id})(\bigvee_k S^1)$ is nilpotent of degree n. $\qquad\square$

Lemma 7.5   *The groups $\pi_s P_n(\mathrm{id})(\bigvee_k S^1)$ are finite for $s \geq 2$.*

Proof   These groups are finitely generated. So it is enough to prove that the groups $\pi_s P_n(\mathrm{id})(\bigvee_k S^1)$ are torsion above degree $s = 1$. We will prove this by induction along the Goodwillie tower where the case $n = 0$ is obvious, because the space is contractible. Next we know by Lemma 7.3 that the rational homology of $D_n(\mathrm{id})(\bigvee_k S^1)$ is concentrated in degree 1. Since $D_n(\mathrm{id})(\bigvee_k S^1)$ is an infinite loop space, a form of the Hurewicz theorem [33, Theorem 9.6.20] tells us that also the groups $\pi_s D_n(\mathrm{id})(\bigvee_k S^1)$ are torsion for $s > 1$. The result now follows from the long exact homotopy sequence of the Goodwillie tower. $\qquad\square$

Corollary 7.6   *There is an isomorphism of groups:*

$$
\pi_1 \operatorname*{holim}_n P_n(\mathrm{id})\left(\bigvee_k S^1\right) \cong \lim_n \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right)
$$

Proof There is the Milnor exact sequence:

$$0 \to \lim_n{}^1 \pi_2 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \pi_1 \operatorname*{holim}_n P_n(\mathrm{id})\left(\bigvee_k S^1\right)$$

$$\to \lim_n \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to 0$$

By Lemma 7.5 the $\lim^1$–term vanishes. $\qquad\square$

The following conjecture is related to the vanishing of $\lim^1 \pi_2 P_n(\mathrm{id})(\bigvee_k S^1)$.

Conjecture 7.7 (Arone–Mahowald–Kuhn) For each prime p the map

$$\pi_s P_{p^n}(\mathrm{id})(S^1)_{(p)} \to \pi_s P_{p^{n-1}}(\mathrm{id})(S^1)_{(p)}$$

is null for $s \geq 2$.

8 Relation to ordinary nilpotent groups

Now we can describe the relation of $\mathcal{G}_n$ to the set-valued theory of ordinary n–nilpotent groups Nil_n, whose k–ary operations are given by the free n–nilpotent group on k generators:

$$\mathrm{Nil}_n(k^+, 1^+) = F_k / \Gamma_{n+1} F_k$$

Here F_k is the free group on k generators. We can exhibit this theory by applying π_0 to the theory of homotopy n–nilpotent groups. First observe that for the case $n = \infty$ the statement

$$\pi_0 \mathcal{G}_\infty(k^+, 1^+) = \pi_1\left(\bigvee_k S^1\right) \cong F_k \cong \mathrm{Nil}_\infty(k^+, 1^+)$$

follows from the Seifert–Van Kampen theorem. This isomorphism has an analogue for finite n. There is a map

$$F_k \cong \pi_1\left(\bigvee_k S^1\right) \to \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \cong \pi_0 \mathcal{G}_n(k^+, 1^+)$$

induced by the natural transformation $\mathrm{id} \to P_n(\mathrm{id})$, which factors through

$$\alpha_n \colon F_k / \Gamma_{n+1} F_k \to \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right),$$

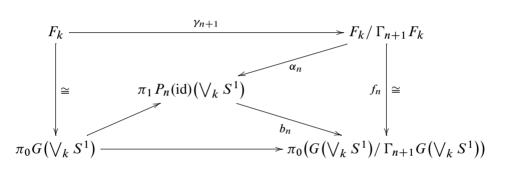because the target is n–nilpotent by Lemma 7.4.

Theorem 8.1 *We have an isomorphism of groups:*

$$\alpha_n \colon \operatorname{Nil}_n(k^+, 1^+) = F_k / \Gamma_{n+1} F_k \xrightarrow{\;\cong\;} \pi_1 P_n(\operatorname{id})\left(\bigvee_k S^1\right) \cong \pi_0 \mathcal{G}_n(k^+, 1^+)$$
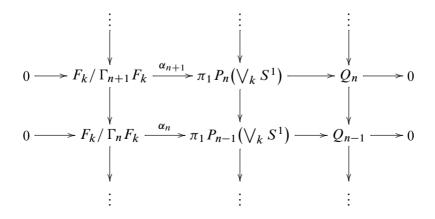
This induces an isomorphism of categories $\operatorname{Nil}_n \cong \pi_0 \mathcal{G}_n$.

