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Quasimorphisms and laws

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Stable commutator length vanishes in any group that obeys a law.

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If G is a group and g is an element of the commutator subgroup $[G,G]$, the *commutator length* of g, denoted $\mathrm{cl}(g)$, is the least number of commutators in G whose product is g. The *stable commutator length*, denoted $\mathrm{scl}(g)$, is the limit $\mathrm{scl}(g) := \lim_{n\to\infty} \mathrm{cl}(g^n)/n$.

A group G is said to obey a *law* if there is a free group F (which may be assumed to have finite rank) and a nontrivial element $w \in F$ so that for every homomorphism $\rho\colon F \to G$, we have $\rho(w) = \mathrm{id}$. For example, abelian (or, more generally, nilpotent or solvable) groups obey laws. The free Burnside groups $B(m,n)$ with $m \ge 2$ generators and odd exponents $n \ge 665$ are perhaps the best known examples of non-amenable groups that obey laws; see for example Adyan [1].

The point of this note is to prove the following:

Main Theorem *Let G be a group that obeys a law. Then $\mathrm{scl}(g) = 0$ for every $g \in [G,G]$.*

The proof is very short, given some basic facts about stable commutator length, which we recall for the convenience of the reader. A basic reference is Bavard's paper [2] or the author's monograph [3], especially Chapter 2.

Definition 1 A *homogeneous quasimorphism* on a group G is a function $\phi\colon G \to \mathbb{R}$ that restricts to a homomorphism on every cyclic subgroup, and for which there is a least number $D(\phi) \ge 0$ (called the *defect*) so that for any $g, h \in G$ there is an inequality $|\phi(gh) - \phi(g) - \phi(h)| \le D(\phi)$.

The defect satisfies the following formula:

Lemma 2 [2, Lemma 3.6] or [3, Lemma 2.24] *Let ϕ be a homogeneous quasimorphism. Then there is an equality*

$$\sup_{g,h\in G} |\phi([g,h])| = D(\phi).$$

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Bavard duality (see [2] or [3, Theorem 2.70]) says that for any $g \in [G,G]$, there is an equality $\mathrm{scl}(g) = \sup_\phi \phi(g)/2D(\phi)$ where the supremum is taken over all homogeneous quasimorphisms ϕ with nonzero defect. In particular, scl is nontrivial on G if and only if G admits a homogeneous quasimorphism with nonzero defect.

On the other hand, there is a topological formula for scl. Let X be a space with $\pi_1(X) = G$, and let $\gamma\colon S^1 \to X$ be a free homotopy class representing the conjugacy class of $g \in G$. If Σ is a compact, oriented surface without sphere or disk components, a map $f\colon \Sigma \to X$ is *admissible* if the map $\partial f\colon \partial\Sigma \to X$ can be factorized as $\partial\Sigma \xrightarrow{d} S^1 \xrightarrow{\gamma} X$. For an admissible map, define $n(\Sigma)$ by the equality $d_*[\partial\Sigma] = n(\Sigma)[S^1]$ in H_1; i.e. $n(\Sigma)$ is the degree with which $\partial\Sigma$ wraps around γ. By reversing the orientation of Σ if necessary, we assume $n(\Sigma) \geq 0$. With this notation, one has the following formula:

Lemma 3 [3, Proposition 2.10] *With notation as above,*

$$\mathrm{scl}(g) = \inf_\Sigma \frac{-\chi(\Sigma)}{2n(\Sigma)}$$

where χ denotes Euler characteristic, and the infimum is taken over all compact, oriented surfaces and all admissible maps.

Notice that both $\chi(\cdot)$ and $n(\cdot)$ are multiplicative under finite covers.

Proof of the Main Theorem Suppose that G obeys a law. Then there is a free group F and a nontrivial word $w \in F$ so that any homomorphism from F to G sends w to id. Let F_2 be free on generators x, y. We can embed F in F_2, and express w as a word v in the generators x, y. Hence any homomorphism from F_2 to G sends v to id.

Let X be a space with $\pi_1(X) = G$. Let Σ be a once-punctured torus. We choose generators for $\pi_1(\Sigma)$, and identify this group with $F_2 = \langle x, y\rangle$. Let α be a loop on Σ whose free homotopy class represents the conjugacy class of v. Then any continuous map $f\colon \Sigma \to X$ sends α to a null-homotopic loop.

Now suppose contrary to the theorem that scl does not vanish on $[G,G]$. By Bavard duality there is a homogeneous quasimorphism ϕ with nonzero defect. Scale ϕ to have $D(\phi) = 1$. Then by Lemma 2, for any $\epsilon > 0$ there are elements g, h in G with $\phi([g,h]) \geq 1 - \epsilon$, and consequently $\mathrm{scl}([g,h]) \geq 1/2 - \epsilon/2$ by Bavard duality.

Let $\gamma\colon S^1 \to X$ be a loop representing the conjugacy class of $[g,h]$. There is a map $f\colon \Sigma \to X$ whose boundary represents the free homotopy class of γ. As above, the loop α on Σ maps to a null-homotopic loop in X. By Scott [4], there is a finite cover

$\widetilde{\Sigma}$ of Σ of degree d (depending on α), so that some lift $\widetilde{\alpha}$ of α is homotopic to an embedded loop α'. Composing the covering map with f gives a map $\widetilde{f}\colon \widetilde{\Sigma} \to X$ for which $\widetilde{f}(\alpha')$ is null-homotopic in X. Since α' is embedded, we can compress $\widetilde{\Sigma}$ along α' to produce a new surface Σ' mapping to X by f'. The map f' is admissible for γ, and satisfies $n(\Sigma') = d$. Moreover, $\chi(\widetilde{\Sigma}) = -d$, and $\chi(\Sigma') = 2 - d$. Consequently, by Lemma 3, we have $\mathrm{scl}([g,h]) \le 1/2 - 1/d$.

Since d is fixed (depending only on the law satisfied by G) but ϵ is arbitrary, we obtain a contradiction. Hence scl vanishes identically on $[G,G]$, as claimed. $\square$

Remark 4 The statement of the Main Theorem may be rephrased positively as saying that if scl is nonzero on G, then for any positive integer n, there are homomorphisms $F_2 \to G$ which are injective on the ball of radius n.

If w is a word in a free group F, define a *w–word* in G to be the image of w under a homomorphism $F \to G$. Let $G(w)$ be the subgroup of G generated by w–words. The *w–length* of $g \in G(w)$, denoted $\mathrm{l}(g|w)$, is the smallest number of w–words and their inverses whose product is g (commutator length is the case $w = xyx^{-1}y^{-1} \in \langle x, y\rangle$), and the *stable w–length*, denoted $\mathrm{sl}(g|w)$ is $\mathrm{sl}(g|w) := \lim_{n\to\infty} \mathrm{l}(g^n|w)/n$.

Question 5 Is there an example of a group that obeys a law, but for which $\mathrm{sl}(\cdot|w)$ is nontrivial for some w?

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