Proof We show first that α_n is injective by constructing a left inverse β_n. According to Corollary 6.9 the functor Γ^n is n–excisive. So there is a natural transformation $P_n(\operatorname{id}) \to \Gamma^n$ under the identity functor. If we evaluate this diagram on $\bigvee_k S^1$ and apply $\pi_1 \cong \pi_0 G$, we obtain a map b_n making the following diagram commutative:

Let $\beta_n = f_n^{-1} b_n$. It follows that $\beta_n \alpha_n \gamma_{n+1} = \gamma_{n+1}$. Since γ_{n+1} is the universal map into an n–nilpotent group, we have $\beta_n \alpha_n = \operatorname{id}$.

But the map α_n is also surjective. Let Q_n be the quotient of α_n, ie the pointed set of left cosets. We obtain a short exact sequence of towers:

From (3) it follows that the vertical maps are surjective for $n \geq 1$. So all $\lim^1$–terms vanish. In the limit we obtain a short exact sequence:

$$0 \to \lim_n F_k / \Gamma_{n+1} F_k \to \lim_n \pi_1 P_n(\mathrm{id})\left(\bigvee_k S^1\right) \to \lim_n Q_n \to 0$$

We can combine the weak equivalence (2) with the isomorphism from Corollary 7.6 to conclude that $\lim_n Q_n \cong *$. In turn we have $Q_n = *$ for all $n \geq 1$, since all tower maps are surjective. So each α_n is an isomorphism. $\qquad\qquad\square$

9 Values of n–excisive functors

We will prove that functors of the form ΩF with F n–excisive naturally take values in the category of homotopy n–nilpotent groups. We take this as a justification of the usefulness of the notion of homotopy n–nilpotent groups.

We need to compose functors. But two functors F and G in $\mathcal{F}$ cannot be composed directly. However, we can extend the functor $F\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}_*$ to a functor $\mathcal{S}_* \to \mathcal{S}_*$ by enriched left Kan extension. By abuse of language we will denote this functor again by F. Then the composition $F \circ G$ is well-defined.

Observe that the functor $\_ \circ G\colon \mathcal{F} \to \mathcal{F}$ commutes with finite limits. And there is a functor

$$P_n(\_ \circ G)\colon \mathcal{F} \to \mathcal{F}, F \mapsto P_n(F \circ G),$$

which also commutes with finite limits. The map $F \to P_n F$ induces a map

$$(5) \qquad\qquad P_n(F \circ G) \to P_n\big((P_n F) \circ G\big)$$

under $F \circ G$.

Lemma 9.1 *The map (5) is an objectwise weak equivalence.*

This is proved directly by Michael Ching in [10, Proposition 6.1(1)]. It is also a consequence of the chain rule (6) which is now a theorem by Arone and Ching [1].

For the next theorem we point out that the language of simplicial algebraic theories and (homotopy) algebras over them makes sense in any cartesian closed symmetric monoidal $\mathcal{S}_*$–model category. The category $\mathcal{F}$ of $\mathcal{S}_*$–functors from $\mathcal{S}_*^{\mathrm{fin}}$ to $\mathcal{S}_*$ equipped with the categorical product as the monoidal structure is such an example.

Theorem 9.2 *Let F be a functor of the form $F = \Omega G$ with an n–excisive functor G in $\mathcal{F}$. Then F is a homotopy $\mathcal{G}_n$–algebra in $\mathcal{F}$.*

Proof A homotopy $\mathcal{G}_n$–algebra in $\mathcal{F}$ is an $\mathcal{S}_*$–functor $\mathcal{G}_n \to \mathcal{F}$ preserving products up to weak equivalence. Given F as above we define the object function of a functor $\underset{\sim}{F} \colon \mathcal{G}_n \to \mathcal{F}$ by

$$k^+ \mapsto F^k = \underset{\sim}{F}(k^+).$$

To see that this actually yields an $\mathcal{S}_*$–enriched functor we observe that this is the object function of the following composition of functors: First we take the full inclusion functor $I_n \colon \mathcal{G}_n \to \mathcal{F}$

$$k^+ \mapsto I_n(k^+) = \prod_k \Omega P_n(\mathrm{id}).$$

Then we precompose with G:

$$\prod_k \Omega P_n(\mathrm{id}) \mapsto \prod_k \Omega P_n(\mathrm{id}) \circ G$$

Finally we apply $P_n \colon \mathcal{F} \to \mathcal{F}$:

$$\prod_k \Omega P_n(\mathrm{id}) \circ G \mapsto P_n\left(\left(\prod_k \Omega P_n(\mathrm{id})\right) \circ G\right)$$

Note that the inclusion I_n as well as the functors $\_ \circ G$ and P_n commute with products up to weak equivalence, and the latter two functors commute both up to homotopy with Ω. We compute using Lemma 9.1:

$$P_n\left(\left(\prod_k \Omega P_n(\mathrm{id})\right) \circ G\right) \simeq \prod_k \Omega P_n\big(P_n(\mathrm{id}) \circ G\big) \simeq \prod_k \Omega P_n(G) \simeq F^k$$

This shows at the same time that the object function of $\underset{\sim}{F}$ is part of an $\mathcal{S}_*$–functor $\mathcal{G}_n \to \mathcal{F}$ and that this functor preserves products up to weak equivalence. Hence $\underset{\sim}{F}$ is indeed a homotopy $\mathcal{G}_n$–algebra in $\mathcal{F}$. $\qquad\square$

Corollary 9.3 *Let F be a functor of the form $F = \Omega G$ with an n–excisive functor G in $\mathcal{F}$. Then for any K in $\mathcal{S}_*^{\mathrm{fin}}$ the space $F(K)$ is a homotopy n–nilpotent group.*

Proof For any space K the evaluation functor $\mathrm{Ev}_K \colon \mathcal{F} \to \mathcal{S}_*$ preserves products. So the statement is a corollary of Theorem 9.2. $\qquad\square$

Work in progress [8] by the two authors shows that the converse of Corollary 9.3 is true: one can associate to any homotopy n–nilpotent group $\underset{\sim}{X}$ an n–excisive functor F such that $F(S^0) \simeq \mathrm{Ev}_{1^+}\underset{\sim}{X}$.

The following statement is also a formal consequence of Theorem 9.2.

Corollary 9.4 *Let G be a functor of the form $G = \Omega H$ with an n–excisive functor H in $\mathcal{F}$. Let F be an arbitrary functor in $\mathcal{F}$. Then $\mathcal{F}(F, G)$ is a homotopy n–nilpotent group naturally in F and G.*

Corollary 9.4 says that the full subcategory of $\mathcal{F}$ of functors of the form ΩF with F being n–excisive is enriched over homotopy n–nilpotent groups. In particular, by Theorem 8.1 we obtain the following statement.

Corollary 9.5 *The full subcategory of the homotopy category of $\mathcal{F}$ given by functors of the form ΩF with F n–excisive is enriched over n–nilpotent groups.*

This generalizes the fact that in the case $n = 1$ the stable homotopy category is enriched over abelian groups.

We would like to finish with two more examples. In work in progress [8] we are going to study closer classifying spaces of homotopy n–nilpotent groups. This point of view is more convenient in the following examples.

Definition 9.6 A space is called a *homotopy n–nilpotent space* if it is given as the delooping or classifying space of a homotopy n–nilpotent group.

Remark 9.7 One can easily see that a homotopy n–nilpotent space is exactly a homotopy algebra over the simplicial theory $\mathcal{P}_n$ given by

$$\mathcal{P}_n(k^+) := \prod_k P_n(\mathrm{id})^{\mathrm{inj}} \subset \mathcal{F}.$$

A proof will appear in [8].

Example 9.8 Let X be a pointed connected space whose homotopy groups $\pi_s X$ vanish unless $k + 1 \leq s \leq (n+1)k$ for some $n, k \geq 1$. In the case $n = 1$ it follows from the classical Blakers–Massey theorem that $X \simeq \mathrm{Post}_{2k} \Omega^\infty \Sigma^\infty X$. Here $\mathrm{Post}_s X$ denotes the s–th Postnikov section of X which preserves infinite loop spaces. Hence X is an infinite loop spaces. For general n it follows from the generalized Blakers–Massey theorem due to Ellis and Steiner [17] or Goodwillie [19, Theorem 2.3], that

$$X \simeq \mathrm{Post}_{(n+1)k} P_n(\mathrm{id})(X).$$

By Corollary 9.3 the space $P_n(\mathrm{id})X$ has a structure as homotopy n–nilpotent space. Postnikov sections of homotopy n–nilpotent groups remain homotopy n–nilpotent groups. This can be proved eg by observing that the Moore model for Postnikov sections [18, VI.3.4.] commutes with products on the nose. It follows, that X can be given the structure of a homotopy n–nilpotent space. Its loop space ΩX is canonically a homotopy n–nilpotent group.

For the next example we rely on the chain rule in Goodwillie's calculus of homotopy functors. Let $\partial^j F$ denote the j–th derivative of F at $*$ and note that all the derivatives $\partial^* F$ together form a symmetric sequence in spectra. It is known by Ching [11] that the derivatives of the identity $\partial^* \mathrm{id}$ form an operad. In [1] Arone and Ching prove that the derivatives of any homotopy functor F in $\mathcal{F}$ form a left and right module over $\partial^* \mathrm{id}$ and that the canonical map

$$(6) \qquad\qquad \partial^*(G \circ F) \simeq \partial^* G \circ_{\partial^* \mathrm{id}} \partial^* F$$

is a weak equivalence. The circle product on the right side is induced by the composition product of symmetric sequences. For a different approach in the spectrum-valued case we refer to Ching [10]. A simple application is the following statement.

Corollary 9.9 Let $n, k \geq 1$. Let $G \to H$ be an n–excisive equivalence in $\mathcal{F}$ and F be a functor with $\partial^j F \simeq *$ for $j \leq k$. Then the induced map

$$G \circ F \to H \circ F$$

is an $(nk + n + k)$–excisive equivalence.

Example 9.10 Let F be a functor in $\mathcal{F}$ with $\partial^j F \simeq *$ unless $k + 1 \leq j \leq nk + n + k$ for some $n, k \geq 1$. Then F factors through the category of homotopy n–nilpotent spaces.

As in the proof of Theorem 9.2 we will use homotopy $\mathcal{P}_n$–algebras in $\mathcal{F}$. We note that the s–th Goodwillie section $P_s L$ of a functor L in $\mathcal{F}$ remains a homotopy $\mathcal{P}_n$–algebra if L itself was one, because P_s commutes with finite homotopy limits in $\mathcal{F}$ by [20, 1.7.]. The map $\mathrm{id} \to P_n(\mathrm{id})$ is an n–excisive equivalence. Precomposing with F yields an equivalence

$$F \to P_{nk+n+k} F \to P_{nk+n+k}\big(P_n(\mathrm{id}) \circ F\big)$$

by Corollary 9.9. By the proof of Corollary 9.3 the functor $P_n(\mathrm{id}) \circ F$ is a homotopy $\mathcal{P}_n$–algebra, and so is F. In particular F takes values in homotopy n–nilpotent spaces. By looping the setup we return to homotopy n–nilpotent groups.

The case $n = 1$ of the previous example was proved in [2, Theorem 4.2].

Appendix A   An auxiliary lemma about realizations

Let us quickly review some notions introduced by Goodwillie in [19, Definition 4.1].

Definition A.1 A functor F in $\mathcal{F}$ is said to satisfy condition $E_n(c, \kappa)$ if there exist constants c and κ both in $\mathbb{Z}$ and $\kappa \geq -1$ such that for every strongly homotopy cocartesian $(n + 1)$–cube $\mathcal{C}$ whose maps $\mathcal{C}(\varnothing) \to \mathcal{C}(\{s\})$ are k_s–connected with $k_s \geq \kappa$ the induced map

$$F\big(\mathcal{C}(\varnothing)\big) \to \underset{S \in P_0(\underline{n+1})}{\mathrm{holim}}\, F\big(\mathcal{C}(S)\big)$$

is $(-c + \Sigma k_s)$–connected. A functor is *stably n–excisive* if it satisfies $E_n(c, \kappa)$ for some c and κ. A functor is *ρ–analytic* if there exists a constant $q \in \mathbb{Z}$ such that F satisfies condition $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

The first author learned the next statement from explanations of the second author and from the unpublished version [28] of Andrew Mauer-Oats' thesis. The published version is [29].

Theorem A.2 *Let F be a reduced finitary ρ–analytic functor from spaces to spaces. If $X_\bullet$ is a simplicial object in k–connected spaces with $k \geq \max(\rho, -q)$ then the canonical map*

$$|FX_\bullet| \xrightarrow{\simeq} F|X_\bullet|$$

is a weak equivalence.

A sketch of the proof goes as follows: First one observes that homogeneous functors with connective coefficient spectrum commute with realizations. Then the theorem follows by induction up the Goodwillie tower. All along one checks that the connectivity estimates allow one to apply theorem [9, B.4] that gives sufficient conditions for the realization functor to commute with pullbacks.

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